MULTIPLE RECURRENCE OF MARKOV SHIFTS AND OTHER INFINITE MEASURE PRESERVING TRANSFORMATIONS

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Abstract. We discuss the concept of multiple recurrence, considering an ergodic version of a conjecture of Erdös. This conjecture applies to infinite measure preserving transformations. We prove a result stronger than the ergodic conjecture for the class of Markov shifts and show by example that our stronger result is not true for all measure preserving transformations.

arithmetic progressions and a conjecture of Erdös

An **arithmetic progression of length** d in $\mathbb{N}$ is a d-tuple

$$(x_1, x_2, \ldots, x_d) \in \mathbb{N}^d \text{ such that } x_k = x_1 + (k-1)y \quad (2 \le k \le d).$$

The **gap** of the arithmetic progression $x + (k-1)y \quad (2 \le k \le d)$ is y. Analogous definitions can be made in an arbitrary commutative semigroup.

Evidently $(x_1, x_2, \ldots, x_d) \in \mathbb{N}^d$ is an arithmetic progression iff $x_k + x_{k+2} = 2x_{k+1}\ \forall\ 1 \le k \le d-2$. One of the longstanding problems in the subject is to give "size" conditions on a subset $K \subset \mathbb{N}$ which ensure existence of arithmetic progressions in K. For example, Szemerédi's theorem (see [17]) states that a subset of positive density contains arithmetic progressions of all lengths; and Roth's theorem (see [16])) states that a subset $K \subset \mathbb{N}$ with $|K \cap [1,n]| > \frac{n}{\log\log n}$ contains arithmetic progressions of length 3.

Recall that Szemerédi's theorem came as a partial answer to a conjecture of Erdös ([6]):

$$K \subset \mathbb{N},\ \sum_{n \in K} \frac{1}{n} = \infty \ \Rightarrow K \text{ contains arithmetic progressions of all lengths.}$$

It is not at present known whether $K \subset \mathbb{N}$, $\sum_{n\in K} \frac{1}{n} = \infty$ implies $K \supset$ arithmetic progressions of length 3. In the sequel, we shall consider an ergodic version of Erdös' conjecture.

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The first methods of constructing progression-free subsets of $\mathbb{N}$ were the so-called **d-greedy algorithms** ($d \in \mathbb{N}$). The d-greedy algorithm constructs a subset $G_d \subset \mathbb{N}$ without arithmetic progressions of length d by successively including every number, except for those which complete an arithmetic progression of length d.

For $d \in \mathbb{N}$ prime, $G_d = K_d := \{\sum_{k=0}^{\infty} a_k d^k\colon\ a_k \in \{0,1,\dots,d-2\},\ a_k \to 0\}$. This is because a) each $n \notin K_d$ completes an arithmetic progression of length d in $K_d \cap [0,n] \cup \{n\}$, and b) for d prime, K_d contains no arithmetic progressions of length d.

Remarks

1) Let $B_n := (1_{K_d}(0),\dots,1_{K_d}(d^n-1))$, then

$$B_1 = 1,\ \ B_{n+1} = \underbrace{B_n,\dots,B_n}_{d-1\text{-times}}\underbrace{0,\dots,0}_{d^n\text{-times}}.$$

This concatenation also defines a cutting and stacking construction of a measure preserving transformation (see [7]) to which we shall return.

2) The d-greedy algorithms do not provide large progression-free sets: $|G_d \cap [1,n]| \asymp n^{\frac{\log(d-1)}{\log d}}$, whereas Behrend (see [3]) has constructed a progression-free subset $B \subset \mathbb{N}$ with $|B \cap [1,n]| >> \frac{n}{e^{c\sqrt{\log n}}}$ for some $c > 0$.

3) It is possible that some kind of a random greedy algorithm may provide larger progression-free sets.

d-recurrence

Let $(X,\mathcal{B},m,T)$ be a non-singular transformation and let $B \in \mathcal{B}_+$ (here and throughout for $\mathcal{A} \subset \mathcal{B}$ we denote $\mathcal{A}_+ := \{A \in \mathcal{A}\colon\ m(A) > 0\}$). For $x \in X$ consider the collection of visit times to B $V_{B,x} := \{n \ge 1\colon\ T^n x \in B\}$ and for $d \in \mathbb{N}$ let

$$\begin{aligned}B_d &= B_d(T)\\ &:= \{x \in X\colon\ V_{B,x} \text{ contains an arithmetic progression of length } d+1\}\\ &= \bigcup_{k,n\ge 1}\{x \in B\colon\ V_{B,x} \supset \{k, k+n, \dots, k+dn\}\\ &= \bigcup_{k=1}^{\infty} T^{-k}\bigcup_{n=1}^{\infty}\bigcap_{j=0}^{d} T^{-jn}B.\end{aligned}$$

Evidently $B_d = \varnothing$ iff $B \in \mathcal{B}$ is a **d-wandering set** in the sense that $B \cap T^{-k}B \cap \cdots \cap T^{-dk}B = \varnothing \mod m\ \forall\ k \ge 1$.

Using the non-singular property of T, we see easily that $m(B_d) > 0$ if and only if $m(B \cap T^{-n}B \cap \cdots \cap T^{-dn}B) > 0$ for some $n \ge 1$.

Accordingly, we call the non-singular transformation $(X, \mathcal{B}, m, T)$ ***d*-recurrent** if for every $B \in \mathcal{B}_+$, $\exists\ n \geq 1$ such that

$$m(B \cap T^{-n}B \cap \cdots \cap T^{-dn}B) > 0.$$

Note that conservativity (Poincaré recurrence) is 1-recurrence.

If the non-singular transformation $(X, \mathcal{B}, m, T)$ is not d-recurrent, then $\exists$ a d-wandering set of positive measure, and indeed (see the Hopf d-decomposition below), if T is conservative and ergodic, then X is a union of such sets $\bmod m$.

We call $(X, \mathcal{B}, m, T)$ **multiply recurrent** if it is d-recurrent $\forall\ d \geq 1$.

If $(X, \mathcal{B}, m, T)$ is an ergodic probability preserving transformation and $B \in \mathcal{B}_+$, then by Birkhoff's ergodic theorem, $V_{B,x}$ has positive density in $\mathbb{N}$ for a.e. $x \in X$ and therefore by Szemerédi's theorem contains arithmetic progressions of all lengths. This shows that $m(B_d) = 1\ \forall\ d \geq 1$ and that $(X, \mathcal{B}, m, T)$ is multiply recurrent. Furstenberg has given an ergodic proof that probability preserving transformations are multiply recurrent and deduced Szemerédi's theorem from this (see [10] and [8]).

The question now arises as to which infinite measure preserving transformations are multiply recurrent.

Roth's theorem has an ergodic version: if $(X, \mathcal{B}, m, T)$ is a conservative, ergodic measure preserving transformation such that

$$\limsup_{n\to\infty} \frac{\log\log n}{n} \sum_{k=0}^{n-1} 1_A \circ T^k > 0$$

$$\text{a.e. for some (and hence all)} \quad A \in \mathcal{B}\ 0 < m(A) < \infty,$$

then T is 2-recurrent. This is proved by applying Roth's theorem to a.e. $V_{A,x}$.

We now return to the measure preserving transformation defined by the cutting and stacking (see [7]) specified by the d-greedy algorithm" (d prime) mentioned in remark 1 (above). This is a piecewise translation $T: \mathbb{R} \to \mathbb{R}$ defined in stages starting with the 0^{th} stage where we have the unit interval $E_1(0) = I$. At the n^{th} stage, we have a "column" of disjoint intervals $C = (E_1(n), \ldots, E_{d^n}(n))$, each of length $\frac{1}{(d-1)^n}$ and a piecewise translation $T: E_k(n) \to E_{k+1}(n) \quad (1 \leq k \leq d^n - 1)$. At the next stage, we extend the definition of T by cutting the column into $d-1$ columns $C_j := (E_1^{(j)}(n), \ldots, E_{d^n}^{(j)}(n)) \quad (1 \leq j \leq d-1)$ where each $E_k^{(j)}(n)$ is an interval of length $\frac{1}{(d-1)^{n+1}}$ and $T: E_k^{(j)}(n) \to E_{k+1}^{(j)}(n) \quad (0 \leq k \leq d^{n-1},\ 1 \leq j \leq d)$.

The $(n+1)^{\text{st}}$ column is

$$C' := (C_1, C_2, \ldots, C_{d-1}, D_n) := (E_1(n+1), \ldots, E_{d^{n+1}}(n+1))$$

where $D_n = (D_1(n), \ldots, D_{d^n}(n))$ is a column of disjoint intervals, disjoint from each of the $E_k(n)$ $(1 \le k \le d^n)$ and each of length $\frac{1}{(d-1)^{n+1}}$. The definition of T is extended by defining $T: E_k(n+1) \to E_{k+1}(n+1)$ $(1 \le k \le d^{n+1} - 1)$ as a translation where it was not already defined at stage n: i.e. for $E_k(n+1) = E^{(j)}_{d^n}(n)$ $(1 \le j \le d)$ and $E_k(n+1) = D_j(n)$ $(1 \le j \le d^n - 1)$. The union of all the intervals used has infinite length and can be assumed to be $\mathbb{R}$. The resulting piecewise translation $T: \mathbb{R} \to \mathbb{R}$ is a conservative, ergodic, measure preserving transformation.

