DISTRIBUTIONAL LIMITS FOR HYPERBOLIC, INFINITE VOLUME GEODESIC FLOWS

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1. Geodesic flows on surfaces of constant negative curvature.

Let H denote the two-dimensional hyperbolic space and φ^t ($t \in \mathbb{R}$) the geodesic flow on $H \times \mathbb{T}$, where $\mathbb{T}$ is the natural identification of directions. Throughout this note we work with the model of the Poincaré disk instead of the Poincaré upper half-plane (sometimes also called the Lobachevsky plane). So we consider $H = \{z \in \mathbb{C} : |z| < 1\}$.

Let Γ be a discrete group of isometries of H, and let H/Γ denote the surface defined by Γ equipped with the metric induced by the hyperbolic metric ρ (see [Be]). The space of line elements of H/Γ is $X_\Gamma := (H/\Gamma) \times \mathbb{T} = (H \times \mathbb{T})/\Gamma$ (also equipped with the induced metric) and the geodesic flow transformations on X_Γ are defined by

$$\varphi_\Gamma^t \Gamma(\omega) = \Gamma \varphi^t(\omega).$$

We consider the measure m =hyperbolic area $\times$ normalised Lebesgue measure on $H \times \mathbb{T}$ and the corresponding induced measure m_Γ on $(H/\Gamma) \times \mathbb{T}$.

For a compact surface the dynamical system $(H/\Gamma, (\varphi_\Gamma^t)_{t \in \mathbb{R}})$ is an Anosov system ([An1], [An2]), the measure m_Γ is finite and φ_Γ is a Bernoulli flow. This is proven by using the existence of expanding and contracting flow invariant foliations (also used by Anosov and Sinai to show that φ_Γ is a K-flow (cf. [An2])) and applying the Ornstein isomorphism theory (Ornstein, Weiss [O-W]). Here we are mainly interested in the non-compact case and our result holds for conservative geodesic flows φ_Γ. The following characterisation of these dynamical systems is given in the work of Hopf and Tsuji (cf. [Ho1], [Ho2], [Ts1] and [Ts2]), which uses methods from potential theory.

Theorem HT.

The geodesic flow φ_Γ is either totally dissipative, or conservative and ergodic.

The geodesic flow is conservative iff

$$\sum_{\gamma \in \Gamma} e^{-\rho(x,\gamma y)} = \infty.$$

This research was supported by a grant from G.I.F., the German-Israel Foundation for Scientific Research and Development.

As in Tr. Mat. Inst. Steklova 216 (1997), Din. Sist. i Smezhnye Vopr., 181–192.

A new formulation of these conditions in terms of Brownian motion has been given by Sullivan (see [Su]).

The series $\sum_{\gamma\in\Gamma} e^{-\rho(x,\gamma y)}$ is called the *Poincaré series.* The *asymptotic Poincaré series* is defined by

$$a_\Gamma(x,y;t) := \sum_{\gamma\in\Gamma;\rho(x,\gamma y)\le t} e^{-\rho(x,\gamma y)} = \int_0^t a_\Gamma(x,y;ds),$$

where $a_\Gamma(x,y;\cdot)$ denotes the distribution function of the measure on $\mathbb{R}$ putting mass $|\{\gamma\in\Gamma:\rho(x,\gamma y)=s\}|\exp[-s]$ on the point $s\in\mathbb{R}$. The asymptotic Poincaré series is up to asymptotic equivalence independent of x and y and denoted by $a_\Gamma(t)$. Here we use $a_\Gamma(ds)=a_\Gamma(0,0,ds)$. For surfaces H/Γ of finite volume, the Poincaré series always diverges, and indeed, $a_\Gamma(t)\propto t$ (as can be deduced from the ergodic theorem). There are also surfaces of infinite volume with divergent Poincaré series, and the following theorem is shown in [A-S]:

Theorem AS. *Any conservative geodesic flow φ_Γ is rationally ergodic with return sequence proportional to $a_\Gamma(t)$.*

In fact, the proportionality factor turns out to be 8π, when the specific measure $dA\times d\theta$ (as introduced above) is used. This can be deduced from our proofs below. The proof of the theorem relies on the estimate

$$\forall\ x,y\in H/\Gamma,\ \epsilon>0,\ \exists\ M\ge 0$$

$$0\le\int_{\Delta(y,\epsilon)} S_t(1_{\Delta(x,\epsilon)})^2 dm_\Gamma \le M\left(\int_{\Delta(y,\epsilon)} S_t(1_{\Delta(x,\epsilon)})dm_\Gamma\right)^2\quad\forall\ t>0,$$

where $\Delta(z,\epsilon)$ is defined as below. In this note, we prove an extension of this estimate for p-th moments, and this provides the distributional limit theorem:

Theorem AD. *Let φ_Γ be a conservative geodesic flow, whose return sequence $a(t)$ is regularly varying with index $\alpha\in[0,1]$. Then for any $f\in L^1_+(m_\Gamma)$ the sequence $S_t(f)/a(t)$ converges in distribution to a random variable $Y_\alpha\int_{X_\Gamma} f\,dm_\Gamma$ where*

$$S_t(f) := \int_0^t f\circ\varphi_\Gamma^s ds,$$

and Y_α has the Mittag-Leffler distribution of order α given by

$$E\big(\exp(zY_\alpha)\big) = \sum_{n=0}^\infty \frac{\Gamma(1+\alpha)^n z^n}{\Gamma(1+\alpha n)}$$

for $z\in\mathbb{C}$ when $0<\alpha\le 1$ and for $|z|<1$ when $\alpha=0$.

