ASSOCIATED ACTIONS AND UNIQUENESS OF COCYCLES

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Abstract. We use the methods of orbital ergodic theory to show the existence of many strange cocycles. Any conservative ergodic flow is the associated action for some recurrent cocycle of an ergodic probability preserving transformation, and this cocycle is determined uniquely up to cohomology via orbit equivalence. As in Algorithms, Dynamics and Fractals, Ed. Y. Takahashi, Plenum Press, New York. 1995

§0 Introduction

Let $(X, \mathcal{B}, m)$ be a non-atomic Lebesgue probability space, and let Z be a countable group of invertible, nonsingular transformations of X. Let G be a locally compact topological group G. A (G-valued) *cocycle* of Z is a measurable function $\varphi : Z \times X \to G$ which satisfies the *cocycle equation*:

$$\varphi(zz', x) = \varphi(z, z'x)\varphi(z', x) \qquad (z, z' \in Z,\ x \in X).$$

If $Z = \mathbb{Z} = \{S^n : n \in \mathbb{Z}\}$ where $S : X \to X$ is an invertible nonsingular transformation of X, then given any measurable function $\varphi : X \to G$ we may define $\varphi : \mathbb{Z} \times X \to G$ (the *cocycle* of S) by

$$\varphi(n, x) = \begin{cases} \varphi(S^{n-1}x)\varphi(S^{n-2}x)\ldots\varphi(x) \quad n \geq 1, \\ e \quad n = 0, \\ \varphi(S^n x)\varphi(S^{n+1}x)\ldots\varphi(S^{-1}x) \quad n \leq -1. \end{cases}$$

The *skew product* action of Z on $X \times G$ is defined by $z_\varphi(x, y) = (zx, \varphi(x)y),\ (z \in Z)$. See [Sch].

The transformations $\{z_\varphi : z \in Z\}$ are nonsingular with respect to the measure $m \times m_G$ where m_G is left Haar measure on G, and the cocycle φ is called *recurrent* if the skew product action Z_φ is conservative.

Recall that the *invariant factor* of a countable group of non-singular transformations $\mathcal{T}$ of the Lebesgue probability space $(Y, \mathcal{A}, p)$ is a Lebesgue probability space $\Omega(\mathcal{T}) = (\Omega, \mathcal{A}', p')$ equipped with a factor map $\pi : Y \to \Omega$ such that $p \circ \pi^{-1} \sim p'$, and

$$\pi^{-1}\mathcal{A}' = \mathcal{I}(\mathcal{T}) := \{A \in \mathcal{A} : \tau^{-1}A = A \mod 0 \ \forall\ \tau \in \mathcal{T}\}.$$

The invariant factor is isomorphic to the measure space of ergodic components of p (or *ergodic decomposition* of $\mathcal{T}$).

Define a G-action on $X \times G$ by $Q_g(x, y) = (x, yg)$. Clearly,

$$z_\varphi \circ Q_g = Q_g \circ z_\varphi, \ \forall\ g \in G, \ z \in Z,$$

and so $Q_g\mathcal{I}(Z_\varphi) = \mathcal{I}(Z_\varphi)$, whence Q acts on the invariant factor of Z_φ. It follows from the ergodicity of Z on X, that $\{z_\varphi \circ Q_g : g \in G, \ z \in Z\}$ acts ergodically on $X \times G$, whence Q is ergodic on $\Omega(Z_\varphi)$.

We call this action of Q on Ω, the *associated group action of the cocycle*, (the G-action of φ). It seems to have been considered first in [Ma], and is called the *Mackey range* of the cocycle in [B-G],[G-S1], and [G-S2].

Remarks

(1) The associated $\mathbb{R}$-action of a non-negative cocycle of an automorphism is precisely the so called *special flow* (see §3) built under the cocycle with that automorphism as base. In this case, the cocycle is non-recurrent. The concept of associated group action has been used in [Ka] to generalise the notion of special flow for multidimensional group actions.

(2) In the setting where S is an invertible, non-singular transformation, and

$$\varphi = \log S', \qquad (S' := \frac{dm \circ S}{dm}),$$

the $\mathbb{R}$-action of φ has been considered and called the *associated*, or *Krieger* flow. The cocycle φ is recurrent if S is conservative.

Theorem 1. **[B-G]** *Let S be an ergodic probability preserving automorphism, and let G be a locally compact, second countable amenable group.*

For any non-singular, conservative ergodic free action $T = \{T_g : g \in G\}$, there is a recurrent G-valued cocycle of $\{S^n : n \in \mathbb{Z}\}$ whose action is isomorphic to T.

For the convenience of the reader, we sketch a proof of theorem 1 (different from that of [B-G]) for the case $G = \mathbb{R}$. Our main result is that when the $G = \mathbb{R}$, the cocycle appearing in theorem 1 is unique up to cohomology via orbit equivalence. This strengthens theorem 5.12 in [B-G]

Thorem 2. (Uniqueness of cocycle) *Let S and S' be ergodic probability preserving automorphisms of X and X' respectively, and suppose that φ and φ' are two cocycles having associated actions which are conjugate, conservative, ergodic, nonsingular free $\mathbb{R}$ actions.*

Then $\exists$ an orbit equivalence *from S to S', i.e. a measure preserving map $\pi : X \to X'$ such that*

$$\{S'^n\pi(x) : n \in \mathbb{Z}\} = \{\pi S^n(x) : n \in \mathbb{Z}\} \quad a.e.x,$$

and a measurable function $g : X \to \mathbb{R}$ such that

$$\varphi(x) - \varphi'(\pi(x)) = g(x) - g(Sx). \quad a.e.x$$

After completing this article, we were informed by Golodets and Sinel'shchikov that they have also obtained theorem 2 by a different method in [G-S2].

Remarks

(1) It is not hard to show that if two cocycles are cohomologous via orbit equivalence (as in theorem 2), then their actions are isomorphic.
(2) Theorems 1, and 2 may be considered as a "measure preserving analogue" of Krieger's theorem ([Kri], see also [K-W], [H-O1]).
(3) Cocycles having trivial, transitive, and periodic associated actions are also unique up to cohomology via orbit equivalence ([G-S1]). Indeed, in case the associated actions in the theorem are transitive, it is well known (see [Sch]) that both cocycles are coboundaries, and we may assume that $\varphi', \varphi \equiv 0$; the result now following from Dye's theorem [Dye].

Let $\widehat{G}$ denote the group of *characters* of G (continuous homomorphisms $G \to \mathbb{T}$). Clearly if φ is a G-valued cocycle, and $a \in \widehat{G}$, then $a \circ \varphi$ is a $\mathbb{T}$-valued cocycle. It is natural to ask whether, for a recurrent φ of an ergodic probability preserving transformation S, the ergodicity of $S_{a\circ\varphi}$ $\forall a \in \widehat{G}$ implies the ergodicity of S_φ.

This is true for compact, Abelian G, (see [F] and [P]). The (apparently well known fact) that it is false for $G = \mathbb{R}$ follows from theorem 1.

Example For any ergodic probability preserving automorphism S, there is a recurrent cocycle $\varphi : X \to \mathbb{R}$ such that $S_{a\circ\varphi}$ is ergodic for every $a \in \mathbb{R}$, but S_φ is not ergodic.

The example is obtained by choosing a recurrent $\varphi : X \to \mathbb{R}$ with a non-trivial, weakly mixing associated $\mathbb{R}$-action. This means that S_φ is not ergodic (as φ's $\mathbb{R}$-action is non-trivial), and also that there are no measurable solutions $a \in \mathbb{R}, \ \psi : X \to \mathbb{T}$ of the functional equation

$$e^{ia\varphi} = \frac{\psi \circ S}{\psi},$$

as such a solution would give rise to an eigenvalue of φ's associated $\mathbb{R}$-action, which is assumed weakly mixing. It follows from [F] that $S_{a\circ\varphi}$ is ergodic for every $a \in \mathbb{R}$. An analogous example can be constructed for the case $G = \mathbb{Z}$.

The organisation of the paper is as follows. After reviewing some definitions from orbital ergodic theory in §1, we sketch a proof of theorem 1 in §2. In §3, we study special flow representations of the associated action, laying the foundations for our proof of theorem 2 in §4. In §4, as an introduction to the ideas involved in the proof (copying towers in an appropriate manner), we first sketch a proof of the uniqueness of ergodic cocycles (first established in [G-S1]). Theorem 2 is then reduced to a "relative Dye theorem" which is more easily established.

§1 Nonsingular equivalence relations

Let $(X, \mathcal{B}, m)$ be a non-atomic standard probability space. A measurable equivalence relation $\mathcal{R} \in \mathcal{B} \otimes \mathcal{B}$ on X is said to be *countable* if $\mathcal{R}_x := \{y \in X : (x, y) \in \mathcal{R}\}$ is countable $\forall \ x \in X$. For example, if Z is a countable group of nonsingular transformations of X, then the equivalence relation *generated* by Z,

$$\mathcal{R}(Z) = \{(x, zx) : x \in X, \ z \in Z\}$$

is countable. It is also "nonsingular" in a reasonable sense which we proceed to explain.

A *partial* non-singular transformation of $(X, \mathcal{B}, m)$ is a triple (ϕ, A, B) where $A, B \in \mathcal{B}$ and $\phi : A \to B$ is an invertible, m-non-singular transformation. It will be natural to sometimes write $\phi = (\phi, A, B)$ and $A = \operatorname{Dom}\phi, \ B = \operatorname{Im}\phi$. A *partial transformation of* $\mathcal{R}$ is a partial transformation ϕ satisfying $(x, \phi x) \in \mathcal{R} \ \forall \ x \in \operatorname{Dom}\phi$. The collection of partial transformations of $\mathcal{R}$ is denoted by $[\mathcal{R}]_*$, and is known as the *groupoid* of $\mathcal{R}$. The *full group* of $\mathcal{R}$ is that subset $[\mathcal{R}] = \{\tau \in [\mathcal{R}]_* : \operatorname{Dom}\tau = \operatorname{Im}\tau = X\}$.