The construction of T is given by the concatenation in remark 1 above. Each interval in each tower is either a subset of, or disjoint from the unit interval I, and for each $n \ge 0$,

$$\big(m(C_1(n)|I), \ldots, m(C_{d^n}(n)|I)\big) \equiv B_n$$

where B_n is as in remark 1.

It follows that (for d prime), $m(I \cap T^{-k} I \cap \cdots \cap T^{-(d-1)k} I) = 0 \ \ \forall\ k \ge 1$ (else K_d would contain an arithmetic progressions of length d) and T is not $(d-1)$-recurrent.

We claim however that T is $(d-2)$-recurrent.

To see this, note first that if $A \in \mathcal{B}(\mathbb{R})$ and $\exists\ N \ge 1,\ K \subset \{1, 2, \ldots, d^N\}$ such that $A = \bigcup_{k \in K} E_k(N)$, then $m(A \cap T^{-d^n} A \cap \cdots \cap T^{-(d-2)d^n} A) = \frac{m(A)}{d-1}$ $\forall\ n \ge N+1$. Since any $B \in \mathcal{B}$ with $m(B) < \infty$ can be approximated arbitrarily well by such sets, we have that

$$m(B \cap T^{-d^n} B \cap \cdots \cap T^{-(d-2)d^n} B) \to \frac{m(B)}{d-1} \text{ as } n \to \infty \ \ \forall\ B \in \mathcal{B},\ m(B) < \infty$$

and that T is $(d-2)$-recurrent. This construction and generalisations thereof are considered in [5] where they are represented as odometers (see §2).

Let $c_n \downarrow$. Recall from [14] that a conservative, ergodic, measure preserving transformation T is $\{c_n\}$-**conservative** if $\sum_{n=1}^{\infty} c_n f \circ T^n = \infty$ a.e. for some, and hence $\forall\ f \in L^1_+$. Note that $\{1\}$-conservativity is the same as conservativity.

The ergodic version of the Erdös question is that $\{\frac{1}{n}\}$-conservative, ergodic measure preserving transformations are multiply recurrent. It is not hard to show that the Erdös conjecture implies the ergodic version. We do not know whether the converse is true.

In §1 we prove the Erdös conjecture for Markov shifts. Indeed for Markov shifts, slightly more is true:
$\{\frac{1}{n^a}\}$-conservativity $\forall\ 0 < a < 1$ implies multiple recurrence.

The proof is accomplished by showing that a Markov shift T is d-recurrent iff $\underbrace{T\times\ldots\times T}_{d\text{-times}}$ is conservative, and then showing that for a $\{\frac{1}{n^a}\}$-conservative Markov shift, this is the case $\forall\ d<\frac{1}{1-a}$.

In §2, we see that the general situation is different, exhibiting some examples of "infinite odometers".

One such exhibit is a conservative, ergodic measure preserving transformation which is $\{\frac{1}{n^a}\}$-conservative $\forall\ 0<a<1$ but not 2-recurrent. This is constructed using Behrend's sequences ([3]).

We conclude this introduction with a "d-analogue" of the basic Hopf decomposition, proving a "Hopf d-decomposition". Recall from [8] that an **IP-set** is a set of form $\{\sum_{k\in F} n_k\colon\ F\subset\mathbb{N}\ |F|<\infty\}$ where $n_1<n_2<\ldots$ is a prescribed sequence.

Proposition "Hopf d-decomposition"

If $(X,\mathcal{B},m,T)$ is a conservative, aperiodic, non-singular transformation and $d\in\mathbb{N}$;
then

1) $X=\mathfrak{C}_d\cup\mathfrak{D}_d$ mod m where :
$\mathfrak{C}_d=\mathfrak{C}_d(T)$ and $\mathfrak{D}_d=\mathfrak{D}_d(T)\in\mathcal{B}$ are disjoint, T-invariant sets,
$\mathfrak{D}_d$ is a countable union of d-wandering sets,
$T|_{\mathfrak{C}_d}$ is d-recurrent and

$$\sum_{k=1}^{\infty} m(B\cap T^{-k}B\cap\cdots\cap T^{-dk}B)=\infty\quad\forall\ B\in\mathcal{B}_+,\ B\subset\mathfrak{C}_d.$$

2) If $A\in\mathcal{B},\ A\subset\mathfrak{C}_d(T)$ and $m(A)>0$, then the collection of d-recurrence times of A: $\{n\geq 1\colon\ m(A\cap T^{-n}A\cap\cdots\cap T^{-dn}A)>0\}$ contains an IP-set.

3) $\mathfrak{C}_d(T^p)=\mathfrak{C}_d(T)\ \forall\ p\geq 1$.

Proof Suppose first that $B\in\mathcal{B}_+$ and that

$$\sum_{k=1}^{\infty} m(B\cap T^{-k}B\cap\cdots\cap T^{-dk}B)<\infty.$$

We show that B has a d-wandering subset of positive measure.

Indeed, for some subset $B_1\in\mathcal{B}_+\cap B,\ \exists\ N\geq 1$ such that

$$m(B_1\cap T^{-k}B_1\cap\cdots\cap T^{-dk}B_1)=0\ \forall\ k\geq N.$$

By Rokhlin's tower theorem (see e.g. [7]), $\exists\ E\in\mathcal{B}$ such that $E,T^{-1}E,\ldots,T^{-N}E$ are disjoint, and

$$m\left(X\setminus\bigcup_{k=0}^{N}T^{-k}E\right)<\frac{m(B_1)}{2}.$$

It follows that $\exists\ 0 \le i \le N$ such that

$$B_2 := B_1 \cap T^{-i}E \in \mathcal{B}_+.$$

Clearly $\forall\ k \ge N$:

$$m(B_2 \cap T^{-k}B_2 \cap \dots \cap T^{-dk}B_2) \le m(B_1 \cap T^{-k}B_1 \cap \dots \cap T^{-dk}B_1) = 0,$$

and for $1 \le k \le N$,

$$m(B_2 \cap T^{-k}B_2 \cap \dots \cap T^{-dk}B_2) \le m(E \cap T^{-k}E) = 0.$$

The collection $\mathcal{W}_d = \mathcal{W}_d(T)$ of d-wandering sets (under T) is a T-invariant, hereditary subcollection of $\mathcal{B}$. A classical exhaustion argument shows that $\exists\ \mathfrak{D}_d \in \mathcal{B}$, a countable union of d-wandering sets, such that any $W \in \mathcal{W}_d$ satisfies $W \subset \mathfrak{D}_d$ mod m. Since $T^{-1}\mathcal{W}_d = \mathcal{W}_d$, we have that $T^{-1}\mathfrak{D}_d \subset \mathfrak{D}_d$ whence by conservativity $T^{-1}\mathfrak{D}_d = \mathfrak{D}_d$ mod m.

By the first part of the proof, if $B \in \mathcal{B}$ and $\sum_{k=1}^{\infty} m(B \cap T^{-k}B \cap \dots \cap T^{-dk}B) < \infty$, then $B \subset \mathfrak{D}_d$ mod m, whence $\mathfrak{C}_d := X \setminus \mathfrak{D}_d$ satisfies statement 1).

To show 2), fix $A \in \mathcal{B},\ m(A) > 0,\ A \subset \mathfrak{C}_d(T)$. Choose $n_1 \ge 1$ such that $m(A \cap T^{-n_1}A \cap \dots \cap T^{-dn_1}A) > 0$ and set $A_1 := A \cap T^{-n_1}A \cap \dots \cap T^{-dn_1}A$. Since $A_1 \subset \mathfrak{C}_d(T),\ \exists\ n_2 > n_1$ such that $m(A_1 \cap T^{-n_2}A_1 \cap \dots \cap T^{-dn_2}A_1) > 0$. Set $A_2 := A_1 \cap T^{-n_2}A_1 \cap \dots \cap T^{-dn_2}A_1$ and continue, finding $n_2 < n_3 < n_4 < \dots$ and $A_3, A_4, \dots \in \mathcal{B}$ such that

$$A_k = A_{k-1} \cap T^{-n_k}A_{k-1} \cap \dots \cap T^{-dn_k}A_{k-1},\ m(A_k) > 0 \quad (k \ge 2).$$

If $F \subset \mathbb{N}$ is finite, write $F = \{k_1 < k_2 < \dots < k_{f-1} < k_f\},\ N_F := \sum_{k \in F} n_k$. We have that $A \cap T^{-N_F}A \cap \dots \cap T^{-dN_F}A \supset A_{k_f}$ whence $m(A \cap T^{-N_F}A \cap \dots \cap T^{-dN_F}A) \ge m(A_{k_f}) > 0$ and N_F is a d-recurrence times of A.