Here, *convergence in distribution* means that for any bounded continuous function $g:[0,\infty]\to\mathbb{R}$ and any probability measure $q\ll m$, we have

$$\int_{X_\Gamma} g\left(\frac{S_t(f)}{a(t)}\right)\,dq \to E(g(Y_\alpha m_\Gamma(f)))$$

as $t \to \infty$ where $m_\Gamma(f) = \int_X f\, dm_\Gamma$.

The theorem is applicable to Abelian covers of compact surfaces. It follows from estimations of numbers of closed geodesics that $a_\Gamma(t) \propto \sqrt{t}$ when H/Γ is a $\mathbb{Z}$-cover of a compact surface, and that $a_\Gamma(t) \propto \log t$ when H/Γ is a $\mathbb{Z}^2$-cover of a compact surface (see [Ad-S], [Ph-S], [La], [Po-S]).

The proof of the theorem uses the method of Darling-Kac [D-K], (see also [Aa]). To our knowledge, this is the first attempt to apply this method in the absence of some convenient Frobenius-Perron operator.

2. Preliminaries from hyperbolic geometry and geodesic flows.

Consider the hyperbolic space $H := \{z \in \mathbb{C} \,:\, |z| < 1\}$ equipped with the arclength

$$ds(u,v) := 2\frac{\sqrt{du^2 + dv^2}}{1 - u^2 - v^2},$$

and the area

$$dA(u,v) := 4\frac{dudv}{(1 - u^2 - v^2)^2}.$$

The *hyperbolic distance* (cf. [Be]) between $x, y \in H$ is denoted by

$$\rho(x,y) = \inf\{\int_\gamma ds : \ \gamma \ \text{ is an arc joining } x \text{ and } y\} = 2\tanh^{-1}\frac{|x-y|}{|1-\overline{x}y|}.$$

Note that with this metric H/Γ has curvature -1, while the metric used in [A-S] gives curvature -4.

For $x \in H$, and $\epsilon > 0$, set

$$N_\rho(x,\epsilon) = \{y \in H : \rho(x,y) < \epsilon\}, \quad \Delta(x,\epsilon) := N_\rho(x,\epsilon) \times \mathbb{T}.$$

Consider the angle set subtended by $N_\rho(y,\epsilon)$ at $0 \notin N_\rho(y,\epsilon)$,

$$\Lambda(y,\epsilon) := \{\theta \in [0,2\pi] : \exists\, r \in (0,1) \ \ni \ \rho(y, re^{i\theta}) < \epsilon\}.$$

We note that

$$\Lambda(y,\epsilon) = \{\theta \in [0,2\pi] : \|\theta - \arg y\| < \sin^{-1}\left(\frac{(1-|y|^2)\tanh\frac{\epsilon}{2}}{|y|(1-\tanh^2\frac{\epsilon}{2})}\right)\}, \tag{1}$$

where $\|\theta\| := \theta \wedge (2\pi - \theta) \ \ \theta \in [0,2\pi)$.

In order to see this, let $\delta = \tanh\frac{\epsilon}{2}$. Then

$$N_\rho(y,\epsilon) = B\left(\frac{(1-\delta^2)y}{1-\delta^2|y|^2}, \frac{\delta(1-|y|^2)}{1-\delta^2|y|^2}\right)$$

where $B(x,r)$ is the Euclidean ball of radius r,

$$B(x,r) = \{y \in \mathbb{C} \,:\, |x-y| < r\}.$$

We'll write

$$\|\Lambda(y,\epsilon)\| = 2\sin^{-1}\left(\frac{(1-|y|^2)\tanh\frac{\epsilon}{2}}{|y|(1-\tanh^2\frac{\epsilon}{2})}\right)$$
$$\sim \frac{2(1-|y|^2)\tanh\frac{\epsilon}{2}}{1-\tanh^2\frac{\epsilon}{2}} \text{ as } |y|\to 1.$$

We also need the following fact: Suppose that $x, y \in H$, $|y| > |x|$, and $\|\arg y - \arg x\| = \theta$, then

$$\rho(0,y) \geq \rho(0,x) + \rho(x,y) - \frac{2\theta}{1-|x|^2}. \tag{2}$$

This can easily be seen as follows: Let $x' \in H$, $|x'| = |x|$, and $\arg x' = \arg y$, then

$$\rho(0,y) = \rho(0,x') + \rho(x',y) \geq \rho(0,x) + \rho(x,y) - \rho(x',x),$$

and clearly

$$\rho(x',x) \leq \frac{2\theta}{1-|x|^2}.$$

Finally, we mention a last evident fact:

$$A(N(x,\epsilon)) \sim \pi\epsilon^2 \text{ as } \epsilon \to 0. \tag{3}$$

3. Proof of the theorem

Let $A \in \mathcal{B}(X_\Gamma)$, $p \geq 1$ and $t > 0$. Define $a^A(p,t) : X_\Gamma \to \mathbb{R}_+$ by

$$a^A(p,t) = \int \dots \int_{0<t_1<\dots<t_p<t} \prod_{\nu=1}^{p} 1_A \circ \varphi_\Gamma^{t_\nu} dt_1 \dots dt_p.$$