The measurable, countable equivalence relation $\mathcal{R}$ is called *nonsingular* if $\mathcal{R} = \{(x, \phi x) : \phi \in [\mathcal{R}]_*, \ x \in \operatorname{Dom}\phi\}$. It is known (see [Fe-Mo]) that every countable, nonsingular equivalence relation is generated by a countable group of nonsingular transformations. The notions of *conservativity, ergodicity*, and *invariant factor* are defined with reference to the generating group of nonsingular transformations.

A measurable equivalence relation is said to be *of type* II_1 if all its partial transformations are probability preserving, and is said to be *hyperfinite* if it is generated by a single nonsingular automorphism. In particular, if $\mathcal{R}$ is a measurable, countable hyperfinite equivalence relation of type II_1 on X, then
$\exists$ an invertible, probability preserving transformation $S : X \to X$ such that $\mathcal{R} = \{(x, S^n x) : x \in X,\ n \in \mathbb{Z}\}$.

Let G be a locally compact topological group G. If $\mathcal{R} = \mathcal{R}(Z)$ where Z is a freely acting group of automorphisms of X, and $\varphi : Z \times X \to G$, is a cocycle of Z, we may define an *orbit cocycle* of $\mathcal{R}$, $\varphi : \mathcal{R} \to G$ by

$$\varphi(x, zx) := \varphi(z, x),$$

which satisfies

$$\varphi(x, z) = \varphi(x, y)\varphi(y, z) \text{ whenever } (x, y),\ (y, z) \in \mathcal{R}.$$

Orbit cocycles of $\mathcal{R}(Z)$ also give rise to cocycles of Z in this way.

The cocycle φ generates the equivalence relation

$$\mathcal{R}_\varphi := \{((x, u), (y, v)) \in (X\times G)\times(X\times G) : (x, y) \in \mathcal{R},\ vu^{-1} = \varphi(x, y)\}$$

on $X \times G$, and is called *recurrent* if $\mathcal{R}_\varphi$ is conservative. Note that in case $\mathcal{R} = \mathcal{R}(Z)$, we have $\mathcal{R}_\varphi = \mathcal{R}(Z_\varphi)$. As before, the *associated action* of φ is the G-action $(x, y) \mapsto (x, yg)$ restricted to the invariant factor of $\mathcal{R}_\varphi$.

§2 Sketch of a proof of theorem 1

Let T be the given non-singular, conservative, ergodic $\mathbb{R}$-action considered (without loss of generality) acting on X. Let R be a conservative ergodic automorphism of a Lebesgue space $(Y, \mathcal{F}, \rho)$ such that $\tilde{R} : Y \times \mathbb{R} \to Y \times \mathbb{R}$ defined by

$$\tilde{R}(y, u) = (Ry, u - \log R'(y))$$

is ergodic. Here, $R' := \frac{d\rho\circ R}{d\rho}$, and the measure on $Y \times \mathbb{R}$ is $d\mu(x, y) = d\rho(x)e^u du$. Note that $\mu \circ \tilde{R} = \mu$. Such an automorphism R (of *type* III_1) is constructed e.g. in [H-O1].

Define an infinite, σ-finite measure space $W = Y \times \mathbb{R} \times X$ equipped with the measure $d\nu(y, u, x) := d\rho(y)e^u dudm(x)$.

Define measure preserving automorphisms of W by

$$\widehat{R}(y,u,x) := (Ry, u - \log R'(y), x),$$
$$\widehat{T}_q(y,u,x) := (y, u - \log T_q'(y), T_q x) \quad (q \in \mathbb{Q}).$$

The measure preserving automorphisms $\widehat{R}$ and $\widehat{T}_q$ evidently commute. The group $Z := \{\widehat{R}^m \widehat{T}_q : m \in \mathbb{Z},\ q \in \mathbb{Q}\}$ is countable, amenable ($\because$ Abelian), and acts ergodically on W (by the ergodicity of $\tilde{R}$ on $X \times \mathbb{R}$, and $\{T_q : q \in \mathbb{Q}\}$ on X). We have taken the idea for the construction of $\widehat{R}$ and $\widehat{T}_q, q \in \mathbb{Q}$ from [Ham1].

We produce first a recurrent $\mathbb{R}$-valued cocycle of Z whose action is isomorphic to T. The relevant cocycle is defined by

$$\varphi(\widehat{R}^m \widehat{T}_t, (y,u,x)) = -t.$$

We show that φ is recurrent, and that the φ's action is isomorphic to T.

The transformation $\widehat{R}_\varphi = \tilde{R} \times \mathrm{Id}$ on $W \times \mathbb{R} = (Y \times \mathbb{R}) \times (X \times \mathbb{R})$ is evidently conservative, and as $\{\widehat{R}^m : m \in \mathbb{Z}\} \subset Z$, the skew product action of Z is also conservative, whence the recurrence of φ.

Next, we identify the invariant factor of the Z-action on $W \times \mathbb{R}$. It is not hard to see that any invariant measurable set is of form

$$B = \{(T_t x, -t) : t \in \mathbb{R},\ x \in B_0\} = B(B_0)$$

where $B_0 \subset X$ is measurable. Evidently,

$$Q_s B(B_0) = B(T_s B_0)$$

and the action of φ is now clearly isomorphic to T.

To establish the theorem, it is sufficient to find a recurrent cocycle of S whose action is the same as that of φ.

By [C-F-W], $\mathcal{R}$ is hyperfinite, and there is an ergodic measure preserving automorphism U of W such that $[U] = [Z]$, and a recurrent cocycle ψ of U whose action is the same as that of φ.

Fix a measurable set $A \subset W$ such that $\nu(A) = 1$, and consider the induced transformation $(U_\psi)_{A \times \mathbb{R}}$. It follows that

$$(U_\psi)_{A \times \mathbb{R}} = (U_A)_\phi$$

where

$$\phi(x) := \sum_{k=0}^{r_A(x)-1} \psi(U^k x), \quad r_A(x) = \min\{n \geq 1 : U^n x \in A\}.$$

This means that ϕ is a recurrent cocycle of U_A, since $(U_A)_\phi = (U_\psi)_{A \times \mathbb{R}}$ is conservative.

Moreover, the action of ϕ is the isomorphic to that of ψ. The isomorphism is constructed as follows,

for each $(U_\psi)_{A\times\mathbb{R}}$-invariant measurable set B, set

$$B^* = \bigcup_{n=-\infty}^{\infty} (U_\psi)^n B.$$

Obviously, $B^* \in \mathcal{I}(U_\psi)$ and $B^* \cap (A \times \mathbb{R}) = B \mod \nu$. This and the fact that

$$\bigcup_{n=-\infty}^{\infty} (U_\psi)^n (A \times \mathbb{R}) = W \times \mathbb{R} \mod \nu$$

ensure that $B \mapsto B^*$ is a bijection from $\mathcal{I}((U_A)_\phi)$ to $\mathcal{I}(U_\psi)$, and hence that the induced point mapping $\Omega((U_A)_\phi) \to \Omega(U_\psi)$ is a measure space isomorphism.

The fact that

$$(B^*) \circ Q_t \equiv (B \circ Q_t)^* \quad (B \in \mathcal{I}((U_A)_\phi), \ \ t \in \mathbb{R})$$

ensures that the actions of ψ and ϕ are isomorphic.

Lastly, by Dye's theorem ([Dye]) we may suppose without loss of generality that $A = X$, $[U_A] = [S]$, and that ϕ is a cocycle of S.

§3 Lacunarity, and special flow representations of the associated action

Definition Let $(Z, \mathcal{C}, p)$ be a standard σ-finite measure space, let $U : Z \to Z$ be an invertible, nonsingular transformation, and let $f : Z \to (0, \infty)$ be measurable. Define

$$W = \{(z, u) : z \in Z, \ 0 \le u < f(z)\},$$

and for $t \in \mathbb{R}, \ (z, u) \in W$:

$$T_t(z, u) = (U^n z, u + t - f(n, z)) \quad \text{for } f(n, z) \le t + u < f(n+1, z)$$

where $f : \mathbb{Z} \times Z \to \mathbb{R}$ is the cocycle of U determined by f. The flow T on W is nonsingular with respect to the product measure of p with Lebesgue measure.

The triple (Z, U, f) is a *special flow representation* of T with *base transformation* U built under the *height function* f. The flow T is said to be *represented* by (Z, U, f).

An isomorphism of special flow representations (Z, U, f) and (Z', U', f') is a measure space isomorphism $\pi : Z \to Z'$ satisfying $\pi U = U'\pi$ and $f' \circ \pi = f$. Clearly, isomorphism of special flow representations entails isomorphism of the represented flows (but not vice versa).

By the Krengel-Kubo theorem ([Kre], [Ku]), any conservative, nonsingular free $\mathbb{R}$-action T of a Lebesgue space has a special flow representation (Z, U, f) with a nonsingular base automorphism U of a Lebesgue space Z built under a ceiling function f which is bounded away from 0.

A periodic flow has a "trivial" special flow representation (Z, U, f) where U is the identity on the one-point space Z, and f is constant.

In this section, to any recurrent cocycle, we associate a special flow representation of its associated action (the K-representation, see below). Although the associated actions of cohomologous cocycles are isomorphic, their K-representations need not be.