Finally we turn to the proof of 3). Let $p \ge 1$. Evidently $\mathfrak{C}_d(T^p) \subset \mathfrak{C}_d(T)$. To show $\mathfrak{C}_d(T^p) \supset \mathfrak{C}_d(T)$ let $A \in \mathcal{B},\ m(A) > 0,\ A \subset \mathfrak{C}_d(T)$. It suffices to show that $\exists\ n \ge 1$ divisible by p such that $m(A \cap T^{-n}A \cap \dots \cap T^{-dn}A) > 0$.

To do this, let $n_1 < n_2 < \dots$ be as in 2). We claim $\exists\ F \subset \{1, 2, \dots, p+1\}$ such that N_F is divisible by p (else $p \ge |\{\sum_{k=1}^{J} n_k \mod p\colon 1 \le J \le p+1\}| = p + 1$). Thus, $N_F = p\nu$ and we have that $m(A \cap T^{-p\nu}A \cap \dots \cap T^{-pd\nu}A) = m(A \cap T^{-N_F}A \cap \dots \cap T^{-dN_F}A) > 0$. □

Remark

In [9], it is shown that if S is a probability preserving transformation, then the set of d-recurrence times for any set of positive measure intersects with any IP-set.

Thus, if $(X, \mathcal{B}, m, T)$ is d-recurrent, $(\Omega, \mathcal{A}, p, S)$ is a probability preserving transformation and $A \in \mathcal{B}_+,\ B \in \mathcal{A}_+$, then $\exists\ n \ge 1$ such that both $m(A \cap T^{-n}A \cap \dots \cap T^{-dn}A) > 0$ and $p(B \cap S^{-n}B \cap \dots \cap S^{-dn}B) > 0$.

It is therefore natural to ask whether $T \times S$ is d-recurrent; and more generally whether any extension of T is d-recurrent.

§1 Markov shifts

The (two-sided) **Markov shift** $(X, \mathcal{B}, m, T)$ of the stochastic matrix $P: S \times S \to [0,1]$ with invariant distribution $\{\mu_s : s \in S\}$ is defined by

$X = S^{\mathbb{Z}}$, T = the shift, $\mathcal{B}$ the σ-algebra of generated by **cylinders** of form

$$[s_0, \dots, s_n]_k := \{x \in X : x_{k+j} = s_j \ \forall 0 \le j \le n\},$$

and

$$m([s_0, \dots, s_n]_k) = \mu_{s_0} p_{s_0,s_1} \dots p_{s_{n-1},s_n}.$$

It follows that $(X, \mathcal{B}, m, T)$ is a measure preserving transformation. It is well-known that T is conservative and ergodic iff P is irreducible and recurrent (see [4], and [1]). We'll call T **mixing** if the corresponding stochastic matrix P is irreducible, recurrent and aperiodic.

Let $(X, \mathcal{B}, m, T)$ be the conservative, ergodic Markov shift of the stochastic matrix P. For $d \ge 1$, the Cartesian product transformation $\underbrace{T \times \dots \times T}_{d\text{-times}}$ is either conservative or totally dissipative (see [1], [12]). It is the Markov shift of the stochastic matrix ${}^dP: S^d \times S^d \to [0,1]$ defined by

$${}^dp_{(s_1,\dots,s_d),(t_1,\dots,t_d)} = p_{s_1,t_1} \dots p_{s_d,t_d}$$

and therefore $\underbrace{T \times \dots \times T}_{d\text{-times}}$ is conservative iff ${}^dP$ is recurrent, i.e.

$$\sum_{n=1}^{\infty} p_{s,s}^{(n)d} = \infty \text{ for some, and hence all } s \in S.$$

Our main result in this section is

Theorem 1.1

Let $d \ge 2$. A conservative, ergodic Markov shift T is d-recurrent $\Leftrightarrow$ $\underbrace{T \times \dots \times T}_{d\text{-times}}$ is conservative.

The $\Rightarrow$ direction is easy. By the d-decomposition, the d-recurrence of T implies that

$$\sum_{n=1}^{\infty} p_{s,s}^{(n)d} = \frac{1}{\mu_s} \sum_{n=1}^{\infty} m([s] \cap T^{-n}[s] \cap \dots \cap T^{-dn}[s]) = \infty \quad \forall \ s \in S,$$

whence conservativity of $\underbrace{T \times \dots \times T}_{d\text{-times}}$.

The $\Leftarrow$ direction is established using a weak, local d-ergodic theorem on states (below).

Let $(X,\mathcal{B},m,T)$ be the Markov shift of the stochastic matrix $P: S\times S\to[0,1]$. Fix $d\ge 1$, $s\in S$, and let $A=[s]_0$. Normalise so that $m(A)=\mu_s=1$ and write

$$u(n):=m(A\cap T^{-n}A)=p_{s,s}^{(n)},\quad a_d(n)=\sum_{k=1}^{n}u(n)^d.$$

Theorem 1.2 *Suppose that T is mixing, and that $\sum_{k=1}^{\infty}u(n)^d=\infty$, then $\forall\ B_0,\dots B_d\in\mathcal{B}\cap A$,*

$$\frac{1}{a_d(n)}\sum_{k=1}^{n}m(B_0\cap T^{-k}B_1\cap T^{-2k}B_2\cap\cdots\cap T^{-dk}B_d)\ \longrightarrow\ m(B_0)\dots m(B_d). \qquad (*)$$

Corollary 1.3
If $0<a<1$ and T is $\{\frac{1}{n^a}\}$-conservative, then T is d-recurrent $\forall\ d<\frac{1}{1-a}$.

Proof (assuming theorem 1.1)
Fixing $s\in S$ and setting $u(n)=p_{s,s}^{(n)}$, it suffices to show that $\sum_{n=1}^{\infty}u(n)^d=\infty\ \forall\ d<\frac{1}{1-a}$.

To this end, suppose that $d<\frac{1}{1-a}$ and $\sum_{n=1}^{\infty}u(n)^d<\infty$, then $\frac{ad}{d-1}>1$ and by Hölder's inequality,

$$\sum_{n=1}^{\infty}\frac{u(n)}{n^a}\le\left(\sum_{n=1}^{\infty}u(n)^d\right)^{\frac{1}{d}}\left(\sum_{n=1}^{\infty}\frac{1}{n^{\frac{ad}{d-1}}}\right)^{\frac{d-1}{d}}<\infty$$

whence $\sum_{n=1}^{\infty}\frac{1}{n^a}1_A\circ T^n<\infty$ a.e. on A contradicting $\{\frac{1}{n^a}\}$-conservativity of T. $\square$

Proof of theorem 1.1 assuming theorem 1.2
Fix $B\in\mathcal{B}$, $m(B)>0$, then $\exists\ s\in S$ such that $C:=B\cap[s]$ has positive measure. Let the period of s be ν, then $T^{\nu}|_{\bigcup_{n=0}^{\infty}T^{-n\nu}[s]}$ is a $\{\frac{1}{n^a}\}$-conservative, mixing Markov shift.

By conservativity of $\underbrace{T\times\dots\times T}_{d\text{-times}}$, $\sum_{n=1}^{\infty}u(n)^d=\infty$ where $u(n):=p_{s,s}^{(n\nu)}$, and by theorem 1.2,

$$\frac{1}{a_d(n)}\sum_{k=1}^{n}m(C\cap T^{-k\nu}C\cap T^{-2k\nu}C\cap\cdots\cap T^{-dk\nu}C)\ \longrightarrow\ m(C)^{d+1}.$$

$\square$

The rest of this section is a proof of theorem 1.2.

Let

$$\mathcal{C} = \{[s, t_1, \ldots, t_n, s]_0\colon\ n \geq 0,\ t_1, \ldots, t_n \in S\}$$

and

$$\mathcal{A} = \{\bigcup_{k=1}^{N} B_k\colon\ B_1, \ldots, B_N \in \mathcal{C} \text{ disjoint}\}.$$

It follows from the conservativity of T that $\mathcal{A}$ generates $\mathcal{B} \cap A$ in the sense that

$$\forall\ B \in \mathcal{B} \cap A,\ \epsilon > 0,\ \exists\ B' \in \mathcal{A}\colon\ m(B \Delta B') < \epsilon.$$

Lemma 1.4 *$(*)$ holds for $B_0, \ldots B_d \in \mathcal{A}$.*

Proof It is sufficient to show that $(*)$ holds for $B_0, \ldots B_d \in \mathcal{C}$. Suppose that

$$B_j = [s, t_1^{(j)}, \ldots, t_{n_j}^{(j)}, s]_0 \quad (0 \leq j \leq d),$$

and that $k \geq n_j\ \forall\ 0 \leq j \leq d$, then

$$m(B_0 \cap T^{-k} B_1 \cap \cdots \cap T^{-dk} B_d) =$$
$$p_{s,t_1^{(0)}} \ldots p_{t_{n_0}^{(0)},s} p_{s,s}^{(k-n_0)} \cdot\ p_{s,t_1^{(1)}} \ldots p_{t_{n_1}^{(1)},s} p_{s,s}^{(k-n_1)} \cdots\ p_{s,s}^{(k-n_{d-1})} p_{s,t_1^{(d)}} \ldots p_{t_{n_d}^{(d)},s}$$
$$= m(B_0) \ldots m(B_d) u(k - n_0) \ldots u(k - n_{d-1}).$$

To complete the proof of the lemma, we must show that

$$\sum_{k=1}^{n} u(k - n_0) \ldots u(k - n_{d-1}) \sim a_d(n)\ \forall\ n_0, \ldots, n_{d-1} \in \mathbb{N}.$$

By Hölder's inequality,

$$\sum_{k=1}^{n} u(k - n_0) \ldots u(k - n_{d-1}) \lesssim a_d(n).$$

We now establish the reverse asymptotic inequality.

