# PARALLEL COORDINATES : *VISUAL* Multidimensional Geometry and its Applications

Alfred Inselberg(©1992, 2004)

Senior Fellow San Diego SuperComputing Center, CA, USA

Computer Science and Applied Mathematics Departments
Tel Aviv University , Israel
aiisreal@post.tau.ac.il
&
Multidimensional Graphs Ltd
Raanana 43556, Israel

# Multidimensional Lines

### Adjacent Variables Form

In $\mathbb{R}^3$ a line is the intersection of two planes. So a line $\ell$ in $\mathbb{R}^N$ is the intersection of $N-1$ non-parallel hyperplanes. Equivalently, it is the set of points (specified by N-tuples) which satisfy a set of $N-1$ linearly independent equations which after some algebraic manipulations (and a few exceptions) can be put in the following form:

$$\ell \quad : \quad \begin{cases} \ell_{1,2} & : \; x_2 = m_2 x_1 + b_2 \\ \ell_{2,3} & : \; x_3 = m_3 x_2 + b_3 \\ \quad \cdots & \\ \ell_{i-1,i} & : \; x_i = m_i x_{i-1} + b_i \\ \quad \cdots & \\ \ell_{N-1,N} & : \; x_N = m_N x_{N-1} + b_N \, . \end{cases}$$

Each equation contains a pair of *adjacently* labeled variables. In the $x_{i-1}x_i$-plane the relation labeled $\ell_{i-1,i}$ is a line, and by the previous *point* $\leftrightarrow$ *line* duality it can be represented by a point $\bar{\ell}_{i-1,i}$ written in homogeneous coordinates below.

$$\bar{\ell}_{i-1,i} = ((i-2)(1-m_i)+1\, ,\, b_i\, ,\, 1-m_i). \quad (1)$$

There are $N-1$ such points for $i = 2\, , \ldots \, , N$ which represent the line $\ell$.

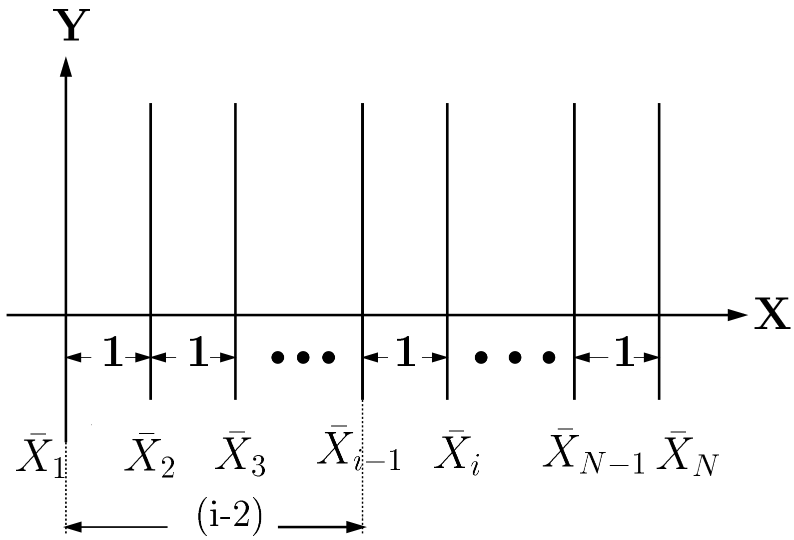


Figure 1: Standard spacing between adjacent axes is taken as 1 unit.

**Base Variable Form**

Another common way of describing a line $\ell \subset \mathbb{R}^N$ is in terms of one, sometimes called the *base*, variable which after appropriate relabeling may be taken as $x_1$. Then

$$\ell \quad : \quad \begin{cases} \ell_{1,2} & : \; x_2 = m_2^1 x_1 + b_2^1 \\ \ell_{1,3} & : \; x_3 = m_3^1 x_1 + b_3^1 \\ \quad \cdots & \\ \ell_{1,i} & : \; x_i = m_i^1 x_1 + b_i^1 \\ \quad \cdots & \\ \ell_{1,N} & : \; x_N = m_N^1 x_1 + b_N^1 \end{cases} ,$$

and the $N-1$ points representing it are :