Let $(X, \mathcal{B}, m)$ be a standard probability space, and let $\mathcal{R} \in \mathcal{B} \otimes \mathcal{B}$ be a countable, measurable equivalence relation. A cocycle $\varphi : \mathcal{R} \to \mathbb{R}$ is called *lacunary* if

$$\tilde{f}(x) := \inf\{\varphi(x, y) : \varphi(x, y) > 0, \ (x, y) \in \mathcal{R}\}$$

is bounded below (i.e. $\exists\ \epsilon > 0$ such that $\tilde{f} > \epsilon$ a.e.). Suppose that $\varphi : \mathcal{R} \to \mathbb{R}$ is a recurrent, lacunary cocycle. Define the subrelation $\mathcal{S} = \mathcal{S}_\varphi$ of $\mathcal{R}$ by setting

$$\mathcal{S} = \{(x, y) | \varphi(x, y) = 0\}.$$

As before, let $\Omega(\mathcal{S})$ be the invariant factor of $\mathcal{S}$, and $\mathcal{I}(\mathcal{S})$ be the $\mathcal{S}$-invariant sets in X. The invariant factor map is $\pi : X \to \Omega(\mathcal{S})$ such that $x \in \pi(x)$, and $\pi^{-1}\mathcal{B}(\Omega(\mathcal{S})) = \mathcal{I}(\mathcal{S})$. Let $\overline{m} = m \circ \pi^{-1}$, and let $\{m_z : z \in \Omega\}$ denote the induced conditional probabilities:

$$\int_A m_z(B) d\overline{m} = m(B \cap \pi^{-1}A) \quad (A \in \mathcal{B}(\Omega), \ B \in \mathcal{B}(X)).$$

Proposition 3.1 *If φ is recurrent, then the conditional probabilities $\{m_z : z \in \Omega\}$ are $\overline{m}$-almost all non-atomic.*

Proof Suppose otherwise, that for some $\delta > 0$, m_z has an atom of mass at least δ for each $z \in A \in \mathcal{B}(\Omega)$, where $\overline{m}(A) > 0$. Then, $\exists\ \alpha : A \to X$ measurable such that $m_z(\{\alpha(z)\}) \geq \delta \ \ \forall\ z \in A$. The set $E = \alpha(A) \in \mathcal{B}(X)$ mod 0, and $m(E) > 0$. Moreover for a.e. $z \in \Omega$, $\#(E \cap \pi^{-1}\{z\}) \leq \frac{1}{\delta}$, whence $\#(E \cap \mathcal{S}_x) \leq \frac{1}{\delta}$ for a.e. $x \in X$.

Let $\epsilon > 0$ be such that

$$\min\{|\varphi(x, y)| : \ (x, y) \in \mathcal{R}, \ \varphi(x, y) \neq 0\} > 2\epsilon.$$

Let $(x,u)\in\widetilde{E}:=E\times(-\epsilon,\epsilon)$ and $(x,y)\in\mathcal{R}$. If $(y,u+\varphi(x,y))\in\widetilde{E}$, then, $\varphi(x,y)=0$ and hence, $y\in\mathcal{S}_x\cap E$. This contradicts the recurrence of φ, as $\#(\widetilde{E}\cap(\mathcal{R}_\varphi)_{(x,u)})\le\frac{1}{\delta}$ for $(x,u)\in\widetilde{E}$. □

As a corollary of proposition 3.1, we have

Hopf equivalence on ergodic components
If $A,\ B\in\mathcal{B}(X)_+$, and $m_z(A)=m_z(B)$ for $\overline{m}$-a.e. $z\in\Omega(\mathcal{S})$, then $\exists\ g\in[\mathcal{S}]_*$ such that $gA=B$ mod m,

the proof being Hopf's classical exhaustion argument which works in this situation because of proposition 3.1.

Note that the function $\tilde{f}$ (involved in the definition of lacunarity above) is $\mathcal{S}$-invariant. Choose a suitable measurable $f:\Omega(\mathcal{S})\to\mathbb{R}$ so that $\tilde{f}=f\circ\pi$.

Theorem 3.2 (c.f. [Katz],[Kri]) *Let $\mathcal{R}$ be an ergodic hyperfinite equivalence relation of type II_1, and let $\varphi:\mathcal{R}\to\mathbb{R}$ be a lacunary, recurrent cocycle of $\mathcal{R}$ whose action T is free, then*
there is a nonsingular automorphism U of $(\Omega(\mathcal{S}),\overline{m})$ such that

$$U(\pi x)=\pi y \text{ where } \varphi(x,y)=-f(\pi x),$$

and that $(\Omega(\mathcal{S}),U,f)$ is a special flow representation for T.

Proof Let $\epsilon>0$ be such that

$$\min\{|\varphi(x,y)|\quad:\quad(x,y)\in\mathcal{R},\varphi(x,y)\neq0\}>2\epsilon.$$

We note that for a.e. x,

$$|\varphi(y,x)-\varphi(z,x)|=|\varphi(y,z)|>2\epsilon\ \text{ or }\ =0\quad((y,x),(z,x)\in\mathcal{R})$$

whence, there exists a sequence of measurable functions $\xi(k,x),-\infty<k<\infty$, such that

$$\begin{aligned}rl\xi(0,x)&=0,\\ \xi(k,x)+2\epsilon&\le\xi(k+1,x),\quad-\infty<k<\infty\\ \{\varphi(y,x)|(y,x)\in\mathcal{R}\}&=\{\xi(k,x)|-\infty<k<\infty\}\ \text{ a.e.}x\end{aligned}$$

The cocycle property implies that for a.e. x,

$$\xi(l+k,x)=\xi(k,x)+\xi(l,y),\quad\forall l\in\mathbb{Z},\forall y:(y,x)\in\mathcal{R}$$

where k is determined by

$$\xi(k,x)=\varphi(y,x).$$

Obviously each $\xi(l,x)$ is an $\mathcal{S}$-invariant function. By definition of the function f,

$$f(\pi(x)) = \xi(1,x).$$

Firstly we will show the existence of an automorphism U on $(\Omega,\overline{m})$ satisfying for a.e.x,

$$U\pi(x) = \pi(y)$$

where

$$\varphi(y,x) = f(\pi(x)).$$

Let

$$(\widehat{X},\widehat{m}) = (X \times \mathbb{Z}, m \times n),$$

where n is the counting measure on $\mathbb{Z}$. Define an equivalence relation $\widehat{\mathcal{R}}$ on $\widehat{X}$ by

$$((x,i),(y,j)) \in \widehat{\mathcal{R}} \text{ if } \varphi(y,x) = \xi(i-j,x).$$

Define a map $\widehat{\pi} : \widehat{X} \to \Omega$ by setting

$$\widehat{\pi}(x,i) = \pi(y)$$

where

$$((x,i),(y,0)) \in \widehat{\mathcal{R}}.$$

We note that a measurable subset $\widehat{A} \subset \widehat{X}$ of positive measure is $\widehat{\mathcal{R}}$-invariant if and only if $\widehat{A}$ is of the form :

$$\widehat{A} = \cup_{i=-\infty}^{\infty} A_i \times \{i\}$$

where each A_i is an $\mathcal{S}$-invariant set of positive measure, and for any i and j and for a.e.$x \in A_i, \exists y \in A_j$ such that

$$((x,i),(y,j)) \in \widehat{\mathcal{R}}.$$

In this case

$$\widehat{\pi}(\widehat{A}) = \pi(A_0) \text{ and } \widehat{A} = \widehat{\pi}^{-1}(\pi(A_0)),$$

whence, $\widehat{\pi}$ induces a measure space isomorphism from the invariant factor space of $\widehat{\mathcal{R}}$ onto the invariant factor space of $\mathcal{S}$. So, the map $\widehat{\pi}$ is considered to be an invariant factor map of $\widehat{\mathcal{R}}$.

We note $\xi(n,x) = f(n,\pi(x))$. Since the automorphism $(x,i) \to (x,i-1)$ of $\widehat{X}$ commutes with $\widehat{\mathcal{R}}$-equivalence relation, it also acts on the invariant factor of $\widehat{\mathcal{R}}$. We denote this factor automorphism by U,

$$U\widehat{\pi}(x,i) = \widehat{\pi}(x,i-1).$$

In other words, U satisfies

$$U\pi(x) = \pi(y)$$

where

$$\varphi(y,x) = \xi(1,x) = f(\pi(x)),$$

(see [H-O2]).

Next we show that (Ω, U, f) is a special flow representation of T. Let

$$W = \{(z,u)|z \in \Omega, 0 \le u < f(z)\}.$$

Let $(x,u) \in X \times I\!R$ and $\xi(n-1,x) \le u < \xi(n,x)$. Choose $y \in X$ so that

$$\varphi(y,x) = \xi(n-1,x)$$

and define a map $\pi : X \times I\!R \to W$ by setting

$$\pi(x,u) = (\pi(y), u - \xi(n-1,x)) = (U^{n-1}\pi(x), u - f(n-1,\pi(x))).$$

This map π is well-defined and is an invariant factor map of $\mathcal{R}_\varphi$ onto W. Obviously, $\pi(x, u+t) = T_t\pi(x,u)$. □

In the sequel, we'll denote $U = U_\varphi$, and $f = f_\varphi$ and call $(\Omega(\mathcal{S}_\varphi), U_\varphi, f_\varphi)$ the *K-representation* of the action corresponding to φ.

The next proposition says that any special flow representation of the action is isomorphic to the K-representation of some cohomologous, lacunary cocycle.

Proposition 3.3 *Suppose that (Z, U, f) is a special flow representation of the free, conservative action of the recurrent cocycle φ, and* $\inf f > 0$, *then there is a lacunary cocycle ψ, cohomologous to φ, and a measure space isomorphism $\pi : \Omega(\mathcal{S}_\psi) \to Z$ such that $\pi \circ U_\psi = U \circ \pi$, and $f \circ \pi = f_\psi$.*

Remark If the flow is periodic, then the result is proven in [Sch], and the lacunary cocycle ψ satisfies the following properties:

(1) For a.e.x, $\{\psi(y,x) : (y,x) \in \mathcal{R}\} = \{n\lambda : n \in Z\!\!Z\}$ where $\lambda > 0$ is the period of the flow.
(2) $\mathcal{S}_\psi = \{(x,y) \in \mathcal{R} : \psi(x,y) = 0\}$ is ergodic.