The Cartesian product transformation $S := \underbrace{T \times \ldots \times T}_{d\text{-times}}$ is a measure preserving transformation of the Cartesian product space $(X^d, \mathcal{B}_d, \mu)$ where $\mathcal{B}_d := \underbrace{\mathcal{B} \otimes \cdots \otimes \mathcal{B}}_{d\text{-times}}$ and $\mu := \underbrace{m \times \ldots \times m}_{d\text{-times}}$. It is also a Markov shift of an irreducible, aperiodic transition matrix.

The condition $\sum_{n=1}^{\infty} u(n)^d = \infty$ implies that S is conservative and ergodic (its stochastic matrix being irreducible and recurrent), whence rationally ergodic with return sequence $a_d(n)$ (see [1]). Since $A^d :=$

$\underbrace{A\times\ldots\times A}_{d\text{-times}}$ is the event of being in a certain state at time 0, we have ([1]) that

$$\sum_{k=0}^{n-1}\mu(B\cap S^{-k}C)\gtrsim\mu(B)\mu(C)a_d(n)\quad\forall\ B,C\in\mathcal{B}_d.$$

Choosing $C=A^d$ and $B=T^{-n_0}A\times T^{-n_1}A\times\ldots T^{-n_{d-1}}A$ gives

$$\sum_{k=1}^{n}u(k-n_0)\ldots u(k-n_{d-1})\sim\sum_{k=0}^{n-1}\mu(B\cap S^{-k}C)\gtrsim a_d(n).$$

□

Next, for $0\le\nu\le d$, let

$$\psi_n^{(\nu)}:=\sum_{k=1}^{n}\prod_{i=1}^{\nu}1_A\circ T^{-ik}\cdot\prod_{j=1}^{d-\nu}1_A\circ T^{jk}.$$

Note that

$$\int_A\psi_n^{(\nu)}dm=\sum_{k=1}^{n}m(T^{\nu k}A\cap\cdots\cap T^kA\cap A\cap T^{-k}A\cap\cdots\cap T^{-(d-\nu)k}A)=a_d(n).$$

Lemma 1.5

$$\int_A(\psi^{(\nu)})^2dm=O\Big(a_d(n)^2\Big)\ \textit{as}\ n\to\infty\ \forall\ 0\le\nu\le d.$$

The proof of lemma 1.5 is given after the proof of theorem 1.2.

Proof of theorem 1.2 Our first claim is

¶1 *(∗) holds for the sets $B_0,\ldots,B_d$ whenever $B_1,\ldots,B_d\in\mathcal{A}$ and $B_0\in\mathcal{B}\cap A$.* Fix $B_1,\ldots,B_d\in\mathcal{A}$, and let

$$\phi_n:=\sum_{k=1}^{n}\prod_{j=1}^{d}1_{B_j}\circ T^{jk}.$$

It is sufficient to show that

$$\frac{\phi_n}{a_d(n)}\to m(B_1)\ldots m(B_d)\text{ weakly in }L^2(A).$$

By lemma 1.4,

$$\frac{1}{a_d(n)}\int_B\phi_n dm\to m(B)m(B_1)\ldots m(B_d)\ \forall\ B\in\mathcal{A}.$$

By lemma 1.5,

$$\int_A(\phi_n)^2dm\le\int_A(\psi^{(0)})^2dm=O\Big(a_d(n)^2\Big),$$

whence for every subsequence $n_k \to \infty$ there is a subsequence (also denoted) $n_k \to \infty$ and $q \in L^2(A)$ such that

$$\frac{1}{a_d(n_k)}\phi_{n_k} \to q \text{ weakly in } L^2(A).$$

It follows that

$$\int_B q dm = m(B)m(B_1)\dots m(B_d) \ \forall \ B \in \mathcal{A},$$

whence $q = m(B_1)\dots m(B_d)$, and

$$\frac{1}{a_d(n)}\phi_n \to m(B_1)\dots m(B_d) \text{ weakly in } L^2(A).$$

□

Our next claim is:
¶2 *for each* $0 \le \nu \le d$, *(∗) holds for the sets* $B_0, \dots, B_d$ *whenever* $B_{\nu+1}, \dots, B_d \in \mathcal{A}$ *and* $B_0 \dots, B_\nu \in \mathcal{B} \cap A$. For each ν, call the claim "Claim ν". We prove the claims by induction on ν.

Claim 0 is ¶1, and established. Assume Claim $\nu-1$, and let $B_{\nu+1}, \dots, B_d \in \mathcal{A}$ and $B_0 \dots, B_{\nu-1} \in \mathcal{B} \cap A$. Set

$$\phi_n := \sum_{k=1}^{n} \prod_{i=1}^{\nu} 1_{B_{\nu-i}} \circ T^{-ik} \prod_{j=1}^{d-\nu} 1_{B_{\nu+j}} \circ T^{jk}.$$

It is sufficient to show that

$$\frac{\phi_n}{a_d(n)} \to m(B_0)\dots m(B_{\nu-1})m(B_{\nu+1})\dots m(B_d) \text{ weakly in } L^2(A).$$

By Claim $\nu - 1$,

$$\frac{1}{a_d(n)}\int_B \phi_n dm \to m(B)m(B_0)\dots m(B_{\nu-1})m(B_{\nu+1})\dots m(B_d) \ \forall \ B \in \mathcal{A}.$$

By lemma 1.5,

$$\int_A (\phi_n)^2 dm \le \int_A (\psi^{(\nu)})^2 dm = O\Big(a_d(n)^2\Big),$$

whence for every subsequence $n_k \to \infty$ there is a subsequence (also denoted) $n_k \to \infty$ and $q \in L^2(A)$ such that

$$\frac{1}{a_d(n_k)}\phi_{n_k} \to q \text{ weakly in } L^2(A).$$

It follows that

$$\int_B q dm = m(B)m(B_0)\dots m(B_{\nu-1})m(B_{\nu+1})\dots m(B_d) \ \forall \ B \in \mathcal{A},$$

whence $q = m(B_0)\dots m(B_{\nu-1})m(B_{\nu+1})\dots m(B_d)$, and

$$\frac{1}{a_d(n)}\phi_n \to m(B_0)\dots m(B_{\nu-1})m(B_{\nu+1})\dots m(B_d) \text{ weakly in } L^2(A)$$

☐Evidently, theorem 1.2 follows from ¶2 when $\nu = d$. ☐

Proof of lemma 1.5. Throughout, we use the Markov property for $\{T^n A\}_{n\in\mathbb{Z}}$:
if $b(1),\dots,b(\kappa) \in \mathbb{Z}$ and $b(1) \le b(2) \le \dots \le b(\kappa)$ then

$$m\left(\bigcap_{r=1}^{\kappa} T^{-b(r)}A\right) = m\left(\bigcap_{r=1}^{\kappa} T^{b(r)}A\right) = \prod_{r=2}^{\kappa} m(A \cap T^{-(b(r)-b(r-1))}A).$$

Set

$$\epsilon_k(\nu) := \prod_{-\nu\le j\le d-\nu,\ j\ne 0} 1_A \circ T^{jk},$$

then

$$\psi_n^{(\nu)} = \sum_{k=1}^{n} \epsilon_k(\nu), \text{and } \int_A (\psi_n^{(\nu)})^2 dm \le 2\sum_{k=1}^{n}\sum_{\ell=k}^{n}\int_A \epsilon_k(\nu)\epsilon_\ell(\nu)dm.$$

The form of $\int_A \epsilon_k(\nu)\epsilon_\ell(\nu)dm$ depends on the orders of the sets $\{ik, j\ell\colon 1 \le i,j \le \nu\}$ and $\{ik, j\ell\colon 1 \le i,j \le d-\nu\}$.

