CHARACTERISTIC FUNCTIONS OF RANDOM VARIABLES ATTRACTED TO 1-STABLE LAWS

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Abstract. The domain of attraction of a 1-stable law on $\mathbb{R}^d$ is characterised by the expansions of the characteristic functions of its elements.

§0 introduction

Let $X_1, X_2, \dots$ be $\mathbb{R}^d$-valued, independent, identically distributed random variables. The distributional limits of $\frac{S_n - A_n}{B_n}$ where $A_n \in \mathbb{R}^d$, $B_n > 0$ are constants, and $S_n = \sum_{k=1}^n X_k$, are given by the well known stable laws. ([Le], [G-K], [I-L]).

A probability distribution function F on $\mathbb{R}^d$ is called *stable* if for all $a, b > 0$ there are $c > 0$ and $v \in \mathbb{R}^d$ such that

$$F_a * F_b(x) = F_c(x - v) \quad (x \in \mathbb{R}^d)$$

where $F_s(x) = F(x/s)$ $(x \in \mathbb{R}^d, s > 0)$, and *strictly stable* if this is true with $v = 0$.

In this case ([Le]) necessarily $a^p + b^p = c^p$ for some $0 < p \le 2$, and p is called the *order* of the stable law F.

A distribution G on $\mathbb{R}^d$ belongs to the *domain of attraction* of the stable law F if there are constants $A_n \in \mathbb{R}^d$ and $B_n > 0$ such that the distributions $\frac{S_n - A_n}{B_n}$ converge weakly to F where $S_n = X_1 + ... + X_n$ and $X_1, X_2, ...$ are i.i.d. with distribution G.

For $p \in (0, 2]$ and $d \in \mathbb{N}$ we let $\mathrm{DA}(p, d)$ be the collection of distribution functions in the domain of attraction of some stable law on $\mathbb{R}^d$ of order p.

In this paper, we obtain expansions of the characteristic functions of distributions on $\mathbb{R}^d$ which are in the domain of attraction of a stable law.

In §1 we deal with the case $d = 1$. The first partial results are in [G-Kor]. The expansions are given fully in [I-L] in case $p \neq 1$ (see theorem 1 below).

Our main result is theorem 2 (below) giving the expansions in case $p = 1$.

In §2 we obtain as corollaries expansions in case $d \ge 2$. Other results in this case are to be found in [R], [Me], [K-M], [A-G1] and [A-G2].

1991 *Mathematics Subject Classification.* 60E07, 60E10, 60F05.

This research was supported by a grant from G.I.F., the German-Israel Foundation for Scientific Research and Development.

As in Ann. Probab. 26 (1998), no. 1, 399–415

A stable law of order p on $\mathbb{R}$ has a characteristic function ψ of form

$$\log\psi(t) = it\gamma - c|t|^p[1 - i\beta\mathrm{sgn}(t)\tan(\frac{p\pi}{2})] \quad (p \neq 1),$$

and

$$\mathrm{Re}\log\psi(t) = -c|t|, \quad \mathrm{Im}\log\psi(t) = t\left(\gamma + \frac{2\beta c}{\pi}\log(1/|t|)\right) \quad (p = 1)$$

where $c > 0$, $\beta, \gamma \in \mathbb{R}$ are constants ([Le]).

The form of the characteristic functions of stable laws on $\mathbb{R}^d$ was obtained by Feldheim (see [Fe], [Le] and theorem 2.3.1 in [S-T]):

To each stable law of order p on $\mathbb{R}^d$ there corresponds a finite measure ν on S^{d-1} (called the *spectral* measure) and $\mu \in \mathbb{R}^d$ (called the *translate*) so that the characteristic function ψ has the form

$$\text{(1a)} \qquad \log\psi(u) = i\langle u, \mu\rangle - \int_{S^{d-1}} |\langle u, s\rangle|^p(1 - i\mathrm{sgn}(\langle s, u\rangle)\tan(\frac{p\pi}{2}))\nu(ds)$$

for $p \neq 1$ and

$$\text{(1b)} \qquad \log\psi(u) = i\langle u, \mu\rangle - \int_{S^{d-1}} |\langle u, s\rangle|\left(1 + i\frac{2}{\pi}\mathrm{sgn}(\langle u, s\rangle)\log(|\langle u, s\rangle|)\right)\nu(ds)$$

for $p = 1$. Evidently a stable law on $\mathbb{R}^d$ has a density if and only if the support of its spectral measure is not contained in a proper subspace of $\mathbb{R}^d$, and in this case we say that both the stable law, and the spectral measure are *nondegenerate.*

Clearly, the stability of a $\mathbb{R}^d$-valued random variable Z implies that of its inner products $\langle Z, u\rangle$, $(u \in \mathbb{R}^d)$.

An example of Marcus ([Ma]) shows that the converse of this is false without additional assumptions.

According to theorems 2.1.2 and 2.1.5 in [S-T], the $\mathbb{R}^d$-valued random variable Z is strictly stable (stable with index ≥ 1) if its inner products $\langle Z, u\rangle$, $(u \in \mathbb{R}^d)$ are strictly stable on $\mathbb{R}$ (stable on $\mathbb{R}$ with index ≥ 1).

The first characterisations of domains of attraction were in terms of the tails of the distributions concerned.

In the unidimensional case ([G-K]), for $p < 2$, the (right continuous) distribution function $G \in$DA$(p, 1)$ iff there is a function $L : \mathbb{R}_+ \to \mathbb{R}_+$, slowly varying at ∞ (see [F]), and constants $c_1, c_2 \geq 0$, $c_1 + c_2 > 0$ such that

$$\text{(2)} \qquad \begin{aligned} L_1(x) &:= x^p(1 - G(x)) = (c_1 + o(1))L(x) \\ L_2(x) &:= x^pG(-x) = (c_2 + o(1))L(x) \qquad \text{as } x \to +\infty. \end{aligned}$$

The results of [G-K] were generalised to $\mathbb{R}^d$ in [R] (see also [Me]), to Hilbert space in [K-M], and to Banach space in [A-G1].