Proof Set

$$W := \{(z,t) : z \in Z, 0 \le t < f(z)\}.$$

Let $\phi : X \times I\!R \to W$ be the invariant factor map of $\mathcal{R}_\varphi$, i.e. such that

$$\phi(x, y+t) = T_t\phi(x,y), \ \& \ \phi^{-1}\mathcal{B}(W) = \mathcal{I}(\mathcal{R}_\varphi).$$

There are measurable maps $\zeta : X \to Z$, and $\eta : X \to I\!R$ such that $0 \le \eta(x) < f(\zeta x)$, and

$$\phi(x,0) = (\zeta x, \eta(x)) \quad \text{for } x \in X.$$

Now, if $(x,y) \in \mathcal{R}$ then

$$\begin{aligned} rl(\zeta x, \eta(x)) &= \phi(x,0) \\ &= \phi(y, \varphi(x,y)) \\ &= T_{\varphi(x,y)}\phi(y,0) \\ &= (U^n\zeta y, \eta(y) + \varphi(x,y) - f(n,\zeta y)) \end{aligned}$$

where $n = n(x,y) \in Z\!\!\!Z$ is defined by

$$f(n,\zeta y) \le \eta(y) + \varphi(x,y) < f(n+1,\zeta y).$$

Since $n(x,y)$ is also the unique $n \in Z\!\!\!Z$ such that $\zeta x = U^n\zeta y$, it's clear that $n : \mathcal{R} \to Z\!\!\!Z$ is an orbit cocycle.

Now define $\psi : \mathcal{R} \to I\!R$ by

$$\psi(x,y) = f(n(x,y), \zeta y).$$

Claim 1 *The function ψ is a lacunary orbit cocycle, and the cocycles φ and ψ are cohomologous.*

Proof Suppose $(x,y),(y,z) \in \mathcal{R}$, and $n(x,y) = k,\ n(y,z) = \ell$.

Then $\zeta y = U^\ell \zeta z$, and

$$\begin{aligned} rl\psi(x,y) + \psi(y,z) &= f(k,\zeta y) + f(\ell,\zeta z) \\ &= f(k, U^\ell\zeta z) + f(\ell,\zeta z) \\ &= f(k+\ell, \zeta z) \\ &= \psi(x,z) \end{aligned}$$

since $k + \ell = n(x,z)$. Thus, ψ is cocycle.

Lacunarity follows from $\inf f > 0$.

Lastly, equating second coordinates in the equation derived above from $\phi(x,0) = \phi(y,\varphi(x,y))$, we have

$$\eta(x) = \eta(y) + \varphi(x,y) - f(n(x,y),\zeta y) \Rightarrow \ \varphi(x,y) - \psi(x,y) = \eta(x) - \eta(y).$$

□

Claim 2

$$\min\{\psi(x,y) : y \in X, \psi(x,y) > 0\} = f(U^{-1}\zeta x), \quad a.e.x$$

Proof We prove that for a.e.$x, \exists y$ such that $(x,y) \in \mathcal{R}$, and $n(x,y) = 1$ which suffices, since

$$\psi(x,y) = f(n(x,y), \zeta y) = f(n(x,y), U^{-n(x,y)}\zeta x).$$

To do this, we must show that for a.e. $x, \exists y$ such that $(x,y) \in \mathcal{R}$, and $\zeta x = U\zeta y$. To show this, note that

$$\phi(x, -f(U^{-1}\zeta x)) = T_{-f(U^{-1}\zeta x)}(\zeta x, \eta(x)) = T_{\eta(x)}(U^{-1}\zeta x, 0)$$

whence $\exists y \in X$, for which $(x,y) \in \mathcal{R}$ and $\zeta y = U^{-1}\zeta x$. This establishes the claim.

□

The map $\zeta : X \to Z$ is apparently $\mathcal{S}_\psi$-invariant, and hence induces a measure space isomorphism $\pi : \Omega(\mathcal{S}_\psi) \to Z$. It follows from claims 1 and 2 that

$$\pi U_\psi = U\pi, \ \& \ f \circ \pi = f_\psi.$$

□

Corollary 3.4 *For any measurable subset $A \subset X$ of positive measure and for each integer n, there exists a partial transformation $g \in [\mathcal{R}]_*$ such that*

$$cDom(g) \subset A$$
$$n(gx,x) = n, \quad \text{and} \quad \psi(gx,x) = f(n, \pi(x)), \quad (x \in Dom(g)),$$
$$\pi(gx) = U^n\pi(x), \quad (x \in Dom(g)).$$

Remarks

(1) Suppose that $\mathcal{R}$ is measure preserving (indeed that $m \circ g = m \forall\ g \in [\mathcal{R}]$). In general, it is not possible to find $g \in [\mathcal{R}]$ such that $\pi(gx) = U\pi x$. If this were the case, then $\overline{m}$ would be U-invariant. By theorem 1, the action may be of type III, whence the absence of U-invariant, absolutely continuous probabilities.
(2) It is not hard to show, using Hopf equivalence, that if $\overline{m}$ is U-invariant, then $\exists\ g \in [\mathcal{R}]$ such that $\pi(gx) = U\pi x$.
(3) As the examples below show, it may be that there is a U-invariant, probability $\mu \sim \overline{m}$, but $\mu \neq \overline{m}$, whence again there is no $g \in [\mathcal{R}]$ such that $\pi(gx) = U\pi x$.

Example For $a_n \geq 4 \ \ (n \in I\!N)$ let

$$X = \prod_{n=1}^{\infty}\{0, 1, \dots, a_n - 1\}, \quad m = \prod_{n=1}^{\infty}\{\frac{1}{a_n}, \dots, \frac{1}{a_n}\},$$

and

$$\mathcal{R} = \{(x,y) \in X \times X : \ \#\{k \in I\!N : x_k \neq y_k\} < \infty\}.$$

For $d_n > 0 \quad (n \in I\!N)$ such that $d_{n+1} > \sum_{k=1}^n d_k$, and $2 \le b_n < a_n$, define, for $n \in I\!N$, $\beta_n : \{0,1,\dots,a_n - 1\} \to I\!R$ by

$$\beta_n(k) = d_n 1_{[0,b_n-1]}(k).$$

Now define $\varphi : \mathcal{R} \to I\!R$ by

$$\varphi(x,y) = \sum_{n=1}^{\infty} (\beta_k(x_k) - \beta_k(y_k)).$$

We have that φ is recurrent since $\#(\mathcal{S}_\varphi)_x = \infty \, \forall \, x \in X$, and lacunary with

$$f(\pi x) = \inf\{\varphi(x,y) : (x,y) \in \mathcal{R}, \ \varphi(x,y) > 0\} = d_{\ell(x)} - \sum_{k=1}^{\ell(x)-1} d_k$$

where

$$\ell(x) = \min\{k \ge 1 : \varphi_k(x_k) = d_k\}.$$

Also

$$\Omega(\mathcal{S}_\varphi) = \{0,1\}^{I\!N}, \quad \overline{m} = \prod_{n=1}^{\infty} \{\frac{b_n}{a_n}, \frac{a_n - b_n}{a_n}\},$$

and

$$U(1,\dots,1,0,\dots) = (0,\dots,0,1,\dots), \text{ the adding machine.}$$

Set

$$\mu = \prod_{n=1}^{\infty} \{\frac{1}{2}, \frac{1}{2}\}.$$

There is a $\overline{m}$-a.c. U-invariant probability iff $\mu \sim \overline{m}$ iff

$$\sum_{n=1}^{\infty} |\frac{2b_n}{a_n} - 1| < \infty$$

and in this case, setting

$$p_0(n) = \frac{b_n}{a_n}, \ p_1(n) = 1 - p_0(n)$$

we have that

$$h(x) = \frac{d\overline{m}}{d\mu}(x) = \prod_{n=1}^{\infty} 2p_{x_n}(n).$$

Fixing, for example $a_n = 2^{n+1} + 1, \ b_n = 2^n, \ \& \ d_n = 2^n$, we obtain $\overline{m} \sim \mu$ but $h \neq 1$.

A *normalizer* of $\mathcal{R}$ is an automorphism of X such that

$$(Ry, Rx) \in \mathcal{R} \text{ for a.e.}(y, x) \in \mathcal{R}.$$

By $N[\mathcal{R}]$ we denote the set of all normalizers of $\mathcal{R}$. By $N[\mathcal{R}]_*$, we denote the set consisting of all partial transformations R satisfying

$$(Rx, Ry) \in \mathcal{R} \text{ for a.e.}(x, y) \in \mathcal{R}_{\mathrm{Dom}\, R}.$$

Lemma 3.5 *Let $R \in N[\mathcal{S}]_* \cap [\mathcal{R}]_*$, then $\exists$ a measurable function $n : \pi(Im(R)) \to I\!N$ such that for a.e. $z \in \pi(Im(R))$ and for m_z-a.e.$x \in Im(R)$*

$$\pi(R^{-1}x) = U^{-n(z)}z.$$

This is uniquely determined if U is aperiodic.