To simplify matters, set

$$\epsilon_k^{\pm}(\nu) = \prod_{j=1}^{\nu} 1_A \circ T^{\pm jk},$$

then $\epsilon_k(\nu) = \epsilon_k^-(\nu)\epsilon_k^+(d-\nu)$, and

$$\int_A \epsilon_k(\nu)\epsilon_\ell(\nu)dm = \int_A (e_k^-(\nu)\epsilon_\ell^-(\nu))(\epsilon_k^+(d-\nu)\epsilon_\ell^+(d-\nu))dm$$

and it follows from the Markov property that

$$\int_A (e_k^-(\nu)\epsilon_\ell^-(\nu))(\epsilon_k^+(d-\nu)\epsilon_\ell^+(d-\nu))dm = \int_A \epsilon_k^+(\nu)\epsilon_\ell^+(\nu)dm \int_A \epsilon_k^+(d-\nu)\epsilon_\ell^+(d-\nu)dm.$$

Accordingly, set

$$\Omega(k,\ell) = \Omega_d(k,\ell) := \{ik,\ j\ell\colon\ 1 \le i,j \le d\} \subset \mathbb{N}_{2d}.$$

Define $N_{(k,\ell)}\colon \mathbb{N}_d \times \{0,1\} \to \Omega_d(k,\ell)$ by $N_{k,\ell}(j,\epsilon) = (1-\epsilon)jk + \epsilon j\ell$.

Definition

A bijection $\omega\colon \mathbb{N}_d \times \{0,1\} \to \mathbb{N}_{2d}$ which satisfies $\omega(i,\epsilon) < \omega(i+1,\epsilon)$ $(i < d-1,\ \epsilon = 0,1)$ is called **admissible**.

Let $\mathfrak{b}_d$ denote the collection of admissible bijections $\omega\colon \mathbb{N}_d \times \{0,1\} \to \mathbb{N}_{2d}$.

An admissible bijection $\omega \in \mathfrak{b}_d$ **orders** $\Omega_d(k,\ell)$ if $ik \le j\ell$ iff $\omega(i,0) < \omega(j,1)$.

For $\omega \in \mathfrak{b}_d$, set

$$D(\omega) := \{(k,\ell) \in \mathbb{N}^2\colon\ k \le \ell,\ \omega \text{ orders } \Omega_d(k,\ell)\}.$$

To describe $D(\omega)$, let $F_d := \{\frac{p}{q}\colon 0 \le p \le q \le d\}$ be the **Farey sequence** of order d. Write $F_d = \{0 := r_0^{(d)} < r_1^{(d)} < \cdots < r_{N_d}^{(d)} = 1\}$. We claim first that

$$\exists\ j < N_d,\ D(\omega) = \{(k,\ell) \in \mathbb{N}^2\colon\ \frac{k}{\ell} \in (r_j, r_{j+1}]\}. \tag{1}$$

To see this let

$$a(\omega) = \max_{i,j\in\mathbb{N}_d,\ \omega(i,0)>\omega(j,1)} \frac{j}{i}\ (\ge 0),\ \&\ b(\omega) = \min_{i,j\in\mathbb{N}_d,\ \omega(i,0)<\omega(j,1)} \frac{j}{i}.$$

Evidently, $a(\omega) < b(\omega)$ are neighbouring elements of F_d, and by definition,

$$\begin{aligned}
&D(\omega)\\
&= \{(k,\ell) \in \mathbb{N}^2\colon\ k \le \ell,\ \frac{k}{\ell} \le \frac{j}{i}\ \forall \omega(i,0) < \omega(j,1),\ \frac{k}{\ell} > \frac{j}{i}\ \forall\ \omega(i,0) > \omega(j,1)\}\\
&= \{(k,\ell) \in \mathbb{N}^2\colon\ k \le \ell,\ a(\omega) < \frac{k}{\ell} \le b(\omega)\}.
\end{aligned}$$

□

Suppose that $1 \le d' < d$. It follows from (1) that $\forall\ \omega \in \mathfrak{b}_d,\ \exists\ \omega' \in \mathfrak{b}_{d'}$ such that $D(\omega) \subset D(\omega')$.

Given $\omega \in \mathfrak{b}_d,\ (k,\ell) \in D(\omega)$, define $\pi_{(k,\ell)}^{(\omega)}\colon \mathbb{N}_{2d} \to \Omega_d(k,\ell)$ by $\pi_{(k,\ell)}^{(\omega)} = N_{(k,\ell)} \circ \omega^{-1}$.

Setting $\omega^{-1}(j) = (\kappa_j, \epsilon_j)$, we have

$$\pi_{(k,\ell)}^{(\omega)}(j) = N_{(k,\ell)} \circ \omega^{-1}(j) = \kappa_j\big[(1-\epsilon_j)k + \epsilon_j\ell\big].$$

Next, for $1 \le j \le 2d$,

$$\begin{aligned}
\phi_{(k,\ell)}^{(\omega)}(j) &:= \pi_{(k,\ell)}^{(\omega)}(j) - \pi_{(k,\ell)}^{(\omega)}(j-1)\\
&= \kappa_j\big[(1-\epsilon_j)k + \epsilon_j\ell\big] - \kappa_{j-1}\big[(1-\epsilon_{j-1})k + \epsilon_{j-1}\ell\big]\\
&= \langle a_j, (k,\ell)\rangle
\end{aligned}$$

where $\pi_{(k,\ell)}^{(\omega)}(0) := 0,\ a_1 = (\kappa_1(1-\epsilon_1), \kappa_1\epsilon_1)$ and

$$a_j = a_j(\omega) := (\kappa_j(1-\epsilon_j) - \kappa_{j-1}(1-\epsilon_{j-1}), \kappa_j\epsilon_j - \kappa_{j-1}\epsilon_{j-1}) \qquad (j \ge 2).$$

Our next claim is

$$\int_A \epsilon_k^+(d)\epsilon_\ell^+(d)dm = \prod_{j=1}^{2d} u(\langle a_j,(k,\ell)\rangle)\ \forall\ (k,\ell)\in D(\omega),\ \omega\in\mathfrak{b}_d. \qquad (2)$$

To see this

$$\begin{aligned}\int_A \epsilon_k^+(d)\epsilon_\ell^+(d)dm &= m\left(A\cap\bigcap_{j=1}^{2d} T^{-\pi^{(\omega)}_{(k,\ell)}(j)}A\right)\\ &= \prod_{j=1}^{2d} m(A\cap T^{-(\phi^{(\omega)}_{(k,\ell)}(j))}A)\\ &= \prod_{j=1}^{2d} u(\langle a_j,(k,\ell)\rangle).\end{aligned}$$

□

The vectors $\{a_j(\omega)\}_{j=1}^{2d}$ are non-zero. Indeed, if $a_1 = 0$ then $\epsilon_1 = 1 = 0$, and if $a_j(\omega) = 0$ for some $j\ge 2$ then it follows from the definition of a_j that $\omega^{-1}(j) = \omega^{-1}(j-1)$ contradicting the bijectivity of ω.

If a_i and a_j are linearly dependent, then $a_i \propto a_j$ in the sense that $a_i = qa_j$ for some $q\in\mathbb{Q}$.

We need to know that

$$\forall\ j_0\in\mathbb{N}_{2d},\ |\{j\in\mathbb{N}_{2d}\colon\ a_j\propto a_{j_0}\}|\le d. \qquad (3)$$

Indeed, the vectors occuring as a_j are of form $(1,0)$, $(0,1)$, $(r,-s)$ and $(-r,s)$ where $1\le r,s\le d$, and we have
$a_j = (1,0)$ when $\pi^{(\omega)}_{(k,\ell)}(j) = \kappa k,\ \pi^{(\omega)}_{(k,\ell)}(j+1) = (\kappa+1)k$;
$a_j = (0,1)$ when $\pi^{(\omega)}_{(k,\ell)}(j) = \kappa\ell,\ \pi^{(\omega)}_{(k,\ell)}(j+1) = (\kappa+1)\ell$;
$a_j = (r,-s)$ when $\pi^{(\omega)}_{(k,\ell)}(j) = s\ell,\ \pi^{(\omega)}_{(k,\ell)}(j+1) = rk$;
$a_j = (-r,s)$ when $\pi^{(\omega)}_{(k,\ell)}(j) = rk,\ \pi^{(\omega)}_{(k,\ell)}(j+1) = s\ell$.

In case e.g. $a_{j_1}, a_{j_2},\dots,a_{j_N}\propto(r,-s)$ then
$\exists\ p_1,p_2\dots p_N\ge 1,\ p_nu\ne p_{\nu'}\ \ (nu\ne\nu')$
such that
$\pi^{(\omega)}_{(k,\ell)}(j_\nu) = p_\nu s\ell,\ \pi^{(\omega)}_{(k,\ell)}(j_\nu+1) = p_\nu rk$
whence $Nr, Ns\le d$ and $N\le\frac{d}{r\vee s}$. □

Consequently, for each $\omega\in\mathfrak{b}_d$,

$$\{a_j(\omega)\colon\ 1\le j\le 2d\} = \{a_j^{(1)}(\omega), a_j^{(2)}(\omega)\colon\ 1\le j\le d\}$$

where $a_j^{(1)}(\omega)$ and $a_j^{(2)}(\omega)$ are linearly independent $\forall\ 1\le j\le d\}$.