The authors would like to thank I.A. Ibragimov for a helpful conversation.

§1 UNIDIMENSIONAL CHARACTERISATION

The characteristic function ψ of $G \in$DA$(p,1)$ is considered in [G-Kor] and [I-L].

In [G-Kor], DA$(p,1)$ is characterised in terms of $\psi(t)$.

In [I-L], the asymptotic expansion of $\log\psi(t)$ around 0 is established with error small when compared to

$$\text{Prob.}\,(|Z| > 1/|t|) = |t|^p(L_1(1/|t|) + L_2((1/|t|)+) = |t|^p(c_1 + c_2 + o(1))L(1/|t|)$$

as $t \to 0$. Here, Z is a G-distributed random variable, and $G \in$DA$(p,1)$ $(p \neq 1)$ satisfies (2) with the slowly varying functions L, L_1, L_2 and constants $c_1, c_2 \geq 0$, $c_1 + c_2 > 0$. Specifically:

Theorem 1 (theorem 2.6.5 in [I-L]**).**
Suppose that G satisfies (2) with $p \neq 1$, then

$$\log\psi(t) = it\gamma - c|t|^pL(|t|^{-1})[1 - i\beta\operatorname{sgn}(t)\tan(\frac{p\pi}{2})] + o(|t|^pL(|t|^{-1}))$$

where

$$\beta = \frac{c_1 - c_2}{c_1 + c_2}, \quad c = \Gamma(1-p)(c_1+c_2)\cos(\frac{p\pi}{2}), \quad \gamma = \begin{cases} 0 & 0 < p < 1 \\ \int xG(dx) & 1 < p \leq 2. \end{cases}$$

The expansion of the characteristic function when $p = 1$ is also treated in [I-L] for a limited class of slowly varying functions L, namely those where

$$\int_0^\lambda \frac{xL(x)dx}{1+x^2} = L(\lambda)(\log\lambda + o(1))$$

as $\lambda \to \infty$ (c.f. theorem 2 here, theorem 2.6.5 there, and formula (2.6.34) there). As can be easily checked, the functions $L(x) \sim (\log x)^a$ $(a \in \mathbb{R})$, and $L(x) \sim e^{(\log x)^a}$ $(0 < a < 1)$ are slowly varying functions not in this class.

Theorem 2.
Suppose that G satisfies (2) with $p = 1$, then

$$\begin{aligned} &\operatorname{Re}\log\psi(t) = -c|t|L(|t|^{-1}) + o(|t|L(|t|^{-1})), \\ &\operatorname{Im}\log\psi(t) = t\gamma + \frac{2\beta c}{\pi}CtL(1/|t|) + t(H_1(1/|t|) - H_2(1/|t|)) + o(|t|L(|t|^{-1})), \end{aligned}$$

as $t \to 0$, where

$$H_j(\lambda) = \int_0^\lambda \frac{xL_j(x)dx}{1+x^2} \qquad (j = 1,2),$$

$$C = \int_0^\infty \left(\cos y - \frac{1}{1+y^2}\right)\frac{dy}{y},$$

and the constants $c > 0$, $\beta, \gamma \in \mathbb{R}$ are defined by

$$\beta = \frac{c_1 - c_2}{c_1 + c_2}, \quad c = \frac{(c_1+c_2)\pi}{2},$$

$$\gamma = \int_{-\infty}^{\infty} \left(\frac{x}{1+x^2} + \operatorname{sgn}(x) \int_0^{|x|} \frac{2u^2}{(1+u^2)^2} du \right) G(dx)$$

Remark 1. Note that $H_1(\lambda) = \int_0^\lambda \frac{x^2 P(Z>x)dx}{1+x^2}$, whence

$$H_1(\lambda) - H_2(\lambda) = E\bigg(\big[|Z| \wedge \lambda - \tan^{-1}(|Z| \wedge \lambda) \big] \operatorname{sgn}(Z) \bigg) = E((|Z| \wedge \lambda) \operatorname{sgn}(Z)) + O(1)$$

as $\lambda \to \infty$ where Z is G-distributed and $H_1,\ H_2$ are as in theorem 2.

Remark 2.

From this representation of the characteristic function of distributions in $\mathrm{DA}(p,1)$ one deduces existence of a p-stable random variable Y, and constants $A_n, B_n \in \mathbb{R},\ B_n > 0$ so that $\frac{S_n - A_n}{B_n} \to Y$ in distribution. These constants (unique up to $o(B_n)$ as $n \to \infty$) are given by

$$nL(B_n) = B_n^p, \quad A_n = \begin{cases} 0 \quad 0 < p < 1, \\ \gamma n \quad 1 < p \le 2, \\ \gamma n + n(H_1(B_n) - H_2(B_n)) \qquad p = 1. \end{cases}$$

To see this in case $p = 1$ write

$$\log E(e^{it(\frac{S_n - A_n}{B_n})}) = -\frac{itA_n}{B_n} + n \log \psi(\frac{t}{B_n}) := \alpha_n(t) + i\beta_n(t),$$

then

$$\alpha_n(t) = -c\frac{n|t|}{B_n} L(\frac{B_n}{|t|}) + o\bigg(\frac{n|t|L(B_n/|t|)}{B_n}\bigg) \to -c|t| \qquad \text{as } n \to \infty,$$

and

$$\beta_n(t) = \frac{t(H_1(B_n/|t|) - H_1(B_n))}{L(B_n)} - \frac{t(H_2(B_n/|t|) - H_2(B_n))}{L(B_n)} + \frac{2\beta ctCL(B_n/|t|)}{\pi L(B_n)} + o\bigg(\frac{n|t|L(B_n/|t|)}{B_n}\bigg).$$