Proof Since $R \in [\mathcal{R}]_*$, we see that for a.e. $x \in \mathrm{Im}\,(R)$, $\exists$ an integer $n = n(x)$ such that

$$\pi(R^{-1}x) = U^{-n}z.$$

We show that n depends only on $\pi(x)$. If not, there are integers k , l and measurable subsets $A \subset X, B \subset X$, and $W \subset \Omega$ of positive measure with the following properties:

(1) For $z \in W$, $m_z(A) > 0$, and $m_z(B) > 0$.
(2) For any $z \in W$,

$$\pi(R^{-1}x) = \begin{cases} U^{-k}z & \pi(x) = z, \quad x \in A, \\ U^{-l}z & \pi(x) = z, \quad x \in B. \end{cases}$$

Since $\mathcal{R}$ is ergodic, there exists $\phi \in [\mathcal{R}]_*$ such that

$$\mathrm{Dom}\,(\phi) \subset A \cap \mathrm{Im}\,(R), \quad \mathrm{Im}\,(\phi) \subset B \cap \mathrm{Im}\,(R).$$

If $x \in$Dom(ϕ) and $z = \pi(x)$ then

$$U^{-k}z = \pi(R^{-1}x) = \pi(R^{-1} \circ \phi(x)) = U^{-l}z.$$

Contradiction. □

Lemma 3.6 *Assume that $\mathcal{R}$ is an ergodic equivalence relation of type II_1. Let $R \in N[\mathcal{S}]_* \cap [\mathcal{R}]_*$ and E be a measurable subset of Dom(R). Then, for a.e. $z \in \pi(Im(R))$,*

$$m_z(RE) = \frac{d\overline{m}U^{-n(z)}}{d\overline{m}}(z) m_{U^{-n(z)}z}(E)$$

where $n(z)$ is as in Lemma 3.5.

Proof

Let $g(z) \in L^\infty(\Omega, \overline{m})$. We may suppose that E satisfies for some integer n ,

$$\pi(Rx) = U^n \pi(x) \quad (x \in E),$$

an arbitrary measurable subset $E \subset$Dom(R) being a countable disjoint union of such sets. Then, for $f \in L^\infty(Z)$, $[g \neq 0] \subset \operatorname{Im} R$,

$$\begin{aligned}
\int g(z) m_z(R(E)) d\overline{m}(z) &= \int g(\pi(x)) 1_{R(E)}(x) m(dx) \\
&= \int g(\pi(Ry)) 1_E(y) m(dy) \quad \text{(use that } R \text{ is } m\text{-preserving)} \\
&= \int g(U^n \pi(y)) 1_E(y) m(dy) \\
&= \int g(U^n z) \overline{m}(dz) \int_{\pi(y)=z} 1_E(y) m_z(dy) \\
&= \int g(z) \frac{d\overline{m} U^{-n}}{d\overline{m}}(z) \overline{m}(dz) \int_{\pi(y)=U^{-n}z} 1_E(y) m_{U^{-n}z}(dy) \\
&= \int g(z) \frac{d\overline{m} U^{-n}}{d\overline{m}}(z) m_{U^{-n}z}(E) \overline{m}(dz).
\end{aligned}$$

□

Remark A relevant idea is seen in [Ham2].

§4 Orbit equivalences, and the proof of Theorem 2

In order to establish theorem 2, we need to construct an orbit equivalence. This will be done by appropriately copying generating sequences of *towers.*

Recall (from [Ham2]), that a *tower* ξ of an equivalence relation $\mathcal{R}$ on X consists of a finite partition $\mathcal{P}_\xi = \{E_\alpha : \alpha \in \Lambda\}$ of X, and a finite family of partial transformations $\mathcal{T}_\xi = \{e_{\alpha,\beta} \in [\mathcal{R}]_* : \alpha, \beta \in \Lambda\}$ satisfying

$$\operatorname{Dom}(e_{\alpha,\beta}) = E_\beta, \quad \operatorname{Im}(e_{\alpha,\beta}) = E_\alpha,$$

and

$$e_{\alpha,\beta} e_{\beta,\gamma} = e_{\alpha,\gamma}, \quad e_{\alpha,\alpha} = \mathrm{Id}|_{E_\alpha}.$$

In order to introduce the method, and use of such copyings (see [K-W] and [H-O1]) we first show that any two $\mathbb{R}$-valued ergodic cocycles of countable, hyperfinite, equivalence relations of type II_1 are cohomologous via orbit equivalence. This was first established in [G-S1]. One needs the following

Lemma*Suppose that φ is an ergodic $\mathbb{R}$-valued cocycle of the ergodic, hyperfinite equivalence relation $\mathcal{R}$ of type II_1. If $\mathcal{P} = \{E_\alpha : \alpha \in \Lambda\}$ is a measurable partition of $Y \in \mathcal{B}$ into sets of equal measure, $r_{\beta,\alpha} \in \mathbb{R}$, $(\alpha, \beta \in \Lambda)$, and $\epsilon > 0$, then there is a tower ξ of $\mathcal{R}_Y$ such that*

$$\mathcal{P}_\xi = \mathcal{P},$$

and

$$|\varphi(e_{\beta,\alpha}x, x) - r_{\beta,\alpha}| < \epsilon \text{ a.e. on } E_\alpha, \ \forall\ \alpha, \beta \in \Lambda.$$

Sketch of proof First note that the tower ξ can be split into a disjoint union of towers $\xi_i = \xi \cap Y_i$ $(i \in \mathbb{N})$ so that

$$|\varphi(e_{\beta,\alpha}x, x) - \varphi(e_{\beta,\alpha}y, y)| < \epsilon \ \ \forall\ \alpha, \beta \in \Lambda,\ x, y \in E_\alpha \cap Y_i, \ \ i \in \mathbb{N}.$$

Because of the ergodicity of $\varphi : \mathcal{R} \to \mathbb{R}$, we have that (see [Sch]) for any $r \in \mathbb{R}$, $\epsilon > 0$, and $A, B \in \mathcal{B}_+$

$$\exists\, R \in [\mathcal{R}]_* \ni \text{ Dom}\, R \subset A, \ \&\, m(\{x \in A :\ Rx \in B,\ |\varphi(Rx, x) - r| < \epsilon\}) > 0,$$

whence, by Hopf equivalence, if $r \in \mathbb{R}$, $\epsilon > 0$, and $A, B \in \mathcal{B}_+$, $m(A) = m(B)$ then

$$\exists\, R \in [\mathcal{R}]_* \ni \text{ Dom } R = A,\ Im\, R = B,\ \&\, |\varphi(Rx, x) - r| < \epsilon \text{ a.e. on } A.$$

The proof is completed by Hopf's exhaustion method. □

Now suppose that $\mathcal{R}$ and $\mathcal{R}'$ are countable, hyperfinite, equivalence relations of type II_1 and that $\varphi : \mathcal{R} \to \mathbb{R}$, $\varphi' : \mathcal{R}' \to \mathbb{R}$ are ergodic cocycles. The lemma is used (as in [K-W],[H-O1]) to obtain isomorphic sequences of towers $\xi_n = (\mathcal{P}_n, \mathcal{T}_n)$ for $\mathcal{R}$, and $\xi'_n = (\mathcal{P}'_n, \mathcal{T}'_n)$ for $\mathcal{R}'$, where

$$\mathcal{P}_n = \{E_\alpha : \alpha \in \Lambda_n\},\ \mathcal{T}_n = \{e_{\alpha,\beta} : \alpha, \beta \in \Lambda_n\},$$

which are *generating* in the sense that the σ-algebra $\mathcal{B}$ is generated by the sets $E_\alpha \in \mathcal{P}_n$, $(n \geq 1)$, and that

$$\mathcal{R} = \bigcup_{n \geq 1} \{(y, x) | \text{for some } \alpha \text{ and } \beta \text{ in } \Lambda_n, x \in E_\alpha,\ y = e_{\beta,\alpha}x\}.$$

These towers are obtained together with a sequence of parameters $\alpha_n \in \Lambda_n$ and satisfy the following.

(1) The tower ξ_0 is trivial in the sense that $|\Lambda_0| = 1$.

(2) ξ_{n+1} refines ξ_n in the sense that

$$\Lambda_{n+1} = \Lambda_n \times \Gamma_{n+1} \quad (\Gamma_{n+1} \text{ a finite set and } \Lambda_1 = \Gamma_1\ )$$

$$E_\alpha = \bigcup_{\gamma \in \Gamma_{n+1}} E_{(\alpha,\gamma)} \quad (\alpha \in \Lambda_n)$$

$$e_{(\alpha,\gamma),(\beta,\gamma)} = e_{\alpha,\beta} \quad \text{on } E_{(\beta,\gamma)}, \quad (\ \alpha, \beta \in \Lambda_n, \text{and } \gamma \in \Gamma_{n+1})$$

(3)
$$E_{\alpha_{n+1}} \subset E_{\alpha_n}$$

(4) For each $n \geq 1$ and for each $\gamma \in \Gamma_{n+1}$, $\exists\ r_{\alpha_{n+1},(\alpha_n,\gamma)} \in I\!R$ such that
$$|\varphi(e_{\alpha_{n+1},(\alpha_n,\gamma)}x, x) - r_{\alpha_{n+1},(\alpha_n,\gamma)}| < \frac{1}{2^n} \quad (\text{ for a.e. } x \in E_{(\alpha_n,\gamma)})$$

(5) The towers ξ'_n satisfy the analogous properties (1')-(4') with the parameter sets Λ_n.

The towers are obtained by means of the following refinement process ([K-W], [H-O1]). A product refinement ξ_{n+1} of ξ_n is obtained by choosing a "base element" E_{α_n} of ξ_n, constructing a tower $\{E_{(\alpha_n,\gamma)} : \gamma \in \Gamma_n\}$ which generates $\mathcal{R}_{E_{\alpha_n}}$ up to some fixed precision, and such that
$$|\varphi(e_{(\alpha_n,\gamma'),(\alpha_n,\gamma)}x, x) - \varphi(e_{(\alpha_n,\gamma'),(\alpha_n,\gamma)}y, y)| < \frac{1}{2^{n+1}} \text{ a.e. on } E_{(\alpha_n,\gamma)}\ \forall\, \gamma, \gamma' \in \Gamma_n.$$
This refinement is copied in a measure preserving way to obtain a refinement ξ'_{n+1} of ξ'_n, which refinement is then refined to ξ'_{n+2} by the same process, and then copied back.