Then

$$(S_t(1_A))^p = p! a^A(p,t),$$

and

$$a^A(p+1,t)(\omega) = \int_0^t 1_A(\varphi_\Gamma^s \omega) a^A(p,t-s)(\varphi_\Gamma^s\omega) ds.$$

Set

$$\overline{a}^A(p,t) = \int_A a^A(p,t) dm_\Gamma,$$

and, for $\lambda > 0$,

$$u^A(p,\lambda) = \int_0^\infty a^A(p,t) e^{-\lambda t} dt$$
$$\overline{u}^A(p,\lambda) = \int_0^\infty \overline{a}^A(p,t) e^{-\lambda t} dt.$$

In case $A = \Delta(x,\epsilon) = N_\rho(x,\epsilon) \times \mathbb{T}$ we shall omit the index A. Also note that in this case (see [A-S] or the proof of the Geometric Lemma below)

$$\overline{a}^\Delta(1,t) \sim 8\pi A(N)^2 a_\Gamma(t).$$

Our goal is the

Main Lemma.

$$\exists M > 0 \ \ni \forall t > 0, p \geq 1 \qquad \overline{a}^{\Delta}(p,t) \leq M^p a_\Gamma(t)^p \tag{4}$$

and

$$\overline{u}^{\Delta}(p,\lambda) \sim \frac{1}{\lambda}\overline{u}(\lambda)^p m_\Gamma(\Delta)^{p+1} \ \textit{as}\ \lambda \to 0 \ \ \forall\ p \in I\!N \tag{5}$$

where $\overline{u}(\lambda) = \frac{1}{m_\Gamma(\Delta)^2}\int_0^\infty m_\Gamma(\Delta \cap \varphi_\Gamma^{-t}\Delta)e^{-\lambda t}dt.$

We first show how Theorem AD is obtained from the Main Lemma.

Proof of Theorem AD. Since $a_\Gamma(t)$ is regularly varying, it has a representation $a_\Gamma(t) = t^\alpha L(t)$ for some $\alpha > 0$ and some slowly varying function L. An application of Karamata's Tauberian theorem to the Laplace transform in (5) (as in the proof of theorem 1 in [D-K]) shows for $p \geq 1$

$$\int_\Delta \Big(S_t(1_\Delta)\Big)^p dm_\Gamma \sim m_\Gamma(\Delta)^{p+1} p! \frac{\Gamma(1+\alpha)^p}{\Gamma(1+p\alpha)} a_\Gamma(t)^p$$

as $t \to \infty$. Since the bound in (4) is uniform over p we have

$$\int_\Delta e^{zS_t(1_\Delta)a_\Gamma^{-1}(t)}dm_\Gamma = \sum_{p=0}^\infty \frac{1}{p!}\int_\Delta \Big(zS_t(1_\Delta)a_\Gamma^{-1}(t)\Big)^p dm_\Gamma$$
$$\to m_\Gamma(\Delta)\sum_{p=0}^\infty z^p m_\Gamma(\Delta)^p \frac{\Gamma(1+\alpha)^p}{\Gamma(1+p\alpha)} = m_\Gamma(\Delta)E(\exp(zY_\alpha)).$$

The proof of the Main Lemma follows from the following two facts:

Geometric Lemma. *Let $x \in H$. There is a function $\eta(\epsilon) \to 0$ as $\epsilon \to 0$ such that for every $\epsilon > 0$, $p \geq 1$ and all t sufficiently large*

$$\overline{a}(p+1,t) = e^{\pm\eta(\epsilon)} 8\pi m_\Gamma(\Delta)\int_0^t \overline{a}(p,t-s)a_\Gamma(ds),$$

where $\Delta = \Delta(x,\epsilon)$.

The lemma can be strengthened to the following form: $\forall \epsilon > 0,\ p \geq 1,\ t > 0$

$$\overline{a}(p+1,t) = e^{\pm\eta(\epsilon)}e^{\pm\kappa(t)} 8\pi m_\Gamma(\Delta)\int_0^t \overline{a}(p,t-s)a_\Gamma(ds),$$

where $\kappa(t) \to 0$ as $t \to \infty$.

Probabilistic Lemma. *Let $\epsilon_0 > 0$, and let $\Delta = \Delta(x,\epsilon_0)$. Then for $A \in \mathcal{B}(X_\Gamma) \cap \Delta$ and $p \geq 1$*

$$\int_A u^A(p,\lambda)dm_\Gamma \sim \frac{m_\Gamma(A)^{p+1}}{m_\Gamma(\Delta)^{p+1}}\int_\Delta u^\Delta(p,\lambda)dm_\Gamma$$

as $\lambda \to 0$.

Proof of the Main Lemma. (4) follows immediately from an iterated application of the Geometric Lemma.

