LOCAL LIMIT THEOREMS FOR GIBBS-MARKOV MAPS

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Abstract. We prove conditional local limit theorems for Gibbs-Markov processes whose marginals are in the domain of attraction of a stable law with order in $(0, 2)$.

Introduction

Given a $\mathbb{R}$-valued stationary stochastic sequence $X_1, X_2, \dots$ defined on a probability space $(\Omega, \mathcal{F}, P)$, we consider *local limits* of the partial sums $S_n := X_1 + \cdots + X_n$, that is the existence of constants $A_n, B_n \in \mathbb{R}$, $B_n \to +\infty$ such that $\forall\ \kappa \in \mathbb{R}$ and $I \subset \mathbb{R}$ (an interval),

$$B_n P(S_n - k_n \in I) \to |I| g(\kappa) \text{ as } \frac{k_n - A_n}{B_n} \to \kappa \tag{LL}$$

where g is a continuous probability density on $\mathbb{R}$.

These local limits are connected to *distributional limits* where

$$\frac{S_n - A_n}{B_n} \xrightarrow{\mathcal{D}} Y \tag{DL}$$

for some limit random variable Y. Indeed it can be shown that if the convergence in (LL) is uniform in $\kappa \in$ compact subsets of $\mathbb{R}$ and $g = f_Y$ (the density of Y) then (DL) is satisfied with the same constants A_n, B_n. This is essentially the De Moivre- Laplace proof of the classical central limit theorem (see [16], [17]).

In the case where $X_1, X_2, \dots$ are independent, (identically distributed) the forms of the possible limit random variables Y are known ([30], [18], [23]) and are distributed according to the well known *stable* laws which have smooth densities.

The local limits for independent random variables satisfying (DL) can be found in [23] (see also references therein and [11]).

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The distributional limits (DL) have been established for wide classes of non-independent stationary stochastic sequences (see [9] and [23] and references therein).

In contrast, the local limits (LL) are only known for various classes of stationary stochastic sequences which are close to Markov chains (see below).

Using methods developed by Markov ([31]) and Doeblin ([13]) to prove (DL) for certain Markov chains, Kolmogorov ([26], see also [17]) obtained the local limits (LL) for Markov chains with finite state space.

Nagaev ([35]) considered a wide class of stationary Markov chains with infinite state space, obtaining local limits in the normal case, and distributional limits in the stable case. Aleshkyavichene ([8]) obtained local limits for certain stationary Markov chains in the non-normal, stable case.

In general, a stationary stochastic sequence $(X_1, X_2, \dots)$ is generated by a dynamical system and a measurable function:
$X_n = f \circ T^n$ where T is a probability preserving transformation of some probability space Ω, and $f : \Omega \to \mathbb{R}$ is measurable.

Local limits were obtained in the normal case for stationary sequences generated by Lasota-Yorke maps of the interval and functions of bounded variation by Rousseau-Egele ([38], see also [33], [34]), and for Lipschitz continuous functions of mixing finite state topological Markov shifts under Gibbs measures ([20], see also [19]; and [12] for multidimensional extensions).

One feature of the results of Aleshkyavichene and Nagaev which interested us, is that the constants A_n and B_n appearing in (DL) and (LL) are completely determined by the marginal distributions of the X_n in case the limit stable random variable Y is not normal.

We show here that this phenomenon persists for stationary sequences generated by mixing Gibbs-Markov maps (see §1) together with aperiodic (see §3), Lipschitz continuous functions. The limit theorems are given in §6.

In fact, the results of §6 are established with respect to the sequence of conditional measures on the fibres of T^n given by the Frobenius-Perron operators (defined in §1). This enables an application to infinite ergodic theory in §7 where we establish pointwise dual ergodicity of certain skew products, including certain interval maps. The results are also applied in [5].

The methods of this paper closely follow those of [35] (as do those of [20], [33], [34] and [38]) working through the spectral theory of Frobenius-Perron operators satisfying the Doeblin-Fortet inequality (see below) and relying on perturbation theory (see [25]). As far as we know this is the first application of such to establish (LL) in the non-normal, stable case.

The main underlying idea is that a stationary sequence $(X_1, X_2, \dots)$ generated by a mixing Gibbs-Markov map together with a Lipschitz continuous function has the property that
$\exists\ 0 < r < 1,\ \epsilon > 0$ and a function $\lambda : (-\epsilon, \epsilon) \to B_{\mathbb{C}}(0,1)$ such that

$$\sup_{|t|<\epsilon} \text{ess-sup}_{y\in\Omega} |E(e^{itS_n}|T^n(\cdot) = y) - \lambda(t)^n p_t(y)| = O(r^n) \text{ as } n \to \infty \tag{N}$$

where $p_t \to 1$ uniformly as $t \to 0$.

This property is established in §4 using the spectral theory of characteristic function (or perturbation) operators (see §2) which are perturbations of the Frobenius-Perron operator (see §1).

The asymptotic expansion of the function $\lambda : (-\epsilon, \epsilon) \to B_{\mathbb{C}}(0,1)$ is obtained in §5 when the distribution of X_1 is in the domain of attraction of a p-stable law $(0 < p < 2)$. It is the same as that of the characteristic function of X_1 (see remark 1 after theorem 5.1). The convergence (LL) (and (DL)) is then established as in the independent case and the constants A_n and B_n appearing in (LL) and (DL) are completely determined by the marginal distributions of the X_n as remarked above.

The classical case where the distribution of X_1 is in the normal domain of attraction of the Gaussian law (i.e. $E(X_1^2) < \infty$) can be obtained by straightforward modification of [20] or [38] and is only included in the discussion in §7.

There is some discussion of stationary sequences generated by mixing Gibbs-Markov maps together with *periodic* (= non-aperiodic) Lipschitz continuous functions in §3 and §7.

§1 Preliminaries on Markov maps

Let $(X, \mathcal{B}, m, T)$ denote a nonsingular transformation of a standard probability space. It is called a *Markov* map if there is a measurable partition α such that $Ta \in \sigma(\alpha) \mod m\ \ \forall\, a \in \alpha$, which generates $\mathcal{B}$ under T in the sense that $\sigma(\{T^{-n}\alpha :\ n \geq 0\}) = \mathcal{B}$ and which satisfies $T|_a$ invertible and nonsingular for $a \in \alpha$. Markov maps are called *Markov fibred systems* in [6].

Write $\alpha = \{a_s : s \in S\}$ and endow $S^{\mathbb{N}}$ with its canonical (Polish) product topology. Let

$$\Sigma = \{s = (s_1, s_2, \dots) \in S^{\mathbb{N}} : \ m(\bigcap_{k=1}^{n} T^{-k} a_{s_k}) > 0 \ \ \forall \ n \geq 1\},$$

then Σ is a closed, shift invariant subset of $S^{\mathbb{N}}$, and there is a measurable map $\phi : \Sigma \to X$ defined by $\{\phi(s_1, s_2, \dots)\} := \bigcap_{k=1}^{\infty} T^{-(k-1)} a_{s_k}$. If $m' = m \circ \phi^{-1} \in \mathcal{P}(S^{\mathbb{N}})$ (the set of probability measures on $S^{\mathbb{N}}$) then Σ is the closed support of m', and ϕ is a conjugacy of $(X, \mathcal{B}, m, T)$ with $(\Sigma, \mathcal{B}(\Sigma), m', \text{shift})$. Thus there is no loss of generality in assuming that $X = \Sigma$, T is the shift, and $\alpha = \{[s] : s \in S\}$.

Throughout this paper, we fix $r \in (0, 1)$ and define the metric $d = d_r$ on X by $d(x, y) = r^{t(x,y)}$ where $t(x, y) = \min\{n \geq 1 : \ x_n \neq y_n\} \leq \infty$, then (X, d) is a Polish space and $T : X \to X$ is Lipschitz continuous on each $a \in \alpha$.

Remark A function $f : X \to \mathbb{R}$ is called *Hölder continuous* on X if $\exists \ \theta \in (0, 1)$ and $M > 0$ such that

$$|f(x) - f(y)| \leq M\theta^{t(x,y)} \ \forall \ x, y \in X.$$

Thus Hölder continuous functions are in fact Lipschitz continuous with respect to the appropriate metric $d = d_\theta$.

For $n \geq 1$, there are m-nonsingular inverse branches of T denoted $v_a : T^n a \to a$ ($a \in \alpha_0^{n-1}$) with Radon Nikodym derivatives

$$v_a' := \frac{dm \circ v_a}{dm}.$$

Since $T\alpha \subset \sigma(\alpha)$, $T^n \alpha_0^{n-1} = T\alpha$, and $\exists$ a (finite or countable) partition β coarser than α so that $\sigma(T\alpha) = \sigma(\beta)$.

We'll assume throughout that T is *topologically mixing* in the sense that

$$\forall \ a, b \in \alpha, \ \exists \ n_{a,b} \ \ni \ T^n a \supset b \ \forall \ n \geq n_{a,b}.$$

The *Frobenius-Perron* operators $P_{T^n} : L^1(m) \to L^1(m)$ defined by

$$\int_X P_{T^n} f \cdot g dm = \int_X f \cdot g \circ T^n dm$$

have the form

$$P_{T^n}f = \sum_{b\in\beta} 1_b \sum_{a\in\alpha_0^{n-1},\ T^n a\supset b} v_a' \cdot f\circ v_a.$$

As mentioned in the introduction, they give the conditional probabilities

$$E(f|T^n(\cdot) = y) = P_{T^n}f(y).$$

Definitions

1) Let $\mathcal{C},\ \mathcal{L}$ be Banach spaces such that $\mathcal{C}\supset\mathcal{L}$ and $\|\cdot\|_{\mathcal{C}}\le\|\cdot\|_{\mathcal{L}}$. We say that the pair $(\mathcal{C},\mathcal{L})$ is *adapted* if $\mathcal{L}$-bounded sets are precompact in $\mathcal{C}$.

2) Let $(\mathcal{C},\mathcal{L})$ be an adapted pair of Banach spaces. A linear operator $P:\mathcal{C}\to\mathcal{C}$ is said to be a *D-F operator on* $(\mathcal{C},\mathcal{L})$ if $\exists\ \theta\in(0,1),\ M>0,\ n\in\mathbb{N}$ such that

$$\|P^n f\|_{\mathcal{L}}\le\theta\|f\|_{\mathcal{L}}+M\|f\|_{\mathcal{C}}\ \ \forall\ f\in\mathcal{L}.$$

We'll call this latter inequality a *D-F inequality.*

It follows from the Arzela-Ascoli theorem that if X is a compact metric space, then $(C(X),L(X))$ is an adapted pair where $C(X)$ and $L(X)$ are the continuous-, and Lipschitz-continuous real valued functions on X (respectively). The terminologies "D-F inequality" and "D-F operator" are in honour of W. Doeblin and R. Fortet who first considered such operators (in [14]).

Recall that a linear operator A on a Banach space $\mathcal{L}$ is *quasi compact* (on $\mathcal{L}$) if $\exists\ N\ge 1,\ \theta\in(0,1)$, projections $E_1,\dots,E_N\subset\mathcal{L}$ onto finite dimensional subspaces and $\lambda_1,\dots,\lambda_N\in S^1:=\{z\in\mathbb{C}:\ |z|=1\}$ such that

$$\|Af-\sum_{k=1}^{N}\lambda_k E_k f\|_{\mathcal{L}}\le M\theta^n\|f\|_{\mathcal{L}}\ \forall\ f\in\mathcal{L}.$$

It was established in [14] that a D-F operator on $(C(X),L(X))$ is quasi compact on $L(X)$ and this was generalised in [24] to show that a D-F operator on an adapted pair $(\mathcal{C},\mathcal{L})$ is quasi compact on $\mathcal{L}$. The proof of this uses *inter alia* that if A is a D-F operator on $(\mathcal{C},\mathcal{L})$, and $\mathfrak{r}(A)$ denotes the spectral radius of $A:\mathcal{L}\to\mathcal{L}$, then $\mathfrak{r}(A)\le 1$ with equality iff $\exists\ f\in\mathcal{L}$ and $\lambda\in S^1$ satisfying $Af=\lambda f$. It is also shown in [14] and [24] that if $f\in\mathcal{C}$ and $\lambda\in S^1$ satisfy $Af=\lambda f$, then $f\in\mathcal{L}$.

Definitions

3) A function $f:X\to\mathbb{R}^d$ is *Lipschitz continuous on* $A\subset X$ if

$$D_A f:=\sup_{x,y\in A}\frac{|f(x)-f(y)|}{d(x,y)}<\infty,$$

and *Lipschitz continuous at* $x \in X$ if it is Lipschitz continuous on some neighbourhood of x.

Given a partition ρ of X into open-closed sets and $1 \le q \le \infty$, we consider $\mathrm{Lip}_{q,\rho} \subset L^q(m)$, consisting of functions $X \to \mathbb{C}$ which are Lipschitz continuous on each $a \in \rho$. The norm on $\mathrm{Lip}_{q,\rho}$ is defined by

$$\|f\|_{\mathrm{Lip}_{q,\rho}} := \|f\|_q + D_\rho f$$

where

$$D_\rho f := \sup_{a \in \rho} D_a f.$$

If ρ is infinite and $q < \infty$, then not all functions in $\mathrm{Lip}_{q,\rho}$ are bounded, but if $f \in \mathrm{Lip}_{q,\rho}$ and $a \in \rho$, then

$$\sup_{x \in a} |f(x)| \le \frac{1}{m(a)} \int_a |f| dm + r D_a f.$$

This is because $|f(x) - f(y)| \le r D_a f \ \forall \ x, y \in a$.

We show that for $1 \le q' < q$, $(L^{q'}(m), \mathrm{Lip}_{q,\rho})$ is an adapted pair on which, for large enough $n \ge 1$, P_{T^n} is a D-F operator.

The following (standard) version of the classical Arzela-Ascoli theorem implies that for $1 \le q' < q$, $(L^{q'}(m), \mathrm{Lip}_{q,\rho})$ is indeed an adapted pair.

Theorem *If* $f_n \in \mathrm{Lip}_{q,\rho}$, *and* $\sup_{n \ge 1} \|f_n\|_{\mathrm{Lip}_{q,\rho}} < \infty$, *then* $\exists \ n_k \to \infty$ *and* $g \in \mathrm{Lip}_{q,\rho}$ *such that*

$$f_{n_k}(x) \to g(x) \quad \textit{as } k \to \infty \ \forall \ x \in X,$$

$$\|g\|_{Lip_{q,\rho}} \le \liminf_{n \to \infty} \|f_n\|_{Lip_{q,\rho}},$$

and

$$\|f_{n_k} - g\|_{q'} \to 0 \quad \textit{as } k \to \infty \ \forall \ 1 \le q' < q.$$

A Markov map $(X, \mathcal{B}, m, T, \alpha)$ is *Gibbs* (Gibbs-Markov) if the two additional assumptions are satisfied:

$$\inf_{a \in \alpha} m(Ta) > 0$$

(we call this the *big image* property), and

($\mathfrak{G}$)

$$\exists \ M > 0 \ \ni \ \left|\frac{v_a'(x)}{v_a'(y)} - 1\right| \le M d(x, y) \ \forall \ n \ge 1, \ a \in \alpha_0^{n-1}, \ x, y \in T^n a.$$

Example 1. Markov chains

Let S be a countable set, $P : S \times S \to [0,1]$ be an aperiodic, irrreducible stochastic matrix and $\pi \in \mathcal{P}(S),\ \pi_s > 0\ \forall\ s \in S$. Let $T : S^{\mathbb{N}} \to S^{\mathbb{N}}$ be the shift and define a Markovian probability $m \in \mathcal{P}(S^{\mathbb{N}})$ by

$$m([s_1, s_2, \dots, s_n]) := \pi_{s_1} p_{s_1,s_2} \dots p_{s_{n-1},s_n}.$$

Let $X = \{x \in S^{\mathbb{N}} :\ m([x_1, \dots, x_n]) > 0\ \forall\ n \geq 1\}$ and let $\alpha = \{[s] : s \in S\}$. Evidently $(X, \mathcal{B}, m, T, \alpha)$ is a Markov map.

It can be shown that $(X, \mathcal{B}, m, T, \alpha)$ is Gibbs-Markov iff $\exists\ M > 1$ such that

$$\frac{1}{M} \leq \frac{p_{s,t}}{\pi_t} \leq M \text{ whenever } s, t \in S \text{ and } p_{s,t} > 0.$$

Example 2. Markov interval maps

Let $I := [0,1]$ and let m be Lebesgue measure on I. Let α be a partition of I mod m into open intervals and let $T : I \to I$ a non-singular transformation so that $T : a \to Ta$ is a homeomorphism $\forall\ a \in \alpha$.