Now for $j = 1, 2$ and $k > 1$ (see (5) in lemma 3 below),

$$H_j(k\lambda) - H_j(\lambda) = c_j L(\lambda) \log k \ + o(L(\lambda)) \qquad \text{as } \lambda \to \infty.$$

Thus with $k = 1/|t|$

$$\beta_n(t) \to t(c_1 - c_2) \log \frac{1}{|t|} + \frac{2\beta cCt}{\pi} = \frac{2\beta ct}{\pi} \left(\log \frac{1}{|t|} + C \right) \qquad \text{as } n \to \infty.$$

Thus, the above representation is a characterization of $\mathrm{DA}(p,1)$.

Remark 3.

We note that the expansion of $\psi(t)$ around 0 up to $o(|t|^p L(1/|t|))$ is determined entirely by the asymptotic equivalence class of the slowly varying function L and the constants $c_1, c_2 \geq 0$ for G satisfying (2) with $p \neq 1$.

This is not the case when $p = 1$ as shown by the following examples.

There is a distribution G so that

$$L_1(x) := x(1 - G(x)) = (\log x)^2 + (\log x)^{\frac{3}{2}} + O(1)$$
$$L_2(x) := xG(-x) = (\log x)^2 + O(1) \qquad \text{as } x \to +\infty.$$

Here, $L(\lambda) = (\log \lambda)^2$, $p = c_1 = c_2 = 1$, and one calculates from theorem 2 that

$$\operatorname{Im} \log \psi(t) = \frac{4t}{5\pi} L\left(\frac{1}{|t|}\right)^{\frac{5}{4}} + o\left(|t| L\left(\frac{1}{|t|}\right)\right) \text{ as } t \to 0.$$

On the other hand, there is a symmetric distribution satisfying

$$L_1(x) = L_2(x) = (\log x)^2 + O(1) \qquad \text{as } x \to +\infty$$

for which also $L(\lambda) = (\log \lambda)^2$, and $p = c_1 = c_2 = 1$; but here (owing to symmetry)

$$\operatorname{Im} \log \psi(t) \equiv 0.$$

Proof of theorem 2. Assume that G is represented in the form (2).

For $x > 0$ define distribution functions G_j $(j = 1, 2)$ on $\mathbb{R}_+$ by

$$G_1(x) = G(x) - G(0), \text{ and } G_2(x) = G(0) - G(-x).$$

We have that

$$G_j(\infty) - G_j(x) = \frac{L_j(x)}{x} = \frac{(c_j + o(1))L(x)}{x}.$$

Write

$$\int (1 - e^{itx} + \frac{itx}{1 + x^2})\, G(dx)$$
$$= \int_0^\infty (1 - e^{itx} + \frac{itx}{1 + x^2})\, G_1(dx) + \int_0^\infty (1 - \frac{itx}{1 + x^2} - e^{-itx})\, G_2(dx)$$

and let

$$\gamma_j = \int_0^\infty \frac{2x^2}{(1 + x^2)^2} (G_j(\infty) - G_j(x)) dx = \int_0^\infty \frac{2x L_j(x) dx}{(1 + x^2)^2}.$$

Integration by parts gives

$$\int_0^\infty (1 - e^{-(-1)^j itx} - (-1)^j \frac{itx}{1 + x^2}) G_j(dx)$$
$$= (-1)^j it \int_0^\infty \left(e^{-(-1)^j itx} - \frac{1 - x^2}{(1 + x^2)^2}\right) \frac{L_j(x) dx}{x}$$
$$= |t| \int_0^\infty \sin(|t|x) \frac{L_j(x) dx}{x} + (-1)^j it \int_0^\infty \left(\cos(tx) - \frac{1 - x^2}{(1 + x^2)^2}\right) \frac{L_j(x) dx}{x}.$$

Changing variables, we obtain that

$$\int_0^\infty \sin(|t|x)\frac{L_j(x)dx}{x} = \int_0^\infty \sin(x)\frac{L_j(x/|t|)dx}{x},$$

$$\int_0^\infty \left(\cos(tx) - \frac{1}{1+(tx)^2}\right)\frac{L_j(x)dx}{x} = \int_0^\infty \left(\cos(x) - \frac{1}{1+x^2}\right)\frac{L_j(x/|t|)dx}{x}.$$

By lemma 1 (below) we see that

$$\int_0^\infty \sin(|t|x)\frac{L_j(x)dx}{x} = (1+o(1))L_j(\frac{1}{|t|})\frac{\pi}{2}.$$

Now

$$\int_0^\infty \left(\cos(tx) - \frac{1-x^2}{(1+x^2)^2}\right)\frac{L_j(x)dx}{x}$$
$$= \int_0^\infty \left(\cos(tx) - \frac{1}{1+(tx)^2}\right)\frac{L_j(x)dx}{x} + \int_0^\infty \frac{x(1-t^2)L_j(x)dx}{(1+x^2)(1+(tx)^2)} + \int_0^\infty \frac{2xL_j(x)dx}{(1+x^2)^2}$$
$$= \int_0^\infty \left(\cos(tx) - \frac{1}{1+(tx)^2}\right)\frac{L_j(x)dx}{x} + \int_0^\infty \frac{x(1-t^2)L_j(x)dx}{(1+x^2)(1+(tx)^2)} + \gamma_j.$$

By lemma 2 (below)

$$\int_0^\infty \left(\cos(tx) - \frac{1}{1+(tx)^2}\right)\frac{L_j(x)dx}{x} = CL_j\left(\frac{1}{|t|}\right) + o\left(L\left(\frac{1}{|t|}\right)\right).$$

Set

$$\tilde{H}_j(\lambda) := \int_0^\infty \frac{xL_j(x)dx}{(1+x^2)(1+\frac{x^2}{\lambda^2})}.$$

By lemma 3 (below), $\tilde{H}_j(\lambda) = H_j(\lambda) + o(L(\lambda))$ as $\lambda \to \infty$.