In particular, choosing $A = N(x, \epsilon') \times \mathbb{T}$ for ϵ' small,

$$\overline{u}^A(p+1, \lambda) = e^{\pm\eta(\epsilon')} 8\pi m_\Gamma(A) \int_0^\infty \int_0^t \overline{a}^A(p, t-s) e^{-\lambda t} a_\Gamma(ds) dt$$
$$= e^{\pm\eta(\epsilon')} 8\pi m_\Gamma(A) \int_0^\infty \int_0^\infty \overline{a}^A(p, r) e^{-\lambda(r+s)} a_\Gamma(ds) dr$$
$$= e^{\pm\eta(\epsilon')} 8\pi m_\Gamma(A) \overline{u}^A(p, \lambda) \int_0^\infty e^{-\lambda s} a_\Gamma(ds).$$

Iterating this estimate gives

$$\overline{u}^A(p, \lambda) = e^{\pm p\eta(\epsilon')} m_\Gamma(A)^p \overline{u}^A(0, \lambda) \left(8\pi \int_0^\infty e^{-\lambda s} a_\Gamma(ds)\right)^p.$$

By the Probabilistic Lemma,

$$\overline{u}(p, \lambda) \sim \frac{m_\Gamma(\Delta)^{p+1}}{m_\Gamma(A)^{p+1}} \overline{u}^A(p, \lambda)$$
$$= e^{\pm p\eta(\epsilon')} \frac{m_\Gamma(\Delta)^{p+1}}{m_\Gamma(A)} \overline{u}^A(0, \lambda) \left(8\pi \int_0^\infty e^{-\lambda s} a_\Gamma(ds)\right)^p.$$

The lemma now follows from

$$\int_0^\infty e^{-\lambda t} a_\Gamma(dt) = \lambda \int_0^\infty a_\Gamma(t) e^{-\lambda t} dt$$
$$\sim \frac{\lambda}{8\pi m_\Gamma(\Delta)^2} \int_0^\infty \int_0^t m(\Delta \cap \phi_\Gamma^{-s}(\Delta)) ds e^{-\lambda t} dt$$
$$= \frac{1}{8\pi m_\Gamma(\Delta)^2} \int_0^\infty m(\Delta \cap \phi_\Gamma^{-s}(\Delta)) e^{-\lambda s} ds$$

and

$$\overline{u}^A(0, \lambda) = \int_0^\infty \overline{a}(0, t) e^{-\lambda t} dt = m_\Gamma(\Delta) \int_0^\infty e^{-\lambda t} dt.$$

4. Proof of the Geometric Lemma.

For $\underline{\gamma} \in \Gamma^p$ (resp. $\underline{t} \in \mathbb{R}^p$) we denote its coordinates by γ_k (resp. t_k), $k = 1, ..., p$. For $t \in \mathbb{R}_+$ define

$$I_p(t) = \{\underline{t} \in \mathbb{R}^p : 0 < t_1 < ... < t_p < t\}.$$

Let $\epsilon > 0$ be fixed and $N = \Delta \times \mathbb{T}$ as before, where $\Delta = \Delta(x, \epsilon)$. We assume ϵ to be sufficiently small.

First observe that

$$\begin{aligned}\overline{a}(p,t) &= \int_\Delta a(p,t)dm_\Gamma \\ &= \int_\Delta \int_{I_p(t)} \prod_{\nu=1}^p 1_\Delta \circ \varphi_\Gamma^{t_\nu} d\underline{t}dm_\Gamma \\ &= \sum_{\underline{\gamma}\in\Gamma^p} \int_\Delta \int_{I_p(t)} \prod_{\nu=1}^p 1_{\gamma_\nu\Delta} \circ \varphi^{t_\nu} d\underline{t}dm \\ &= \sum_{\underline{\gamma}\in\Gamma^p} \int_N \int_0^1 \int_{I_p(t)} \prod_{\nu=1}^p 1_{\gamma_\nu N\times\mathbb{T}} \circ \varphi^{t_\nu}(z,\theta)d\underline{t}d\theta dA(z).\end{aligned}$$

Set

$$\begin{aligned}\psi_p(t,z) &= \sum_{\underline{\gamma}\in\Gamma^p} \int_0^1 \int_{I_p(t)} \prod_{\nu=1}^p 1_{\gamma_\nu N\times\mathbb{T}} \circ \varphi^{t_\nu}(z,\theta)d\underline{t}d\theta \\ &= \sum_{\underline{\gamma}\in\Gamma^p} \int_0^1 \int_{I_p(t)} \prod_{\nu=1}^p 1_{\varphi_z^{-1}\gamma_\nu N}(\tanh t_\nu e^{i\theta})d\underline{t}d\theta \\ &= \int_0^1 \int_{I_p(t)} \sum_{\underline{\gamma}\in\Gamma^p} \prod_{\nu=1}^p 1_{\varphi_z^{-1}\gamma_\nu N}(\tanh t_\nu e^{i\theta})d\underline{t}d\theta, \end{aligned} \tag{6}$$

so $\overline{a}(p,t) = \int_N \psi_p(t,z)A(dz)$.