Clearly $(I, \mathcal{B}(I), m, T, \alpha)$ is a Markov map iff $Ta \in \sigma(\alpha)\ \forall\ a \in \alpha$.

Now suppose that $\forall\ a \in \alpha$, there is a C^2-diffeomorphism $v_a : \overline{Ta} \to \overline{a}$ such that $T|_a = v_a^{-1}$. Assume in addition that T is uniformly expanding in the sense that $\exists\ c > 1$ such that $|T'| \geq c$, and that T has the so-called Adler property (see [7])

$$\sup_{x \in a \in \alpha} \frac{|T''(x)|}{T'(x)^2} := M < \infty.$$

A calculation (see R. Adler's afterword in [10]) shows that

$$\frac{|T^{n\prime\prime}(x)|}{T^{n\prime}(x)^2} \leq M_1 := \frac{Mc}{c-1}\ \forall\ n \geq 1,\ x \in a \in \alpha_0^{n-1}$$

whence

$$\left|\frac{v_a'(x)}{v_a'(y)} - 1\right| \leq M_1|x-y|\ \ \forall\ n \geq 1,\ a \in \alpha_0^{n-1},\ x, y \in T^n a.$$

We claim that $(\mathfrak{G})$ holds with $d = d_r,\ r = \frac{1}{c}$. This is because if $x, y \in I$ and $r^{n+1} < d(x,y) \leq r^n$, then $\exists\ b \in \alpha_0^{n-1}$ such that $x, y \in b$, whence

$$|x - y| \leq \text{diam.}\, b = \text{diam.}\, v_b(T^n b) \leq \|v_b'\|_\infty \leq r^n \leq r^{-1} d(x,y),$$

and

$$\left|\frac{v_a'(x)}{v_a'(y)} - 1\right| \leq r^{-1} M_1 d(x,y).$$

Thus if T is a uniformly expanding piecewise C^2 Markov map having the big image property and the Adler property, then $(I, \mathcal{B}(I), m, T, \alpha)$ is a Gibbs-Markov map.

The following shows when the Gibbs-Markov property is preserved under passage to an equivalent measure.

Proposition 1.1 *Suppose that $(X, \mathcal{B}, m, T, \alpha)$ is a Gibbs-Markov map, and that $\mu \sim m$, $\log \frac{d\mu}{dm} \in \mathrm{Lip}_{\infty,\alpha}$;*
then, $(X, \mathcal{B}, \mu, T, \alpha)$ is a Gibbs-Markov map.

Henceforth, unless stated otherwise, $(X, \mathcal{B}, m, T, \alpha)$ will denote a mixing Gibbs-Markov map.

Proposition 1.2 (Renyi's property)

$$\forall\ n \geq 1,\ a \in \alpha_0^{n-1},\ v_a' = M''^{\pm 1} m(a)\ a.e.\ on\ T^n a.$$

Here, and throughout $a = c^{\pm 1} b$ (where $a, b, c > 0$) means $c^{-1} b \leq a \leq cb$.

Proofs of proposition 1.2 and the next corollary (a descendant of Renyi's theorem [37]) can be found in §2 and §3 of [6] (see also [3] chapter 4).

Corollary
1) T is exact, and $\exists\ h \in L^\infty(m)$ such that $h > 0$ a.e., and $P_T h = h$.
2) If $|T\alpha| < \infty$, then $\log h \in L^\infty(m)$.

We now consider P_T acting on the space $L := \mathrm{Lip}_{\infty,\beta}$.

Lemma 1.3 *Suppose that $g \in Lip_{1,\beta}$ and $a \in \alpha_0^{n-1}$, then*

$$|v_a'(x) g(v_a(x)) - v_a'(y) g(v_a(y))| \leq M'' d(x,y) \left(M \int_a |g| dm + (M+1) m(a) r^n D_a g \right).$$

Proof

$$\begin{aligned} &|v_a'(x) g(v_a(x)) - v_a'(y) g(v_a(y))| \\ &\leq v_a'(x) |g(v_a(x))| \left| \frac{v_a'(y)}{v_a'(x)} - 1 \right| + v_a'(y) |g(v_a(x)) - g(v_a(y))| \\ &= I\ +\ II. \end{aligned}$$

We have that

$$|g(v_a(x))| \leq \frac{1}{m(a)} \int_a |g| dm + r^n D_a g \ \ \forall\ x \in X.$$

Hence, by Renyi's property, and $(\mathfrak{G})$,

$$\begin{aligned}I &\le MM''d(x,y)m(a)|g(v_a(x))| \\ &\le MM''d(x,y)\Big(\int_a |g|dm + m(a)r^n D_a g\Big).\end{aligned}$$

Also by Renyi's property,

$$II \le M''m(a)d(v_a(x),v_a(y))D_a g = M''d(x,y)m(a)r^n D_a g.$$

The result is that

$$I + II \le M''d(x,y)\Big(M\int_a |g|dm + (M+1)m(a)r^n D_a g\Big).$$

$\square$

Proposition 1.4 (D-F Inequality) *For $f \in \mathrm{Lip}_{1,\beta}$,*

$$\|P_{T^n} f\|_L \le M''\left((M+2)r^n D_\beta f + (M+1)\|f\|_1\right).$$

In particular, $P_T : \mathrm{Lip}_{1,\beta} \to L$.

Proof Let $g \in \mathrm{Lip}_{1,\beta}$, then

$$P_{T^n} g = \sum_{a\in\alpha_0^{n-1}} 1_{T^n a} v_a' \cdot g \circ v_a.$$

For each $n \ge 1$, and $a \in \alpha_0^{n-1}$ we have

$$|g(v_a(x)) - \frac{1}{m(a)}\int_a g dm| \le D_a g\ r^n\ \forall\ x \in T^n a$$

whence, using Renyi's property and $D_a g \le D_\beta g$,

$$\begin{aligned}|P_{T^n} g(x)| &\le \sum_{a\in\alpha_0^{n-1}} 1_{T^n a}(x) v_a'(x)\Big(\frac{1}{m(a)}\int_a |g|dm + D_\beta g\ r^n\Big) \\ &\le M'' \sum_{a\in\alpha_0^{n-1}} m(a)\Big(\frac{1}{m(a)}\int_a |g|dm + D_\beta g r^n\Big) \\ &= M''D_\beta g\ r^n + M''\|g\|_1.\end{aligned}$$

It follows that

$$\|P_{T^n} g\|_\infty \le M''D_\beta g\ r^n + M''\|g\|_1.$$

For $g \in L$, $x, y \in b \in \beta$,

$$\begin{aligned}
&|P_{T^n}g(x) - P_{T^n}g(y)| \\
&\le \sum_{a\in\alpha_0^{n-1},\ T^n a\supset b} |v_a'(x)g(v_a(x)) - v_a'(y)g(v_a(y))| \\
&\le M''d(x,y) \sum_{a\in\alpha_0^{n-1},\ T^n a\supset b} \left(M \int_a |g|dm + (M+1)m(a)r^n D_\beta g \right) \text{ by lemma 1.3} \\
&\le M''d(x,y)(M\|g\|_1 + r^n(M+1)D_\beta g).
\end{aligned}$$

□

Corollary 1.5 ([21]) *Let $h \in L^\infty(m)$ satisfy $P_T h = h$, then $h \in L$.*

This follows from [14] and [24] as remarked above.

Theorem 1.6

$$P_T = \mu + Q$$

in Hom (L, L) *where* $\mu f = \int_X f dm \cdot h,\ Q\mu = \mu Q = 0,$ *and* $\mathfrak{r}(Q) < 1.$

Proof By the theorem of [24],

$$P_T = \mu + Q$$

where $Q\mu = \mu Q = 0$, $\mathfrak{r}(Q) < 1$ and $\exists\ N \in \mathbb{N},\ \lambda_1, \dots, \lambda_N \in \mathbb{C},\ |\lambda_k| = 1$ and finite dimensional subspaces $E_1, \dots, E_N \subset \mathcal{L}$ such that $\mu = \sum_{k=1}^N \lambda_k P_{E_k}$ where P_{E_k} is a projection onto E_k $(1 \le k \le N)$.

By the corollary, T is exact and so $E_k = \{0\}$ whenever $\lambda_k \ne 1$. It follows that $\mu f = \int_X f dm \cdot h$. □

Henceforth, unless stated otherwise, we shall assume that the mixing Gibbs-Markov map $(X, \mathcal{B}, m, T, \alpha)$ is probability preserving, in particular $h = 1$.

It is now possible to obtain a (well-known) strengthening of the exactness part of Renyi's theorem known as "exponential decay of correlations" (see [39]):

By theorem 1.6, $\exists\ \theta \in\ (0,1)$ and $K > 1$ such that

$$\|P_{T^n} f - \int_X f dm\|_L \le K\theta^n \|f\|_L \ \ \forall\ n \ge 1,\ \ f \in \text{Lip}_{1,\beta}.$$

Since $\forall\ n \ge 1,\ a \in \alpha_0^{n-1}$ we have $P_{T^n} 1_a = v_a'$ and $\|v_a'\|_L \le (M + 1)M''m(a)$, it follows that (T, α) is *continued fraction mixing* (see [2], [3] and [6]):

for some $K' > 0$,

$$|m(a\cap T^{-(n+k)}B)-m(a)m(B)| \le K'\theta^n m(a)m(B) \ \forall\, n,k \ge 1,\ a \in \alpha_0^{k-1},\ B \in \mathcal{B}.$$

§2 CHARACTERISTIC FUNCTION OPERATORS

Let $(X, \mathcal{B}, m, T, \alpha)$ be a mixing, probability preserving Gibbs-Markov map.

For $\omega : X \to S^1$ measurable, define

$$P_\omega f := P(\omega f) \quad (f \in L^1(m))$$

where $P = P_T$, and for $\phi : X \to \mathbb{R}^d$ $(\phi = (\phi^{(1)}, \dots, \phi^{(d)}))$ measurable, $t \in \mathbb{R}^d$ set $P_t := P_{\chi_t(\phi)}$ where $\chi_t(y) := e^{i\langle t,y\rangle}$.

In *the independent case* where ϕ is α-measurable and $\alpha, T^{-1}\alpha, \dots$ are independent,

$$P_t 1 = E(e^{i\langle t,\phi\rangle})$$

which is why the P_t are sometimes called *characteristic function operators.*

We'll use these operators in §5 to obtain local (and distributional) limits. The relevant spectral theory is developed in §4. In this section, we establish the necessary basic properties of these operators as perturbations of P.

Proposition 2.1 (D-F inequality) *Suppose that $\omega : X \to S^1$ is Lipschitz continuous on each $a \in \alpha$, and that $D_\alpha\omega < \infty$, then for $f \in L$,*

$$\|P_\omega^n f\|_L \le M''\Bigg(\bigg(M + \frac{rD_\alpha\omega}{1-r} + 1\bigg)\|f\|_1 + \bigg(M + \frac{rD_\alpha\omega}{1-r} + 2\bigg) r^n D_\beta f\Bigg).$$

Proof Note that

$$P_\omega^n(f) = P_{T^n}(\omega_n f) \text{ where } \omega_n(x) := \prod_{k=0}^{n-1}\omega(T^k x).$$

It follows that

$$P_\omega^n(f)(x) = \sum_{b\in\beta} 1_b(x) \sum_{a\in\alpha_0^{n-1},\ T^n a\supset b} v_a'(x)\omega_n(v_a(x))f(v_a(x)).$$

We have that $|P_\omega^n(f)(x)| \le P_{T^n}(|f|)(x) \le M''\left(\|f\|_1 + r^n D_\beta f\right)$ as in the proof of proposition 1.4.

For $x, y \in b \in \beta$,

$$
\begin{aligned}
&|P_\omega^n(f)(x) - P_\omega^n(f)(y)| \\
&\le \sum_{a \in \alpha_0^{n-1},\ T^n a \supset b} |v_a'(x)\omega_n(v_a(x))f(v_a(x)) - v_a'(y)\omega_n(v_a(y))f(v_a(y))| \\
&\le \sum_{a \in \alpha_0^{n-1},\ T^n a \supset b} \Big(I_a + II_a\Big)
\end{aligned}
$$

where

$$
\begin{aligned}
I_a &= |\omega_n(v_a(x))\big(f(v_a(x))v_a'(x) - f(v_a(y))v_a'(y)\big)| \\
&\le M'' d(x,y)(M \int_a |f| dm + m(a)(M+1)r^n D_\beta f)
\end{aligned}
$$

by lemma 1.3, and

$$
\begin{aligned}
II_a &= v_a'(y)|\omega_n(v_a(x))f(v_a(y)) - \omega_n(v_a(y))f(v_a(y))| \\
&= v_a'(y)|f(v_a(y))||\omega_n(v_a(x)) - \omega_n(v_a(y))| \\
&\le M''(\int_a |f| dm + r^n m(a) D_\beta f)|\omega_n(v_a(x)) - \omega_n(v_a(y))|
\end{aligned}
$$

by the Renyi property, and Lipschitz continuity.

Now

$$
\begin{aligned}
|\omega_n(v_a(x)) - \omega_n(v_a(y))| &= \sum_{k=0}^{n-1} |\omega(T^k v_a(x)) - \omega(T^k v_a(y))| \\
&\le \sum_{k=0}^{n-1} r^{n-k} D_\alpha \omega \, d(x,y) \\
&\le \frac{r D_\alpha \omega}{1-r} d(x,y),
\end{aligned}
$$

so

$$
II_a \le M'' \frac{r D_\alpha \omega}{1-r} d(x,y)(\int_a |f| dm + r^n m(a) D_\beta f);
$$

and the conclusion is

$$\begin{aligned}
&D_\beta(P_\omega^n(f)) \\
&\qquad \le \sum_{a\in\alpha_0^{n-1}} M''\Bigg(\Bigg(M+\frac{rD_\alpha\omega}{1-r}\Bigg)\int_a |f|dm + \Bigg(M+\frac{rD_\alpha\omega}{1-r}+1\Bigg)r^n m(a)D_\beta f\Bigg) \\
&\qquad = M''\Bigg(\Bigg(M+\frac{rD_\alpha\omega}{1-r}\Bigg)\|f\|_1 + \Bigg(M+\frac{rD_\alpha\omega}{1-r}+1\Bigg)r^n D_\beta f\Bigg).
\end{aligned}$$

□

Corollary 2.2 *Suppose that $\omega : X \to S^1$ is Lipschitz continuous on each $a \in \alpha$, and that $D_\alpha\omega < \infty$.*

If $g : X \to \mathbb{C}$ is measurable, and $g \circ T = \lambda\omega g$ for some $\lambda \in \mathbb{C}$, then $g \in L$.

Proof Since T is conservative (being probability preserving) and ergodic, we have that $\lambda \in S^1$, and $|g|$ is constant and hence integrable. Also

$$P_\omega g = P(\omega g) = P(\overline{\lambda} g \circ T) = \overline{\lambda} g.$$

It now follows from proposition 2.1 and [14], [24] (as remarked before) that $g \in L$. □

Corollary 2.3 *Suppose that $\phi : X \to \mathbb{R}^d$ is Lipschitz continuous on each $a \in \alpha$, and that $D_\alpha\phi < \infty$.*

If $g : X \to \mathbb{R}^d$ is measurable, and $\phi = g \circ T - g$, then g is Lipschitz continuous on each Ta, $(a \in \alpha)$.

If $|T\alpha| < \infty$, then $g \in L$ (whence also $\phi \in L$).

Proof By corollary 2.2, $\chi_t(g) \in L\ \forall\ t \in \mathbb{R}^d$, and so g is continuous. Thus $\exists\ M > 0$ such that $\forall\ w \in X,\ \exists\ n \ge 1$ such that

$$|g(y) - g(z)| \le Md(y,z)\ \ \forall\ y,z \in [w_1,\dots,w_n].$$

By possibly increasing $M > 0$, we ensure that in addition, $r(D_\alpha\phi + M) \le M$.