Putting everything together we obtain

$$\int_0^\infty (1 + \frac{itx}{1+x^2} - e^{itx})G_1(dx) + \int_0^\infty (1 - \frac{itx}{1+x^2} - e^{-itx})G_2(dx)$$
$$= L(\frac{1}{|t|})|t|(c_1+c_2)\pi/2 - itL(1/|t|)(c_1-c_2)C$$
$$- it(\tilde{H}_1(1/|t|) - \tilde{H}_2(1/|t|)) - it(\gamma_1-\gamma_2) + o\left(|t|L(\frac{1}{|t|})\right)$$
$$= L(\frac{1}{|t|})|t|(c_1+c_2)\pi/2 - itL(1/|t|)(c_1-c_2)C$$
$$- it(H_1(1/|t|) - H_2(1/|t|)) - it(\gamma_1-\gamma_2) + o\left(|t|L(\frac{1}{|t|})\right)$$

and hence theorem 2. □

We conclude this section by collecting the lemmas on slowly varying functions needed for theorem 2.

Assume that $h : \mathbb{R}_+ \to \mathbb{R}_+$ is locally integrable, slowly varying at infinity, and such that $u \mapsto \frac{h(u)}{u}$ is a non-increasing function. Recall that h has a representation

$$h(x) = \eta(x)\exp\left[\int_1^x \frac{\epsilon(s)}{s}ds\right]$$

for some functions $\eta(s) \to K \in \mathbb{R}$ and $\epsilon(s) \to 0$ as $s \to \infty$ (see [F]).

Lemma 1.

$$\int_0^\infty \frac{\sin y}{y} h(\frac{y}{t})dy = (1+o(1))h(\frac{1}{t})\frac{\pi}{2}.$$

Proof. As the proof of lemma 2.6.1 in [I-L]. □

Lemma 2.

$$\int_0^\infty \left[\cos y - \frac{1}{1+y^2}\right]\frac{1}{y}h(\frac{y}{t})dy = (1+o(1))h(\frac{1}{t})\int_0^\infty \left[\cos y - \frac{1}{1+y^2}\right]\frac{1}{y}dy.$$

Proof. We first split the region of integration into four parts: $I_1 = [\Delta_1, \infty)$, $I_2 = [\delta, \Delta_1)$, $I_3 = [t\Delta_2, \delta)$ and $I_4 = [0, t\Delta_2)$ where $\delta < 1 < \Delta_1 = (N - \frac{1}{2})\pi \quad (N \in \mathbb{N})$.

Since $\left| \int_{[\Delta_1+n\pi, \Delta_1+(n+1)\pi]} \cos y \frac{h(y/t)dy}{y} \right|$ decreases in n,

$$\left| \int_{I_1} \cos y \frac{h(y/t)dy}{y} \right| \le \frac{\pi h(\Delta_1/t)}{\Delta_1} \sim \frac{\pi h(1/t)}{\Delta_1}.$$

Also,

$$\int_{I_1} \frac{1}{1+y^2}\frac{h(y/t)dy}{y} \le \frac{h(\Delta_1/t)}{\Delta_1}\pi \sim \frac{\pi h(1/t)}{\Delta_1}.$$

Since for $x \in [\Delta_2 t, \delta)$

$$\frac{h(x/t)}{h(1/t)} = (1+o(1))\exp[\int_{x/t}^{1/t} \frac{\epsilon(s)}{s}ds] = \exp[o(-\log x)] \le x^{-1/2}$$

for t small enough and Δ_2 large enough,

$$\left| \int_{I_3} \left(\frac{1}{1+y^2} - \cos y\right) h(y/t)\frac{dy}{y} \right| = O\left(h(1/t)\int_0^\delta \left|\frac{1}{1+y^2} - \cos y\right| y^{-3/2}dy\right)$$
$$= O\left(h(1/t)\delta^{3/2}\right).$$

Since the function h is locally integrable, it follows that for t small enough

$$\left| \int_{I_4} \left(\frac{1}{1+y^2} - \cos y\right) h(y/t)\frac{dy}{y} \right| = \left| \int_0^{\Delta_2} \left(\frac{1}{1+t^2z^2} - \cos tz\right) h(z)\frac{dz}{z} \right|$$
$$= O\left(t^2\Delta_2 \int_0^{\Delta_2} |h(z)|dz\right) = O(t^2) = o(h(1/t))$$

For $\delta \le x \le \Delta_1$ we have (uniformly in x) by the slow variation property of h

$$\lim_{t\to 0} \frac{h(x/t)}{h(1/t)} = 1.$$

It follows that

$$\begin{aligned}&\left|\int_{I_2}\left(\frac{1}{1+y^2}-\cos y\right)[h(y/t)-h(1/t)]\frac{dy}{y}\right|\\&\le 2h(1/t)\left[\sup_{\delta\le x\le\Delta_1}\left|\frac{h(x/t)}{h(1/t)}-1\right|\right]\int_\delta^{\Delta_1}\frac{dy}{y}\\&=o(h(1/t)).\end{aligned}$$

Applying the estimates for I_1, I_3, and I_4 with $h = 1$ it follows that

$$\int_0^\infty\left(\frac{1}{1+y^2}-\cos y\right)\frac{h(y/t)-h(1/t)}{y}dy = o(h(1/t)) + O\Big(h(1/t)(\delta^{3/2}+\Delta_1^{-1})\Big).$$

