# PARALLEL COORDINATES : VISUAL Multidimensional Geometry and its Applications

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# Curves

### **Point-Curves and Line-Curves**

 $\mathcal{R}$ ecall the fundamental duality in the plane point-to-line correspondence :

$$P: (p_1, p_2, p_3) \longrightarrow \bar{P}: [(p_1 - p_2), dp_3, -dp_1], \qquad (1)$$

where the distance between the  $x_1$  and  $x_2$  axes is d, and as usual, the triples within [...] and within (...) denote line and point homogeneous coordinates respectively. For regular (i.e. in the Euclidean plane) points

$$P: (p_1, p_2, 1) \longrightarrow \overline{P}: [(p_1 - p_2), d, -dp_1].$$

The second half of the duality is the line-to-point correspondence :

$$\ell : [a_1, a_2, a_3] \longrightarrow \bar{\ell} : (da_2, -a_3, a_1 + a_2), \tag{2}$$

where the  $a_i$ , i = 1, 2 are the coefficients of the  $x_i$  in the equation of  $\ell$  and  $a_3$  is the constant. When  $a_2 \neq 0$ , the slope of  $\ell$  is  $m = -\frac{a_1}{a_2}$  and the intercept  $b = -\frac{a_3}{a_2}$  so :

$$\ell: [m, -1, b] \longrightarrow \overline{\ell}: (d, b, 1 - m).$$
(3)

A way to obtain (2) from (1) is to find the *envelope* of all the lines  $\overline{P}$  which are the images of the points  $P \in \ell$ . Applied to each point of a smooth point-curve c results in the line-curve  $\overline{c}$  shown in Fig. 1.

$$point - curve \leftrightarrow line - curve$$
.



Figure 1: **Point-curve** and their **line-curve** images.

Separation in the *xy*-plane



Figure 2: The "above" and "below" relations between points and lines switch at m = 1.

**Lemma** P is on, below, above a line  $\ell$  whose slope  $m < 1(m \ge 1) \iff \overline{P}$  is on, below(above), above(below)  $\overline{\ell}$ . Let

$$M_c(I) = max\{m_c(P) | P(x_1) \in c, x_1 \in I\},$$
(4)

where  $m_c(P)$  is the slope of the tangent of the curve c at the point P. Further, by considering at each vertex the supporting line with maximum slope instead of the tangent in the above



Figure 3: Curve segments r flip cu  $\leftrightarrow$  cd for  $M_r < 1$  but not for  $M_r \ge 1$ .

definition,  $M_c(I)$  can be defined for a complete piece-wise smooth curve. Clearly then, tracing the above/below relation in the image  $\bar{c}$  of a curve c depends on whether  $M_c(I)$  goes through the value 1. There are some straight-forward consequences of the Lemma which, for convenience are listed separately. They are true since the statements are true point-wise. **corollary** 

- 1. For  $M_c < 1$ , c is below (above) a point  $P \Leftrightarrow \bar{c}$  is below(above)  $\bar{P}$ .
- 2. For  $M_c \ge 1$ , c is below (above) a point  $\mathbf{P} \Leftrightarrow \bar{c}$  is above (below)  $\bar{P}$ .

#### corollary

- 1. If r is a cd (cu) curve segment with  $M_r < 1 \Rightarrow \bar{r}$  is cu (cd)
- 2. If r is a cd (cu) curve segment with  $M_r \ge 1 \Rightarrow \bar{r}$  is cd (cu).

Cusps, Inflection Points and Duality



Figure 4: Cusp  $\leftrightarrow$  Inflection point duality is independent of the curves' orientation.

#### **Point-Curves from Point-Curves**

 $\mathcal{E}$ arly in the development (1980) of  $\parallel$ -coords the direct construction of the a curve's image as a **point curve** was accomplished as outlined below. Among benefits this when applied judiciously avoids **over-plotting** by the plethora of the lines which are the tangents at the *non-convex* portions of the image curve.

Consider a general planar curve c given by :

$$c: F(x_1, x_2) = 0, (5)$$

Substituting in eq. (3) yields the point-coordinates

$$x = \frac{\partial F/\partial x_2}{(\partial F/\partial x_1 + \partial F/\partial x_2)}, \ y = \frac{(x_1 \partial F/\partial x_1 + x_2 \partial F/\partial x_2)}{(\partial F/\partial x_1 + \partial F/\partial x_2)}.$$
 (6)



Figure 5: Obtaining the *point-curve*  $\bar{c}$  directly from the *point-curve* c.