$$\bar{\ell}_{1,i} = (i-1,\; b_i^1,\; 1-m_i^1) , \qquad (2)$$

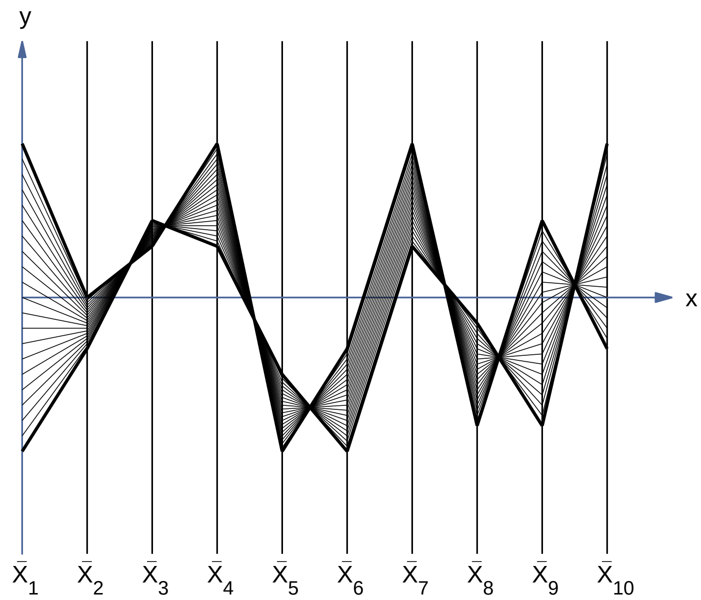


Figure 2: Line interval in 10-D – the thicker polygonal lines represent it's end-points. The adjacent variables representation, consisting of nine properly indexed points, is obtained by the sequential intersections of the polygonal lines' linear portions. Note that $\bar{\ell}_{1,2}$ is to the right of the $X_2$-axis and $\bar{\ell}_{6,7}$ is an ideal point. The remaining points are in between the corresponding pairs of axes.

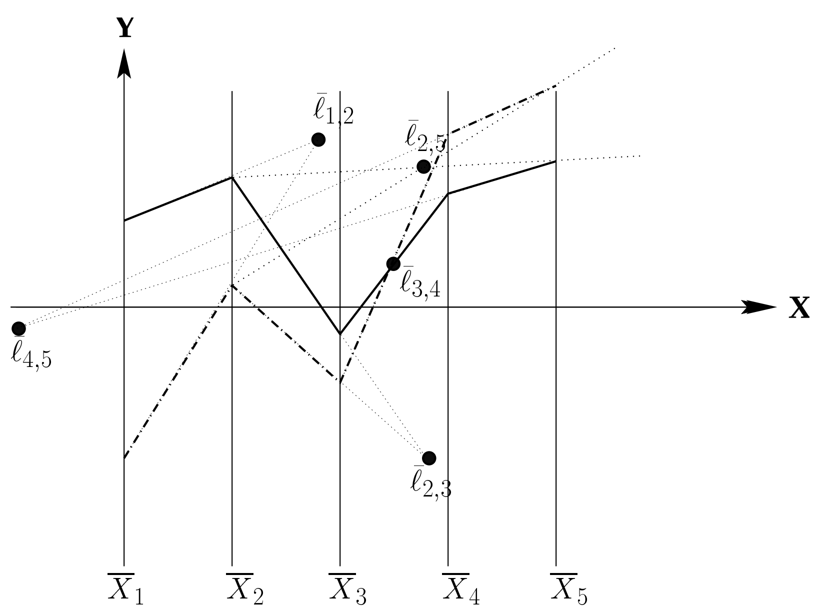


Figure 3: Algorithm for constructing a pairwise linear relation, in this case $\bar{\ell}_{25}$, given the $N-1$ points, $\bar{\ell}_{i-1,i}$ , representing the line.