Letting $\Delta_1 \to \infty$ and $\delta \to 0$ as $t \to 0$, the lemma follows. □

Lemma 3. *Let*

$$H(\lambda) := \int_0^\lambda \frac{xh(x)dx}{1+x^2};$$

then H is slowly varying at infinity,

$$\frac{h(\lambda)}{H(\lambda)} \to 0 \quad \textit{as } \lambda \to \infty, \tag{3}$$

$$\tilde{H}(\lambda) := \int_0^\infty \frac{xh(x)dx}{(1+x^2)(1+\frac{x^2}{\lambda^2})} = H(\lambda) + o(h(\lambda)) \qquad \textit{as } \lambda \to \infty, \tag{4}$$

and

$$H(k\lambda) - H(\lambda) \sim h(\lambda)\cdot\log k \qquad \textit{as } \lambda \to \infty. \tag{5}$$

Remark. Slow variation of H, (3), and (5) are established in lemma 1 of [Par].

Proof.
We first show (5):

$$\begin{aligned}H(k\lambda)-H(\lambda) &= \int_\lambda^{k\lambda}\frac{xh(x)dx}{1+x^2} \sim \int_\lambda^{k\lambda}\frac{h(x)dx}{x}\\&= \int_1^k \frac{h(\lambda x)dx}{x} \sim \log k\, h(\lambda).\end{aligned}$$

Next, we see that (3) follows from (5) as $\forall\ M > 1$,

$$\frac{H(\lambda)}{h(\lambda)} = \frac{H(e^M e^{-M}\lambda)}{h(\lambda)} \ge \frac{H(e^M e^{-M}\lambda) - H(e^{-M}\lambda)}{h(\lambda)} \sim \frac{h(e^{-M}\lambda)M}{h(\lambda)} \to M \text{ as } \lambda \to \infty.$$

It follows from (3) and (5) that H is slowly varying at ∞.

To continue, we claim that

$$\tilde{H}(\lambda) = \int_0^\lambda \frac{xh(x)dx}{(1+x^2)(1+\frac{x^2}{\lambda^2})} + \frac{\log 2}{2} h(\lambda) + o(h(\lambda)) \qquad \text{as } \lambda \to \infty. \tag{6}$$

To see this, note that

$$\int_\lambda^\infty \frac{xh(x)dx}{(1+x^2)(1+\frac{x^2}{\lambda^2})} = \int_1^\infty \frac{xh(\lambda x)dx}{(\frac{1}{\lambda^2}+x^2)(1+x^2)}$$
$$= h(\lambda)\int_1^\infty \frac{xdx}{(\frac{1}{\lambda^2}+x^2)(1+x^2)} + h(\lambda)\int_1^\infty (\frac{h(\lambda x)}{h(\lambda)} - 1)\frac{xdx}{(\frac{1}{\lambda^2}+x^2)(1+x^2)}$$
$$= \frac{\log 2}{2} h(\lambda) + o\Big(h(\lambda)\Big)$$

as $\lambda \to \infty$ by the dominated convergence theorem since $|\frac{h(\lambda x)}{h(\lambda)} - 1| \to 0$ as $\lambda \to \infty\ \forall\ x > 1$ and $|\frac{h(\lambda x)}{h(\lambda)} - 1| \le x\ \forall\ x$ large enough. This establishes (6).

To finish the proof of (4), we note that

$$\frac{xh(x)}{(1+x^2)(1+\frac{x^2}{\lambda^2})} = \frac{\lambda^2}{\lambda^2-1}\left(\frac{xh(x)}{x^2+1} - \frac{xh(x)}{x^2+\lambda^2}\right),$$

whence in view of (6),

$$\tilde{H}(\lambda) = \frac{\lambda^2}{\lambda^2-1}\int_0^\lambda \frac{xh(x)dx}{x^2+1} - \frac{\lambda^2}{\lambda^2-1}\int_0^\lambda \frac{xh(x)dx}{x^2+\lambda^2} + \frac{\log 2}{2}h(\lambda) + o(h(\lambda))$$

Now

$$\frac{\lambda^2}{\lambda^2-1}\int_0^\lambda \frac{xh(x)dx}{x^2+1} = H(\lambda) + O(\frac{H(\lambda)}{\lambda^2}) = H(\lambda) + o(h(\lambda)) \qquad \text{as } \lambda \to \infty$$

because both h and H are slowly varying at ∞; and

$$\frac{\lambda^2}{\lambda^2-1}\int_0^\lambda \frac{xh(x)dx}{x^2+\lambda^2} \sim \int_0^\lambda \frac{xh(x)dx}{x^2+\lambda^2} = \int_0^1 \frac{xh(\lambda x)dx}{x^2+1} \sim \frac{\log 2}{2}h(\lambda) \qquad \text{as } \lambda \to \infty.$$

Thus

$$\tilde{H}(\lambda) = H(\lambda) + o(h(\lambda)) \qquad \text{as } \lambda \to \infty$$

which is (4). □

§2 MULTIDIMENSIONAL CHARACTERISATION

Corollary 1. *Let $0 < p < 2,\ p \ne 1$ and G be a distribution function on $\mathbb{R}^d$. The following are equivalent:*

(A) G belongs to the domain of attraction of the nondegenerate stable law of order p, spectral measure ν and translate μ.