We have now established the necessary machinery to complete the proof of lemma 1.5.

Assume that $\nu \geq d-\nu$. For each $\omega \in \mathfrak{b}_\nu$, let $\omega' \in \mathfrak{b}_{d-\nu}$ be such that $D(\omega) \subset D(\omega')$.

Since $\{(k,\ell) \in \mathbb{N}^2\colon k \leq \ell\} = \bigcup_{\omega \in \mathfrak{b}_\nu} D(\omega)$ (a disjoint union), we have:

$$\begin{aligned}\int_A (\psi_n^{(\nu)})^2 dm &\leq 2\sum_{k=1}^{n}\sum_{\ell=k}^{n}\int_A \epsilon_k(\nu)\epsilon_\ell(\nu)dm \\ &= 2\sum_{\omega\in\mathfrak{b}_\nu}\ \sum_{(k,\ell)\in D(\omega),\ k,\ell\leq n}\int_A \epsilon_k(\nu)\epsilon_\ell(\nu)dm.\end{aligned}$$

For each $\omega \in \mathfrak{b}_\nu$,

$$\begin{aligned}&\sum_{(k,\ell)\in D(\omega)\cap\mathbb{N}_n^2}\int_A \epsilon_k(\nu)\epsilon_\ell(\nu)dm \\ &= \sum_{(k,\ell)\in D(\omega)\cap\mathbb{N}_n^2}\int_A \epsilon_k^+(\nu)\epsilon_\ell^+(\nu)dm\int_A \epsilon_k^+(d-\nu)\epsilon_\ell^+(d-\nu)dm \\ &= \sum_{(k,\ell)\in D(\omega)\cap\mathbb{N}_n^2}\prod_{j=1}^{2\nu} u(\langle a_j(\omega),(k,\ell)\rangle)\prod_{j=1}^{2(d-\nu)} u(\langle a_j(\omega'),(k,\ell)\rangle) \\ &= \sum_{(k,\ell)\in D(\omega)\cap\mathbb{N}_n^2}\prod_{j=1}^{d} u(\langle a_j^{(1)},(k,\ell)\rangle)u(\langle a_j^{(2)},(k,\ell)\rangle)\end{aligned}$$

where

$$\{a_j^{(1)},a_j^{(2)}\}_{j=1}^{2d} = \{a_j^{(1)}(\omega),a_j^{(2)}(\omega)\}_{j=1}^{2\nu}\cup\{a_j^{(1)}(\omega'),a_j^{(2)}(\omega')\}_{j=1}^{2(d-\nu)}.$$

Consider $B_j\colon\mathbb{R}^2\to\mathbb{R}^2$ defined by $(B_jx)_i := \langle x, a_j^{(i)}\rangle \quad (i=1,2)$ which is injective. Let $K>0$ be such that $\|B_jx\|_\infty \leq K\|x\|_\infty\ \forall\ x,j$.

By Hölder's inequality,

$$\begin{aligned}&\sum_{(k,\ell)\in D(\omega)\cap\mathbb{N}_n^2}\int_A \epsilon_k(\nu)\epsilon_\ell(\nu)dm \\ &\leq \prod_{j=1}^{d}\left(\sum_{(k,\ell)\in D(\omega)\cap\mathbb{N}_n^2} u(\langle a_j^{(1)},(k,\ell)\rangle)^d u(\langle a_j^{(2)},(k,\ell)\rangle)^d\right)^{\frac{1}{d}} \\ &= \prod_{j=1}^{d}\left(\sum_{(k,\ell)\in B_j(D(\omega)\cap\mathbb{N}_n^2)} u(k)^d u(\ell)^d\right)^{\frac{1}{d}} \\ &\leq \sum_{(k,\ell)\in\mathbb{N}_{Kn}^2} u(k)^d u(\ell)^d \\ &= a_d(Kn)^2.\end{aligned}$$

To complete the proof of the lemma, we must show that $a_d(Kn) = O(a_d(n))$ as $n\to\infty$.

To see this, note first that $v_k = u_k^d$ is a recurrent renewal sequence and so $\exists\ 1 = c_0 \geq c_1 \geq \cdots \geq c_n \downarrow 0$ such that $\sum_{k=0}^{n} v_k c_{n-k} = 1\ \forall\ n \geq 0$. It can be shown that

$$1 \leq \frac{a_d(n)L(n)}{n} \leq e^2\ \forall\ n \geq 1$$

where $L(n) := \sum_{k=0}^{n-1} c_k$. It follows that for $K > 1$,

$$a_d(Kn) \leq \frac{Ke^2 n}{L(Kn)} \leq \frac{Ke^2 n}{L(n)} \leq Ke^2 a_d(n).$$

□

§2 INFINITE ODOMETERS

Definition: $(b_1, b_2, \ldots)$-adic odometer For $b_k \geq 1$ define

$$\Omega = \Omega(b_1, b_2, \ldots) := \prod_{k=1}^{\infty} \{0, 1, \ldots, b_k - 1\}$$

Define addition on Ω by

$$(\omega + \omega')_n = \omega_n + \omega'_n + \epsilon_n \mod b_n$$

where

$$\epsilon_n = \begin{cases} 0 & n = 1 \text{ or } \omega_{n-1} + \omega'_{n-1} + \epsilon_{n-1} < b_{n-1}, \\ 1 & n \geq 2 \text{ and } \omega_{n-1} + \omega'_{n-1} + \epsilon_{n-1} \geq b_{n-1}, \end{cases}$$

It follows (see [11]) that Ω equipped with the product topology is a compact topological group.

It is called the (group of) **$(b_1, b_2, \ldots)$-adic integers** since

$$\mathbb{Z}_+ \cong \Omega_0 := \{\omega \in \Omega\colon\ \omega_n \to 0\} \text{ by } \omega \leftrightarrow \sum_{n=1}^{\infty} B(n)\omega_n$$

where $B(1) = 1,\ B(n) = b_1 b_2 \ldots b_{n-1}\ \ (n \geq 2)$,

$$-1 \leftrightarrow\ (b_1 - 1, b_2 - 1, \ldots)$$

and

$$-\mathbb{N} \cong \{\omega \in \Omega\colon\ b_n - 1 - \omega_n \to 0\} = (b_1 - 1, b_2 - 1, \ldots) - \Omega_0.$$

The symmetric product probability measure is a Haar measure on Ω.

The $(b_1, b_2, \ldots)$**-adic adding machine** (or **odometer**) $\tau\colon \Omega \to \Omega$ is $\tau x = x + 1$ where $1 := (1, \overline{0})$.

Now let $1 \leq b_n\ \ (n \geq 1)$ and let T be the $(b_1, b_2, \ldots)$-adic odometer on $\Omega(b_1, b_2, \ldots)$. Suppose that $0 \in K_n \subset \mathbb{Z}_+ \cap [0, b_n - 1]\ \ (n \geq 1)$ and let $W := \{x \in \Omega(b_1, b_2, \ldots)\colon\ x_n \in K_n\ \forall\ n \geq 1\}$.

Our first result in this section is that all points of W excepting possibly one return to W under positive iterations of T, and that the first return transformation on W is itself isomorphic to an odometer.

Let $a_n := |K_n|$ and write:

$$K_n = \{0 = t_0(n) < \cdots < t_{a_n-1}(n)\},$$

$$\alpha(n,k) = \begin{cases} t_{k+1}(n) - t_k(n) & k < a_n - 1, \\ b_n - t_{a_n-1}(n) & k = a_n - 1. \end{cases}$$

Note that $\Omega(a_1, a_2, \dots) \cong W$ by $x = (x_1, x_2, \dots) \leftrightarrow t(x) = (t_{x_1}(1), t_{x_2}(2), \dots)$. Accordingly, define $A(n)\ \ (n \geq 1)$ by $A(1) = 1,\ A(n) = a_1 a_2 \dots a_{n-1}\ \ (n \geq 2)$.

Proposition 2.1

Suppose that $x \in \Omega(a_1, a_2, \dots)$ *and that* $\ell(x) := \min\{n \geq 1\colon\ x_n < a_n - 1\} < \infty$*, then*

$$\varphi(t(x)) := \min\{n \geq 1\colon\ T^n(t(x)) \in W\} = \varphi(\ell(x), x_{\ell(x)})$$

where

$$\varphi(k,j) = \sum_{i=1}^{k-1} B(i)\alpha(i, a_i - 1) + B(k)\alpha(k,j)$$

and

$$T_W(t(x)) := T^{\varphi(t(x))} t(x) = t(\tau x)$$

where τ *is the* $(a_1, a_2, \dots)$*-adic odometer on* $\Omega(a_1, a_2, \dots)$*.*

Thus, the adding machine T with digits $b_1, b_2, \dots$ equipped with the σ-finite invariant measure m with $m(W) = 1$ is isomorphic to a tower over τ (equipped with Haar measure on Ω) with height function φ as above (see [13]).