Since $g(Tx) = g(x)+\phi(x)$, we have $\forall\ y,z \in [w_2,\dots,w_n] = T[w_1,\dots,w_n]$

$$\begin{aligned}
|g(y) - g(z)| &\le (D_\alpha\phi + M)d(w_1y, w_1z) \\
&= r(D_\alpha\phi + M)d(y,z) \le Md(y,z).
\end{aligned}$$

Doing this n times, we get

$$|g(y) - g(z)| \le Md(y,z)\ \ \forall\ y,z \in T[w_n].$$

□

Theorem 2.4 (Continuity) *Suppose that $\phi : X \to \mathbb{R}^d$ is Lipschitz continuous on each $a \in \alpha$, and that $D_\alpha\phi < \infty$, then*

$$\|P_s - P_t\|_{Hom(L,L)} \leq$$

$$M''\Big((2+2M+|t|D_\alpha\phi)E|1-\chi_{t-s}(\phi)|+(3+2M+|t|D_\alpha\phi)|s-t|D_\alpha\phi\Big).$$

Proof

For $g \in L$ and $t \in \mathbb{R}^d$ we have

$$P_t g = P(e^{i\langle t,\phi\rangle}g) = \sum_{a\in\alpha} \chi_t(\phi \circ v_a) 1_{Ta} v_a' \cdot g \circ v_a,$$

whence

$$(P_t - P_s)g = \sum_{a\in\alpha} 1_{Ta}\chi_t(\phi \circ v_a)(1 - \chi_{s-t}(\phi \circ v_a))v_a' \cdot g \circ v_a.$$

We'll use that for $x, y \in a \in \alpha$,

$$|\chi_t(\phi(x)) - \chi_t(\phi(y))| \leq |t|D_a\phi d(x,y)$$

whence

$$|\chi_t(\phi(v_a(x))) - \frac{1}{m(a)}\int_a \chi_t(\phi)dm| \leq |t|D_a\phi \quad \forall\ x \in Ta,$$

and

$$|1 - \chi_t(\phi \circ v_a)| \leq \frac{1}{m(a)}\int_a |1 - \chi_t(\phi)|dm + |t|D_a\phi \text{ on } Ta. \qquad (1)$$

Also, using (1), $\forall\ x, y \in a \in \alpha$,

(2)

$$\begin{aligned}
&|(\chi_t(\phi(x)) - \chi_s(\phi(x))) - (\chi_t(\phi(y)) - \chi_s(\phi(y)))| \\
&\leq |\chi_{s-t}(\phi(y)) - \chi_{s-t}(\phi(x))| + |1 - \chi_{s-t}(\phi(y))||\chi_t(\phi(x)) - \chi_t(\phi(y))| \\
&\leq d(x,y)\Big(|s-t|D_a\phi(1+|t|D_a\phi) + |t|D_a\phi\frac{1}{m(a)}\int_a |1-\chi_{s-t}(\phi)|dm\Big).
\end{aligned}$$

In particular, we have that, $\forall\ x,y \in Ta,\ a \in \alpha$,

$$\begin{aligned}
&|(P_s - P_t)g(x)| \\
&\quad \le \sum_{a\in\alpha} 1_{Ta}(x)|1-\chi_{s-t}(\phi(v_a(x)))|v_a'(x)|g(x)| \\
&\quad \le M'' \sum_{a\in\alpha} 1_{Ta}(x)\bigg(\int_a |1-\chi_{s-t}(\phi)|dm + m(a)|s-t|D_a\phi\bigg)\|g\|_\infty \text{ by (1)} \\
&\quad \le M''\bigg(E(|1-\chi_{s-t}(\phi)|) + |s-t|D_a\phi\bigg)\|g\|_\infty.
\end{aligned}$$

For $x, y \in b \in \beta$,

$$\begin{aligned}
&|(P_s - P_t)g(x) - (P_s - P_t)g(y)| = \\
&\bigg|\sum_{\substack{a\in\alpha\\ Ta\supset b}} (\chi_t(\phi(v_a(x))) - \chi_s(\phi(v_a(x))))v_a'(x)g(v_a(x)) - \\
&\qquad\qquad (\chi_t(\phi(v_a(y))) - \chi_s(\phi(v_a(y))))v_a'(y)g(v_a(y))\bigg| \le \\
&\sum_{\substack{a\in\alpha\\ Ta\supset b}} \bigg(v_a'(x)|g(v_a(x))||(\chi_t(\phi(v_a(x))) - \chi_s(\phi(v_a(x)))) - (\chi_t(\phi(v_a(y))) - \chi_s(\phi(v_a(y))))| \\
&+ |1-\chi_{s-t}(\phi(v_a(y)))||v_a'(x)g(v_a(x)) - v_a'(y)g(v_a(y))|\bigg) \\
&= \sum_{a\in\alpha,\ Ta\supset b} (I_a + II_a).
\end{aligned}$$

By lemma 1.3 and (1)

$$\begin{aligned}
&II_a = |v_a'(x)g(v_a(x)) - v_a'(y)g(v_a(y))||1-\chi_{s-t}(\phi(v_a(x)))| \\
&\quad \le M''d(x,y)((M+1)rD_\beta g + M\|g\|_\infty)\bigg(\int_a |1-\chi_{s-t}(\phi)|dm + m(a)|s-t|D_\alpha\phi\bigg)
\end{aligned}$$

whence

$$\sum_{a\in\alpha,\ Ta\supset b} II_a \le M''d(x,y)((M+1)rD_\beta g + M\|g\|_\infty)\bigg(E|1-\chi_{s-t}(\phi)| + |s-t|D_\alpha\phi\bigg).$$

Using Renyi's property and (2),

$$\begin{aligned}
&I_a = v_a'(y)|g(v_a(y))||(\chi_t(\phi(v_a(x))) - \chi_s(\phi(v_a(x)))) - (\chi_t(\phi(v_a(y))) - \chi_s(\phi(v_a(y))))| \\
&\le M''\|g\|_\infty d(x,y)\bigg(m(a)|s-t|D_\alpha\phi(1+|t|D_\alpha\phi) + |t|D_\alpha\phi\int_a |1-\chi_{s-t}(\phi)|dm\bigg)
\end{aligned}$$

and

$$\sum_{a\in\alpha,\ Ta\supset b} I_a \le M''\|g\|_\infty d(x,y)\Bigg(|s-t|D_\alpha\phi(1+|t|D_\alpha\phi)+|t|D_\alpha\phi E(|1-\chi_{s-t}(\phi)|)\Bigg).$$

The conclusion is that

$$\begin{aligned}
&\|(P_s-P_t)g\|_L\\
&\le M''\Bigg(E(|1-\chi_{s-t}(\phi)|)+|s-t|D_\alpha\phi\Bigg)\|g\|_\infty\\
&+M''((M+1)rD_\beta g+M\|g\|_\infty)\Bigg(E|1-\chi_{s-t}(\phi)|+|s-t|D_\alpha\phi\Bigg)\\
&+M''\|g\|_\infty\Bigg(|s-t|D_\alpha\phi(1+|t|D_\alpha\phi)+|t|D_\alpha\phi E(|1-\chi_{s-t}(\phi)|)\Bigg)\\
&\le M''\|g\|_L\Bigg((2+2M+|t|D_\alpha\phi)E|1-\chi_{s-t}(\phi)|+(3+2M+|t|D_\alpha\phi)|s-t|D_\alpha\phi\Bigg).
\end{aligned}$$

$\square$

Remark

In case $\phi\in \mathrm{Lip}_{2,\alpha}$ then (as in [38]), $t\mapsto P_t$ is C^2 $(\mathbb{T}\to \mathrm{Hom}(L,L))$ with $\frac{\partial P_t}{\partial t_j}f=P(i\phi^{(j)}e^{i\langle\phi,t\rangle}f)$ and $\frac{\partial^2 P_t}{\partial t_j\partial t_k}=-P(\phi^{(j)}\phi^{(k)}e^{i\langle\phi,t\rangle}f)$.

§3 Periodic and aperiodic cocycles

Suppose that $(X,\mathcal{B},m,T)$ is an ergodic probability preserving transformation, G is a locally compact, Abelian, second countable topological group, and $\phi:X\to G$ is measurable.

Define the *skew product* transformation $T_\phi: X\times G\to X\times G$ by $T_\phi(x,y)=(Tx,y+\phi(x))$. Evidently T_ϕ preserves the measure $m\times m_G$ on $X\times G$ where m_G is Haar measure on G.

We say that ϕ is *aperiodic* if there is no character $\gamma\in\widehat{G}$ so that $\gamma\circ\phi$ is T-cohomologous to a constant, i.e. the equation

$$\gamma\circ\phi=\frac{\lambda g(x)}{g(Tx)}$$

has no solution $\lambda\in S^1:=\{z\in\mathbb{C}:\ |z|=1\}$ (the multiplicative unit circle), $g:X\to S^1$ measurable other than $\lambda=1,\ g\equiv 1$. We say that ϕ is *periodic* if it is not aperiodic.

In case T_ϕ is ergodic, aperiodicity is the same as demanding that any eigenfunction for T_ϕ is the lifting of an eigenfunction for T (defined on X).

In this section, we consider a probability preserving, mixing Gibbs-Markov maps $(X, \mathcal{B}, m, T, \alpha)$ and we discuss the aperiodicity of Lipschitz continuous functions $\phi : X \to G$. We begin with sufficient conditions for aperiodicity. The results are analogous to those of [27] and [28].

Theorem 3.1
Let $(X, \mathcal{B}, m, T, \alpha)$ be a probability preserving, mixing Gibbs-Markov map, and suppose that $\varphi : X \to S^1$ is α-measurable.

If $g : X \to S^1$ is measurable, $\lambda \in S^1$ and $\varphi = \lambda \cdot g \cdot \overline{g} \circ T$, then g is α_*-measurable where α_* is the finest partition with the property that each Ta is contained in an atom of α_*.

We'll say that a Gibbs-Markov map $(X, \mathcal{B}, m, T, \alpha)$ is *almost onto* if $\alpha_* = \{X\} \bmod m$; equivalently $\forall\, b, c \in \alpha,\ \exists\, n \geq 1,\ b = a_0, a_1, \ldots, a_n = c \in \alpha$ such that $Ta_k \cap Ta_{k+1} \neq \emptyset \ \ (0 \leq k \leq n-1)$.

Corollary 3.2 *Let $T : X \to X$ be a probability preserving, almost onto Gibbs-Markov map with respect to the partition α.*

1) Suppose that $\varphi : X \to S^1$ is α-measurable and $\lambda \in S^1$. If $g : X \to S^1$ is measurable, and $\varphi = \lambda \overline{g} \cdot g \circ T$, then g is constant.

2) Suppose that $\phi : X \to G$ is α-measurable, then either ϕ is aperiodic, or $\exists\, \gamma \in \widehat{G},\ \lambda \in S^1$ such that

$$\gamma \circ \phi \equiv \lambda.$$

To prove this theorem, we consider skew products over $(X, \mathcal{B}, m, T, \alpha)$ as in [27] and [28].

Let $(Y, \mathcal{C}, \mu)$ be a standard σ-finite measure space and suppose that $\{T_a :\ a \in \alpha\}$ are non-singular transformations of Y (i.e. $\mu \circ T_a^{-1} \sim \mu$).

Define the *Y-skew product over α* $\tau : X \times Y \to X \times Y$ by

$$\tau(x, y) = (T(x), T_{a(x)} y)$$

where $x \mapsto a(x)$ is the so-called α-name of x (i.e. $x \in a(x) \in \alpha$).

We have that

$$\tau^n(x, y) = (T^n x, T_{a_n(x)} y)$$

where $x \mapsto a_n(x) \in \alpha_0^{n-1}$ is the α_0^{n-1}-name of x, and $T_{(a_0, \ldots, a_{n-1})} := T_{a_{n-1}} \circ \cdots \circ T_{a_0}$.

The Frobenius-Perron operators $P_{\tau^n} : L^1(m \times \mu) \to L^1(m \times \mu)$ satisfy

$$P_{\tau^n} f(x, y) = \sum_{b \in \beta} 1_b(x) \sum_{a \in \alpha_0^{n-1},\ \tau^n a \supset b} v_a'(x) P_{T_a}(f(v_a(x), \cdot))(y)$$

where $P_{T_a} : L^1(\mu) \to L^1(\mu)$ is the Frobenius-Perron operator corresponding to T_a.

Definition ([28]) A Markov map $(X, \mathcal{B}, m, T, \alpha)$ is called *quasi-Markov* if whenever τ is a Y-skew product over α, $g \in L^1(m \times \mu)_+$ satisfies $P_\tau g = g$, then $[g > 0]$ is $\alpha \times \mathcal{C}$-measurable.

We prove

Theorem 3.3 *Any probability preserving, mixing Gibbs-Markov map is quasi-Markov*

and deduce theorem 3.1 from it.

For $f : X \times Y \to \mathbb{R}$ measurable and $y \in Y$ let $f_y : X \to \mathbb{R}$ be defined by $f_y(x) := f(x, y)$. Set

$$|||f|||_{\mathcal{L}} := \int_Y \|f_\cdot\|_L d\mu = \int_Y \|f_\cdot\|_\infty d\mu + \int_Y \left(\sup_{x,x' \in b \in \beta} \frac{|f(x,\cdot) - f(x',\cdot)|}{d(x,x')} \right) d\mu,$$

$$\|f\|_{\mathcal{L}} := \inf \{|||g|||_{\mathcal{L}} : \ g = f \ m \times \mu - \text{ a.e.}\},$$

and let

$$\mathcal{L} = \{f : X \times Y \to \mathbb{R} : \ \|f\|_{\mathcal{L}} < \infty\}.$$

Clearly $f \in \mathcal{L}$ entails the existence of $g = f$ a.e. such that $g_y \in L \ \forall \ y \in Y$.

Proposition 3.4 *If $f_n \in \mathcal{L}, \ f \in L^1(m \times \mu)$, and $f_n \xrightarrow{m\times\mu} f$, then*

$$\liminf_{n\to\infty} \|f_n\|_{\mathcal{L}} \geq \|f\|_{\mathcal{L}}.$$

Proof Suppose without loss of generality that

$$\|f_n\|_{\mathcal{L}} = \int_Y \|(f_n)_\cdot\|_L d\mu \leq M \ \forall \ n \geq 1.$$

It follows that for $\liminf_{n\to\infty} \|(f_n)_\cdot\|_L =: M(\cdot) < \infty$ and that for μ-a.e. $y \in Y$, $\exists$ a subsequence $n_k(y) \to \infty$ so that $\|(f_{n_k(y)})_y\|_L \to M(y)$ as $k \to \infty$.

By the version of the Arzela-Ascoli theorem in §1, there are further subsequences $n_k'(y) \to \infty$ (of $\{n_k(y)\}_{k\geq 1}$), and functions $g_y \in L$ so that

$$f_{n_k'(y)}(x, y) = (f_{n_k'(y)})_y(x) \to g_y(x) \ \forall \ x \in X, \ \& \ \|g_y\|_L \leq M(y).$$

Since $f(x,y) = g_y(x)$ a.e., we have that

$$\begin{aligned}\|f\|_{\mathcal{L}} &\le \int_Y \|g.\|_L d\mu \\ &\le \int_Y \liminf_{n\to\infty} \|(f_n).\|_L d\mu \\ &\le \liminf_{n\to\infty} \int_Y \|(f_n).\|_L d\mu \text{ by Fatou's lemma} \\ &= \liminf_{n\to\infty} \|f_n\|_{\mathcal{L}}.\end{aligned}$$

□

Now let $T : X \to X$ be a Gibbs-Markov map with respect to the partition α, and let τ be a skew product over (T,α).

Lemma 3.5

$$\|P_{\tau^n} f\|_{\mathcal{L}} \le M''((M+1)\|f\|_1 + (M+2)r^n\|f\|_{\mathcal{L}}) \ \ \forall\ f \in \mathcal{L},\ n \ge 1$$

where M and M'' are as in §1.