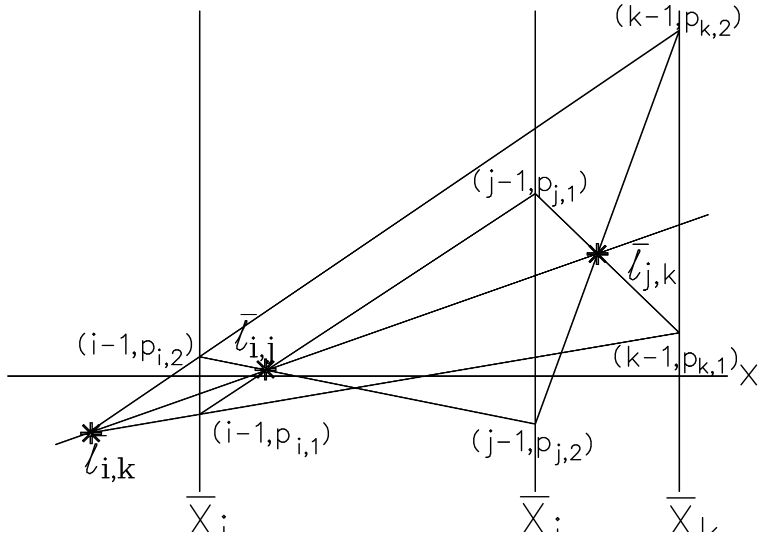


Figure 4: The Collinearity for the 3 points $\bar{\ell}_{i,j}, \ \bar{\ell}_{j,k}, \ \bar{\ell}_{i,k} \quad i \neq j \neq k \in (1, 2, ..., N)$. Desargues Theorem with the two triangles being in perspective with respect to the ideal point in vertical direction. The y-axis is offscale.

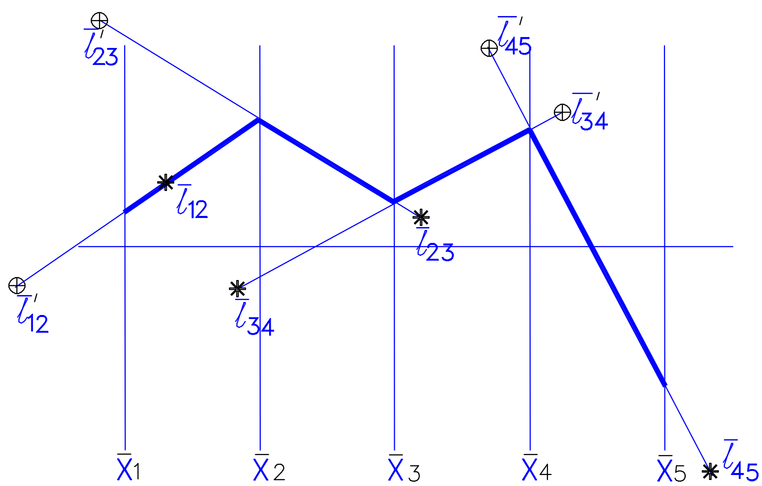


Figure 5: Two intersecting lines $\ell$ and $\ell'$ in $\mathbb{R}^5$. The points representing one line are marked with * and the other with circles $\circ$. Points with the same subscripts are joined $i, i+1$ with $i, i+1$ then $i+1, i+2$ with $i+1, i+2$. If at any stage the lines do not intersect the common $\bar{X}_i$ axis at the same point that shows non-intersection and algorithm terminates. Otherwise, as above, the construction continues to the last pair the output being the polygonal representing the point of intersection. Alternatively, one of the lines $\ell$ rotated about one of it's points shown here by the polygonal. This corresponds to the translation of the $\bar{\ell}$s to the new positions $\bar{\ell}'$s on the same polygonal line.