We call the measure preserving transformation $(\Omega(b_1, b_2, \dots), \mathcal{B}, m, T)$ the **infinite odometer** with **digits** $b_1, b_2, \dots$ and **base sets** $K_1, K_2, \dots$.

Remarks

1) The measure preserving transformation "defined by the d-greedy algorithm" is isomorphic to an infinite odometer with digits $b_n = d$ and base sets $K_n = \{0, 1, \dots, d-2\}\ \ \forall\ n \geq 1$.

2) The infinite odometer with digits $b_1, b_2, \dots$ and base sets $K_1, K_2, \dots$ is isomorphic to the cutting and stacking construction defined by

$$B_0 := 1,\ B_{n+1} = B_n(1_{K_n}(0)), B_n(1_{K_n}(1)), \dots, B_n(1_{K_n}(b_n - 1))$$

where $B_n(1) := B_n$ and $B_n(0) := 0^{|B_n|}$.

3) It can be shown that an infinite odometer is of **positive type** in the sense that $\limsup_{n\to\infty} m(A \cap T^{-n}A) > 0\ \ \forall\ A \in \mathcal{B}\ m(A) > 0$ (see

[15]) iff

$$\limsup_{n\to\infty}\sup_{t\in\mathbb{N}}\frac{1}{|K_n|}|\{x\in K_n:\ x+t\in K_n\}|>0.$$

This is evidently the case when $\liminf_{n\to\infty}|K_n|<\infty$, in which case it can be shown that the infinite odometer enjoys the stronger property of **partial rigidity** in the sense of [2].

By corollary 1.4 of [2], all Cartesian products $\underbrace{T\times\ldots\times T}_{d\text{-times}}$ $(d\ge 1)$ of a partially rigid measure preserving transformation T are conservative.

The next proposition generalises this.

Proposition 2.2

Suppose that $(X,\mathcal{B},m,T)$ is an invertible, conservative, ergodic measure preserving transformation of positive type, then $\underbrace{T\times\ldots\times T}_{d\text{-times}}$ is of positive type (and hence conservative) $\forall\ d\ge 1$.

Proof

Fix $d\ge 1$ and let $S:=\underbrace{T\times\ldots\times T}_{d\text{-times}}$- a measure preserving transformation of the Cartesian product space $(X^d,\mathcal{B}_d,\mu)$ where $\mathcal{B}_d:=\underbrace{\mathcal{B}\otimes\cdots\otimes\mathcal{B}}_{d\text{-times}}$ and $\mu:=\underbrace{m\times\ldots\times m}_{d\text{-times}}$.

Let

$$\mathcal{Z}_d:=\{A\in\mathcal{B}_d:\ \mu(A)<\infty,\ \mu(A\cap S^{-n}A)\to 0 \text{ as } n\to\infty\}.$$

A classical exhaustion argument shows that $\exists\ Z_d\in\mathcal{B}$, a countable union of sets in $\mathcal{Z}_d$, such that any $A\in\mathcal{Z}_d$ satisfies $A\subset Z_d$ mod μ. It follows that

$$\mu(B\cap S^{-n}C)\to 0\ \ \forall\ B,C\in\mathcal{B}_d\cap Z_d\ \mu(B),\mu(C)<\infty$$

whence $\{A\in\mathcal{B}:\ A\subset Z_d\ \mu(A)<\infty\}=\mathcal{Z}_d$.

Since $T^{n_1}\times\ldots\times T^{n_d}\mathcal{Z}_d=\mathcal{Z}_d\ \forall\ (n_1,\ldots,n_d)\in\mathbb{Z}^d$, we have that

$$T^{n_1}\times\ldots\times T^{n_d}Z_d=Z_d\mod\mu\ \ \forall\ (n_1,\ldots,n_d)\in\mathbb{Z}^d.$$

The ergodicity of this $\mathbb{Z}^d$ action shows that either $Z_d=X^d\mod\mu$ or $\mu(Z_d)=0$.

Since sets of form A^d $(A\in\mathcal{B},\ 0<m(A)<\infty)$ are not in $\mathcal{Z}_d$, we must have that $\mu(Z_d)=0$. Thus S is of positive type. □

Thus, all Cartesian products of positive-type infinite odometers are conservative. The next proposition (2.3) shows however, that this does not imply their $\{c_n\}$-conservativity for any $\{c_n\}$.

Proposition 2.3

For any $c_n \downarrow 0$, $\exists$ a positive-type infinite odometer which is $\{c_n\}$-dissipative.

Proof

Choose $b_n \geq 2$ such that $c_{B(n)} \leq \frac{1}{4^n}$ (where $B(n+1) := b_1 \dots b_n$), and let T be the infinite odometer with digits b_n and base sets $K_n = \{0,1\}$; which is of positive-type by remark 3 above.

On W, we have

$$\sum_{n=1}^{\infty} c_n 1_W \circ T^n = \sum_{n=1}^{\infty} c_{\varphi_n} = \sum_{k=0}^{\infty} \sum_{n=2^k}^{2^{k+1}-1} c_{\varphi_n} \leq \sum_{k=0}^{\infty} 2^k c_{\varphi_{2^k}}$$

where $\varphi_n := \sum_{k=0}^{n-1} \varphi \circ T_W^n$.

Now,

$$\varphi(x) = \varphi(\ell(x), x_{\ell(x)}) = \begin{cases} 1 & \ell(x) = 1, \\ \sum_{k=1}^{\ell(x)-1} B(k)(b_k - 1) + B(\ell)(x) & \text{else,} \end{cases}$$

so $\varphi(x) = 2B(\ell(x)) - 1 \geq B(\ell(x))$ and

$$\varphi_{2^n}(x) \geq \sum_{(\epsilon_1,\dots,\epsilon_n)\in\{0,1\}^n \setminus \{\underline{1}\}} B(\ell(\epsilon)) = \sum_{k=1}^{n} 2^{n-k} B(k) \geq B(n),$$

whence

$$\sum_{n=1}^{\infty} c_n 1_W \circ T^n \leq \sum_{k=0}^{\infty} 2^k c_{B(k)} < \infty.$$

□

Proposition 2.4

Suppose that $d \geq 2$ and for each $n \geq 1$, $K_n \subset [0, (b_n - 1)/2]$ and K_n has no arithmetic progressions of length $d+1$ in $\mathbb{N}$,

then

W has no arithmetic progressions of length $d+1$ in $\Omega(b_1, b_2, \dots)$ and $W \in \mathcal{W}_d(T)$.

Proof

Suppose first that $x, y, z \in W$ and that $z - y = y - x$ in $\Omega(b_1, b_2, \dots)$, equivalently $x + z = 2y$. Since $\omega_n \leq (b_n - 1)/2$ $\forall$ $n \geq 1$, we have that $(x+z)_n = x_n + z_n$ and $(y+y)_n = 2y_n$ $\forall$ $n \geq 1$. Thus $x_n + z_n = 2y_n$ $\forall$ $n \geq 1$.

Next, suppose that $N \geq 1$ and $x \in \bigcap_{k=0}^{d} T^{-kN}W$. Set $x(k) = T^{kN}x = x + kN \in W$. We have that $x(k+2) - x(k+1) = x(k+1) - x(k) = N$ in $\Omega(b_1, b_2, \dots)$, equivalently:

$$x(k) + x(k+2) = 2x(k+1) \qquad (0 \leq k \leq d-2).$$

By the above, $\forall\ n \geq 1, 0 \leq k \leq d-2$: $x_n(k) + x_n(k+2) = 2x_n(k+1)$, equivalently: $x_n(k+2) - x_n(k+1) = x_n(k+1) - x_n(k)$ and $x_n(0), \dots, x_n(d)$ are in arithmetic progression. It follows from the assumption that $x_n(0) = \cdots = x_n(d)\ \forall\ n \geq 1$, whence $x(0) = \cdots = x(d)$ and $N = 0$ contradicting $N \geq 1$. □

The rest of the section is devoted to the advertised construction of an infinite odometer which is $\{\frac{1}{n^a}\}$-conservative $\forall\ 0 < a < 1$, but not 2-recurrent.