Proof

$$\begin{aligned}|P_{\tau^n} f(x,y)| &\le \sum_{b\in\beta} 1_b(x) \sum_{a\in\alpha_0^{n-1},\ \tau^n a\supset b} v_a'(x) P_{T_a}(|f(v_a(x),\cdot)|)(y) \\ &\le \sum_{b\in\beta} 1_b(x) \sum_{a\in\alpha_0^{n-1},\ \tau^n a\supset b} v_a'(x) P_{T_a}(\frac{1}{m(a)}\int_a |f.|dm + r^n\|f.\|_L)(y) \\ &\le M'' \sum_{a\in\alpha_0^{n-1}} P_{T_a}(\int_a |f.|dm + r^n m(a)\|f.\|_L)(y).\end{aligned}$$

Hence, using $\int_Y P_{T_a} f d\mu = \int_Y f d\mu$,

$$\int_Y \|(P_{T^n} f).\|_\infty d\mu \le M''(\|f\|_1 + r^n\|f\|_{\mathcal{L}}).$$

Next, we have for $x, x' \in b \in \beta$,

$$
\begin{aligned}
&|P_{\tau^n} f(x,\cdot) - P_{\tau^n} f(x',\cdot)| \\
&= \Big| \sum_{a\in\alpha_0^{n-1},\ T^n a\supset b} P_{T_a}\Big(v_a'(x) f(v_a(x),\cdot) - v_a'(x') f(v_a(x'),\cdot)\Big)\Big| \\
&\le \sum_{a\in\alpha_0^{n-1},\ T^n a\supset b} P_{T_a}\Big(|v_a'(x) f(v_a(x),\cdot) - v_a'(x') f(v_a(x'),\cdot)|\Big) \\
&\le \sum_{a\in\alpha_0^{n-1}} P_{T_a}\Big(M'' d(x,x')\big(M \int_a |f_\cdot| dm + (M+1) r^n m(a) \|f_\cdot\|_L\big)\Big) \text{ by lemma 1.3,}
\end{aligned}
$$

and it follows that

$$\int_Y \Big(\sup_{x,x'\in b\in\beta} \frac{|P_{\tau^n} f(x,\cdot) - P_{\tau^n} f(x',\cdot)|}{d(x,x')}\Big) d\mu \le M''\Big(M\|f\|_1 + (M+1) r^n \|f\|_{\mathcal{L}}\Big).$$

$\square$

Proposition 3.6 *If $h \in L^1(m\times\mu)_+$ satisfies $P_\tau h = h$, then $h \in \mathcal{L}$.*

Proof It follows from lemma 3.5 that $\exists\ M''' > 0$ such that

$$\|P_{\tau^n} f\|_{\mathcal{L}} \le M''' \|f\|_{\mathcal{L}}\ \forall\ f \in \mathcal{L}.$$

Suppose that $h \in L^1(m\times\mu)_+$ satisfies $P_\tau h = h$. Given $0 < \epsilon < 1$, choose $f_\epsilon \in \mathcal{L}$ with $\|f_\epsilon - h\|_1 < \epsilon$. By the stochastic ergodic theorem ([29]),

$$\frac{1}{n}\sum_{k=0}^{n-1} P_{\tau^k} f_\epsilon \overset{m\times\mu}{\longrightarrow} h_\epsilon$$

where $P_\tau h_\epsilon = h_\epsilon \in L^1(m\times\mu)$.

It follows from Fatou's lemma that

$$\|h - h_\epsilon\|_1 \le \liminf_{n\to\infty} \|h - \frac{1}{n}\sum_{k=0}^{n-1} P_{\tau^k} f_\epsilon\|_1 \le \|f_\epsilon - h\|_1 < \epsilon.$$

We claim that $h_\epsilon \in \mathcal{L}$. This follows from proposition 3.4, as

$$\|h_\epsilon\|_{\mathcal{L}} \le \liminf_{n\to\infty} \|\frac{1}{n}\sum_{k=0}^{n-1} P_{\tau^k} f_\epsilon\|_{\mathcal{L}} \le M''' \|f_\epsilon\|_{\mathcal{L}}.$$

Again, by lemma 3.5, we have

$$\begin{aligned}\|h_\epsilon\|_{\mathcal{L}} &= \|P_{\tau^n} h_\epsilon\|_{\mathcal{L}} \ \forall\ n \geq 1\\ &\leq M''(r^n\|h_\epsilon\|_{\mathcal{L}} + \|h_\epsilon\|_1)\\ &\longrightarrow M''\|h_\epsilon\|_1 \text{ as } n \to \infty\\ &\leq M''(\|h\|_1 + 1).\end{aligned}$$

To finish, set $H_n := h_{\frac{1}{n}}$, then $H_n \in \mathcal{L}$ and $H_n \xrightarrow{m\times\mu} h$ and $\sup_n \|H_n\|_{\mathcal{L}} < \infty$. By proposition 3.4, $h \in \mathcal{L}$.

□

Proof of theorem 3.3 Let $(X, \mathcal{B}, m, T, \alpha)$ be a probability preserving, mixing Gibbs-Markov map. Suppose that τ is a Y-skew product over α and that $h \in L^1(m \times \mu)_+$ satisfies $P_\tau h = h$.

Let $A := [h > 0]$. Clearly $\tau^{-1}A \supset A \bmod m \times \mu$. We show that A is $\alpha \times \mathcal{C}$-measurable.

To this end, let

$$A_1 := \{(x, y) \in A :\ m([x_1] \setminus A_y) = 0\},$$

then $A_1 = \bigcup_{a\in\alpha} a \times B_a$ where $B_a = \{y \in Y :\ m(a \setminus A_y) = 0\}$. We show that $A_1 = A \mod m \times \mu$.

We first claim that if this is not the case, then $\exists\ N \geq 2,\ s = [s_0, \ldots, s_N] \in \alpha_0^{N-1}$ and $C \in \mathcal{C},\ \mu(C) > 0$ such that $s \times C \subset A \setminus A_1$.

To see this we note that by proposition 3.6, $h \in \mathcal{L}$ and so for μ-a.e. $y \in Y$, $x \mapsto h(x, y)$ is continuous $X \to \mathbb{R}$.

It follows that for $m \times \mu$-a.e. $(x, y) \in A,\ \exists\ n \geq 1\ x \in a \in \alpha_0^{n-1}$ such that $m(a \setminus A_y) = 0$ whence $A = \bigcup_{n\geq 1,\ a\in\alpha_0^{n-1}} a \times C_a$ where $C_a = \{y \in Y :\ m(a \setminus A_y) = 0\}$; and our claim is established.

Let $p \ll m \times \mu$ be that finite (τ-invariant) measure with density $\frac{d\,p}{dm\times\mu} = h$ then $(A, \mathcal{B} \times \mathcal{C}, p, \tau)$ is conservative and so $\exists\ n > N$ such that $p((s \times C) \cap \tau^{-n}(s \times C)) > 0$. It follows that also $p((s \times C) \cap \tau^n(s \times C)) > 0$.

Since $n > N$, $T^n s \in \sigma(\alpha)$ and $(s_0 \times Y) \cap \tau^n(s \times C) = s_0 \times C'$ for some $C' \in \mathcal{C}$. We have that $\mu(C \cap C') > 0$ because $(s \times C) \cap \tau^n(s \times C) = s \times (C \cap C')$.

The conclusion is that $s_0 \times C' \subset \tau^n A \subset A$ whence $s_0 \times C' \subset A_1$; and $p(A_1 \cap (s \times C)) \geq p((s_0 \times C') \cap (s \times C)) > 0$ contradicting $s \times C \subset A \setminus A_1$.

□

Proof of theorem 3.1 This proof is that of lemma 3 in [28]. Let $Y = [0, 1)$ equipped with $\mu =$ Lebesgue measure. Define the transformations $T_a : Y \to Y$ by $T_a y = y + \psi(a) \mod 1$ where $\varphi = \lambda e^{2\pi i\psi}$ and let τ

be the corresponding skew product. The probability $p = m \times \mu$ is τ-invariant. If $h(x,y) = g(x)e^{2\pi iy}$ then

$$h \circ \tau(x,y) = h(Tx, y + \psi(x)) = g(Tx)\overline{\lambda}\varphi(x)e^{2\pi iy} = h(x,y).$$

We claim that h is $\alpha \times \mathcal{C}$-measurable. To see this, note that if $A \in \mathcal{B} \times \mathcal{C}$ is τ-invariant, then $P_\tau 1_A = 1_A$ and by the quasi Markov property, $A \in \alpha \times \mathcal{C}$. It follows that any measurable, τ-invariant function is $\alpha \times \mathcal{C}$-measurable; for example h.

It now follows that g is α-measurable and an easy computation using that $g \circ T$ is also α-measurable shows that g is indeed α_*-measurable. $\square$

To complete this section, we now turn to the periodic case.

Let $(X, \mathcal{B}, m, T, \alpha)$ be a mixing Gibbs-Markov map with respect to the partition α and invariant probability m, and let $\phi : X \to \mathbb{R}^d$ be Lipschitz continuous on each $a \in \alpha$ with $D\phi := \sup_{a\in\alpha} D_a\phi < \infty$.

Recall from §2 that $P_t \in \mathrm{Hom}\,(L, L)$ is a D-F operator, whence $\mathfrak{r}(P_t) \leq 1$.

Proposition 3.7

Let $t \in \mathbb{R}^d$, the following are equivalent:
1) $\mathfrak{r}(P_t) = 1$
2) $\exists\, g : X \to S^1$ Lipschitz continuous and $z \in S^1$ such that $P_t(g) = zg$,
3) $\chi_t(\phi)$ is cohomologous to a constant.

Proof

The equivalence of 1) and 2) are shown in [24]. We show equivalence with 3).

Suppose that $\chi_t(\phi) = z\overline{g}g \circ T$ where $z \in S^1$ and $g : X \to \mathbb{C}$. By ergodicity of T, we may assume that $g : X \to S^1$. It follows that $P_t(g) = P(g\chi_t(\phi)) = P(zg \circ T) = zg$, whence by corollary 2.2 g is Lipschitz continuous. Conversely, suppose that $\exists\, g : X \to S^1$ Lipschitz continuous and $z \in S^1$ such that $P_t(g) = zg$ whence $| < \chi_t \circ \phi g, zg \circ T > | = | < P_t g, zg > | = \|g\|_2$. By the Cauchy-Schwartz inequality, $\chi_t \circ \phi g$ and $g \circ T$ are linearly dependent and $\chi_t(\phi)$ is cohomologous to the constant z. $\square$

Set

$$\mathfrak{Q} := \{t \in \mathbb{R}^d :\ \chi_t(\phi) \text{ is cohomologous to a constant}\}$$

Proposition 3.8 *$\mathfrak{Q}$ is a closed subgroup of $\mathbb{R}^d$ and $\exists\ z \in \widehat{\mathfrak{Q}}$ such that $\chi_t(\phi)$ is cohomologous to, and only to $z(t)$.*

Proof Evidently $\mathfrak{Q}$ is a subgroup of $\mathbb{R}^d$. Also, for each $t \in \mathfrak{Q}$, $\exists\,!\, z(t) \in S^1$ such that $\chi_t(\phi)$ is cohomologous to $z(t)$ (as non-unicity would imply $\exists\, z \in S^1,\ z \neq 1$ such that z is cohomologous to 1, an impossibility due to the weak mixing of T).

To see that $\mathfrak{Q}$ is closed suppose that $t_n \in \mathfrak{Q}$ satisfy $P_{t_n} g_n = z(t_n) g_n$ where $g_n \in L, |g_n| = 1$ a.e.. Suppose also that $t_n \to t \in \mathbb{R}^d$ and $z(t_n) \to \zeta \in S^1$.

By proposition 2.1 $\exists\, M, k \geq 1,\ 0 < \theta < 1$ such that
$\|P_{t_n}^k h\|_L \leq \theta \|h\|_L + M\|h\|_1\ \forall\, n \geq 1,\ h \in L$

and it follows that
$\|g_n\|_L = \|P_{t_n}^k g_n\|_L \leq \theta \|g_n\|_L + M\ \forall\, n \geq 1$ whence $\|g_n\|_L \leq \frac{M}{1-\theta}$.

By the Arzela-Ascoli theorem $\exists\, n_k \to \infty,\ g \in L$ such that $\|g_{n_k} - g\|_1 \to 0$. Clearly $|g| = 1$ a.e., and to see that $t \in \mathfrak{Q}$, we must show that $P_t g = \zeta g$ by proposition 3.7.

By theorem 2.4, $P_{t_n} \overset{\mathrm{Hom}(L,L)}{\longrightarrow} P_t$, whence

$$\begin{aligned}\|P_t g - \zeta g\|_1 &\leq \|P_t g - P_t g_n\|_1 + \|P_t g_n - P_{t_n} g_n\|_L + \|z(t_n) g_n - \zeta g\|_1 \\ &\leq \|g - g_n\|_1 + \frac{M}{1-\theta} \|P_t - P_{t_n}\|_{\mathrm{Hom}(L,L)} + |z(t_n) - \zeta| + \|g - g_n\|_1 \\ &\to 0\end{aligned}$$

as $n \to \infty$. □

In particular, if $\mathfrak{Q} = \mathbb{R}^d$ then $\exists\, c \in \mathbb{R}^d$ such that $e^{it(\phi - c)}$ is a coboundary in $S^1\ \forall\, t \in \mathbb{R}^d$. By [22] and [32], $\phi - c$ is a coboundary in $\mathbb{R}^d$. By corollary 2.3, the transfer function is Lipschitz continuous on each $a \in \alpha$.

More generally,

Proposition 3.9 *If $\mathfrak{Q}$ is not a discrete subgroup of $\mathbb{R}^d$ then $\exists$ a vector subspace $V \subset \mathfrak{Q}$, $v \in V$, and Lipschitz continuous functions $g : X \to V$, $\psi : X \to V^\perp$, Lipschitz continuous on each $a \in \alpha$, such that $\phi = g \circ T - g + v + \psi$ and such that $\mathfrak{Q}(\psi) \cap V^\perp$ is a discrete subgroup of $V^\perp$.*

Proof For each $t \in \mathbb{R}^d$, $\{u \in \mathbb{R} :\ ut \in \mathfrak{Q}\}$ is a closed subgroup of $\mathbb{R}$ and hence either discrete or $= \mathbb{R}$. Set $V := \{t \in \mathbb{R}^d :\ ut \in \mathfrak{Q}\ \forall\, u \in \mathbb{R}\}$. It follows that $V \subset \mathfrak{Q}$ is a vector subspace of $\mathbb{R}^d$ and that $\mathfrak{Q}(\phi) \cap V^\perp$ is a discrete subgroup of $V^\perp$.

Taking projections we obtain functions $\phi_0 : X \to V,\ \psi : X \to V^\perp$ Lipschitz continuous on each $a \in \alpha$ such that $\phi = \phi_0 + \psi$.

We check that $\mathfrak{Q}(\psi) \cap V^\perp = \mathfrak{Q}(\phi) \cap V^\perp$ and hence is a discrete subgroup of $V^\perp$. Also $\mathfrak{Q}(\phi_0) \cap V = \mathfrak{Q}(\phi) \cap V = V$ whence by the above $\exists\ v \in V$ and $g : X \to V$ Lipschitz continuous on each $a \in \alpha$, such that $\phi_0 = g \circ T - g + v$. $\square$

§4 A Nagaev-type Spectral theorem

Let $(X, \mathcal{B}, m, T, \alpha)$ be a mixing Gibbs-Markov map with respect to the partition α and invariant probability m, and $\phi : X \to \mathbb{R}^d$ be Lipschitz continuous on each $a \in \alpha$, and $D_\alpha \phi := \sup_{a\in\alpha} D_a\phi < \infty$.

Theorem 4.1

1) There are constants $\epsilon > 0,\ K > 0$ and $\theta \in (0,1)$; and functions $\lambda : B(0,\epsilon) \to B_{\mathbb{C}}(0,1),\ N : B(0,\epsilon) \to Hom(L,L)$ such that

$$\|P_t^n h - \lambda(t)^n N(t)h\|_L \le K\theta^n \|h\|_L \quad \forall\ |t| < \epsilon,\ n \ge 1,\ h \in L$$

where $\forall |t| < \epsilon$, $N(t)$ is a projection onto a one-dimensional subspace (spanned by $g(t) := N(t)1$) and $g(t)$ satisfies

$$\|g(t) - 1\|_L \le K(|t| + E(|e^{it\phi} - 1|)).$$

2) If ϕ is aperiodic, then $\forall\ \widetilde{M} > 0,\ \epsilon > 0,\ \exists\ K' > 0$ and $\theta' \in (0,1)$ such that

$$\|P_t^n h\|_L \le K'\theta'^n \|h\|_L \quad \forall\ \epsilon \le\ |t| \le \widetilde{M},\ h \in L.$$

By theorem 2.4, $t \mapsto P_t$ is continuous $\mathbb{R}^d \to \text{Hom}\,(L,L)$, and by proposition 2.1 P_t is a D-F operator $\forall\ t \in \mathbb{R}^d$. The proof of the theorem is established by two lemmas about D-F operators.

The next two lemmas are well known. Similar statements can be found in [35], [36] and [38]. We suppose that $(\mathcal{C}, \mathcal{L})$ is adapted, and write $\|P\| := \|P\|_{\text{Hom}\,(\mathcal{L},\mathcal{L})}$ for $P \in \text{Hom}\,(\mathcal{L},\mathcal{L})$.