There is an important special case when the original point-curve is given explicitly by  $x_2 = g(x_1)$ . Then eq. (6) reduces to :



Figure 6: Horizontal position of  $\overline{\ell}$  depends only on the slope m of  $\ell$ .

# **Curve Plotting**

T he image of a piecewise smooth curve can be computed and plotted via eq. (6). Qualitatively we can learn quite a bit to sketch the curve's image to sketch using some considerations of the duality by reviewing Fig. 6 which we saw earlier.

Later (p. 16) it is pointed out that the image  $\bar{c}$  of an algebraic (i.e. described by a polynomial) curve c of degree n is also algebraic with degree  $n^*$  as given by the Plücker class formula eq. (14) where s, d are the number of cusps and double-crossing points respectively. For the curve c on the left of Fig. 7, n = 3, d = 0, s = 0, hence for the image curve (right) is  $n^* = 6$ . The analysis is facilitated by Fig. 8. The curve c has slope m = 1 at the points  $A_L$ and  $A_R$  causing the image curve  $\bar{c}$  to split the tangents there mapping to ideal points. The portion  $c_I$  of c between  $A_L$  and  $A_R$  has the the right branch of  $\bar{c}$  as its the image, i.e.  $\bar{c}_I$  The inflection point ip I is mapped into the cusp  $\overline{i}$  of  $\overline{c}$  in between the two axes since its slope  $m_i$  is negative. On  $c_I$  the curve's slope m are  $m_i \ge m < 1$  and for this reason  $\bar{c}_I$  opens to the right intersecting the  $X_2$  at two points; the images of the tangents at the extrema where m = 0 the higher for the higher intercept (i.e. b) in this case the maximum  $M_a$  of c. The left portion of  $c_I$  being cd maps into the upper portion of the of  $\bar{c}_I$  which is cu and approaching  $A_L$  assymptotically. Similarly the right portion of  $c_I$  being cu maps into the lower portion which is cd approaching  $A_R$  asymptoically. Similarly, the left portion of  $\bar{c}$  approaches the  $\bar{X}_1$ -axis asymptotically for asqq  $|x_1| \to \infty$  the curve's slope  $m \to \infty$ . The upper portion of the left branch is cu being the image of the portion  $-\infty < x_1 < x_1(A_L)$  which is also cu. Note that the symmetry of c with respect to the tangent i at I is transformed to symmetry





Figure 8: (Right)Analysis of the image curve above in terms of the slope (left) at the cubic's important points.

with respect to the line  $\overline{I}$  through the cusp  $\overline{i}$  the point image of the tangent *i* (see exercises below).

The curve c shown in Fig. 9 is prescribed by an implicit polynomial with n = 3, s = d = 0.





Figure 10: The image of the curve  $c: F(x_1, x_2) = x_1^3 + x_{\text{xt}}^2$ <del>0 also has</del> degree 6.  $x_{1|_5}$ 

Function

Demos Sampling Show Axes Reset Again  $\bar{c}$  has degree  $a^*_{\text{netic}} = 6$  but finite how causes stemming from the 2 ip of c. The image of the right branch  $c_{R'}$  of c is the portion containing the cusps in the slope  $p_{Auto b}$  as  $x_1 \xrightarrow{h_0} \infty$  with  $\bar{c}_R$  approaching the  $\bar{X}_1$ -axis asyptotically. The (two) points of c where the slope m = 1 are on the oval spliting its image to the hyperbola-like part. Proceeding with another curve c(left) Fig. 10 also prescribed by an implicit polynomial. Again n = 3, s = d = 0 and the image curve  $\bar{c}$  has degree  $n^* = 6$  There are two points of c where m = 1 responsible for the split of  $\bar{c}$ . The cusp to the right of the  $\bar{X}_2$ -axis is the image of the lower *ip* of *c* where the





Figure 1,  $t_{5}^{2}$ : The image of the parametric polynomial  $x(t_{1}) = t_{5}^{2} \cdot x_{2}(t)_{t_{1}} t_{5}^{3}$  has degree 3.