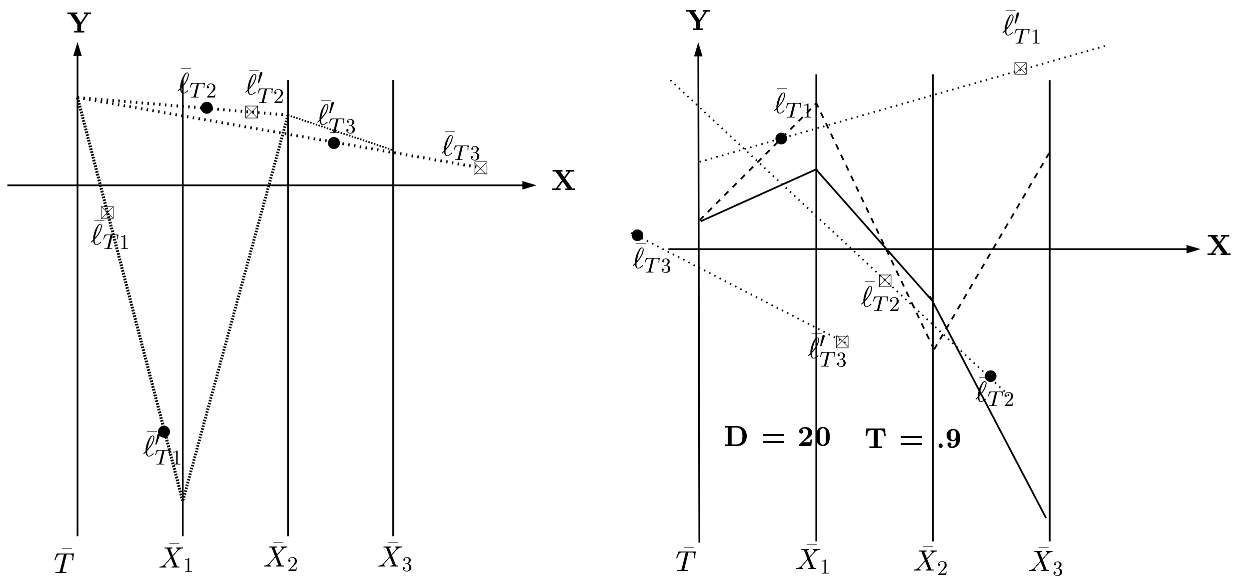


Figure 6: On the left is the intersection, for the base-variable T description, of two lines in 4-D. They are the equations of the trajectories of two particles moving with constant velocities. Intersection is the space **and time T** coordinates of the collision. On the right is non-intersection between two lines in 4-D. The gap between the upper and lowest intersections (lines joining points with the same indices) on the $\bar{T}$ axis indicates the minimum distance (20) between the two moving particles. and time = .9 when it occurs. Note the maximum gap on the $\bar{T}$-axis formed by the lines joining the $\bar{\ell}$'s with the same subscript. The polygonal lines represent the points where the minimum distance occurs at the *same* value of T are shown i.e. the time T and the positions of the particles where they are closest.

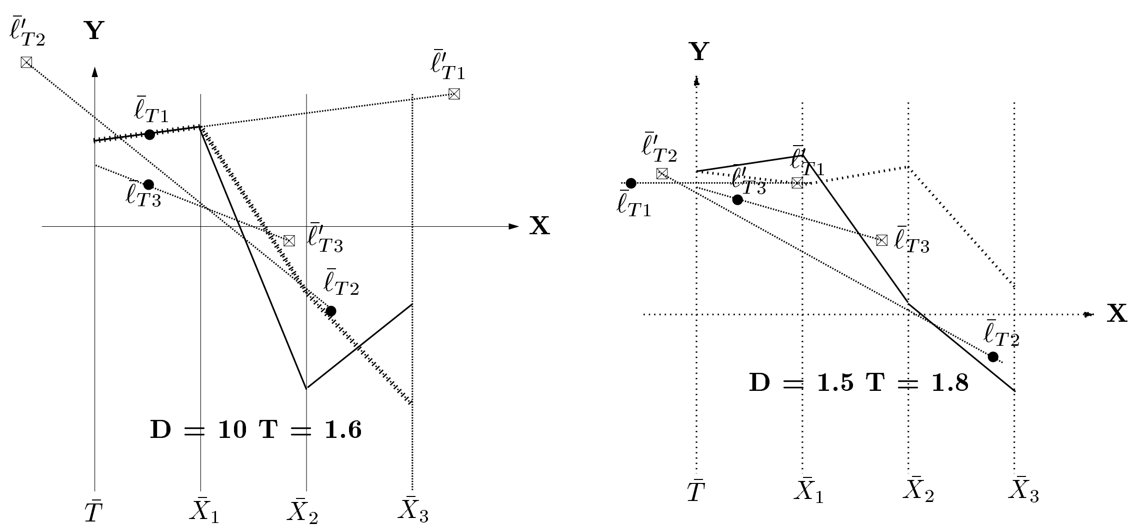


Figure 7: Diminishing the minimum distance with a near collision shown on the right.