(B) The characteristic function ψ of G has the form

$$\log\psi(tu) = \begin{cases} -t^p L(\frac{1}{t})\Phi(u) + it\langle u,\mu\rangle + o(t^p L(\frac{1}{t})) & \text{if } p > 1 \\ -t^p L(\frac{1}{t})\Phi(u) + o(t^p L(\frac{1}{t})) & \text{if } p < 1 \end{cases}$$

as $t \to 0^+$, $\forall\ u \in S^{d-1}$, *where* $\mu \in \mathbb{R}^d$, L *is slowly varying at infinity,* ν *is a nondegenerate finite measure on* S^{d-1} *and*

$$\Phi(u) := \int_{S^{d-1}} |\langle u, s\rangle|^p (1 - i\mathrm{sgn}\langle s, u\rangle \tan(\frac{p\pi}{2}))\nu(ds).$$

Proof of corollary 1. (A)⇒(B).

Let $X_1, X_2, ...$ be i.i.d. with distribution G and $A_n \in \mathbb{R}^d$, $B_n > 0$ such that $\frac{S_n - A_n}{B_n} \to Z$ weakly where Z is p-stable. Let $u \in \mathbb{R}^d$. It follows from Feldheim's theorem that $\langle u, Z\rangle$ has a 1-dimensional p-stable distribution with parameters $\gamma'_u = \langle u, \mu\rangle$, $c'_u = \int_{S^{d-1}} |\langle u, s\rangle|^p \nu(ds)$ and

$$\beta'_u = \frac{1}{c'_u} \int_{S^{d-1}} |\langle u, s\rangle|^p \mathrm{sgn}(\langle u, s\rangle)\nu(ds).$$

The characteristic function $\psi(tu)$ of $\langle u, X_1\rangle$ has a form

$$\log \psi(tu) = it\gamma_u - |t|^p L_u(1/|t|)\left(1 - i\beta_u \mathrm{sgn}(t) \tan\left(\frac{\pi p}{2}\right)\right)$$

as in theorem 1 with some slowly varying function L_u and parameters γ_u and β_u (we normalize L_u so that $c_u = 1$). Hence

$$\begin{aligned} &it\left(\frac{n\gamma_u}{B_n} - \frac{\langle u, A_n\rangle}{B_n}\right) - |t|^p \frac{n}{B_n^p} L_u\left(\frac{B_n}{|t|}\right)\left(1 - i\beta_u \mathrm{sgn}(t) \tan\left(\frac{p\pi}{2}\right)\right) \\ &\quad \to it\gamma'_u - c'_u|t|^p\left(1 - i\beta'_u \mathrm{sgn}(t) \tan\left(\frac{p\pi}{2}\right)\right). \end{aligned}$$

The parameter γ_u must be linear in u if $p > 1$, since $\frac{n\gamma_u - \langle u, A_n\rangle}{B_n} \to \langle u, \mu\rangle$ and $\frac{n}{B_n} \to \infty$. In case $p < 1$, γ_u can be arbitrary since $\frac{n}{B_n} \to 0$. Moreover, $\frac{n}{B_n^p} L_u(B_n)$ converges to c'_u and $\beta_u = \beta'_u$. Setting $L(t) = \frac{1}{c'_u} L_u(t)$ for some fixed u we obtain for $v \in I\!R^d$

$$\lim_{n\to\infty} \frac{L(B_n)}{L_v(B_n)} = \lim_{n\to\infty} \frac{(n/B_n^p)L_u(B_n)}{c'_u(n/B_n^p)L_v(B_n)} = 1/c'_v,$$

hence $L_v(\lambda) \sim c'_v L(\lambda)$ as $\lambda \to \infty$.

(B)⇒(A).

Conversely, if the characteristic function ψ of G is as in (B), then for every $u \in \mathbb{R}^d$ the characteristic functions of $Y_n^{(u)} = B_n^{-1} \sum_{k=1}^n (\langle u, X_k\rangle - \langle A_n, u\rangle)$ converges, where $X_1, X_2, ...$ are i.i.d. with distribution G, where B_n is defined by $nL(B_n) = B_n^p$ and where $A_n = 0$ if $p < 1$ and $A_n = n\mu$ if $p > 1$.

It follows that the characteristic functions of $\frac{S_n - A_n}{B_n}$ converge (necessarily to a characteristic function), such that the limit variable Z has all distributions $\langle u, Z\rangle$ ($u \in I\!R^d$) p-stable. Thus Z is stable itself if $p > 1$. In case $p < 1$ we note that Z has a characteristic function of the form (1a) with $\mu = 0$ and is strictly stable. □

If G is a distribution function on $\mathbb{R}^d$ we define $G_u(\cdot)$ to be the distribution function of $\langle u, Z\rangle$ where Z is a random variable with distribution G.

Corollary 2. *(A) If a distribution function G on $\mathbb{R}^d$ belongs to the domain of attraction of the nondegenerate stable law of order* 1*, spectral measure ν and translate μ, then its characteristic function ψ has the form*

$$\begin{aligned} &\operatorname{Re}\log\psi(tu) = -tL(\frac{1}{t})\int_{S^{d-1}} |\langle u,s\rangle|\nu(ds) + o(tL(\frac{1}{t})), \\ (7)\quad &\operatorname{Im}\log\psi(tu) = tH_u(\frac{1}{t}) + tL(\frac{1}{t})\frac{2C}{\pi}\int_{S^{d-1}} \langle u,s\rangle\nu(ds) + t\gamma_u + o\Big(tL(\frac{1}{t})\Big)\end{aligned}$$

as $t \to 0^+$ $\forall$ $u \in S^{d-1}$, where : L is slowly varying at infinity, $C = \int_0^\infty(\cos y - \frac{1}{1+y^2})\frac{dy}{y}$, and where

$$H_u(x) = \int_0^x \frac{v(1-G_u(v)-G_u(-v))}{1+v^2}dv$$

has a representation

$$(8)\qquad H_u(\lambda) = \langle u,\Gamma_\lambda\rangle - \frac{2L(\lambda)}{\pi}\int_{S^{d-1}}\langle u,s\rangle\log(|\langle u,s\rangle|)\nu(ds) - \gamma_u + o(L(\lambda))$$

for some $\Gamma_\lambda \in \mathbb{R}^d$ and satisfies

$$(9)\qquad H_u(k\lambda) - H_u(\lambda) \sim \frac{2}{\pi}L(\lambda)\int_{S^{d-1}}\langle u,s\rangle\nu(ds)\log k$$

as $\lambda \to \infty$.