Note that it follows from property (4) that for each $n \geq 1$ and for each $\beta, \beta' \in \Lambda_n$, $\exists r_{\beta',\beta} \in I\!R$ such that
$$|\varphi(e_{\beta',\beta}x, x) - r_{\beta',\beta}| < 3 \ \text{ a.e. on } E_\beta.$$

The natural correspondences between the towers ξ_n and ξ'_n generate an orbit equivalence Φ of $\mathcal{R}$ with $\mathcal{R}'$ such that
$$\Phi E_\beta = E'_\beta, \quad \Phi \circ e_{\alpha,\beta} = {e'}_{\alpha,\beta} \circ \Phi, \quad (\alpha, \beta \in \Lambda_n, n \geq 1).$$
It follows that for a.e. $(x, y) \in \mathcal{R}$,
$$|\varphi'(\Phi y, \Phi x) - \varphi(y, x)| < 6,$$
whence $\exists\ \eta : X \to I\!R$ bounded and measurable such that
$$\varphi'(\Phi y, \Phi x) - \varphi(y, x) = \eta(y) - \eta(x).$$
In case φ and φ' are ergodic $Z\!\!\!Z$ valued cocycles, an adjustment of the above shows that there is an orbit equivalence $\Phi : X \to X'$ such that
$$\varphi'(\Phi x, \Phi y) = \varphi(x, y) \text{ a.e. on } \mathcal{R}.$$
In case φ and φ' have isomorphic periodic actions, they are also cohomologous via orbit equivalence. To see this, we may suppose that φ and φ' satisfy the conditions in the remark after proposition 3.3. The result now reduces to the uniqueness of ergodic $Z\!\!\!Z$-valued cocycles.

We now turn to the

Proof of theorem 2.
Let $\mathcal{R}$ and $\mathcal{R}'$ be hyperfinite of type II_1. Let φ and φ' be recurrent orbit cocycles of $\mathcal{R}$ and $\mathcal{R}'$ respectively, having isomorphic associated actions. By proposition 3.3, we may assume without loss of generality, that φ and φ' are lacunary, and have isomorphic K-representations.
Let φ and φ' have (respectively):
kernels $\mathcal{S} = \mathcal{S}_\varphi$, and $\mathcal{S}' = \mathcal{S}'_\varphi$;
K-representations $(\Omega, U, f) := (\Omega(\mathcal{S}), U_\varphi, f_\varphi)$ and $(\Omega', U', f') := (\Omega(\mathcal{S}'), U_{\varphi'}, f_{\varphi'})$.
Suppose that $\mu \sim \overline{m}$ and $\mu' \sim \overline{m}'$ are probabilities, and that $\theta : (\Omega, \mathcal{B}(\Omega), \mu) \to (\Omega', \mathcal{B}(\Omega'), \mu')$ is a measure space isomorphism satisfying

$$\theta \circ U = U' \circ \theta, \quad f' \circ \theta = f, \ \& \quad \mu \circ \theta^{-1} = \mu'.$$

Let:

(1) $\pi : X \to \Omega$ and $\pi' : X' \to \Omega'$ be the invariant factor maps,
(2) $h = \frac{d\overline{m}}{d\mu}$, $h' = \frac{d\overline{m'}}{d\mu'}$ where $\overline{m} = m \circ \pi^{-1}$ and $\overline{m'} = m' \circ \pi'^{-1}$.

One way to establish the theorem would be to obtain an orbit equivalence $X \to X'$ extending $\theta : \Omega \to \Omega'$. For this to be possible, we would need

$$\overline{m}' \circ \theta^{-1} = \overline{m} \quad (\Leftrightarrow \ h' \circ \theta = h).$$

Indeed, our first task will be to reduce to this situation, which will yield an orbit equivalence as above, and establish the theorem without coboundary (see lemma 4.1 below).

The reduction will be done by restricting to subsets $Y \in \mathcal{B}(X),\ Y' \in \mathcal{B}(X')$ in such a way as to deform the measures appropriately.
We now describe this process of restriction. For $Y \in \mathcal{B}(X)$ such that $\pi(Y) = \Omega$,

(1) let $\mathcal{R}_Y = \mathcal{R} \cap (Y \times Y),\ \varphi_Y := \varphi|_{\mathcal{R}_Y}$;
(2) note that the kernel of φ_Y is given by $\mathcal{S}_{\varphi_Y} = \mathcal{S}_\varphi \cap (Y \times Y) := \mathcal{S}_Y$,
(3) and the invariant factor for $\mathcal{S}_Y$ is $\Omega(\mathcal{S}_Y) = \Omega$, with invariant factor map $\pi_Y = \pi|_Y$ (since $\pi(Y) = \Omega$).

Note also that (since $\pi(Y) = \Omega$)

$$\inf \{\varphi_Y(x, y) : (x, y) \in \mathcal{R}_Y,\ \varphi_Y(x, y) > 0\} = f(\pi x) \ \forall\ x \in Y.$$

Set also

$$m_Y(\cdot) = \frac{m(\cdot \cap Y)}{m(Y)},\ \overline{m_Y} = m_Y \circ \pi^{-1},\ \&\ h_Y = \frac{d\overline{m_Y}}{d\mu}.$$

New conditional probabilities $\{(m_Y)_z : z \in \Omega\}$ on $(X, \mathcal{B}(X))$ are induced as before by the invariant factor map $\pi_Y = \pi : Y \to \Omega(\mathcal{S}_Y) = \Omega$, and these are defined by

$$\int_A (m_Y)_z(B) d\overline{m_Y}(z) = m_Y(\pi^{-1}A \cap B), \quad (A \in \mathcal{B}(\Omega). \ B \in \mathcal{B}(X),$$

whence it follows that

$$(m_Y)_z(B) = \frac{m_z(B \cap Y)}{m_z(Y)} \ \ (B \in \mathcal{B}(X), \ z \in \Omega).$$

For $A \in \mathcal{B}(\Omega)$,

$$\int_A h_Y d\mu = \int_A d\overline{m_Y} = \frac{m(\pi^{-1}A \cap Y)}{m(Y)} = \int_A \frac{m_z(Y)}{m(Y)} d\overline{m}(z) = \int_A \frac{m_z(Y)}{m(Y)} h(z) d\mu(z),$$

whence,

$$h_Y(z) = \frac{m_z(Y)}{m(Y)} h(z).$$

The reduction

Set

$$rl Z_1 = \{z \in \Omega : h'(\theta z) > h(z)\},$$
$$Z_2 = \{z \in \Omega : h'(\theta z) \le h(z)\}.$$

Choose $Y \in \mathcal{B}(X)$ and $Y' \in \mathcal{B}(X')$ such that

$$m_z(Y) = 1, \quad m'_{\theta z}(Y') = \frac{h(z)}{h'(\theta z)} \text{ for } \ z \in Z_1,$$

and

$$m_z(Y) = \frac{h'(\theta z)}{h(z)}, \quad m'_{\theta z}(Y') = 1 \text{ for } z \in Z_2.$$

It follows that for $z \in \Omega$,

$$h(z) m_z(Y) = h'(\theta z) m'_{\theta z}(Y'),$$

consequently:

$$\begin{aligned} m(Y) &= m'(Y'), \\ h'_{Y'}(\theta z) &= \frac{m'_{\theta z}(Y')}{m'(Y')} h'(\theta z) \\ &= \frac{m_z(Y)}{m(Y)} h(z) \\ &= h_Y(z), \end{aligned}$$

and

$$\overline{m_Y} \circ \theta^{-1} = \overline{m'_{Y'}},$$

hence, for $k \in \mathbb{Z}$,

$$\frac{d\overline{m'_{Y'}}U'^k}{d\overline{m'_{Y'}}}(\theta z) = \frac{d\overline{m_Y}U^k}{d\overline{m_Y}}(z)$$

Lemma 4.1

There is a measure preserving and measure space isomorphism $\Phi :$ $(Y, \mathcal{B}(Y), m_Y) \to (Y', \mathcal{B}(Y'), m'_{Y'})$ *with the following properties:*

$$\pi' \circ \Phi = \theta \circ \pi, \quad (x \in Y), \tag{1}$$

$$(m'_{Y'})_{\theta z}(\Phi(A)) = (m_Y)_z(A), \quad (A \subset Y, z \in \Omega), \tag{2}$$

$$(\Phi(x), \Phi(y)) \in \mathcal{R}'_{Y'} \quad \textit{iff} \quad (x, y) \in \mathcal{R}_Y, \tag{3}$$

$$\varphi'_{Y'}(\Phi(x), \Phi(y)) = \varphi_Y(x, y), \quad ((x, y) \in \mathcal{R}_Y). \tag{4}$$

Proof This lemma can be thought of as a "relative" version of Dye's theorem ([Dye]) in that it establishes the existence of an orbit equivalence extending a given factor space isomorphism (conditions (1),(2), and (3)). Condition (4) will follow automatically as θ is an isomorphism of K-representations. The method of proof is to show that towers of $\mathcal{R}_Y$ can be copied as towers of $\mathcal{R}'_{Y'}$.