slope 0 < m < 1 and also,  $\sin \frac{c}{k!0! = \frac{1}{2} \times 20} = \frac{1}{2} \text{ period}$  as  $x_{1} = \frac{5}{5} \propto \frac{5}{5} \text{ bind} \text{ Axes}$  for  $\bar{c}$  approaches the  $\bar{X}_{1} = \frac{1}{2} \times 20^{\circ} = \frac{1}{2} \text{ period}$  as  $x_{1} = \frac{5}{5} \propto \frac{5}{5} \text{ bind} \text{ Axes}$  for  $\bar{c}$  approaches the  $\bar{X}_{1} = \frac{1}{2} \times 20^{\circ} = \frac{1}{2} \text{ period}$  in the next example shown in  $\frac{1}{5} = \frac{1}{5} \propto \frac{1}{5} \text{ the image curve has degree}$   $n^{\circ} = \frac{3}{2}, d = 1, s = 0$ . Both points of c where  $m = \frac{1}{5}$  are to the right of the double-crossing point with the part  $c_{R}$  to the right of these points is mapped into the upper portion of c. The remaining part  $c_{L}$  of c containing the double-point is mapped into two



Figure  $1_{2:1}^{2:1}$  The single of the exponential curve  $c : x_2 \times 1^{2} c_1 + c_2 \times 1^{2} c_2 + c_3 + c_4 + c_4 + c_4 + c_5 + c_4 + c$ 

Function	Demos			Sampling	Show Axes	Reset
⊙ Explicit O Implicit O Parametric	Choose function	•	×1:	.5 -> 5		1(
x2(x1) = exp(x1)		9	delt	a: 0.001	🗹 Auto Dualize	Inro
	Go			Set Defaults	Dualize Now!	Close



Figure 15: The image of 5e; 5e;

m	= 1 is mapped to the ide	al point	; al <b>omg</b> t	he <i>x</i> -axi	5.	Sampling	Show Axes	Reset
	👁 Explicit 🔍 Implicit 🔍 Parametric	Choose fun	iction	•	X1:	-6.3> 6.3	]	
	x2(x1) = sin(x1)				delt	a: 0.0001	✓ Auto Dualize	Info
			Go	10		Set Defaults	Dualize Now!	Close



called *bitangent*. The algebraic curve c shown in Fig. 12, is specified parametrically, and has n = 3, s = 1, d = 0 so that  $\bar{c}$  has degree  $n^* = 3$ . The two branches of c are tangent to the  $x_1$ -axis at the cusp and, therefore, it maps to the inflection point where  $\bar{c}$  crosses the  $\bar{X}_2$ -axis. As in the previous examples due to  $m \to \infty$  as  $x_1 \to \infty \bar{c}$  approaches the positive  $\bar{X}_1$ -axis asymptoically on either side.

It is very easy to sketch the image of the exponential curve seen in Fig. 13. There is one point with slope m = 1 spliting the image curve  $\bar{c}$  into a part left of  $\bar{X}_1$ , corresponding to the the part on the right part where  $1 < m < \infty$ , and the left part of c where 0 < m < 1whose image is to the right of the  $\bar{X}_2$ -axis. In the absence of negative slopes of c,  $\bar{c}$  does not have a portion in between the  $\parallel$ -axes.

It is interesting to trace the image of oscillatory curves starting with the trigonometric function  $x_2 = sin(x_1)$  plotted in Fig. 14 for the interval  $x_1 \in [-\pi, 0]$ . The *ip* at  $x_1 = \pi$  maps into the cusp which is in between the axis since the tangents' slopes m < 0 in its vicinity. In the remainder the slope  $0 < m(x_1) = cos(x_1) \le 1$  and hence  $\bar{c}$  opens to the right the upper portion approaching the x-axes asyptotically due the ideal point from  $x_1 = 0$  where the slope m = 1. The graph for the interval  $x_1 \in [0, 2\pi]$  is the mirror image the curve above, Fig. 15, since the slopes  $m(x_1) = cos(-x_1)$ . Altogether then the *ip* at  $x_1 = \pm \pi$  with m = 1are mapped into the two cusps with x = 1/2 and the *ip* at the origin, having slope m = 1, is mapped to the ideal point along the x-axis. The mirror symmetry is preserved for the image of the curve  $c : x_2 = x_1 cos(x_1)$  in Fig. 16 and for the analogous reasons.