Lemma 4.2

Suppose that $P_0 \in Hom(\mathcal{L},\mathcal{L})$ satisfies $P_0 = \mu_0 + Q_0$ where $\mu_0^2 = \mu_0$, $dim \mu_0 L = 1$, $\mu_0 Q_0 = Q_0 \mu_0 = 0$ and such that the spectral radius of Q_0, $\mathfrak{r}(Q_0) < 1$, then
$\exists\ \epsilon > 0,\ \lambda : B(P_0,\epsilon) \to \mathbb{C},\ N_1,\ Q : B(P_0,\epsilon) \to Hom(\mathcal{L},\mathcal{L})$ holomorphic, such that

$$P^n = \lambda(P)^n N_1(P) + Q(P)^n \quad (n \ge 1)$$

and where $N_1(P)$ is a projection onto a 1-dimensional subspace. Moreover, $|\lambda(P)| \le 1$ and $\exists\ K \in \mathbb{R}_+,\ \theta \in (0,1)$ such that $\|Q(P)^n\| \le K\theta^n\ \ \forall\ n \ge 1,\ P \in B(P_0,\epsilon)$.

The proof of lemma 4.2 is standard using [15], chapter VII, §3.6.

Lemma 4.3

Suppose that $\mathcal{K} \subset Hom(\mathcal{L}, \mathcal{L})$ is a compact set of D-F operators, none of which has an $\mathcal{L}$-eigenvalue on S^1 (the unit circle), then $\exists\ K \in \mathbb{R}_+$ *and* $\theta \in (0,1)$ *such that*

$$\|P^n\| \leq K\theta^n \quad \forall\ n \geq 1,\ \ P \in \mathcal{K}.$$

Proof

We first show that $\max_{P \in \mathcal{K}} \mathfrak{r}(P) < 1$.

For $P \in \mathcal{K}$ and $z \in \rho(P)$

$$R_P(z) = (zI - P)^{-1}.$$

For $b > \mathfrak{r}(P)$,

$$M(P,b) := \sup_{|z| \geq b} \|R_P(z)\| < \infty.$$

If $P' \in \mathrm{Hom}(\mathcal{L}, \mathcal{L})$ and $\|P - P'\| < M(P,b)^{-1}$ then $\forall\ |z| > b$

$$\sum_{n=1}^{\infty} \|\big((P' - P)R_P(z)\big)^n\| < \infty,$$

whence

$$R_P(z) \sum_{n=0}^{N} \big((P' - P)R_P(z)\big)^n \to (zI - P')^{-1}$$

in $\mathrm{Hom}(\mathcal{L}, \mathcal{L})$ as $N \to \infty$ and $B(0,b)^c \subset \rho(P')$ which implies $\mathfrak{r}(P') \leq b$.

For each $P \in \mathcal{K}$, choose $r_P \in (\mathfrak{r}(P), 1)$. As above, for each $P \in \mathcal{K}$, $\exists\ \epsilon_P = M(P, r_P)^{-1}$ such that

$$\mathfrak{r}(Q) \leq r_P \quad \forall \quad Q \in B(P, \epsilon_P).$$

By compactness of $\mathcal{K}$, $\ \exists\ P_1, \ldots, P_N \in \mathcal{K}$ such that

$$\mathcal{K} \subset \bigcup_{k=1}^{N} B(P_k, \epsilon_{P_k})$$

with the consequence that

$$\mathfrak{r}(P) \leq r_0 := \max_{1 \leq k \leq N} r_{P_k} < 1 \quad \forall\ P \in \mathcal{K}.$$

To complete the proof choose

$$\max_{P \in \mathcal{K}} \mathfrak{r}(P) < b < 1.$$

We have that $(z, P) \mapsto (zI - P)^{-1}$ is continuous $\{|z| = b\} \times \mathcal{K} \to \mathrm{Hom}(\mathcal{L}, \mathcal{L})$.

Therefore

$$\sup_{|z|=b,\ P\in\mathcal{K}} \|(zI-P)^{-1}\| =: K < \infty.$$

Now, for $n \geq 1$,

$$P^n = \frac{1}{2\pi i}\oint_{|z|=b}(zI-P)^{-1}z^n dz = \frac{1}{2\pi}\int_0^{2\pi}(be^{it}I-P)^{-1}b^{n+1}e^{i(n+1)t}dt$$

whence

$$\|P^n\| \leq \frac{1}{2\pi}\int_0^{2\pi}\|(be^{it}I-P)^{-1}\|b^{n+1}dt \leq K\theta^{n+1}.$$

$\square$

Proof of theorem 4.1

The first statement follows from lemma 4.2, and theorem 2.4.

By proposition 3.7, the aperiodicity of ϕ implies that for $t \neq 0,\ P_t$ has no eigenvalue on S^1.

By theorem 2.4, $\{P_t :\ \epsilon \leq |t| \leq \widetilde{M}\}$ is compact in $\text{Hom}(L, L)$, and so by lemma 4.3, $\exists\ \theta \in (0,1)$ and $K > 0$ so that $\|P_t^n\|_{\text{Hom}(L,L)} \leq K\theta^n$ for $n \geq 1,\ \epsilon \leq |t| \leq \widetilde{M}$. $\square$

Remark

It follows from lemma 4.2 and the remark at the end of §2 that if $\phi \in \text{Lip}_{2,\alpha}$, then $t \mapsto \lambda(t) = \lambda(P_t)$ is C^2. It can be shown as in [38] that $\frac{\partial\lambda}{\partial t_j}(0) = iE(\phi^{(j)})$ and $\frac{\partial^2\lambda}{\partial t_j \partial t_k}(0) = -\lim_{n\to\infty}\frac{E(\phi_n^{(j)}\phi_n^{(k)})}{n}$. Thus, $\lambda(t) = 1 + iE(\langle\phi, t\rangle) + \frac{t^*At}{2} + o(|t|^2)$ as $t \to 0$ where $A_{j,k} = \frac{\partial^2\lambda}{\partial t_j \partial t_k}(0)$.

The next section is devoted to an analogue of this when the marginal distributions are in the domain of attraction of a non-normal, stable distribution.

§5 Expansion of the eigenvalue

Definition

A random variable X on $\mathbb{R}$ is called *stable* if for all $a, b > 0$ there are $c > 0$ and $v \in \mathbb{R}$ such that $aX + bX' \overset{d}{\to}= cX + v$ where X' is an independent copy of X and $Y \overset{d}{\to}= Z$ means that the random variables Y and Z have the same distribution.

In this case necessarily $a^p + b^p = c^p$ for some $0 < p \leq 2$, and p is called the *order* of X.

We'll denote a stable random variable with order $p \in (0,2]$ by X_p. It is known that (up to translation) X_p has a characteristic function of form

$$g_{X_p}(t) := E\left(e^{itX_p}\right) = \begin{cases} e^{-c|t|^p(1-i\beta \text{sgn}(t)\tan(\frac{p\pi}{2}))} & p \neq 1, \\ e^{-c|t|+i\frac{2\beta c}{\pi}t\log\frac{1}{|t|}} & p = 1 \end{cases}$$

where $c > 0$, $\beta \in \mathbb{R}$ are constants, and an absolutely continuous distribution function with smooth, positive density f_{X_p}.

Throughout this section, we let $(X, \mathcal{B}, m, T, \alpha)$ be a mixing, probability preserving Gibbs-Markov map, and let

$$\phi : X \to \mathbb{R}$$

be Lipschitz continuous on each $a \in \alpha$, with $D_\alpha\phi := \sup_{a\in\alpha} D_a\phi < \infty$ and distribution G in the domain of attraction of a stable law with order $0 < p < 2$ equivalently ([16], [18], [23], [30]):

$$L_1(x) := x^p(1-G(x)) = (c_1+o(1))L(x), \;\; L_2(x) := x^pG(-x) = (c_2+o(1))L(x)$$

as $x \to +\infty$ where L is a slowly varying function on $\mathbb{R}_+$ and where $c_1, c_2 \geq 0$, $c_1 + c_2 > 0$.

Let the operator $P_t : L \to L$ be defined (as in §4) by $P_tf = P_T(\chi_t(\phi)f)$, let $\epsilon > 0$ and $\lambda(t) := \lambda(P_t)$ $(|t| < \epsilon)$ be as in theorem 4.1, and let $E(e^{it\phi}) = \widehat{G}(t)$.

Theorem 5.1 **(Expansion of the eigenvalue)**

$$\text{Re}\log\lambda(t) = -c|t|^pL(|t|^{-1})(1+o(1)),$$

and

$$\text{Im}\log\lambda(t)$$
$$= \begin{cases} t\gamma + c\beta|t|^p\text{sgn}(t)\tan(\frac{p\pi}{2}) + o\Big(|t|^pL(1/|t|)\Big) & (p \neq 1), \\ t\gamma + \frac{2\beta c}{\pi}CtL(1/|t|)) + t(H_1(1/|t|) - H_2(1/|t|)) + o\Big(|t|L(1/|t|)\Big) & (p = 1) \end{cases}$$

as $t \to 0$, *where*

$$H_j(\lambda) = \int_0^\lambda \frac{xL_j(x)dx}{1+x^2} + o(L(\lambda)) \quad \textit{as } \lambda \to \infty \quad (j = 1, 2),$$

$$C = \int_0^\infty \left(\cos y - \frac{1}{1+y^2}\right)\frac{dy}{y},$$

and the constants $c > 0,\ \beta, \gamma \in \mathbb{R}$ are defined by

$$\beta = \frac{c_1 - c_2}{c_1 + c_2}, \quad c = \begin{cases} (c_1 + c_2)\Gamma(1-p)\cos(\frac{p\pi}{2}) & p \neq 1, \\ \frac{(c_1+c_2)\pi}{2} & p = 1; \end{cases}$$

$$\gamma = \begin{cases} 0 & p < 1, \\ \int_{-\infty}^{\infty} \left(\frac{x}{1+x^2} + \operatorname{sgn}(x) \int_0^{|x|} \frac{2u^2}{(1+u^2)^2} du \right) G(dx) & p = 1, \\ \int_{-\infty}^{\infty} x G(dx) & p > 1. \end{cases}$$

Remark

The expansion of $\widehat{G}$ is given by theorem 2.6.5 in [23] in case $p \neq 1$, and by theorem 2 in [4] in case $p = 1$. As a corollary, we obtain that under the conditions of theorem 5.1

$$|\log \lambda(t) - \log \widehat{G}(t)| = o\left(|t|^p L(1/|t|) \right) \text{ as } t \to 0.$$

Lemma 5.2

$$E(|1 - e^{it\phi}|) = \begin{cases} O(|t|) & 1 < p < 2, \\ O\left(\sqrt{|t| L(\frac{1}{|t|})} \right) & p = 1 \end{cases}$$

as $t \to 0$.

Proof These estimates follow from the expansion of $\widehat{G}$ (see theorem 2.6.5 in [23] in case $p \neq 1$ and theorem 2 in [4] in case $p = 1$). □

In the next 5 lemmata, $h : \mathbb{R}_+ \to \mathbb{R}_+$ is locally integrable and slowly varying at ∞; $\eta > 0$, and $g : [-\eta, \eta] \times \mathbb{R}_+ \to \mathbb{R}$ satisfies

$$\lim_{t \to 0} \sup_{x \in \mathbb{R}} |g(t, x) - K| = 0$$

for some constant $K \geq 0$. We'll denote $\|g\| := \sup_{x \in \mathbb{R},\ |t| \leq \eta} |g(t, x)|$.

Lemmas 5.3-5 with g constant follow from lemmas 2.6.1-2 in [23], and lemma 5.6 with g constant corresponds to lemma 2 in [4].

The proofs of the lemmas can be easily extracted from their corresponding prototypes (with g constant) and so are not given.

Lemma 5.3 *For $0 < p < 2$, if the function $u \to u^{-p} h(u) g(t, u)$ is decreasing for every fixed t, then*

$$\int_0^\infty \frac{\sin(y)}{y^p} h(\frac{y}{t}) g(t, \frac{y}{t}) dy = \begin{cases} (K + o(1)) h(\frac{1}{t}) \Gamma(1-p) \cos(\frac{p\pi}{2}) & \text{if } p \neq 1 \\ (K + o(1)) h(\frac{1}{t}) \frac{\pi}{2} & \text{if } p = 1. \end{cases}$$

Lemma 5.4 *For $0 < p < 1$, if the function $u \to u^{-p}h(u)g(t,u)$ is decreasing for every fixed t, then*

$$\int_0^\infty \frac{\cos(y)}{y^p} h(\frac{y}{t})g(t,\frac{y}{t})dy = (K+o(1))h(\frac{1}{t})\Gamma(1-p)\sin(\frac{p\pi}{2}).$$

Lemma 5.5 *Let $1 < p < 2$, then*

$$\int_0^\infty \frac{\cos(y)-1}{y^p} h(\frac{y}{t})g(t,\frac{y}{t})dy = (K+o(1))h(\frac{1}{t})\Gamma(1-p)\sin(\frac{p\pi}{2}).$$

Lemma 5.6 *Suppose that the function $u \to u^{-1}h(u)g(t,u)$ is decreasing for every fixed t, then*

$$\int_0^\infty \left[\cos y - \frac{1}{1+y^2}\right]\frac{1}{y} h(\frac{y}{t})g(t,\frac{y}{t})dy = (K+o(1))h(\frac{1}{t})\int_0^\infty \left[\cos y - \frac{1}{1+y^2}\right]\frac{1}{y}dy.$$

Lemma 5.7 *Let*

$$\tilde{H}_g(\lambda) := \int_0^\infty \frac{xh(x)g(\frac{1}{\lambda},x)dx}{(1+x^2)(1+\frac{x^2}{\lambda^2})}, \quad H(\lambda) := \int_0^\lambda \frac{xh(x)dx}{1+x^2},$$

and suppose that $\exists\ \epsilon > 0$ such that $\sup_{x\in\mathbb{R}_+} |g(\frac{1}{\lambda},x)-1| = O(\lambda^{-\epsilon})$ as $\lambda \to \infty$, then

$$\tilde{H}_g(\lambda) = H(\lambda) + o(h(\lambda)) \text{ as } \lambda \to \infty.$$

Proof This corresponds to lemma 3 in [4], and is deduced from it. Set

$$\tilde{H}(\lambda) := \int_0^\infty \frac{xh(x)dx}{(1+x^2)(1+\frac{x^2}{\lambda^2})}.$$

It is shown in lemma 3 of [4] that

$$\tilde{H}(\lambda) = H(\lambda) + o(h(\lambda)) \text{ as } \lambda \to \infty.$$

The lemma is therefore a consequence of

$$|\tilde{H}_g(\lambda) - \tilde{H}(\lambda)| = o(h(\lambda)) \text{ as } \lambda \to \infty.$$

To see this

$$\begin{aligned}|\tilde{H}_g(\lambda) - \tilde{H}(\lambda)| &\le \int_0^\infty |g(\frac{1}{\lambda}, x) - 1| \frac{xh(x)}{(1+x^2)(1+\frac{x^2}{\lambda^2})} \\ &\le \sup_{x\in\mathbb{R}_+} |g(\frac{1}{\lambda}, x) - 1| \; H(\lambda) \\ &= O(H(\lambda)\lambda^{-\epsilon}) \\ &= o(h(\lambda)) \text{ as } \lambda \to \infty\end{aligned}$$

because both h and H are slowly varying at ∞. $\square$

Proof of theorem 5.1 Let $\tilde{g}_t = (\int_X g(t)dm)^{-1} g(t)$ denote the eigenfunction of P_t with eigenvalue $\lambda(t)$ satisfying $\int_X \tilde{g}_t \, dm = 1$, then

$$\lambda(t) = \lambda(t) \int_X \tilde{g}_t \, dm = \int_X P_t \tilde{g}_t \, dm = \int_X P(\tilde{g}_t e^{it\phi}) \, dm = \int_X \tilde{g}_t e^{it\phi} \, dm.$$

By theorem 4.1

$$\|g(t) - 1\|_L = O\left(|t| + E|1 - e^{it\phi}|\right) \quad \text{as } t \to 0,$$

whence

$$\|\tilde{g}_t - 1\|_\infty = O\left(|t| + E|1 - e^{it\phi}|\right) \quad \text{as } t \to 0.$$

By lemma 5.2,

$$\|\tilde{g}_t - 1\|_\infty = \begin{cases} O(|t|) & 1 < p < 2, \\ O\left(\sqrt{|t| L(\frac{1}{|t|}}\right) & p = 1 \end{cases} \text{ as } t \to 0.$$

Denote by $\mathcal{F}_0$ the σ-algebra generated by ϕ and let $\hat{g}_t \circ \phi = E(\tilde{g}_t | \mathcal{F}_0)$, then

$$\lambda(t) = \int_X \hat{g}_t \circ \phi \exp[it\phi] \, dm = \int_{-\infty}^\infty \hat{g}_t(x) \exp[itx] G(dx),$$

$$\sup_{x\in\mathbb{R}} |\hat{g}_t(x) - 1| \le \|\tilde{g}_t - 1\|_\infty = \begin{cases} O(|t|) & 1 < p < 2, \\ O\left(\sqrt{|t| L(\frac{1}{|t|}}\right) & p = 1 \end{cases} \text{ as } t \to 0,$$

and

$$\int_{-\infty}^\infty \hat{g}_t(x) \; G(dx) = 1 \;\forall\; t \in \mathbb{R}.$$

For $|t|$ small enough, $g_t^r := \mathrm{Re}\, \hat{g}_t \ge 0$ and we may write

$$\hat{g}_t = g_t^r + i g_t^+ - i g_t^-$$

where $g_t^\pm := (\pm \mathrm{Im}\, \hat{g}_t) \wedge 0 \ge 0$.