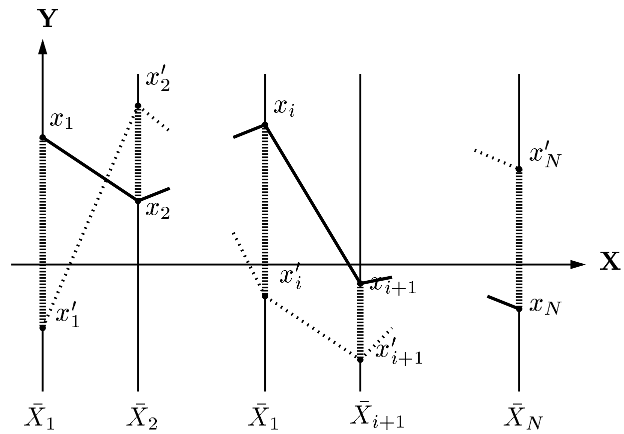


Figure 8: $L_1$ distance between the points $P = (x_1, ..., x_i, ..., x_N)$ and $P' = (x'_1, ..., x'_i, ..., x'_N)$.

**Theorem**(Constrained Min-Dist) – The unique minimum value of the $L_1(x_1)$ distance is attained at $x_1 = \alpha_i$ for at least one $i = 2, \ldots, N$ .

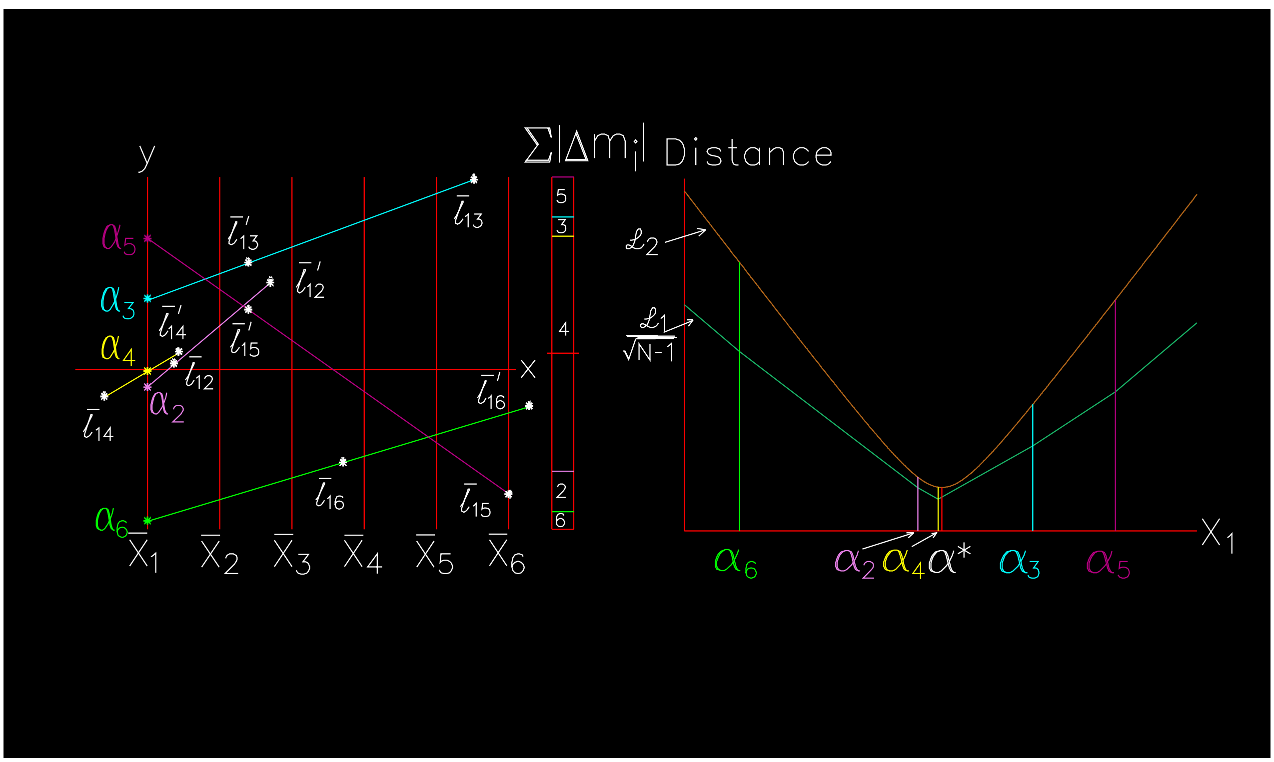


Figure 9: Constructing the $x_1 = \alpha_I$ minimizing the $L_1$ distance between two lines which here occurs at $x_1 = \alpha_4$. For comparison the minimum $L_2$ distance occurs at $x_1 = \alpha*$.