#### **Conic Transforms**

 $\mathcal{T}$  he treatment is particularly pleasing for the conic sections which are described by the quadratic function

$$F(x_1, x_2) = A_1 x_1^2 + 2A_4 x_1 x_2 + A_2 x_2^2 + 2A_5 x_1 + 2A_6 x_2 + A_3 = = (x_1, x_2, 1) \begin{pmatrix} A_1 & A_4 & A_5 \\ A_4 & A_2 & A_6 \\ A_5 & A_6 & A_3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ 1 \end{pmatrix},$$
(8)

where the type of conic is determined by the sign of the discriminant  $\Delta = (A_4^2 - A_1A_2)$ . The coefficient matrix is denoted by A and its determinant, which plays an important role in the development, is

$$detA = A_3(A_1A_2 - A_4^2) - A_1A_6^2 - A_2A_5^2 + 2A_4A_5A_6.$$
(9)

For conics, using the identity that for a polynomial F of degree  $n F(\mathbf{x}) = 0 \Rightarrow \nabla F \cdot \mathbf{x} = \nabla F \cdot \mathbf{x} - nF$  with the second expression being linear, eq. (6) and becomes

$$x = \frac{A_4 x_1 + A_2 x_2 + A_6}{[(A_1 + A_4) x_1 + (A_2 + A_4) x_2 + (A_5 + A_6)]}$$

$$y = -\frac{A_5 x_1 + A_6 x_2 + A_3}{[(A_1 + A_4) x_1 + (A_2 + A_4) x_2 + (A_5 + A_6)]}.$$
(10)

These are *Mobius*<sup>1</sup> transformations which form a group (see any good book in modern Algebra) [1]). This observation enables substantial simplifications of the earlier treatment of conics and their transforms (see [3] and [4]). The *inverse*, expressing  $x_1$  and  $x_2$  in terms of x and y, is a Mobius transformation of the form

$$x_1 = \frac{a_{11}x + a_{12}y + a_{13}}{a_{31}x + a_{32}y + a_{33}} , \ x_2 = \frac{a_{21}x + a_{22}y + a_{23}}{a_{31}x + a_{32}y + a_{33}} , \tag{11}$$

The result obtained is

$$f(x, y) = (x y 1)\mathbf{a} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = 0.$$
(12)

The conclusion then is that

conics in the xy - plane  $\mapsto$  conics in the  $x_1x_2$  - plane

The specific result obtained is

$$f(x,y) = (x \ y \ 1)\mathbf{a} \begin{pmatrix} x\\ y\\ 1 \end{pmatrix} = 0.$$
(13)

with **a** is a 3x3 matrix whose elements are given in terms of the coefficients in eq. 8 enabling the classification of the conic transforms into six cases.

<sup>&</sup>lt;sup>1</sup>Also called *linear rational* transformations.



Figure 17: Ellipses always map into hyperbolas. Each assymptote is the image of a point where the tangent has slope 1.

## Classification of the Conic Transforms



Figure 18: A parabola whose ideal point does not have direction with slope 1 **always** transforms to a hyperbola with a vertical asymptote. The other asymptote is the image of the point where the parabola has tangent with slope 1.



Figure 19: A parabola whose ideal point has direction with slope 1 transforms to a parabola - self-dual.



Figure 20: Hyperbola to ellipse – dual of case shown in Fig. 17



Figure 21: Hyperbola to parabola. This occurs when one of the assymptotes has slope 1- dual of case shown in Fig. 18



Figure 22: Hyperbola to hyperpola – self-dual case.

## Transforms of Algebraic Curves

Conic transforms are studied for two reasons. One is the ease of use of the Mobius transformations which is completely general for Quadrics, the surfaces prescribed by quadratic equations, in *any dimension*. The other is that their detailed analysis serves as a model and guide in the study of the related images of far more complex curves, regions and their ramifications in the next section.

Algebraic curves, described by polynomial equations, are studied in Algebraic Geometry. An algebraic curve c has many invariants which are properties independent of the particular coordinate system used. Examples are the number of components in its graph, degree n, number of double-points d (points where the curve crosses itself once), cusps s, inflection points ip and bitangents b (i.e. tangents at two points). The dual curve  $c^*$  is the image of c under a point  $\leftrightarrow$  line duality. The same symbols with the \* superscript denote the dual's invariants. Algebraic curves and their duals have been studied extensively starting in 1830 by the mathematician and physicist Julius Plücker who also made other important contributions to the field. His results apply to the class of Plücker curves i.e.

- 1. c is irreducible and of degree  $n \ge 2$ , and
- 2. the singularities of  $c, \bar{c}$  are at most simple double-points (i.e. point where curve crosses itself once) and cusps.