(B) Let the characteristic function ψ of a distribution G on $\mathbb{R}^d$ satisfy (7) for some $\gamma_u \in \mathbb{R}$, some finite measure ν on S^{d-1}, some slowly varying function L and some functions H_u with representation (8) and satisfying (9). Then G belongs to the domain of attraction of a nondegenerate stable law of order 1.

Proof of corollary 2. (A) As before, let $X_1, X_2, ...$ be i.i.d. with distribution G and $A_n \in \mathbb{R}^d$, $B_n > 0$ such that $\frac{S_n - A_n}{B_n} \to Z$ weakly where Z is 1-stable. Let $u \in \mathbb{R}^d$. It follows from Feldheim's theorem that $\langle u, Z\rangle$ has a 1-dimensional 1-stable distribution with parameters

$$\gamma'_u = \langle u,\mu\rangle - \frac{2}{\pi}\int_{S^{d-1}}\langle u,s\rangle\log(|\langle u,s\rangle|)\nu(ds)$$

$$c'_u = \int_{S^{d-1}}|\langle u,s\rangle|\nu(ds), \quad \beta'_u = \frac{1}{c'_u}\int_{S^{d-1}}\langle u,s\rangle\nu(ds).$$

By theorem 2, the characteristic function $\psi(tu)$ of $\langle u, X_1\rangle$ has a form

$$\begin{aligned}&\log\psi(tu) = -|t|L_u\left(\frac{1}{|t|}\right) + \\ &it\gamma_u + it\frac{2\beta_u C}{\pi}L_u\left(\frac{1}{|t|}\right) + it\Big(H_{1u}(1/|t|) - H_{2u}(1/|t|)\Big) + o\Big(|t|L_u(1/|t|)\Big)\end{aligned}$$

where

$$H_{ju}(\lambda)=\int_0^\lambda \frac{xL_{ju}(x)}{1+x^2}dx$$
$$L_{ju}(x)=\begin{cases} x(1-G_u(x)) & \text{if } j=1\\ xG_u(-x) & \text{if } j=2\end{cases}$$

for some parameters γ_u, β_u and slowly varying functions L_u (normalised so that $c_u=1$), L_{ju}. Also note that by theorem 2 $L_{ju}(x)=(c_{ju}+o(1))L_u(x)$ with $c_{1u}+c_{2u}=2/\pi$. Set $H_u=H_{1u}-H_{2u}$.

From the assumed convergence of characteristic functions, we have that

$$\operatorname{Re} n\log\psi(\frac{tu}{B_n})\sim\frac{nL_u(B_n)|t|}{B_n}\to c'_u|t|.$$

As in the proof of corollary 1, there exists a function L so that $c'_vL\sim L_v$ for all $v\in\mathbb{R}^d$. Moreover, using (5) $\forall\, t\in\mathbb{R}$, as $n\to\infty$,

$$\begin{aligned}&\operatorname{Im} n\log\psi(\frac{tu}{B_n})-\langle A_n,u\rangle\frac{t}{B_n}=\frac{nL_u(B_n)}{B_n}(c_{1u}-c_{2u})t\log\frac{1}{|t|}+\\ &t\Bigg(\frac{n\gamma_u}{B_n}-\frac{\langle A_n,u\rangle}{B_n}+\frac{nH_u(B_n)}{B_n}+\frac{2Cn\beta_uL_u(B_n)}{\pi B_n}\Bigg)+o(1)\\ &\to t\gamma'_u+\frac{2\beta'_uc'_ut}{\pi}\log\frac{1}{|t|}.\end{aligned}$$

Equating coefficients of t, and $t\log\frac{1}{|t|}$, we see that

$$\frac{nL_u(B_n)}{B_n}(c_{1u}-c_{2u})\to\frac{2\beta'_uc'_u}{\pi}$$

and

$$\frac{n}{B_n}\Bigg(H_u(B_n)+\frac{2C\beta_u}{\pi}L_u(B_n)+\gamma_u-\langle u,A_n/n\rangle\Bigg)\to\gamma'_u$$

as $n\to\infty$.

Hence $c'_u(c_{1u}-c_{2u})=c'_u\beta_u2/\pi=c'_u2\beta'_u/\pi$ and $\beta_u=\beta'_u$.

To conclude, we determine the conditions for H_u and γ_u. Since $c'_uL\sim L_u$ and since L_u is slowly varying,

$$\begin{aligned}&H_u(B_n)+\frac{2C\beta'_uc'_u}{\pi}L(B_n)+\gamma_u-\langle u,A_n/n\rangle\\ &-\langle u,\frac{B_n\mu}{n}\rangle+\frac{2B_n}{n\pi}\int_{S^{d-1}}\langle u,s\rangle\log(|\langle u,s\rangle|)\nu(ds)\\ &=o(\frac{B_n}{n}),\end{aligned}$$

or (because $\beta'_uc'_u$ is linear in u and $nL(B_n)\sim B_n$)

$$H_u(B_n)=\langle u,\Gamma_{B_n}\rangle-\frac{2L(B_n)}{\pi}\int_{S^{d-1}}\langle u,s\rangle\log(|\langle u,s\rangle|)\nu(ds)-\gamma_u+o(L(B_n)),$$

where

$$\Gamma_{B_n} = \frac{A_n}{n} + \mu L(B_n) - \frac{2CL(B_n)}{\pi} \int_{S^{d-1}} \langle \cdot, s\rangle \nu(ds).$$