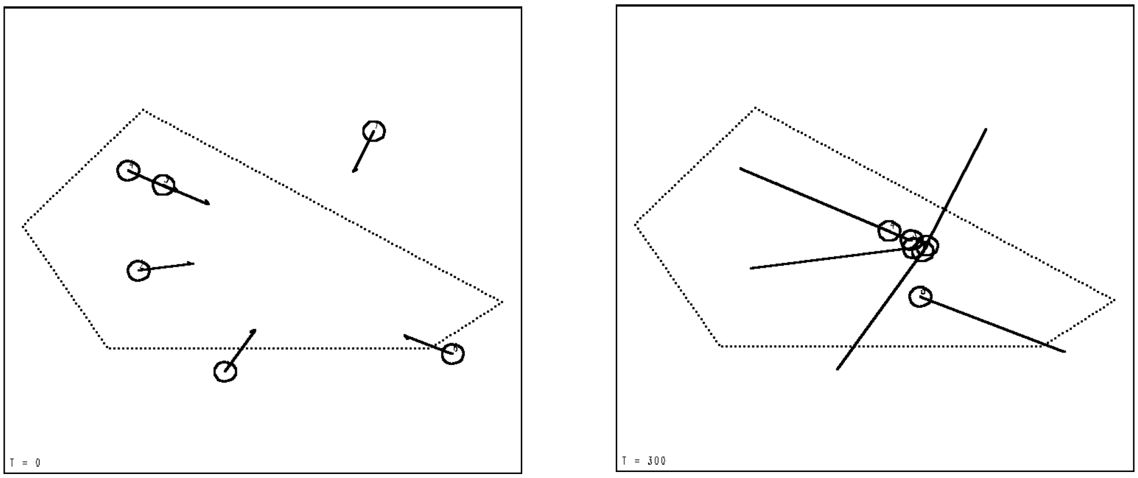


Figure 10: On the left 6 aircraft flying at the same altitude. The positions are at a certain time (taken as 0 seconds and shown on the bottom left). Circles centered at each aircraft are the protected airspaces with the the minimum allowable separation as diameter. The arrows represent the velocities. On the right conflicts, the overlaping circles, occur within the next 5 minutes.

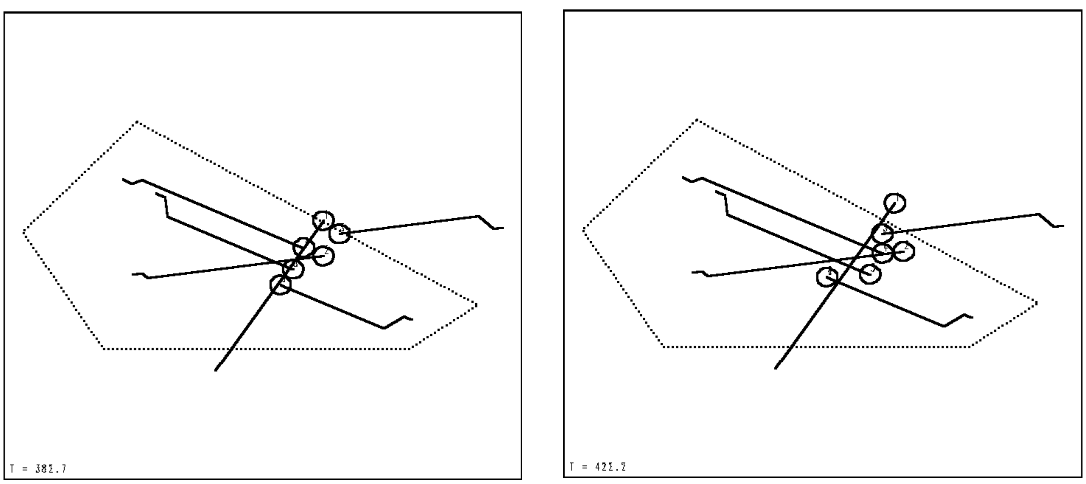

Figure 11: On the left after conflict resolution maneuvers an instance with 3 pairs of tangent circles and on the right a triple tangency. Parallel coordinates are used **internally** by the algorithm.

# References

[1] A. Inselberg and B. Dimsdale. Multidimensional lines i: Representation. *SIAM J. of Applied Math.*, 54-2:559–577, 1994.

[2] A. Inselberg and B. Dimsdale. Multidimensional lines ii: Proximity and applications. *SIAM J. of Applied Math.*, 54-2:578–596, 1994.