Lemma 2.5

Suppose that $\sup K_n \asymp b_n$ *and that* $b_n > 2a_n$, *then*

$$\sum_{j=0}^{a_k-2} \varphi(k,j) \asymp B(k). \tag{1}$$

$$\Gamma(n) := A(n) \sum_{k=1}^{n} \frac{1}{A(k)} \sum_{j=0}^{a_k-2} \varphi(k,j) \asymp B(n). \tag{2}$$

$$\tilde{\Gamma}(n) := \Gamma(n) + \sum_{k=0}^{a_{n+1}-2} \varphi(n+1,k) \asymp B(n+1). \tag{3}$$

$$\varphi_{A(n+1)} = \Gamma(n) + \varphi(a_1-1, \dots, a_n-1, x_{n+1}, \dots) \overset{w.p.1-\frac{1}{a_{n+1}}}{\to} = \Gamma(n) + \varphi(n+1, x_{n+1}). \tag{4}$$

Proof

(1): We have

$$\begin{aligned}\sum_{j=0}^{a_k-2} \varphi(k,j) &= \sum_{j=0}^{a_k-2} \left(\sum_{i=1}^{k-1} B(i-1)\alpha(i, a_i-1) + B(k-1)\alpha(k,j) \right) \\ &= (a_k-1) \sum_{i=1}^{k-1} B(i-1)\alpha(i,a_i-1) + B(k-1) \sum_{j=0}^{a_k-2} \alpha(k,j)\end{aligned}$$

whence

$$\sum_{j=0}^{a_k-2} \varphi(k,j) \geq B(k-1) \sum_{j=0}^{a_k-2} \alpha(k,j) = B(k-1)t_{a_k-1} \asymp B(k),$$

and

$$\sum_{j=0}^{a_k-2}\varphi(k,j)\le a_k\sum_{i=1}^{k-1}B(i)+B(k)$$
$$=B(k)+a_kB(k-1)\sum_{i=1}^{k-1}\frac{B(i)}{B(k-1)}$$
$$\le B(k)+a_kB(k-1)\sum_{i=1}^{k-1}\frac{1}{2^{k-i-1}}$$
$$\le B(k)+2a_kB(k-1)\sim B(k).$$

(2) is seen thus:

$$\Gamma(n)=A(n)\sum_{k=1}^{n}\frac{1}{A(k)}\sum_{j=0}^{a_k-2}\varphi(k,j)$$
$$\asymp A(n)\sum_{k=1}^{n}\frac{B(k)}{A(k)}$$
$$\asymp B(n)\left(1+\sum_{k=1}^{n}\frac{A(n)B(k)}{B(n)A(k)}\right)$$
$$\asymp B(n)$$

since $\frac{A(n)B(k)}{B(n)A(k)}\le\frac{1}{2^{n-k}}$.

(3) is established using (1):

$$\tilde{\Gamma}(n):=\Gamma(n)+\sum_{k=0}^{a_{n+1}-2}\varphi(n+1,k)\asymp B(n)+B(n+1)\asymp B(n+1).$$

To see (4), for $n\ge 1$ write $\Omega_n:=\prod_{k=1}^{n}\{0,1,\ldots,a_k-1\}$, then $\forall\ \omega\in\Omega$ and $n\ge 1$,

$$\{((\tau^k\omega)_1,\ldots(\tau^k\omega)_n):\ 0\le k\le A(n+1)-1\}=\Omega_n.$$

Moreover if $0\le k\le A(n+1)-1$ and $(\tau^k\omega)_j=a_j-1\ \ (1\le j\le n)$ then $(\tau^k\omega)_j=\omega_j\ \forall\ j\ge n+1$. It follows that

$$\varphi_{A(n+1)}=\sum_{(\omega_1,\ldots,\omega_n)\in\Omega_n\setminus\{(a_1-1,\ldots,a_n-1)\}}\varphi(\ell(\omega),\omega_{\ell(\omega)})$$
$$+\varphi(a_1-1,\ldots,a_n-1,\omega_{n+1},\ldots),$$

whence

$$\sum_{(\omega_1,\ldots,\omega_n)\in\Omega_n\setminus\{(a_1-1,\ldots,a_n-1)\}}\varphi(\ell(\omega),\omega_{\ell(\omega)})$$
$$=\sum_{k=1}^{n}|\{(\omega_1,\ldots,\omega_n)\in\Omega_n,\ \ell(\omega)=k\}|\varphi(k,\omega_k)$$
$$=\sum_{k=1}^{n}a_{k+1}\ldots a_n\sum_{j=0}^{a_k-2}\varphi(k,j)=\Gamma(n).$$

□

Proposition 2.6
$\exists\ c>0$ and a conservative, ergodic measure preserving transformation which is $\{\frac{e^{c\sqrt{\log_2 n}}}{n}\}$-conservative and not 2-recurrent.

Remark In particular, this conservative, ergodic measure preserving transformation is $\{\frac{1}{n^a}\}$-conservative $\forall\ 0<a<1$.

Proof
By Behrend's theorem (see [3]), $\exists\ c>0$ and

$$\forall\ n\geq 1,\quad \exists\ K\subset\mathbb{N}\cap[0,n],\ |K|=\frac{n}{L_c(n)}$$

without arithmetic progressions of length 3 where $L_c(x):=2^{c\sqrt{\log_2 n}}$. We use this as follows to define a suitable infinite odometer T.

The infinite odometer will have digits $(b_1,b_2,\ldots)$ and base set $W=K_1\times K_2\times\ldots$ where $|K_n|=a_n$ and $\max K_n<\frac{b_n-1}{2}$. By proposition 2.4 it will not be 2-recurrent.

Set $b_{2n+1}=4$ and $K_{2n+1}=\{0,1\}$.

Next, we define K_{2n} and b_{2n}. For $n\geq 1$ set $\alpha_n:=2^{n^2}$, then $\frac{\alpha_n}{n}\uparrow\infty$ and $\sum_{k=1}^{n}\alpha_k^s\sim\alpha_n^s\ \forall\ s>0$. Set $a_{2n}=e^{\alpha_n}$ and $b_{2n}=a_{2n}L_c(a_{2n})$. Using Behrend's theorem as above, choose sets $K_{2n}\subset\mathbb{N}\ (n\geq 1)$ without arithmetic progressions of length of length 3 such that $|K_{2n}|=a_{2n}$, $\max K_{2n}\leq\frac{b_{2n}}{2}$.

We claim that T is $\{\frac{L_s(n)}{n}\}$-conservative $\forall\ s > c$, indeed,

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{L_s(n)}{n} 1_W \circ T^n &= \sum_{n=1}^{\infty} \frac{L_s(\varphi_n)}{\varphi_n} \\ &\geq \sum_{n=1}^{\infty} \sum_{A(2n)\leq k\leq A(2n+1)} \frac{L_s(\varphi_k)}{\varphi_k} \\ &\geq \sum_{n=1}^{\infty} (A(2n+1) - A(2n)) \frac{L_s(\varphi_{A(2n+1)})}{\varphi_{A(2n+1)}} \\ &\geq \sum_{n=1}^{\infty} \frac{A(2n+1) L_s(\varphi_{A(2n+1)})}{2\varphi_{A(2n+1)}}.\end{aligned}$$

Now, by (4) of lemma 2.5,

$$m([\varphi_{A(2n+1)} \leq \tilde{\Gamma}(2n+1)]) \geq 1 - \frac{1}{a_{2n+1}} = \frac{1}{2}.$$

By the Borel-Cantelli lemma, for a.e. $x \in W$, $\exists\ n_k = n_k(x) \to \infty$ such that $\varphi_{A(2n_k+1)}(x) \leq \tilde{\Gamma}(2n_k+1)\ \forall\ k$.

It follows that

$$\begin{aligned}\frac{A(2n_k+1)L_s(\varphi_{A(2n_k+1)})}{2\varphi_{A(2n_k+1)}} &\geq \frac{A(2n_k+1)L_s(\tilde{\Gamma}(2n_k+1))}{2\tilde{\Gamma}(2n_k+1)} \\ &\asymp \frac{A(2n_k+1)L_s(B(2n_k+2))}{B(2n_k+2)} \text{ by (3) of lemma 2.5} \\ &\asymp \frac{A(2n_k+1)L_s(B(2n_k+1))}{B(2n_k+1)} \quad \forall\ k \text{ since } b_{2n_k+1} = 4.\end{aligned}$$

Now $B(2n+1) = A(2n+1)2^n e^{c\sum_{k=1}^n \sqrt{\alpha_k}}$, whence as $n \to \infty$:

$$\frac{B(2n+1)}{A(2n+1)} = 2^n e^{c\sum_{k=1}^n \sqrt{\alpha_k}} = e^{c\sqrt{\alpha}_n(1+o(1))}$$

and

$$L_s(B(2n+1)) = e^{s\sqrt{\alpha}_n(1+o(1))}$$

since $\log B(2n+1) = \alpha_n(1+o(1))$.

It follows that

$$\frac{A(2n+1)L_s(B(2n+1))}{B(2n+1)} = e^{(s-c)\sqrt{\alpha}_n(1+o(1))} \to \infty$$

whence $\sum_{n=1}^{\infty} \frac{L_s(n)}{n} 1_W \circ T^n = \infty$ a.e. and T is $\{\frac{L_s(n)}{n}\}$-conservative. ☐

Remark

The interested reader may generalise proposition 2.6 (with analogous proof) to show that given an increasing slowly varying function $x \mapsto L(x)$, a sequence $k_n \to \infty$ and sets $K_n \subset [0, k_n]$ with $|K_n \cap [0, k_n]| k_n / L(k_n)$, but without arithmetic progressions of length $d+1$, then for every $\epsilon > 0$ there is an odometer which is $\{\frac{1}{L(n)^{1+\epsilon}}\}$-conservative but has d-wandering sets.

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