For $*=r,+,-$, set $g_t=g_t^*$, then $dG_t:=g_t dG$ is a (positive) measure on $\mathbb{R}$. Note that

$$\lim_{t\to 0}\sup_{x\in\mathbb{R}}|g_t(x)-K|=0$$

where $K=K_*=1$ for $*=r$ and $K=0$ otherwise.

Define distribution functions $G^j,\ G_t^j$ $(j=1,2)$ on $\mathbb{R}_+$ by

$$G_t^1(x):=G_t(x)-G_t(0),\ G_t^2(x):=G_t(0)-G_t(-x),$$

$$G^1(x):=G(x)-G(0),\ \&\ \ G^2(x):=G(0)-G(-x).$$

We have that

$$G_t^j(\infty)-G_t^j(x)=\frac{h_j(x)}{x^p}g_j(t,x),$$

where (as $x\to\infty$)

$$h_j(x):=\begin{cases}x^p(1-G(x)=(c_1+o(1))L(x) & j=1,\\ x^pG(-x)=(c_2+o(1))L(x) & j=2;\end{cases}$$

and

$$g_1(t,x):=\frac{\int_x^\infty g_t(u)\ G(du)}{\int_x^\infty G(du)},\ \&\ g_2(t,x):=\frac{\int_{-\infty}^{-x} g_t(u)\ G(du)}{\int_{-\infty}^{-x} G(du)}$$

and it follows that $\sup_{x\in\mathbb{R}}|g_j(t,x)-K|\to 0$ as $t\to 0$.

First let $0<p<1$, then

$$\begin{aligned}&\int_{\mathbb{R}}(1-e^{itx})\ G_t(dx)\\ &=\int_0^\infty(1-e^{itx})\ G_t^1(dx)+\int_0^\infty(1-e^{-itx})\ G_t^2(dx).\end{aligned}$$

Integration by parts and substitution of the above gives (for $j = 1, 2$)

$$\int_0^\infty (1 - \exp[-(-1)^j itx]) G_t^j(dx) = i(-1)^j t \int_0^\infty \exp[-i(-1)^j tx] g_j(t,x) \frac{h_j(x)}{x^p} dx$$
$$= i(-1)^j \operatorname{sgn}(t) \int_0^\infty \exp[-i(-1)^j y \operatorname{sgn}(t)] g_j(t, y/|t|) \frac{h_j(y/|t|)}{(y/|t|)^p} dy$$
$$= \int_0^\infty \sin[y]\ g_j(t, y/|t|)\ \frac{h_j(y/|t|)}{(y/|t|)^p} dy$$
$$+ i(-1)^j \operatorname{sgn}(t) \int_0^\infty \cos[y]\ g_j(t, y/|t|)\ \frac{h_j(y/|t|)}{(y/|t|)^p} dy$$
$$= |t|^p \int_0^\infty \sin[y]\ g_j(t, y/|t|)\ \frac{h_j(y/|t|)}{y^p} dy$$
$$+ i(-1)^j |t|^p \operatorname{sgn}(t) \int_0^\infty \cos[y]\ g_j(t, y/|t|)\ \frac{h_j(y/|t|)}{y^p} dy.$$

Applying lemmas 5.3 and 5.4 we obtain

$$\int_{\mathbb{R}} (1 - \exp[itx]) G_t(dx)$$
$$= (K + o(1)) h_1(1/|t|) |t|^p \Gamma(1-p) \left[\cos(\frac{p\pi}{2}) - i \operatorname{sgn}(t) \sin(\frac{p\pi}{2})\right]$$
$$+ (K + o(1)) h_2(1/|t|) |t|^p \Gamma(1-p) \left[\cos(\frac{p\pi}{2}) + i \operatorname{sgn}(t) \sin(\frac{p\pi}{2})\right]$$
$$= (K + o(1)) L(\frac{1}{|t|}) |t|^p \Gamma(1-p) \left[(c_1 + c_2)\cos(p\pi/2) - i \operatorname{sgn}(t)\ (c_1 - c_2)\sin(p\pi/2)\right]$$
$$= (K + o(1)) c L(1/|t|) |t|^p (1 - i \operatorname{sgn}(t)\ \beta \tan(p\pi/2)).$$

Finally

$$1 - \lambda(t) = \int (1 - \exp[itx]) g_t^r(x) G(dx)$$
$$+ i \int (\exp[itx] - 1) g_t^+(x) G(dx) - i \int (\exp[itx] - 1) g_t^-(x) G(dx)$$
$$= (1 + o(1)) c L(1/|t|) |t|^p (1 - i \operatorname{sgn}(t)\ \beta \tan(p\pi/2)).$$

The case
$1 < p < 2$ is treated analogously using lemmas 5.3 and 5.5, and replacing

$$\int_{\mathbb{R}} (1 - e^{itx}) G_t(dx) \text{ with } \int (1 - \exp[itx] + itx)\ G_t(dx).$$

We turn to the case $p = 1$. Write

$$\int (1 - e^{itx} + \frac{itx}{1+x^2})\ G_t(dx)$$
$$= \int_0^\infty (1 - e^{itx} + \frac{itx}{1+x^2})\ G_t^1(dx) + \int_0^\infty (1 - \frac{itx}{1+x^2} - e^{-itx})\ G_t^2(dx).$$

Integration by parts gives

$$\int_0^\infty (1 - e^{-(-1)^j itx} - (-1)^j \frac{itx}{1+x^2}) G_t^j(dx)$$
$$= (-1)^j it \int_0^\infty \left(e^{-(-1)^j itx} - \frac{1-x^2}{(1+x^2)^2} \right) \frac{h_j(x) g_j(t,x) dx}{x} =$$
$$|t| \int_0^\infty \sin[|t|x] \frac{h_j(x) g_j(t,x) dx}{x} + (-1)^j it \int_0^\infty \left(\cos[tx] - \frac{1-x^2}{(1+x^2)^2} \right) \frac{h_j(x) g_j(t,x) dx}{x}.$$

Now

$$\int_0^\infty \left(\cos[tx] - \frac{1-x^2}{(1+x^2)^2} \right) \frac{h_j(x) g_j(t,x) dx}{x}$$
$$= \int_0^\infty \left(\cos[tx] - \frac{1}{1+(tx)^2} \right) \frac{h_j(x) g_j(t,x) dx}{x}$$
$$+ \int_0^\infty \frac{x(1-t^2) h_j(x) g_j(t,x) dx}{(1+x^2)(1+(tx)^2)} + \int_0^\infty \frac{2x h_j(x) g_j(t,x) dx}{(1+x^2)^2}$$
$$= \int_0^\infty \left(\cos[tx] - \frac{1}{1+(tx)^2} \right) \frac{h_j(x) g_j(t,x) dx}{x} + (1-t^2) \tilde{H}_{g_j}(1/|t|) + \gamma_t^j$$

where

$$\gamma_t^j := \int_0^\infty \frac{2x^2}{(1+x^2)^2} (G_t^j(\infty) - G_t^j(x)) dx = \int_0^\infty \frac{2x h_j(x) g_j(t,x) dx}{(1+x^2)^2}$$

and

$$\tilde{H}_{g_j}(\lambda) := \int_0^\infty \frac{x h_j(x) g_j(t,x) dx}{(1+x^2)(1+\frac{x^2}{\lambda^2})}.$$

Changing variables, and using lemmas 5.3 and 5.6 respectively, we obtain that

$$\int_0^\infty \sin[|t|x] \frac{h_j(x)) g_j(t,x) dx}{x} = \int_0^\infty \sin[x] \frac{h_j(x/|t|)) g_j(t, x/|t|) dx}{x}$$
$$= \frac{K c_j \pi}{2} L(\frac{1}{|t|}) + o\left(L(\frac{1}{|t|}) \right) \text{ as } t \to 0,$$

and

$$\begin{aligned}&\int_0^\infty \left(\cos[tx] - \frac{1}{1+(tx)^2}\right)\frac{h_j(x)g_j(t,x)dx}{x}\\
&= \int_0^\infty \left(\cos[x] - \frac{1}{1+x^2}\right)\frac{h_j(x/|t|)g_j(t,x/|t|)dx}{x}\\
&= CKc_jL(\frac{1}{|t|}) + o\left(L(\frac{1}{|t|})\right) \text{ as } t\to 0.\end{aligned}$$

By lemma 5.7,

$$\tilde{H}_{g_j}(\lambda) = KH_j(\lambda) + o(L(\lambda)) \text{ as } \lambda\to\infty$$

where $H_j(\lambda) := \int_0^\lambda \frac{xh_j(x)dx}{1+x^2}$, and $t^2H_j(1/|t|) = o\left(L(1/|t|)\right)$ as $t\to 0$ since H_j is slowly varying.

Putting everything together we obtain

$$\begin{aligned}&\int_0^\infty (1+\frac{itx}{1+x^2} - e^{itx})G_t^1(dx) + \int_0^\infty (1-\frac{itx}{1+x^2} - e^{-itx})G_t^2(dx)\\
&= KL(\frac{1}{|t|})|t|(c_1+c_2)\pi/2 - itL(1/|t|)(c_1-c_2)CK\\
&- itK(\tilde{H}_1(1/|t|) - \tilde{H}_2(1/|t|)) - it(\gamma_t^1 - \gamma_t^2) + o\left(|t|L(\frac{1}{|t|})\right)\\
&= L(\frac{1}{|t|})|t|(c_1+c_2)K\pi/2 - itL(1/|t|)(c_1-c_2)CK\\
&- itK(H_1(1/|t|) - H_2(1/|t|)) - it(\gamma_t^1 - \gamma_t^2) + o\left(|t|L(\frac{1}{|t|})\right).\end{aligned}$$

Define

$$\gamma_t := \tilde{\gamma}_t^1 - \tilde{\gamma}_t^2 + \int_{\mathbb{R}} \frac{x}{1+x^2}\hat{g}_t(x)G(dx)$$

where

$$\tilde{\gamma}_t^1 := \int_0^\infty \left(\int_0^x \frac{2u^2}{(1+u^2)^2}du\right)\hat{g}_t(x)G(dx),$$

$$\tilde{\gamma}_t^2 := \int_{-\infty}^0 \left(\int_0^{-x} \frac{2u^2}{(1+u^2)^2}du\right)\hat{g}_t(x)G(dx),$$

then

$$1-\lambda(t)+it\gamma_t = it(\tilde{\gamma}_t^1-\tilde{\gamma}_t^2)+\int(1+\frac{itx}{1+x^2}-\exp[itx])g_t^r(x)G(dx)$$

$$+i\int(1+\frac{itx}{1+x^2}-\exp[itx])g_t^+(x)G(dx)-i\int(1+\frac{itx}{1+x^2}-\exp[itx])g_t^-(x)G(dx)$$

$$=c|t|L(1/|t|)+it(H_1(1/|t|)-H_2(1/|t|))-it\frac{2\beta c}{\pi}CL(1/|t|)+o\Big(|t|L(1/|t|)\Big)$$

where the constants are as in the theorem.

Finally, to complete the proof of theorem 5.1, we note that:
$\gamma_t \equiv 0$ in case $0<p<1$;
in case $1<p<2$,

$$\gamma_t := \int_{\mathbb{R}} x\hat{g}_t(x)G(dx) = \gamma + O(|t|) \text{ as } t\to 0;$$

and in case $p=1$,

$$\gamma_t = \int_{-\infty}^{\infty}\left(\frac{x}{1+x^2}+\operatorname{sgn}(x)\int_0^{|x|}\frac{2u^2}{(1+u^2)^2}du\right)\hat{g}_t(x)G(dx)$$

$$=\gamma+O\left(\sqrt{|t|L(\frac{1}{|t|})}\right) \text{ as } t\to 0.$$

$\square$

§6 Local limit theorems

Again, throughout this section, we let $(X,\mathcal{B},m,T,\alpha)$ be a mixing, probability preserving Gibbs-Markov map. Let

$$\phi : X\to\mathbb{R}$$

be Lipschitz continuous on each $a\in\alpha$, with $D_\alpha\phi := \sup_{a\in\alpha} D_a\phi<\infty$.

We assume that the m-distribution G of ϕ is in the domain of attraction of a stable law with order $0<p<2$ equivalently ([16], [18], [23], [30]):

$$L_1(x) := x^p(1-G(x)) = (c_1+o(1))L(x), \quad L_2(x) := x^pG(-x) = (c_2+o(1))L(x)$$

as $x\to+\infty$ where L is a slowly varying function on $\mathbb{R}_+$ and where $c_1, c_2\ge 0$, $c_1+c_2>0$.

Throughout this section, we use notations from §4 and write:

$$N_j(t) := N_j(P_t) \quad (j=1,2), \quad \lambda(t) := \lambda(P_t).$$

Also, let $\phi_n := \sum_{k=0}^{n-1}\phi\circ T^k \quad (n\ge 1)$.

Theorem 6.1 (Distributional limit theorem)
Under the above conditions,

$$\frac{\phi_n - A_n}{B_n} \xrightarrow{\mathcal{D}} X_p$$

where

$$nL(B_n) = B_n^p, \quad A_n = \begin{cases} 0 \quad 0 < p < 1, \\ \gamma n \quad 1 < p < 2, \\ \gamma n + \frac{2n}{\pi}(H_1(B_n) - H_2(B_n)) \qquad p = 1. \end{cases}$$

Proof We claim first that

$$n \log \lambda(\frac{t}{B_n}) - it\frac{A_n}{B_n} \to \log g_{X_p}(t) \text{ as } n \to \infty.$$

This follows from theorem 5.1 as in the independent case.
Using theorem 4.1 (1),

$$\begin{aligned}\int_X e^{it\left(\frac{\phi_n}{B_n} - \frac{A_n}{B_n}\right)} dm &= e^{-it\frac{A_n}{B_n}} \int_X P^n\left(e^{it\frac{\phi_n}{B_n}}\right) dm \\ &= e^{-it\frac{A_n}{B_n}} \int_X P^n_{\frac{t}{B_n}} 1 dm \\ &= e^{-it\frac{A_n}{B_n}} \lambda(\frac{t}{B_n})^n \int_X g(\frac{t}{B_n}) dm + O(\theta^n).\end{aligned}$$

The theorem follows since $g(s) \xrightarrow{L}\to g(0) \equiv 1$ as $s \to 0$. □

Theorem 6.2 (Conditional lattice local limit theorem)
Suppose that $\phi : X \to \mathbb{Z}$ is aperiodic, let A_n, B_n be as in theorem 6.1, and suppose that $k_n \in \mathbb{Z}$, $\frac{k_n - A_n}{B_n} \to \kappa \in \mathbb{R}$ as $n \to \infty$, then

$$\|B_n P_{T^n}(1_{[\phi_n = k_n]}) - f_{X_p}(\kappa)\|_\infty \to 0 \textit{ as } n \to \infty,$$

and, in particular

$$B_n m([\phi_n = k_n]) \to f_{X_p}(\kappa) \textit{ as } n \to \infty.$$

Proof By theorem 4.1, $\exists\, \delta > 0,\ \theta \in (0,1)$ such that $\forall\ |t| \le \delta$,

$$\|P_t^n 1 - \lambda(t)^n g(t)\|_L = O(\theta^n) \ \ \forall\ |t| \le \delta,$$

and that

$$\|P_y^n 1\|_L = O(\theta^n) \ \ \forall\ \delta \le |y| \le \pi.$$

By theorem 5.1 and since L is slowly varying at infinity, by possibly shrinking $\delta > 0$, we can ensure in addition that

$$-\text{Re}\,\log\lambda(t) \geq \frac{c}{2}|t|^p L(\frac{1}{|t|}) \;\; \forall\; |t| \leq \delta,$$

and that $\exists\; 0 < \epsilon = \epsilon(\delta)$ such that

$$\frac{L(\frac{B_n}{|t|})}{L(B_n)} \geq |t|^\epsilon \;\forall\; n \geq 1,\;\; |t| \leq \delta B_n.$$