We obtain the expansion for $H_u(\lambda)$ ($B_n \le \lambda < B_{n+1}$) from

$$\begin{aligned} &H_u(\lambda) - H_u(B_n) = H_{1u}(\lambda) - H_{1u}(B_n) - [H_{2u}(\lambda) - H_{2u}(B_n)] \\ &\sim \log\left(\frac{\lambda}{B_n}\right)\Big(L_{1u}(\lambda) - L_{2u}(\lambda)\Big) + o(L(\lambda)) = o(L(\lambda)) \end{aligned}$$

and

$$\begin{aligned} H_u(\lambda) &= H_u(B_n) + H_u(\lambda) - H_u(B_n) = H_u(B_n) + o(L(\lambda)) \\ &= \langle u, \Gamma_{B_n}\rangle - \frac{2L(\lambda)}{\pi} \int_{S^{d-1}} \langle u, s\rangle \log(|\langle u, s\rangle|)\nu(ds) - \gamma_u + o(L(\lambda)), \end{aligned}$$

since

$$1 \le \frac{\lambda}{B_n} \le \frac{B_{n+1}}{B_n} \sim \frac{(n+1)L(B_{n+1})}{nL(B_n)} \to 1.$$

(8) follows setting $\Gamma_\lambda = \Gamma_{B_n}$ if $B_n \le \lambda < B_{n+1}$. Finally, (9) holds because

$$\begin{aligned} H_u(k\lambda) - H_u(\lambda) &\sim \log(k)\Big(L_{1u}(\lambda) - L_{2u}(\lambda)\Big) \sim \log(k)(c_{1u} - c_{2u})L_u(\lambda) \\ &\sim \log(k)(c_{1u} - c_{2u})c'_u L(\lambda) = \frac{2}{\pi} c'_u \beta'_u \log(k) L(\lambda). \end{aligned}$$

(B) Conversely, if the characteristic function ψ of G is as in (B), then for every $u \in \mathbb{R}^d$ the characteristic functions of

$$Y_n^{(u)} = B_n^{-1} \sum_{k=1}^{n} (\langle u, X_k\rangle - \langle A_n, u\rangle)$$

converges, where $X_1, X_2, ...$ are i.i.d. with distribution G, where B_n is defined by $nL(B_n) = B_n$ and where

$$A_n = n\Gamma_{B_n} + \frac{2CnL(B_n)}{\pi} \int_{S^{d-1}} \langle \cdot, s\rangle \nu(ds).$$

Let $c'_u = \int_{S^{d-1}} |\langle u, s\rangle| \nu(ds)$ be defined as before. We have that

$$\log\left(\psi\Big(\frac{tu}{B_n}\Big)^n e^{-\frac{it\langle u, A_n\rangle}{B_n}}\right) \to -|t|c'_u - it\frac{2}{\pi} \int_{S^{d-1}} \langle u, s\rangle \log|\langle tu, s\rangle| \nu(ds).$$

□

Example. Let $0 < p < 2$, $\nu \in \mathcal{P}(S^{d-1})$ be nondegenerate, and let L be slowly varying at ∞.

If $Y \in \mathrm{DA}\,(p,1)$, $Y > 0$ with tails given by $P(Y > \lambda) = \frac{2L(\lambda)}{\pi\lambda^p}$, and Z is a ν-distributed random variable on S^{d-1} independent of Y, then $X := YZ$ is in the domain of attraction of a nondegenerate stable law of order p on $\mathbb{R}^d$, and with spectral measure ν.

This follows from (and illustrates) corollaries 1 and 2. Indeed, using the notation $\psi_U(u) := -\log\Bigg(E(e^{i\langle U,u\rangle}\Bigg)$, we have that for $u \in S^{d-1}$ and $t > 0$

$$\psi_X(tu) = E\Bigg(\psi_Y(\langle Z,tu\rangle) + O(\psi_Y(\langle Z,tu\rangle)^2)\Bigg) = E(\psi_Y(\langle Z,tu\rangle)) + o(t^pL(1/t)),$$

as $t \to 0$, whence by [I-L] for $p \neq 1$

$$\psi_X(tu) = it\gamma\langle u,E(Z)\rangle - t^pL(1/t)\int_{S^{d-1}} |\langle u,s\rangle|^p(1 - i\mathrm{sgn}(\langle s,u\rangle)\tan(\frac{p\pi}{2}))\nu(ds) + o(t^pL(1/t))$$

as $t \to 0$,

and by theorem 2 for $p = 1$

$$\mathrm{Re}\,\psi_X(tu) = -tL(1/t)\int_{S^{d-1}} |\langle s,u\rangle| d\nu(s) + o(tL(1/t)),$$

$$\begin{aligned}\mathrm{Im}\,\psi_X(tu) &= t\gamma\langle u,E(Z)\rangle + t(H(1/t) + \frac{2C}{\pi}L(1/t))\int_{S^{d-1}} \langle s,u\rangle d\nu(s) \\ &+ tL(1/t)\frac{2}{\pi}\int_{S^{d-1}} \langle s,u\rangle \log\frac{1}{|\langle s,u\rangle|} d\nu(s) + o(tL(1/t))\end{aligned}$$

as $t \to 0$, where $H(\lambda) := \int_0^\lambda \frac{2xL(x)dx}{\pi(1+x^2)}$ and where $\gamma := E\Bigg(\frac{Y}{1+Y^2} + \int_0^Y \frac{2u^2}{(1+u^2)^2}du\Bigg)$.

If, in the example Y was not chosen positive, but satisfying (2) with constants c, c_1, c_2, then the spectral measure of X is given by

$$\nu^*(A) = c_1\nu(A) + c_2\nu(-A) \quad (A \in \mathcal{B}(S^{d-1}).$$

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