Next consider

$$\Gamma_0 := \{\underline{\gamma} \in \Gamma^p : \int_0^1 \int_{I_p(t)} \prod_{\nu=1}^p 1_{\varphi_z^{-1}\gamma_\nu N}(\tanh t_\nu e^{i\theta})d\underline{t}d\theta > 0\}.$$

Since ϵ is so small so that $\{\gamma N\}_{\gamma\in\Gamma}$ are disjoint and since φ_z is ρ-preserving, it follows that if $(\gamma_1,\ldots,\gamma_p) \in \Gamma_0$, then

$$\rho(\gamma_{k+1}(x),z) \geq \rho(\gamma_k(x),z)\ \forall\ 1 \leq k \leq p-1,$$

Denote $\Lambda_\gamma = \Lambda(\varphi_z^{-1}\gamma(x),\epsilon)$, the angle set subtended by $\varphi_z^{-1}\gamma N$ at 0, then by (1),

$$\|\Lambda_\gamma\| \sim 2\epsilon(1-|\varphi_z^{-1}\gamma(x)|^2) \text{ as } \gamma\to\infty, \text{ and } \epsilon\to 0.$$

Set

$$\begin{aligned}\Gamma_p^\pm := \{&\underline{\gamma} = (\gamma_1,...,\gamma_p) \in \Gamma^p :\ \rho(\gamma_{k+1}(x),z) \geq \rho(\gamma_k(x),z), \\ &\|\arg\varphi_z^{-1}\gamma_k(x) - \arg\varphi_z^{-1}\gamma_{k+1}(x)\| \leq \frac{\|\Lambda_{\gamma_k}\| \pm \|\Lambda_{\gamma_{k+1}}\|}{2}\ \forall\ 1\leq k\leq p-1\}.\end{aligned}$$

We claim that

$$\Gamma_p^- \subseteq \Gamma_0 \subseteq \Gamma_p^+.$$

Clearly, if $(\gamma_1, \ldots, \gamma_p) \in \Gamma_0$, then

$$\bigcap_{k=1}^{p} \Lambda_{\gamma_k} \neq \emptyset,$$

and hence for $1 \le k \le p-1$,

$$\| \arg \varphi_z^{-1}\gamma_k(x) - \arg \varphi_z^{-1}\gamma_{k+1}(x)\| \le \frac{\|\Lambda_{\gamma_k}\| + \|\Lambda_{\gamma_{k+1}}\|}{2}.$$

On the other hand, if $\underline{\gamma} = (\gamma_1, \ldots, \gamma_p) \in \Gamma_p^-$, then all balls $\phi_z^{-1}\gamma_\nu N$ lie in the shadow of the first ball ($\nu = 1$), and hence $\gamma \in \Gamma_0$.

Setting, for $p \ge 1$, $\underline{t} = (t_1, ..., t_p)$, $0 < t_1 < ... < t_p$,

$$\Gamma_0(\underline{t}) = \{\underline{\gamma} = (\gamma_1, ..., \gamma_p) \in \Gamma_0 : |\rho(z, \gamma_k(x)) - t_k| \le \epsilon \quad \forall k\}$$
$$\Gamma_p^{\pm}(\underline{t}) = \{\underline{\gamma} = (\gamma_1, ..., \gamma_p) \in \Gamma_p^{\pm} : |\rho(z, \gamma_k(x)) - t_k| \le \epsilon \quad \forall k\}$$

we have from (6) that

$$\psi_p(t, z) = \int_0^1 \int_{I_p(t)} \sum_{\underline{\gamma} \in \Gamma_0(\underline{t})} \prod_{\nu=1}^{p} 1_{\varphi_z^{-1}\gamma_\nu N}(\tanh t_\nu e^{i\theta}) d\underline{t} d\theta.$$

It follows that

$$\psi_p^-(t, z) \le \psi_p(t, z) \le \psi_p^+(t, z),$$

where

$$\psi_p^{\pm}(t, z) = \int_0^1 \int_{I_p(t)} \sum_{\underline{\gamma} \in \Gamma_p^{\pm}(\underline{t})} 1_{\varphi_z^{-1}\gamma_p N}(\tanh t_p e^{i\theta}) d\underline{t} d\theta.$$

For $\beta \in \Gamma$, let

$$\Gamma^{\pm}(\beta) = \{\gamma \in \Gamma : \ \rho(z, \gamma(x)) \ge \rho(z, \beta(x)) - \epsilon,$$
$$|\arg \varphi_z^{-1}\gamma(x) - \arg \varphi_z^{-1}\beta(x)| < \frac{\|\Lambda_\beta\| \pm \|\Lambda_\gamma\|}{2}\}.$$

It follows that for $\underline{t}' = (t_1, ..., t_{p-1})$, $\underline{t} = (t_1, ..., t_p)$, $0 < t_1 < ... < t_p$

$$\Gamma_p^{\pm}(\underline{t}) := \{\underline{\gamma} \in \Gamma^p : (\gamma_1, ..., \gamma_{p-1}) \in \Gamma^{\pm}(\underline{t}'), \ \gamma_p \in \Gamma^{\pm}(\gamma_{p-1}), |\rho(z, \gamma_p(x) - t_p| \le \epsilon\}.$$