It follows that, uniformly on X,

$$\begin{aligned}
2\pi B_n P_{T^n}(1_{[\phi_n = k_n]}) &= B_n P_{T^n}\left(\int_{-\pi}^{\pi} e^{-itk_n} e^{it\phi_n} dt\right) \\
&= B_n \int_{-\pi}^{\pi} e^{-itk_n} P_{T^n}(e^{it\phi_n}) dt \\
&= B_n \int_{-\pi}^{\pi} e^{-itk_n} P_t^n 1 dt \\
&= B_n \int_{|t|\leq\delta} e^{-itk_n} \lambda(t)^n g(t) dt + O(B_n\theta^n) \\
&= \int_{-\delta B_n}^{\delta B_n} e^{-it\frac{A_n}{B_n}} \lambda(\frac{t}{B_n})^n g(\frac{t}{B_n}) e^{it(\frac{A_n - k_n}{B_n})} dt + o(1) \\
&= \int_{-\delta B_n}^{\delta B_n} e^{-it\frac{A_n}{B_n}} \lambda(\frac{t}{B_n})^n e^{it(\frac{A_n - k_n}{B_n})} dt + o(1) \\
&\to \int_{\mathbb{R}} g_{X_p}(t) e^{-i\kappa t} dt \\
&= 2\pi f_{X_p}(\kappa)
\end{aligned}$$

as $n \to \infty$ by dominated convergence, since for $|t| \leq \delta B_n$,

$$|\lambda(\frac{t}{B_n})^n| \leq e^{-\frac{c}{2}|t|^p \frac{L(\frac{B_n}{|t|})}{L(B_n)}} \leq e^{-\frac{c}{2}|t|^{p+\epsilon}},$$

which latter function is integrable on $\mathbb{R}$. $\square$

Theorem 6.3 (Conditional non-lattice local limit theorem) *Suppose that $\phi : X \to \mathbb{R}$ is aperiodic, let A_n, B_n be as in theorem 6.1, let $I \subset \mathbb{R}$ be an interval, and suppose that $k_n \in \mathbb{Z}$, $\frac{k_n - A_n}{B_n} \to \kappa \in \mathbb{R}$ as $n \to \infty$, then*

$$B_n P_{T^n}(1_{[\phi_n \in k_n + I]}) \to |I| f_{X_p}(\kappa) \;\; \textit{as } n \to \infty$$

where $|I|$ is the length of I, and in particular

$$B_n m([\phi_n \in k_n + I]) \to |I| f_{X_p}(\kappa) \;\; \textit{as } n \to \infty.$$

Proof We use the method of Breiman (see [11]).

Suppose that $h \in L^1(\mathbb{R})$, $\hat{h} \in L^1(\mathbb{R})$, and that $\hat{h} \equiv 0$ off $[-M, M]$.

Arguing as in the proof of theorem 6.2, we obtain $\delta > 0$, and $0 < \theta < 1$ such that, uniformly on X:

$$\begin{aligned}
B_n P_{T^n}(h(\phi_n - k_n)) &= \frac{B_n}{2\pi}\int_{-M}^{M} \hat{h}(x) P_{T^n}(e^{ix(\phi_n - k_n)})dx \\
&= \frac{B_n}{2\pi}\int_{-M}^{M} \hat{h}(x) e^{-ik_n x} P_x^n 1 dx \\
&= \frac{B_n}{2\pi}\int_{|x|\le\delta} \hat{h}(x) e^{-ik_n x}\lambda(x)^n g(x)dx + O(B_n\theta^n) \\
&= \frac{1}{2\pi}\int_{|x|\le\delta B_n} \hat{h}(\frac{x}{B_n}) e^{-i\frac{k_n}{B_n}x}\lambda(\frac{x}{B_n})^n g(\frac{x}{B_n})dx + o(1) \\
&= \frac{1}{2\pi}\int_{|x|\le\delta B_n} \hat{h}(\frac{x}{B_n}) e^{-i\frac{k_n}{B_n}x}\lambda(\frac{x}{B_n})^n dx + o(1) \\
&\to \frac{1}{2\pi}\int_{\mathbb{R}} \hat{h}(0) g_{X_p}(x) e^{-i\kappa x}dx \\
&= \int_{\mathbb{R}} h(x)dx f_{X_p}(\kappa)
\end{aligned}$$

by dominated convergence as again, for some $\epsilon > 0$, $\forall\ |t| \le \delta B_n$, $|\lambda(\frac{t}{B_n})^n| \le e^{-\frac{c}{2}|t|^{p+\epsilon}}$, which latter function is integrable on $\mathbb{R}$.

Let $k(x) = \frac{\sin^2 x}{x^2}$, then $k > 0$, $k \in L^1(m_{\mathbb{R}})$ and $\hat{k}$ has compact support.

It follows from theorem 10.7 in [11] that if U is a vague neighbourhood of $m_{\mathbb{R}}$, then $\exists\ \eta > 0$ and $t_1, \ldots, t_N \in \mathbb{R}$ such that for μ a Radon measure on $\mathbb{R}$:

$$\left|\int_{\mathbb{R}} e^{it_j x}k(x)d\mu - \int_{\mathbb{R}} e^{it_j x}k(x)dx\right| < \eta \quad (1 \le j \le N) \implies \quad \mu \in U.$$

Thus, for $h : \mathbb{R} \to \mathbb{R}$ continuous with compact support,

$$B_n P_{T^n}(h(\phi_n - k_n)) \to \int_{\mathbb{R}} h(x)dx f_{X_p}(\kappa)$$

uniformly on X. The theorem follows from monotone approximation of 1_I by non-negative continuous functions with compact support. □

We conclude this section with a local limit theorem for processes with marginals in the domain of attraction of multidimensional symmetric p-stable distributions.

Definition

A random vector X is called *stable* if for all $a, b > 0$ there are $c > 0$ and $v \in \mathbb{R}$ such that $aX + bX' \overset{d}{\to}= cX + v$ where X' is an independent copy of X.

In this case necessarily $a^p + b^p = c^p$ for some $0 < p \leq 2$, and p is called the *order* of X.

The stable random vector X is called *nondegenerate* if its distribution is absolutely continuous on $\mathbb{R}^d$.

As is well known (see [40])
the random vector $X = (X_1, \ldots, X_d)$ has symmetric p-stable distribution

if and only if

$$E(e^{i\langle u,X\rangle}) = e^{-c_{p,\nu}(u)}.$$

Here $c_{p,\nu}(u) := \int_{S^{d-1}} |\langle u, s\rangle|^p \nu(ds)$ where ν is a symmetric measure on S^{d-1} (called the *spectral measure*). It is known that X is nondegenerate iff the support of ν is not contained in any subspace of $\mathbb{R}^d$ (equivalently $c_{p,\nu}(u) > 0 \ \forall \ u \in \mathbb{R}^d \setminus \{0\}$). In this case, we call the spectral measure nondegenerate.

Definition

The distribution of the symmetric random vector $Y = (Y_1, \ldots, Y_d)$ is in the *strict domain of attraction* of a symmetric p-stable distribution $(0 < p \leq 2)$ if there are constants B_n (necessarily regularly varying with index p) such that

$$\text{dist.}\, \frac{S_n}{B_n} \to \text{dist.}\, Z$$

where Z has symmetric p-stable distribution, and
$S_n = X^{(1)} + \cdots + X^{(n)}$ where $\{X^{(1)}, \ldots, X^{(n)}\}$ are independent and distributed as Y.

As is well known (see [4])
the random vector $X = (X_1, \ldots, X_d)$ is in the *strict domain of attraction* of a symmetric p-stable distribution with spectral measure ν

if and only if

there is a function $L : \mathbb{R}_+ \to \mathbb{R}_+$, slowly varying at ∞ such that

$$E(e^{it\langle u,X\rangle}) = e^{-c_{p,\nu}(u)|t|^p L(\frac{1}{|t|})(1+o(1))} \text{ as } t \to 0 \ \ \forall \ u \in \mathbb{R}^{d-1} \setminus \{0\}.$$

Lemma 6.4 *Let $(X, \mathcal{B}, m, T, \alpha)$ be a probability preserving Gibbs-Markov map, and let*

$$\phi : X \to \mathbb{R}^d$$

be Lipschitz continuous on each $a \in \alpha$ and aperiodic.

If

$$E(e^{i\frac{\langle u,\phi_n\rangle}{B(n)}}) \to e^{-c_{p,\nu}(u)} \forall\ u \in \mathbb{R}^d$$

where $p \in (0,2]$, $c_{p,\nu}(u) > 0 \quad \forall\ u \in \mathbb{R}^d$ *and* $B(n)$ *is* $1/p$*-regularly varying at* ∞, *then*

$$-\log \lambda(u) \sim \frac{c_{p,\nu}(\tilde{u})}{B^{-1}(\frac{1}{|u|})}$$

as $u \to 0$ *where* $\tilde{u} = |u|^{-1}u \in S^{d-1}$.

Proof

To see this note first that

$$E(e^{i\langle u,\phi_n\rangle}) = E(P_u^n 1)$$

and therefore by theorem 4.1,

$$E(e^{i\langle u,\phi_n\rangle}) = \lambda(u)^n E(g(u)) + O(\theta^n)$$

uniformly on $B(0,\delta)$ as $n \to \infty$, whence

$$E(e^{i\frac{\langle u,\phi_n\rangle}{B(n)}}) - \lambda(u/B_n)^n \ \to\ 0$$

as $n \to \infty$ uniformly in $u \in S^{d-1}$.

It follows from our assumption that

$$E(e^{i\frac{\langle u,\phi_n\rangle}{B(n)}}) \to e^{-c_{p,\nu}(u)}$$

as $n \to \infty$ uniformly in $u \in S^{d-1}$, whence

$$\lambda(u/B_n)^n \ \to\ e^{-c_{p,\nu}(u)}$$

as $n \to \infty$ uniformly in $u \in S^{d-1}$.

Since ν is nondegenerate, $\exists\ K \subset (0,1)$ compact such that $e^{-c_{p,\nu}(u)} \in K\ \forall\ u \in S^{d-1}$. The function $z \mapsto \log z$ is uniformly continuous on K, whence

$$-n \log \lambda(u/B_n) \ \to\ c_{p,\nu}(u)$$

as $n \to \infty$ uniformly in $u \in S^{d-1}$.

It follows that

$$-\log \lambda(\frac{1}{B(n)}u) \sim \frac{c_{p,\nu}(u)}{n}$$

as $n \to \infty$, uniformly in $u \in S^{d-1}$.

For $s > 0$, set $N_s = [B^{-1}(\frac{1}{|s|})]$, then

$$s = \frac{1}{B(B^{-1}(\frac{1}{s}))} = \frac{1}{B(\lambda_s N_s)} := \frac{\mu_s}{B(N_s)}$$

where $\lambda_s \to 1$ as $s \to 0+$ and hence also $\mu_s \to 1$ as $s \to 0+$; and we now have that

$$-\log \lambda(su) \sim -\log \lambda\left(\frac{\mu_s}{B(N_s)}u\right) \sim \frac{c_{p,\nu}(u)}{N_s} \sim \frac{c_{p,\nu}(u)}{B^{-1}(\frac{1}{s})}$$

as $s \to 0+$ uniformly in $u \in S^{d-1}$; which proves the lemma. $\square$

Theorem 6.5

Let $(X, \mathcal{B}, m, T, \alpha)$ be a mixing, probability preserving Gibbs-Markov map, and let G be a subgroup of $\mathbb{R}^d$ of form $G = A(\mathbb{R}^k \times \mathbb{Z}^\ell)$ where $k + \ell = d$ and $A \in GL(d, \mathbb{R})$.

Suppose that

$$\phi : X \to G$$

is aperiodic, Lipschitz continuous on each $a \in \alpha$, and that dist. ϕ is in the domain of attraction of a nondegenerate symmetric p-stable distribution for some $0 < p < 2$ with normalising constants B_n and density f_p, then $\forall$ compact neighbourhood V with $m_G(\partial V) = 0$, uniformly on X,

$$B_n^d P_{T^n}(1_{[\phi_n \in V]}) \to f_p(0) m_G(V) \text{ uniformly on } X \text{ as } n \to \infty.$$

Proof It follows from lemma 6.4 that

$$-\log \lambda(tu) = c_{p,\nu}(u)|t|^p L(\frac{1}{|t|})(1 + o(1))$$

as $t \to 0$ uniformly in $u \in S^{d-1}$.

We fix $\delta, \epsilon > 0$ and $c > 0$ such that

$$-\log \lambda(tu) \geq c|t|^p L(\frac{1}{|t|}) \;\; \forall\ t < \delta,\ u \in S^{d-1}$$

and

$$L(\lambda/r) \geq r^\epsilon L(\lambda) \quad \forall\ \lambda \text{ large, and } 0 < r \leq \delta\lambda.$$

We give a proof in the discrete case only, (the other being analogous and using the method of Breiman as in theorem 6.3).

$$\begin{aligned} &P_{T^n}(1_{[\phi_n(\cdot)=z]})(x) \\ &= \int_{\widehat{G}} \overline{\gamma}(z) P_\gamma^n 1(x) d\gamma \\ &= \int_{B(0,\delta)} \overline{\gamma}(z) \lambda(\gamma)^n g(\gamma)(x) d\gamma + O(\theta^n). \end{aligned}$$

Changing to polar coordinates,

$$\begin{aligned}
&\int_{B(0,\delta)} \overline{z}(\gamma)\lambda(\gamma)^n g(\gamma)(x)d\gamma \\
&= \int_{S^{d-1}} \left(\int_0^\delta r^{d-1}\overline{z}(ru)\lambda(ru)^n g(ru)(x)dr \right) d\mu(u) \\
&= \frac{1}{B_n^d} \int_{S^{d-1}} \left(\int_0^{\delta B_n} r^{d-1}\overline{z}(ru/B_n)\lambda(ru/B_n)^n g(ru/B_n)(x)dr \right) d\mu(u)
\end{aligned}$$

where μ is Lebesgue measure on S^{d-1}.

For each $u \in S^{d-1}$, $n \geq 1$ and $r < \delta B_n$,

$$|\lambda(ru/B_n)^n| \leq e^{-c\frac{nr^p}{B_n^p}L(\frac{B_n}{r})} = e^{-c\frac{r^p L(\frac{B_n}{r})}{L(B_n)}} \leq e^{-cr^{p+\epsilon}} \ \forall \ n \text{ large.}$$

The latter function is integrable and so by the dominated convergence theorem,

$$\begin{aligned}
&\int_{S^{d-1}} \left(\int_0^{\delta B_n} r^{d-1}\overline{z}(ru/B_n)\lambda(ru/B_n)^n g(ru/B_n)(x)dr \right) d\mu(u) \ \longrightarrow \\
&\int_{S^{d-1}} \left(\int_0^{\infty} r^{d-1}e^{-r^p c_{p,\nu}(u)}dr \right) d\mu(u) \\
&= \int_{\mathbb{R}^d} e^{-c_{p,\nu}(z)}dz = f_p(0).
\end{aligned}$$

□

§7 SKEW PRODUCTS

Let $(X, \mathcal{B}, m, T, \alpha)$ be a mixing, probability preserving Gibbs-Markov map, and let G be a subgroup of $\mathbb{R}^d$ of form $G = A(\mathbb{R}^k \times \mathbb{Z}^\ell)$ where $k + \ell = d$ and $A \in GL(d, \mathbb{R})$.

Suppose that

$$\phi : X \to G$$

is aperiodic, Lipschitz continuous on each $a \in \alpha$ and $D_\alpha\phi := \sup_{a\in\alpha} D_a\phi < \infty$.

The *skew product* (see [41]) is $T_\phi : X \times G \to X \times G$ defined by

$$T_\phi(x, g) = (Tx, g + \phi(x)).$$

It follows from the ergodicity of T that T_ϕ is either conservative, or totally dissipative ([41], see also e.g. proposition 8.1.1 of [3]). In this section, we use the additional structure of T and ϕ to establish stronger ergodic properties for T_ϕ.