Whereas all irreducible quadratics and cubics are *Plücker* curves, there exists quartics, n = 4, which are not.

Of interest here are the relations a curve's invariants and those of its dual. As indicated in the equalities tabulated below, in addition to the  $ip \leftrightarrow cusp$  duality, there is a *bitangent*  $\leftrightarrow$  *double* - *point* duality which we already met in Fig. 10. This is reasonable for the two tangents at a double-point map into the two points on a bitangent which is the double-point's image.

$\mathbf{Point}(\mathbf{s}) \text{ on curve } c \rightarrow$	map into points of the curve $\bar{c} \Rightarrow$	relation
The 2 points of $c$ on a bitangent	map into a double-point of $\bar{c}$	$b = d^*$ .
A double-point of $c$	maps into two points on a bit angent of $\bar{c}$	$d = b^*$ .
An inflection-point of $c$	maps into a cusp of $\bar{c}$	$ip = s^*$ .
A cusp of $c$	maps into an inflection-point of $\bar{c}$	$s = ip^*$ .

Table 1: Equalities between invariants of Plücker curves and their duals.

The dual of c is an algebraic curve whose degree  $n^*$  depends n and the invariants d, s as given by the *Plücker class formula* :

$$n^* = n(n-1) - 2d - 3s . (14)$$

For n = 2 the *Plücker* class formula yields  $n^* = 2$  and  $s^* = 0$  confirming the conclusions in section .

The polynomial describing the dual  $c^*$  can be found for any point  $\leftrightarrow$  line duality by two different methods. However, with  $n^* = O(n^2)$  and the complexity increasing rapidly with the number of non-zero coefficients of the polynomial specifying c the process is very tedious. All this applies to  $\bar{c}$ , which is the image under the particular  $\parallel$ -coords duality, when c is an algebraic curve c. As pointed out in section, the properties of  $\bar{c}$  are immediately available from *Plücker*'s results. Together with the qualitative considerations discussed and a good curve-plotter gives a complete grasp of  $\bar{c}$  and its properties. This avoids the laborious process of obtaining the polynomial for  $\bar{c}$  and its subsequent computation for plotting.

There exist numerous generalizations of the *Plücker* formulae.

#### Convex Sets and their Relatives

Consider a double-cone, as shown in Fig. 23, whose base is a bounded convex set rather than a circle. The three type of sections shown are generalizations of the conics and are conveniently called  $gconics^2$ . They are either a :

**bounded convex** set is abbreviated by *bc*, or an



Figure 23: Gconics - three types of sections: (left) bounded convex set bc, (right) unbounded convex set uc and (middle) hyperbola-like gh regions.

<sup>&</sup>lt;sup>2</sup>The corresponding regions have been previously referred to as *estars*, *pstars* and *hstars* [10], [11].



Figure 24: A bounded convex set bc always transforms to a gh (generalized hyperbola) – this is the generalization of the case shown in Fig. 17.



Figure 25: An unbounded convex set uc whose ideal points do not have slope 1 transforms to a gh (generalized hyperbola). This is the generalization of the case shown in Fig. 18.

**unbounded convex** set is denoted by *uc* containing a non-empty set of ideal points whose slope *m* is in an interval  $m \in [m_1, m_2]$ , or a



Figure 26: Unbounded convex set uc having ideal point with slope 1 transforms to a uc – self-dual case. This is the generalization of the case shown in Fig. 19.



Figure 27: A gh whose supporting lines have slope  $m \in [m_1, m_2]$  where the  $m_1 < 1 < m_2$  are the assymptotes' slopes transforms to a bounded convex bc set. This is the generalization of the conic case shown in Fig. 20.

**generalized hyperbola** denoted by gh consisting of two full(not segments) lines  $\ell_u$ ,  $\ell_\ell$ , called assymptotes two infinite chains, convex-upward chain  $c_u$  above both assymptotes, and another convex-downward chain  $c_\ell$  below both assymptotes.

# Geonics and their Transforms

Theorem The images of gconics are gconics



Figure 28: A gh with  $1 \notin [m_1, m_2]$ , where the  $m_i$  are the assymptotes' slopes, transforms to a gh – Self-dual case. This is the generalization of the case shown in Fig. 22.



Figure 29: The Convex Union (also called "Convex Merge") of bcs corresponds to the Outer Union of their images (ghs).

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Figure 30: Inner intersection and intersections are dual.

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