Copying Lemma ([K-W])

Given any tower

$$\xi = (\mathcal{P}, \mathcal{T}) \quad \mathcal{P} = \{E_\alpha : \alpha \in \Lambda\}, \ \mathcal{T} = \{e_{\alpha,\beta} : \alpha, \beta \in \Lambda\}$$

of $\mathcal{R}_Y$,

there is a measure preserving and measure space isomorphism

$$\Phi : (Y, \mathcal{B}(Y), m_Y) \to (Y', \mathcal{B}(Y'), m'_{Y'})$$

and a tower

$$\xi' = (\mathcal{P}', \mathcal{T}') \quad \mathcal{P}' = \{E'_\alpha : \alpha \in \Lambda\}, \ \mathcal{T}' = \{e'_{\alpha,\beta} : \alpha, \beta \in \Lambda\}$$

of $\mathcal{R'}_Y$*, satisfying:*

$$\pi' \circ \Phi = \theta \circ \pi \tag{1}$$

$$\Phi(E_\alpha) = E'_\alpha \tag{2}$$

$$\Phi e_{\beta,\alpha} = e'_{\beta,\alpha}\Phi \quad \textit{on} \quad E_\alpha \tag{3}$$

$$\pi'(e'_{\beta,\alpha} \circ \Phi(x)) = U'^{-n}\pi'(\Phi(x)) \quad \textit{if} \quad \pi(e_{\beta,\alpha}x) = U^{-n}\pi(x), \tag{4}$$

where $n \in \mathbb{Z}$.

Proof of the Copying Lemma

Let $\alpha, \beta \in \Lambda$. Then $\pi(e_{\beta,\alpha}x)$ is of the form:

$$\pi(e_{\beta,\alpha}x) = U^{-n}\pi(x) \quad (x \in E_\alpha),$$

where $n = n(\beta, \alpha, x) \in \mathbb{Z}$.

Partition each set E_α into countable disjoint subsets $E_{\alpha,i}, i \geq 1$ so that

$$n(\beta, \alpha, x) = n(\beta, \alpha, i) \text{ (constant) for } x \in E_{\alpha,i},$$

and, for each $\alpha, \beta \in \Lambda$ and $i \geq 1$,

$$E_{\beta,i} = e_{\beta,\alpha}(E_{\alpha,i}).$$

Set

$$Y_i = \bigcup_\beta E_{\beta,i}.$$

Now we have a countable disjoint family of the restrictions $\xi_i = (\mathcal{P} \cap Y_i, \mathcal{T}|_{Y_i})$ of the tower ξ to the sets Y_i . Here, $\mathcal{P} \cap Y_i := \{E_\alpha \cap Y_i : \alpha \in \Lambda\}$, and $\mathcal{T}|_{Y_i} = \{e_{\gamma,\beta,i} := e_{\gamma,\beta}|_{Y_i} : \gamma, \beta \in \Lambda\}$.

As we'll copy each ξ_i individually, and disjointly, we'll drop the subscript i, and "assume" that

$$n(\beta, \alpha, x) = \text{constant} = n_{\beta,\alpha} \quad \text{on} \quad E_\alpha.$$

So, $e_{\beta,\alpha} \in N[\mathcal{S}_Y]_* \cap [\mathcal{R}_Y]_*$, and

$$U^{-n_{\beta,\alpha}} \pi(E_\alpha) = \pi(E_\beta)$$

In order to facilitate notation, we'll denote for the rest of the proof of the copying lemma:

$$\overline{m_Y} = \nu, \ (m_Y)_z = \nu_z, \ \overline{m'_{Y'}} = \nu', \ (m'_{Y'})_{z'} = m'_{z'}, \quad (z \in \Omega, \ z' \in \Omega').$$

We recall that $\nu \circ \pi^{-1} = \nu'$, and note Lemma 3.6 can now be written as:

Lemma 3.6' *If $R \in N[\mathcal{S}_Y]_* \cap [\mathcal{R}_Y]_*, E \subset Dom(R)$, then for a.e. $z \in \pi(Im(R))$,*

$$\nu_z(RE) = \nu_{U^{-n(z)}z}(E) \cdot \frac{d\nu U^{-n(z)}}{d\nu}(z),$$

where $n(z)$ be as in Lemma 3.5.

Choose a finite partition $\{E'_\beta | \beta \in \Lambda\}$ of Y' so that

$$\nu'_{\theta z}(E'_\beta) = \nu_z(E_\beta), \quad (z \in \Omega).$$

Fix $\alpha \in \Lambda$ and take a measure preserving and measure space isomorphism $\Phi = \Phi_\xi : E_\alpha \to E'_\alpha$ such that $\pi' \Phi = \theta \pi$. Write $n_{\beta,\alpha} = n_\beta$. It follows that

$$\nu'_{\theta z}(\Phi(A)) = \nu_z(A), \quad \Big(z \in \Omega, \quad A \subset E_\alpha\Big).$$

It follows from Lemma 3.6' that for a.e.$z \in \pi(E_\beta)$,

$$\nu_z(E_\beta) = \nu_z(e_{\beta,\alpha}E_\alpha) = \nu_{U^{n_\beta}z}(E_\alpha)\frac{d\nu U^{n_\beta}}{d\nu}(z),$$

or for a.e. $z \in E_\alpha$,

$$\begin{aligned}&\nu_{U^{-n_\beta}z}(E_\beta)\\ &= \nu_{U^{-n_\beta}z}(e_{\beta,\alpha}(E_\alpha))\\ &= \nu_z(E_\alpha)\left(\frac{d\nu U^{-n_\beta}}{d\nu}(z)\right)^{-1}.\end{aligned}$$

By corollary 3.4, one can choose $R' \in N[\mathcal{S}'_{Y'}]_*$ such that

$$\mathrm{Dom}\,(R') \subset E'_\alpha \tag{1}$$

$$\nu'_{z'}(\mathrm{Dom}\,(R')) > 0 \quad \text{if and only if} \quad {m'}_{z'}(E'_\alpha) > 0 \quad \text{a.e.} z \tag{2}$$

$$\pi'(R'x') = U'^{-n_\beta}\pi'(x'), \quad (x' \in \mathrm{Dom}\,(R')). \tag{3}$$

We now claim that for a.e.$z' \in \pi'(E'_\beta)$

$$\nu'_z(\mathrm{Im}\,(R')) \leq \nu'_z(E'_\beta).$$

To see this, we notice that $\pi'(\mathrm{Im}\,(R')) = \pi'(E'_\beta)$. By Lemma 3.6', for a.e.$z' \in \pi'(E'_\beta)$

$$\begin{aligned} rl\nu'_z(\mathrm{Im}\,R') &= \nu'_{U'^{n_\beta}z'}(\mathrm{Dom}\,R')(\frac{d\nu' U'^{-n_\beta}}{d\nu'}(U'^{n_\beta}z'))^{-1}\\ &\leq \nu'_{U'^{n_\beta}z'}({E'}_\alpha)\frac{d\nu' U'^{n_\beta}}{d\nu'}(z')\\ &= \nu_{U^{n_\beta}z}(E_\alpha)\frac{d\nu U^{n_\beta}}{d\nu}(z)\\ &= \nu_z(E_\beta)\\ &= \nu'_{z'}(E'_\beta)\end{aligned}$$

where $z = \theta^{-1}(z')$.

For a.e.$z' \in \pi'(E'_\beta)$ define $d = d_\beta = d_\beta(z') \geq 1$ by

$$d_\beta = [\frac{\nu'_{z'}(E'_\beta)}{\nu'_{z'}(\mathrm{Im}\,R')}] = \max\{k \in I\!\!N : k \leq \frac{\nu'_{z'}(E'_\beta)}{\nu'_{z'}(\mathrm{Im}\,R')}\}.$$

Applying Hopf-equivalence, we obtain ${g'}_1, {g'}_2, \cdots, g'_d \in [\mathcal{S}'_{Y'}]_*$ satisfying:

$$\mathrm{Dom}\,(g'_i) \subset \mathrm{Im}\,(R'), \quad \mathrm{Im}\,(g'_i) \subset E'_\beta \quad (i \geq 1). \tag{1}$$

$$\text{The subsets } \mathrm{Im}(g'_i)\text{'s are disjoint.} \tag{2}$$

$$\nu'_{U'^{-n_\beta}z'}(\mathrm{Dom}\,(g'_i)\triangle\mathrm{Im}\,(R')) = 0 \quad (1 \leq i \leq d). \tag{3}$$

$$\nu'_{U'^{-n_\beta}z'}(\mathrm{Dom}\,(g'_{d+1})) = \nu'_{U'^{-n_\beta}z'}(E'_\beta) - d\cdot\nu'_{U'^{-n_\beta}z'}(\mathrm{Im}\,R'). \qquad (4)$$

where $z' \in \pi'(E'_\alpha)$ and $d = d_\beta(U'^{-n_\beta}z')$. Then obviously,

$$\bigcup_i \mathrm{Im}\,(g'_i) = E'_\beta.$$

We are going to show that for a.e.$z' \in \pi'(E'_\alpha)$

$$[\frac{\nu'_{z'}(E'_\alpha)}{\nu'_{z'}(\mathrm{Dom}\,(R'))}] = d \qquad (1)$$

$$\nu'_{z'}(R'^{-1}(\mathrm{Dom}\,(g'_{d+1}))) = \nu'_{z'}(E'_\alpha) - d\cdot\nu'_{z'}(\mathrm{Dom}\,(R')) \qquad (2)$$

where $d = d_\beta(U'^{-n_\beta}z')$.