Recall from [1] (or [3], chapter 3) that a conservative, ergodic, measure preserving transformation S of a σ-finite measure space $(Y, \mathcal{C}, \nu)$ is called *pointwise dual ergodic* if there are constants $a_n \to \infty$ such that

$$\frac{1}{a_n} \sum_{k=0}^{n-1} P_{S^k} f \;\to\; \int_Y f d\nu \quad \text{a.s. } \; \forall\, f \in L^1(\nu).$$

The sequence of constants a_n is unique up to asymptotic equality and is called the *return sequence* of S and is denoted $a_n \sim a_n(S)$.

We suppose in addition throughout that either $E(\phi) = 0$ and $E(\|\phi\|^2) < \infty$; or that dist. ϕ is in the strict domain of attraction of a nondegenerate p-stable distribution for some $0 < p < 2$, specifically suppose that

$$\frac{\phi_n}{B_n} \xrightarrow{\mathcal{D}} X_p$$

where $B_n > 0$ and X_p is nondegenerate p-stable with $0 < p \le 2$ and that $B_n \propto \sqrt{n}$ in case $p = 2$.

Theorem 7.1

1) T_ϕ is totally dissipative if $\sum_{n=1}^\infty \frac{1}{B_n^d} < \infty$, and conservative if $\sum_{n=1}^\infty \frac{1}{B_n^d} = \infty$.

2) If T_ϕ is conservative, then it is pointwise dual ergodic with return sequence

$$a_n(T_\phi) \sim f_{X_p}(0) \sum_{k=0}^{n} \frac{1}{B_k^d}.$$

Proof

1) Let $h \in L$ and let $V \subset G$ be a compact neighbourhood with $m_G(\partial V) = 0$, then, as in the proof of theorem 6.5,

$$P_{T_\phi^n}(h \otimes 1_V)(x,y) = P_{T^n}(h(\cdot)1_V(y - \phi_n(\cdot)))(x) \sim \frac{f_{X_p}(0)E(h)m_G(V)}{B_n^d}$$

uniformly in $x \in X$ and $y \in$ any compact subset of G.

Thus, by [3] §1.3, T_ϕ is conservative iff $\sum_{n=1}^\infty \frac{1}{B_n^d} = \infty$, and totally dissipative otherwise.

2) If T_ϕ is conservative, set $a_n := f_{X_p}(0) \sum_{k=0}^n \frac{1}{B_k^d}$. It follows that

$$\frac{1}{a_n} \sum_{k=0}^{n} P_{T_\phi^k} h \otimes 1_V \to \int_{X \times G} h \otimes 1_V dm \times m_G$$

a.e. $\forall\ h \otimes 1_V$ as above. Fixing $V_0 \subset G$ with $m_G(V_0) = 1$, we have by Hurewicz's ergodic theorem that

$$\frac{1}{a_n}\sum_{k=0}^{n} P_{T_\phi^k} f(x,y) \approx \frac{\sum_{k=0}^{n} P_{T_\phi^k} f(x,y)}{\sum_{k=0}^{n} P_{T_\phi^k} 1 \otimes 1_{V_0}(x,y)} \to \mathcal{E}(f)(x,y)$$

a.e. $\forall\ f \in L^1(m \times m_G)$ where $\mathcal{E}(f) \circ T_\phi = \mathcal{E}(f)$ and $\int_{X\times G} \mathcal{E}(f) 1 \otimes 1_{V_0} dm \times m_G = \int_{X\times G} f dm \times m_G$.

Thus $\mathcal{E}(h \otimes 1_V)$ is constant $\forall\ h \otimes 1_V$ as above. Since the linear span of such $h \otimes 1_V$ is dense in L^1, $\mathcal{E}(f)$ is constant $\forall\ f \in L^1$ and T_ϕ is ergodic, and pointwise dual ergodic.

□

Proposition 7.2 *If G is discrete and T_ϕ is conservative, then T_ϕ is exact.*

Proof In case G is discrete, we have that T_ϕ is an aperiodic Markov fibred system with the Renyi property, and hence exact if conservative (by theorem 3.2 in [6]). □

We finish this section with some

Remarks about the periodic case

Let $(X, \mathcal{B}, m, T, \alpha)$ be a mixing Gibbs-Markov map with respect to the partition α and invariant probability m, and let $\phi : X \to \mathbb{Z}^d$ be Lipschitz continuous on each $a \in \alpha$ with $D\phi := \sup_{a\in\alpha} D_a\phi < \infty$.

By theorem 4.1,
there are constants $\epsilon > 0$, $K > 0$ and $\theta \in (0,1)$; and functions $\lambda : B(0,\epsilon) \to B_{\mathbb{C}}(0,1)$, $N : B(0,\epsilon) \to \mathrm{Hom}(L,L)$ such that

$$\|P_t^n h - \lambda(t)^n N(t)h\|_L \le K\theta^n \|h\|_L \quad \forall\ |t| < \epsilon,\ n \ge 1,\ h \in L$$

and for $g(t) = N(t)1$

$$\|g(t) - 1\|_L \le K(|t| + E(|e^{it\phi} - 1|)).$$

We consider T_ϕ for ϕ periodic (i.e. not aperiodic).

It follows from proposition 3.7 that $|\lambda(t)| = 1\ \forall\ |t|$ small iff ϕ is cohomologous to a constant.

Assume now (as in §5 and §6) that $|\lambda(t)| < 1\ \forall\ 0 < |t| < \epsilon$.

It follows that

$$\mathfrak{q} := \{t \in \mathbb{T}^d :\ \chi_t(\phi) \text{ is cohomologous to a constant}\}$$

is a finite subgroup of $\mathbb{T}^d$.

Let $z \in \widehat{\mathfrak{q}}$ be such that $\chi_a(\phi)$ is cohomologous to $z(a)$ $\forall\ a \in \mathfrak{q}$. It is standard that $\exists\ \zeta : X \to \widehat{\mathfrak{q}}$ such that

$$\chi_a(\phi) = z(a)\overline{\zeta}(a)\zeta(a) \circ T \quad \forall\ a \in \mathfrak{q}.$$

It follows from corollary 2.2 that ζ is Lipschitz continuous on each $b \in \alpha$ with $D_\alpha \zeta < \infty$.

In this situation for $a \in \mathfrak{q}$ and $k \geq 1$ we have

$$\chi_a(\phi_k) = \prod_{j=0}^{k-1} \chi_a(\phi) \circ T^j = \prod_{j=0}^{k-1} (z(a)\overline{\zeta}(a) \circ T^j \zeta(a) \circ T^{j+1}) = z(a)^k \overline{\zeta}(a)\zeta(a) \circ T^k$$

whence $\forall\ t \in \mathbb{T}^d$ and $f \in L$,

$$\begin{aligned} P_{t+a}^k f &= P^k(\chi_t(\phi_k) z(a)^k \overline{\zeta}(a)\zeta(a) \circ T^k f) \\ &= z(a)^k \zeta(a) P^k(\chi_t(\phi_k)\overline{\zeta}(a) f) \\ &= z(a)^k \zeta(a) P_t^k(\overline{\zeta}(a) f) \end{aligned}$$

and for $|t| < \epsilon$,

$$\|P_{t+a}^k f - \zeta(a) z(a)^k \lambda(t)^k N(t)\overline{\zeta}(a) f\|_L \leq K\theta^k \|f\|_L.$$

Now assume that $z(t) \neq 1\ \forall\ t \in \mathfrak{q} \setminus \{0\}$.

For $\epsilon > 0$ so that $\{B(a,\epsilon) :\ a \in \mathfrak{q}\}$ are disjoint, if

$$\rho = \rho(\epsilon) := \max_{t \notin \bigcup_{a \in \mathfrak{q}} B(a,\epsilon)} \mathfrak{r}(P_t) \vee \theta,$$

then $\rho < 1$ and

$$\begin{aligned} P^k(1_{[\phi_k = 0]})(x) &= \int_{\mathbb{T}^d} P_t^k 1(x) dt \\ &= \sum_{a \in \mathfrak{q}} \int_{B(a,\epsilon)} P_t^k 1(x) dt + O(\rho^k) \\ &= \sum_{a \in \mathfrak{q}} z(a)^k \zeta(a)(x) \int_{B(0,\epsilon)} P_t^k(\overline{\zeta}(a))(x) dt + O(\rho^k) \\ &= \sum_{a \in \mathfrak{q}} z(a)^k \zeta(a)(x) \int_{B(0,\epsilon)} \lambda(t)^k N(t)\overline{\zeta}(a)(x) dt + O(\rho^k) \\ &= \sum_{a \in \mathfrak{q}} z(a)^k \zeta(a)(x) \int_{B(0,\epsilon)} \lambda(t)^k g(t)(x)\Lambda(t)(\overline{\zeta}(a)) dt + O(\rho^k), \end{aligned}$$

where $\Lambda(t) \in \mathrm{Hom}(L,\mathbb{R})$, $N(t)h = \Lambda(t)(h)g(t)$.

Since $z(a)^q = 1\ \forall\ a \in \mathfrak{q}$ and $\{z^k :\ 0 \le k \le q-1\} = \hat{\mathfrak{q}}$, it follows that

$$\sum_{k=0}^{q-1} 1_{\{y\in X:\ \zeta(a)(y)=z(a)^k\zeta(a)(x)\}} = 1$$

whence

$$\Lambda(t)(\sum_{k=0}^{q-1} z(a)^k \zeta(a)(x)\overline{\zeta}(a)) = \Lambda(t)(1) = 1.$$

It follows that uniformly on X with $q = |\mathfrak{q}|$,

$$\begin{aligned}\sum_{k=0}^{qn-1} P^k 1_{[\phi_k=0]} &= \sum_{\varkappa=0}^{n-1}\sum_{\ell=0}^{q-1} P^{q\varkappa+\ell}(1_{[\phi_{q\varkappa+\ell}=0]})\\ &= \sum_{\varkappa=0}^{n-1}\sum_{a\in\mathfrak{q}}\int_{B(0,\epsilon)}\Lambda(t)\left(\sum_{\ell=0}^{q-1}\lambda(t)^\ell z(a)^\ell\zeta(a)(x)\overline{\zeta}(a)\right)\lambda(t)^{q\varkappa}g(t)(x)dt + O(1)\\ &= \sum_{\varkappa=0}^{n-1} q\int_{B(0,\epsilon)}\lambda(t)^{q\varkappa}g(t)(x)dt(1+o(1)) + O(1).\end{aligned}$$

Set $u_n = u_n(\epsilon) := \int_{B(0,\epsilon)}\lambda(t)^n dt$, then as in the proof of theorem 6.5,

$$u_n \propto \frac{1}{B_n^d}$$

and it follows that T_ϕ is:
totally dissipative if $\sum_{n=1}^\infty \frac{1}{B_n^d} < \infty$,

and
conservative if $\sum_{n=1}^\infty \frac{1}{B_n^d} = \infty$. Moreover, in case $\sum_{n=1}^\infty \frac{1}{B_n^d} = \infty$, we have that

$$\sum_{k=0}^{qn-1} P^k 1_{[\phi_k=0]} \sim q\sum_{\varkappa=0}^{n-1} u_{q\varkappa}$$

whence T_ϕ is pointwise dual ergodic with

$$a_n(T_\phi) \propto \sum_{\varkappa=0}^{n-1} u_{q\varkappa} \propto \sum_{k=1}^{n}\frac{1}{B_k^d}.$$

Now suppose that

$$\mathfrak{q}_0 := \{t \in \mathbb{T}^d :\ \chi_t(\phi) \text{ is a coboundary}\} = \{t \in \mathfrak{q} :\ z(t) = 1\} \ne \{0\}$$

and set

$$G := \mathfrak{q}_0^\perp := \{n \in \mathbb{Z}^d :\ \chi_t(n) = 1\ \forall\ t \in \mathfrak{q}_0\}$$

a finite index subgroup of $\mathbb{Z}^d$. It is not hard to see that $\exists\ K : X \to \mathbb{Z}^d,\ \psi : X \to G$ Lipschitz continuous on each $b \in \alpha$ such that $\psi =$

$\phi + K - K \circ T$ (indeed, here $\zeta(a) = \chi_a(K)$); and that for no $t \in \hat{G}$ is $\chi_t(\psi)$ a coboundary.

Now let $F \subset \mathbb{Z}^d$ be finite such that $\mathbb{Z}^d = \bigcup_{g \in F}(g + G)$ disjointly. Let $\pi : X \times \mathbb{Z}^d \to X \times \mathbb{Z}^d$ be defined by $\pi(x, n) = (x, n + K(x))$, then $\pi^{-1} \circ T_\phi \pi = T_\psi$, whence each of the sets $\pi^{-1}(X \times (g + G)) \quad (g \in F)$ is T_ϕ-invariant. If T_ϕ is conservative, this is the ergodic decomposition of T_ϕ, moreover:

Theorem 7.3 *Suppose that $\phi : X \to \mathbb{Z}^d$ is Lipschitz continuous on each $a \in \alpha$ and $D_\alpha \phi < \infty$, then T_ϕ is either totally dissipative or conservative according to whether $\sum_{n=1}^{\infty} \frac{1}{B_n^d}$ converges or diverges (respectively).*

If T_ϕ is conservative then each of its ergodic components is pointwise dual ergodic with asymptotic type $a_n(T_\phi) \propto \sum_{k=1}^{n} \frac{1}{B_k^d}$.

Note that it follows from [6] (see also [3]) that in case T_ϕ is conservative, its ergodic components are open.

Now consider $T : \mathbb{R} \to \mathbb{R}$ defined by $Tx = x + v(x)$ where $v : \mathbb{R} \to \mathbb{R}$ is:
odd ($v(-x) = -v(x)$), 1-periodic ($v(x + 1) = v(x)$), and piecewise C^2, increasing and onto. Noting that $T(x + 1) = T(x) + 1$ we can write

$I = [-\frac{1}{2}, \frac{1}{2}]$ and

$$\mathbb{R} \cong I \times \mathbb{Z} \text{ by } (x, n) \longrightarrow n + x.$$

T is conjugate to

$$\tau_\phi : I \times \mathbb{Z} \to I \times \mathbb{Z} \text{ defined by } \tau_\phi(x, n) = (\tau x, n + \phi(x))$$

where $\tau : I \to I$ is defined by

$$\tau x = \big\{(x + v(x)) + \frac{1}{2}\big\} - \frac{1}{2},$$

and $\phi : I \to \mathbb{Z}$ is defined by

$$\phi(x) = \big[(x + v(x)) + \frac{1}{2}\big].$$

Here $\{\,\cdot\,\}$, and $[\,\cdot\,]$ denote fractional and integral parts respectively.

Evidently the map $\tau : I \to I$ is piecewise onto.

Theorem 7.4

Suppose that $v' \geq c > 0$ *and that* $\sup \frac{|v''|}{(1+v')^2} < \infty$, *then there is a probability* $p \sim$ *Lebesgue measure such that* τ *is a piecewise onto Gibbs-Markov map, and* $\phi : (-\frac{1}{2}, \frac{1}{2}) \to \mathbb{Z}$ *is aperiodic.*

Proof We have that $\tau' = 1 + v' \geq 1 + c > 1$, and $\tau'' = v''$, whence τ is an expanding, piecewise onto interval map satisfying Adler's condition (see example 2 in §1). As in example 2, τ is a (piecewise onto) Gibbs-Markov map.

There is an invariant probability $p \sim$ Lebesgue measure with Lipschitz continuous log-density, whence by proposition 1.1, τ is a (piecewise onto) Gibbs-Markov map with respect to p (see e.g. lemma 2.1 in [6] or [3] chapter 4).

Since $\phi : I \to \mathbb{Z}$ is onto, its aperiodicity follows from corollary 3.2. □

Corollary 7.5

1) $\exists\ \mu \sim$ *Lebesgue measure with periodic, Lipschitz continuous log-density.*

2) Either T *is totally dissipative, or* T *is pointwise dual ergodic and exact.*

Proof By theorems 7.1 and 7.4. Note that $\mu(A) = \sum_{n \in \mathbb{Z}} p(A + n)$ where p is the invariant measure for τ. □

Theorem 7.6 *Suppose that*

$$m([v \geq t]) \sim \frac{L(t)}{t^p} \text{ as } t \to \infty$$

where L *is slowly varying at* ∞ *and* $p \in (0, 2)$ *and set*

$$B_n^p = nL(B_n),$$

then T *is conservative iff* $\sum_{n=1}^{\infty} \frac{1}{B_n} = \infty$

and in this case

$$a_n(T) \propto \sum_{k=1}^{n} \frac{1}{B_k}.$$

Proof By theorem 7.2. □

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