Next

$$\begin{aligned}
\psi_p^{\pm}(t, z) &= \int_0^1 \int_{I_p(t)} \sum_{\underline{\gamma} \in \Gamma_p^{\pm}(\underline{t})} 1_{\varphi_z^{-1}\gamma_p N}(\tanh t_p e^{i\theta}) d\underline{t} d\theta \\
&= \int_{I_{p-1}(t)} \sum_{\underline{\gamma} \in \Gamma_{p-1}^{\pm}(\underline{t}')} \int_0^1 \int_{\tanh t_{p-1}}^{\tanh t} \sum_{\substack{\gamma \in \Gamma^{\pm}(\gamma_{p-1}) \\ |\rho(z,\gamma(x)) - \tanh^{-1} r| \le \epsilon}} 1_{\varphi_z^{-1}\gamma N}(re^{i\theta}) \frac{dr d\theta}{1 - r^2} d\underline{t}' \\
&\sim \int_{I_{p-1}(t)} \sum_{\underline{\gamma} \in \Gamma_{p-1}^{\pm}(\underline{t}')} \int_{N_\rho(0,t) \setminus N_\rho(0,t_{p-1})} \frac{1 - |\omega|^2}{4|\omega|} \sum_{\gamma \in \Gamma^{\pm}(\gamma_{p-1})} 1_{\varphi_z^{-1}\gamma N}(\omega) dA(\omega) d\underline{t}' \\
&\sim A(N) \int_{I_{p-1}(t)} \sum_{\underline{\gamma} \in \Gamma_{p-1}^{\pm}(\underline{t}')} \sum_{\substack{\gamma \in \Gamma^{\pm}(\gamma_{p-1}) \\ \rho(z,\gamma(x)) \le t \pm \epsilon}} e^{-\rho(z,\gamma(x))} d\underline{t}' \\
&=: A(N) \Phi_p^{\pm}(t, z).
\end{aligned}$$

The inductive step on $\Phi_p^\pm$ is (with $\underline{t}''=(t_1,...,t_{p-2})$)

$$\Phi_p^\pm(t,z)=\int_{I_{p-1}(t)}\sum_{\underline{\gamma}\in\Gamma_{p-1}^\pm(\underline{t}')}\sum_{\substack{\gamma\in\Gamma^\pm(\gamma_{p-1})\\ \rho(z,\gamma(x))\le t\pm\epsilon}}e^{-\rho(z,\gamma(x))}d\underline{t}'$$

$$=\int_{I_{p-2}(t)}\sum_{\underline{\gamma}\in\Gamma_{p-2}^\pm(\underline{t}'')}\left(\int_{t_{p-2}}^t\sum_{\substack{\beta\in\Gamma^\pm(\gamma_{p-2})\\ \rho(z,|\beta(x))-\tau|\le\epsilon}}\sum_{\substack{\gamma\in\Gamma^\pm(\beta))\\ \rho(z,\gamma(x))\le t\pm\epsilon}}e^{-\rho(z,\gamma(x))}d\tau\right)d\underline{t}''$$

$$\sim 2\epsilon\int_{I_{p-2}(t)}\sum_{\underline{\gamma}\in\Gamma_{p-2}^\pm(\underline{t}'')}\left(\sum_{\substack{\beta\in\Gamma^\pm(\gamma_{p-2})\\ \rho(z,\beta(x))\le t\pm\epsilon}}\sum_{\substack{\gamma\in\Gamma^\pm(\beta))\\ \rho(z,\gamma(x))\le t\pm\epsilon}}e^{-\rho(z,\gamma(x))}\right)d\underline{t}''.$$

Fixing $\beta\in\Gamma$, we have

$$\sum_{\substack{\gamma\in\Gamma^\pm(\beta))\\ \rho(z,\gamma(x))\le t\pm\epsilon}}e^{-\rho(z,\gamma(x))}=\sum_{\substack{\kappa\in\beta^{-1}\Gamma^\pm(\beta))\\ \rho(z,\beta\kappa(x))\le t\pm\epsilon}}e^{-\rho(z,\beta\kappa(x))}$$
$$\sim e^{-\rho(z,\beta(x))}\sum_{\substack{\kappa\in\beta^{-1}\Gamma^\pm(\beta))\\ \rho(x,\kappa(x))\le t-\rho(z,\beta(x)\pm 2\epsilon}}e^{-\rho(x,\kappa(x))}$$

by (2).

Let $\beta\in\Gamma$ and $\Omega(\beta)$ denote the interval in S^1 such that for $\xi\in\Omega(\beta)$ the ray $\beta^{-1}(0)\xi$ intersects $\beta^{-1}N_\rho(\beta(x),\epsilon)$. It is easily seen that