Let $z' \in \pi'(E'_\alpha)$. The first is obtained from

$$\begin{aligned}\frac{\nu'_{U'^{-n_\beta}z'}(E'_\beta)}{\nu'_{U'^{-n_\beta}z'}(\mathrm{Im}\,(R'))} &= \frac{\nu'_{U'^{-n_\beta}z'}(E'_\beta)}{\nu'_{z'}(\mathrm{Dom}\,(R'))}\cdot\frac{d\nu' U'^{-n_\beta}}{d\nu'}(z') \qquad \Big(\text{use Lemma 3.6'}\Big)\\ &= \frac{\nu_{U^{-n_\beta}z}(E_\beta)}{\nu_{z'}(\mathrm{Dom}\,(R'))}\cdot\frac{d\nu U^{-n_\beta}}{d\nu}(z)\\ &= \frac{\nu_z(E_\alpha)}{\nu'_{z'}(\mathrm{Dom}\,(R'))}\\ &= \frac{\nu'_{z'}(E'_\alpha)}{\nu'_{z'}(\mathrm{Dom}\,(R'))}.\end{aligned}$$

The second is that if $z' \in \pi'(E'_\alpha)$ and $d = d_\beta(U'^{-n_\beta}z')$ then

$$\begin{aligned}&\nu'_{z'}(R'^{-1}(\mathrm{Dom}\,(g'_{d+1})))\\ &= \nu'_{U'^{-n_\beta}z'}(\mathrm{Dom}\,(g'_{d+1}))\frac{d\nu' U'^{-n_\beta}}{d\nu'}(z')\\ &= \Big(\nu'_{U'^{-n_\beta}z'}(E'_\beta) - d\cdot\nu'_{U'^{-n_\beta}z'}(\mathrm{Im}\,(R'))\Big)\frac{d\nu' U'^{-n_\beta}}{d\nu'}(z')\\ &= \nu'_{z'}(E'_\alpha) - d\cdot\nu'_{z'}(\mathrm{Dom}\,(R')).\end{aligned}$$

Thus, if $z' \in \pi(E'_\alpha)$ and $d = d_\beta(U'^{-n_\beta}z')$ then

$$\begin{aligned}&\nu'_{z'}(R'^{-1}(\mathrm{Dom}\,(g'_{d+1})))\\ &= \nu'_{z'}(E'_\alpha) - [\frac{\nu'_{z'}(E'_\alpha)}{\nu'_{z'}(\mathrm{Dom}\,(R'))}]\cdot\nu'_{z'}(\mathrm{Dom}\,(R')).\end{aligned}$$

Therefore, using Hopf-equivalence by $[\mathcal{S}'_{Y'}]_*$, we obtain $\rho'_i \in [\mathcal{S}'_{Y'}]_*$ satisfying the following conditions:

$$\mathrm{Dom}\,(\rho'_i) \subset \mathrm{Dom}\,(R'),\quad \mathrm{Im}\,(\rho'_i) \subset E'_\alpha. \qquad (1)$$

$$\text{The sets }\ \mathrm{Im}\,(\rho'_i)\ \text{ are disjoint.} \qquad (2)$$

$$\text{If} \quad z' \in \pi'(E'_\alpha) \text{ and } \quad d = d_\beta(U'^{-n_\beta} z') \quad \text{and} \quad 1 \le i \le d \text{ then} \tag{3}$$

$$\nu'_{z'}(\mathrm{Dom}\,(\rho'_i) \triangle \mathrm{Dom}\,(R')) = 0 \qquad (1 \le i \le d).$$

$$\nu'_{z'}(\mathrm{Dom}\,(\rho'_{d+1}) \triangle R'^{-1}(\mathrm{Dom}\,(g'_{d+1}))) = 0. \tag{4}$$

Then, obviously

$$\bigcup_i \mathrm{Im}\,(\rho'_i) = E'_\alpha.$$

Using g_i's and ρ_i's, let us define a map ${e'}_{\beta,\alpha} : E'_\alpha \to E'_\beta$ by setting for $x' \in \mathrm{Im}(\rho'_i)$,

$${e'}_{\beta,\alpha} x' = g'_i \cdot R' \cdot {\rho'_i}^{-1}(x').$$

(1)

$${e'}_{\beta,\alpha} \in N[\mathcal{S}'_{Y'}]_*,$$

(2)

$$\mathrm{Dom}\,(e'_{\beta,\alpha}) = E'_\alpha, \quad \mathrm{Im}\,(e'_{\beta,\alpha}) = E'_\beta$$

(3)

$$\pi'({e'}_{\beta,\alpha} x') = U'^{-n_\beta} \pi'(x'), \quad (x' \in E'_\alpha)$$

Extend $\Phi : E_\alpha \to E'_\alpha$ by setting for each $\beta \in \Lambda$,

$$\Phi = e'_{\beta,\alpha} \circ \Phi \circ e_{\alpha,\beta}.$$

Set

$$\begin{aligned} e'_{\alpha,\beta} &= {e'_{\beta,\alpha}}^{-1}, \\ e'_{\beta,\epsilon} &= e'_{\beta,\alpha} e'_{\alpha,\epsilon}, \\ \xi' &= (\mathcal{P}', \mathcal{T}') \text{ where} \\ \mathcal{P}' &= \{E'_\beta : \beta \in \Lambda\} \text{ and} \\ \mathcal{T}' &= \{e'_{\beta,\epsilon} | \beta, \epsilon \in \Lambda\}. \end{aligned}$$

We have constructed the tower ξ', and completed the proof of the copying lemma. □

By hyperfiniteness, $\mathcal{B}(Y)$ is generated by a sequence of towers of $\mathcal{R}_Y$, and each $\mathcal{R}_Y$-orbit is a countable increasing union of finite orbits by towers. So, in order to complete the proof of lemma 4.1, we apply the copying lemma to a refinement of ξ' in Y', which approximates $\mathcal{B}(Y')$ and $\mathcal{R}'_{Y'}$ orbits with some fixed precision, obtaining a refinement of ξ in Y, and continue this procedure back and forth as before. In the limit we obtain $\Phi : Y \to Y'$ satisfying conditions (1),(2), and (3) of the lemma. As mentioned above, condition (4) follows automatically. □

We complete the proof of theorem 2 by extending the domain of definition of Φ. Set

$$Z_{2,n} = \{z \in Z_2 | nh'(\theta z) \leq h(z) < (n+1)h'(\theta z)\}, \quad n \geq 1.$$

Choose a countable partition $\{F_{n,i} | n \geq 1, 1 \leq i \leq n\}$ of the set $X \backslash Y$ satisfying

(1) The subsets $\pi(F_{n,i})$ are disjoint and

$$\bigcup_{i=1}^{n} \pi(F_{n,i}) = Z_{2,n}$$

(2)

$$m_z(F_{n,i}) = \frac{h'(\theta z)}{h(z)} m_z(Y), \quad (i < n, z \in Z_{2,n})$$
$$m_z(F_{n,n}) = \frac{h(z) - nh'(\theta z)}{h(z)} \quad (z \in Z_{2,n})$$

For each n and for each $1 \leq i \leq n$, a partial transformation $\alpha_{n,i} \in [\mathcal{S}]_*$ with the domain $F_{n,i}$ and the image which is a subset of Y is defined. Set

$$G_{n,i} = \mathrm{Im}\,(\alpha_{n,i}), \quad G'_{n,i} = \Phi(G_{n,i}).$$

We recall $m(Y) = m'(Y')$ and choose a countable partition $\{F'_{n,i}\}$ of $X' \backslash Y'$ such that

$$m'(F'_{n,i}) = m'(G'_{n,i})$$

Then, by Hopf-equivalence, we obtain partial transformations $\alpha'_{n,i} \in [\mathcal{R}']_*$ with domain $F'_{n,i}$ and image $G'_{n,i}$.

We obtain a measure preserving isomorphism from X onto X' by extending the previous Φ by setting

$$\Phi(x) = \alpha'^{-1}_{n,i} \circ \Phi \circ \alpha_{n,i}(x) \quad (x \in F_{n,i}).$$

Finally let us define a measurable function $\eta(x)$ which cancels the difference of φ and φ' on the set Y^c. Set

$$rl\eta(x) = 0 \quad (x \in Y),$$
$$\eta(x) = \varphi'(\alpha'^{-1}_{n,i} \circ \Phi \circ \alpha_{n,i}(x), \quad \Phi \circ \alpha_{n,i}(x)) \quad (x \in F_{n,i}).$$

Then, it is easily checked that

$$\varphi(x,y) - \varphi'(\Phi(x), \Phi(y)) = \eta(y) - \eta(x).$$

□

Concluding Remarks

(1) The possible lack of $g \in [\mathcal{R}]$ satisfying $\pi(gx) = U\pi x$ (as pointed out in section 3) makes the proof of theorem 2 more difficult. If the hyperfinite equivalence relation $\mathcal{R}$ admits a σ-finite, infinite invariant measure, then such g exist, and a simplification of our proof may establish uniqueness (up to cohomology via orbit equivalence) of a cocycle with a given free, conservative action. Indeed, in this type II_∞ setup, Bezugly and Golodets [B-G] obtained this result using Krieger's cohomology lemma and discrete decomposition theorem, which use the existence of g. However in our setup, the invariant measure is finite and so, the proof must be rigid.

(2) As a consequence of this, it is shown in [B-G] that $\varphi\times\text{Id}$ and $\varphi'\times\text{Id}$ are cohomologous up to orbit equivalence. Here, $\varphi\times\text{Id}$ is the cocycle on the product equivalence relation $\mathcal{R}\times\mathcal{R}_{\mathbb{Z}}$ of type II_∞ where

$$((x,i),\ \ (y,j)) \in \mathcal{R}\times\mathcal{R}_{\mathbb{Z}} \ \text{ iff } \ (x,y)\in\mathcal{R}, \text{ and } i,\ j\in\mathbb{Z}$$

and where

$$(\varphi\times\text{Id})((x,i),\ \ (y,j)) = \varphi(x,y) \qquad ((x,y)\in\mathcal{R}, \text{ and } i,\ j\in\mathbb{Z}).$$

Theorem 2 refines this result.

(3) When $\mathcal{R}$ is of type III, then the tensor product cocycle $(\varphi, \log(\rho))$ taking values in $\mathbb{R}\times\mathbb{R}$ where ρ is the Radon-Nikodym cocycle, and φ has a given action is unique up to cohomology via orbit equivalence. In this setup, the copying lemma will be proved by replacing $\mathcal{S}$ by $\text{Ker}(\varphi)\cap\text{Ker}(\log(\rho))$. This is established by a different method in [B-G].

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