(7) $\quad\beta^{-1}\Gamma^\pm(\beta)\sim\{\gamma\in\Gamma:\arg\gamma(x)\in\Omega(\beta)\}$ where $|\Omega(\beta)|\to\theta(\epsilon)$ as $|\beta(x)|\to 1$,

where $2\pi|\Omega(\beta)|$ denotes the arc length of $\Omega(\beta)$ and where (this can be deduced from $\cosh(\epsilon)=2/|\xi-\eta|$ where ξ,η are the endpoints of a geodesic tangent to the geodesic ball of radius ϵ and center 0)

$$\theta(\epsilon)\sim 4\epsilon \text{ as } \epsilon\to 0. \tag{8}$$

It has been shown in [A-S], that for a suitable measure μ on H/Γ

$$\frac{1}{a_\Gamma(t)}\int_0^t 1_{N_\rho(z,\epsilon)\times\mathbb{T}}(y,\cdot)\circ\varphi^{-s}ds\to\mu(N)$$

weakly in $L^2(\mathbb{T})$. By standard arguments it follows from this that

$$\mu(N)a_\Gamma(t)|I|\sim\int_I S_t(1_\Delta)dt$$
$$\sim\sum_{\gamma:\arg(\gamma)\in I\pm\epsilon;\rho(0,\gamma(0))\le t}\frac{1}{4}(1-|\gamma(0)|^2)\mu(N).$$

Therefore

$$\sum_{\substack{\kappa\in\beta^{-1}\Gamma^\pm(\beta))\\ \rho(x,\kappa(x))\le t-\rho(z,\beta(x))\pm 2\epsilon}}e^{-\rho(x,\kappa(x))}\sim\ |\Omega(\beta)|a_\Gamma(t-\rho(z,\beta(x))\pm 2\epsilon).$$

$$\sum_{\substack{\kappa\in\gamma_{p-2}^{-1}\Gamma^\pm(\gamma_{p-2}))\\ \rho(x,\kappa(x))\le t-\rho(z,\beta(x)\pm 2\epsilon}}e^{-\rho(x,\kappa(x))}\sim\ |\Omega(\gamma_{p-2})|a_\Gamma(t-\rho(z,\beta(x))\pm 2\epsilon). \tag{9}$$

Using (3), (7)–(9) we obtain (with $\rho_0 = \rho(z, \beta(x))$)

$$\begin{aligned}
&\Phi_p^{\pm}(t,z)\\
&= 2\epsilon \int_{I_{p-2}(t)} \sum_{\underline{\gamma}\in\Gamma^{\pm}_{p-2}(\underline{t}')} \Bigg(\sum_{\substack{\beta\in\Gamma^{\pm}(\gamma_{p-2})\\ \rho_0\le t\pm\epsilon}} e^{-\rho_0}|\Omega(\beta)| a_\Gamma(t-\rho_0\pm 2\epsilon)\Bigg) d\underline{t}''\\
&\sim 8\epsilon^2 \int_{I_{p-2}(t)} \sum_{\underline{\gamma}\in\Gamma^{\pm}_{p-2}(\underline{t}')} \Bigg(\sum_{\substack{\beta\in\Gamma^{\pm}(\gamma_{p-2})\\ \rho_0\le t\pm\epsilon}} e^{-\rho_0} a_\Gamma(t-\rho_0\pm 2\epsilon)\Bigg) d\underline{t}''\\
&\sim 2\epsilon \int_{I_{p-2}(t)} \sum_{\underline{\gamma}\in\Gamma^{\pm}_{p-2}(\underline{t}')} \sum_{\substack{\kappa\in\gamma_{p-2}^{-1}\Gamma^{\pm}(\gamma_{p-2})\\ \rho(x,\kappa(x))\le t-\rho_0\pm\epsilon}} e^{-\rho(x,\kappa(x))} \sum_{\beta'} e^{-\rho(x,\gamma_{p-2}^{-1}\beta'(x))} d\underline{t}''\\
&= 8\epsilon^2 m_\Gamma(\Delta)^{-1}\int_0^t \overline{a}(p-1,t-s)a_\Gamma(ds),
\end{aligned}$$

where

$$\sum_{\beta'} e^{-\rho(x,\gamma_{p-2}^{-1}\beta'(x))} = \sum_{\substack{\gamma_{p-2}^{-1}\beta'\in\gamma_{p-2}^{-1}\Gamma^{\pm}(\gamma_{p-2})\\ \rho(x,\gamma_{p-2}^{-1}\beta'(x))\le t-\rho(x,g_{p-2}^{-1}(z))\pm\epsilon}} e^{-\rho(x,\gamma_{p-2}^{-1}\beta'(x))}.$$

The lemma follows from $m_\Gamma(\Delta) = A(N) \sim \pi\epsilon^2$ (see (3)).

5. Proof of the Probabilistic Lemma.

To prove this, we first show for $\Delta := N_\rho(x,\epsilon)\times \mathbb{T}$ that

$$\exists\ M_p \ \ni\ \int_\Delta u^\Delta(p,\lambda)^2 dm_\Gamma \le M_p\Big(\int_\Delta u^\Delta(p,\lambda)dm_\Gamma\Big)^2\ \forall\ \lambda>0. \tag{10}$$

To see this, we note that

$$\begin{aligned}
\int_\Delta u^\Delta(p,\lambda)^2 dm_\Gamma &= \int_\Delta\int_0^\infty\int_0^\infty a(p,s)a(p,t)e^{-\lambda s}e^{-\lambda t}dsdtdm_\Gamma\\
&= \int_0^\infty\int_0^\infty\Big(\int_\Delta a(p,s)a(p,t)dm_\Gamma\Big)e^{-\lambda s}e^{-\lambda t}dsdt.
\end{aligned}$$

Using the Geometric Lemma we have

$$\begin{aligned}
&\int_\Delta a(p,s)a(p,t)dm_\Gamma \le \Big(\int_\Delta a(p,s)^2dm_\Gamma\Big)^{\frac12}\Big(\int_\Delta a(p,t)^2dm_\Gamma\Big)^{\frac12}\\
&= p!^{-2}\sqrt{\int_\Delta S_s^{2p}dm_\Gamma\int_\Delta S_t^{2p}dm_\Gamma} = \frac{(2p)!}{p!^2}\sqrt{\overline{a}(2p,s)\overline{a}(2p,t)}\\
&\le M_p a_\Gamma(s)^p a_\Gamma(t)^p \le M_p'\int_\Delta a(p,s)dm_\Gamma\int_\Delta a(p,t)dm_\Gamma.
\end{aligned}$$

Substituting in the above gives (10).

It suffices to show that for $A \in \mathcal{B},\ A \subset \Delta$,

$$\frac{u^A(p,\lambda)}{\int_\Delta u^\Delta(p,\lambda)dm_\Gamma} \underset{\lambda\to 0}{\longrightarrow} \frac{m_\Gamma(A)^p}{m_\Gamma(\Delta)^{p+1}} \text{ weakly in } L^2(\Delta). \tag{11}$$

We begin by showing this for $A = \Delta$. Using (10), we get that for fixed $p \ge 1$

$$\sup_{\lambda>0} \left\| \frac{u^\Delta(p,\lambda)}{\int_\Delta u^\Delta(p,\lambda)dm_\Gamma} \right\|_{L^2(\Delta)} < \infty. \tag{12}$$

Given $\lambda_k \to 0,\ \exists$ a subsequence $\lambda_k' \to 0$ and $h \in L^2(\Delta)$ such that

$$\frac{u^\Delta(p,\lambda_k')}{\int_\Delta u^\Delta(p,\lambda_k')dm_\Gamma} \underset{k\to\infty}{\longrightarrow} h,$$

and $\exists$ a further subsequence $\lambda_k'' \to 0$ such that

$$\left| \int_\Delta \left(\frac{u^\Delta(p,\lambda_k'')}{\int_\Delta u^\Delta(p,\lambda_k'')dm_\Gamma} - h \right) \left(\frac{u^\Delta(p,\lambda_\ell'')}{\int_\Delta u^\Delta(p,\lambda_\ell'')dm_\Gamma} - h \right) dm_\Gamma \right| < \frac{1}{2^\ell} \quad \forall\ k < \ell,$$

whence

$$\frac{1}{N}\sum_{k=1}^N \left(\frac{u^\Delta(p,\lambda_k'')}{\int_\Delta u^\Delta(p,\lambda_k'')dm_\Gamma} - h \right) \to 0 \text{ a.e. as } N \to \infty,$$

and

$$\frac{1}{N}\sum_{k=1}^N \frac{u^\Delta(p,\lambda_k'')}{\int_\Delta u^\Delta(p,\lambda_k'')dm_\Gamma} \to h \text{ a.e. as } N \to \infty.$$

The set on which this convergence takes place is clearly φ_Γ-invariant, and h is also φ_Γ-invariant, whence the convergence is a.e. on X_Γ, and h is constant. Since, clearly $\int_\Delta h dm_\Gamma = 1$, we have that $h = \frac{1}{m_\Gamma(\Delta)}$.

Now fix $A \in \mathcal{B},\ A \subset \Delta$. By the ratio theorem

$$\frac{u^A(p,\lambda)}{u^\Delta(p,\lambda)} \underset{\lambda\to 0}{\longrightarrow} \frac{m_\Gamma(A)^p}{m_\Gamma(\Delta)^p} \text{ a.e.} \tag{13}$$

Also, we have, by (12) that

$$\sup_{\lambda>0} \left\| \frac{u^A(p,\lambda)}{\int_\Delta u^\Delta(p,\lambda)dm_\Gamma} \right\|_{L^2(\Delta)} \le \sup_{\lambda>0} \left\| \frac{u^\Delta(p,\lambda)}{\int_\Delta u^\Delta(p,\lambda)dm_\Gamma} \right\|_{L^2(\Delta)} < \infty,$$

whence, as above, $\forall\ \lambda_k \to 0,\ \exists$ a subsequence $\lambda_k' \to 0$ and $h \in L^2(\Delta)$ such that

$$\frac{1}{N}\sum_{k=1}^N \frac{u^A(p,\lambda_k')}{\int_\Delta u^\Delta(p,\lambda_k')dm_\Gamma} \to h \text{ a.e. as } N \to \infty.$$

Note that $\lambda_k' \to 0$ can be chosen so that in addition,

$$\frac{1}{N}\sum_{k=1}^N \frac{u^\Delta(p,\lambda_k')}{\int_\Delta u^\Delta(p,\lambda_k')dm_\Gamma} \to \frac{1}{m_\Gamma(\Delta)} \text{ a.e. as } N \to \infty,$$

whence, by (13)

$$h = \frac{m_\Gamma(A)^p}{m_\Gamma(\Delta)^{p+1}}.$$

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