PARALLEL COORDINATES : *VISUAL* Multidimensional Geometry and its Applications

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Planes, p-flats & Hyperplanes

Vertical Line Representation

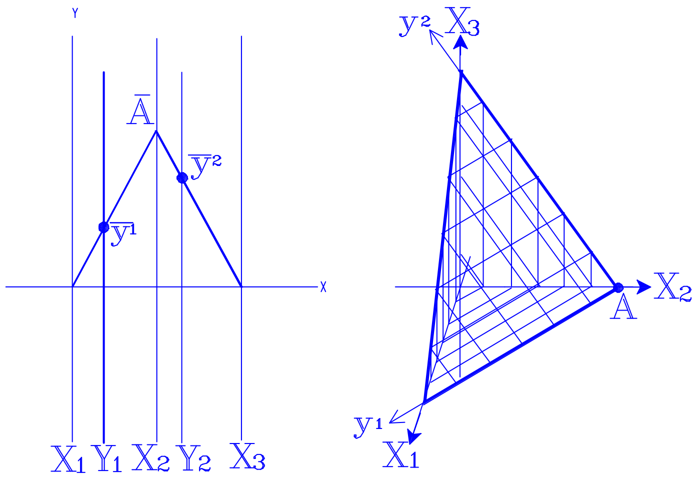


Figure 1: A plane π in $\mathbb{R}^3$ can be represented by two vertical lines and a polygonal line representing one of its points.

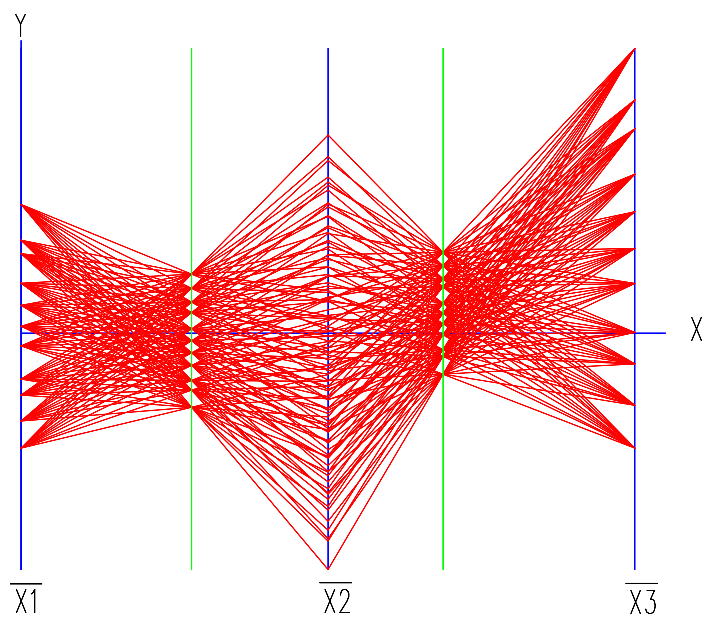


Figure 2: A set of coplanar points on a regular grid in $\mathbb{R}^3$ with the two vertical lines pattern.

This generalized to $\mathbb{R}^N$ where a hyperplane can be represented by $N-1$ vertical lines.

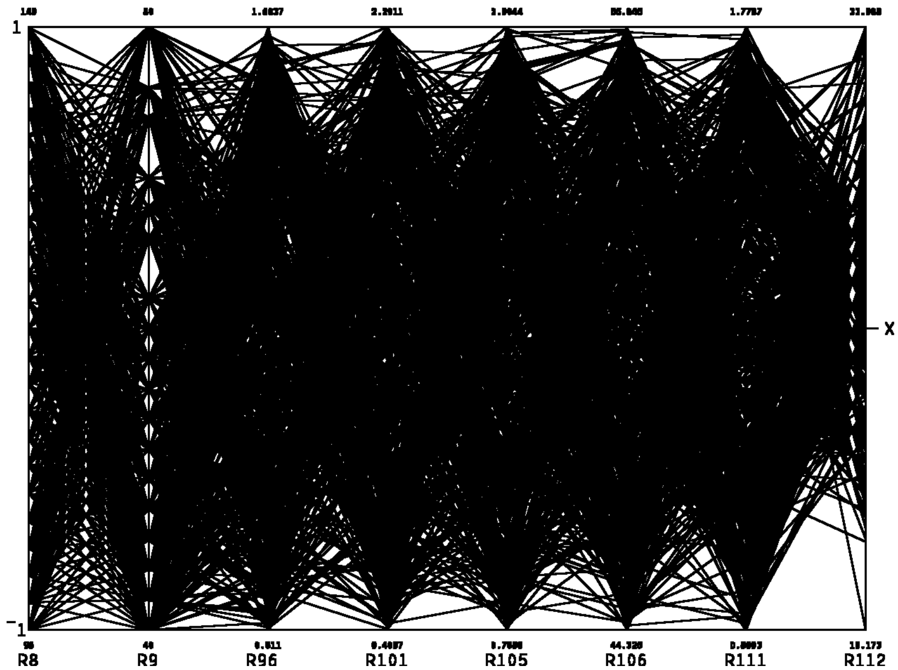


Figure 3: Industrial data. Note pattern between the R111 and R112 axes.

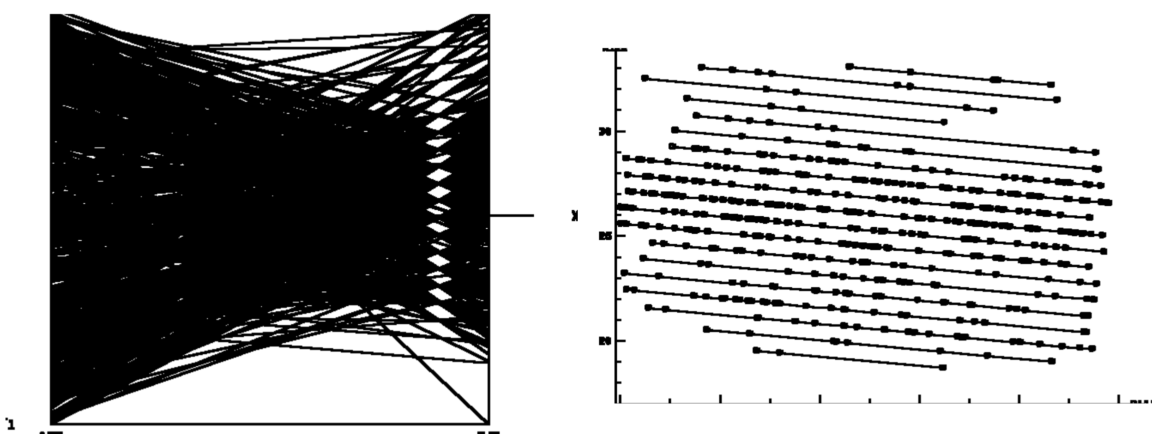


Figure 4: Portion between the R111 and R112 axes in || and cartesian coords. A parameter linearly related to R111 and R112 had not been measured and was discovered as a result of these plots.

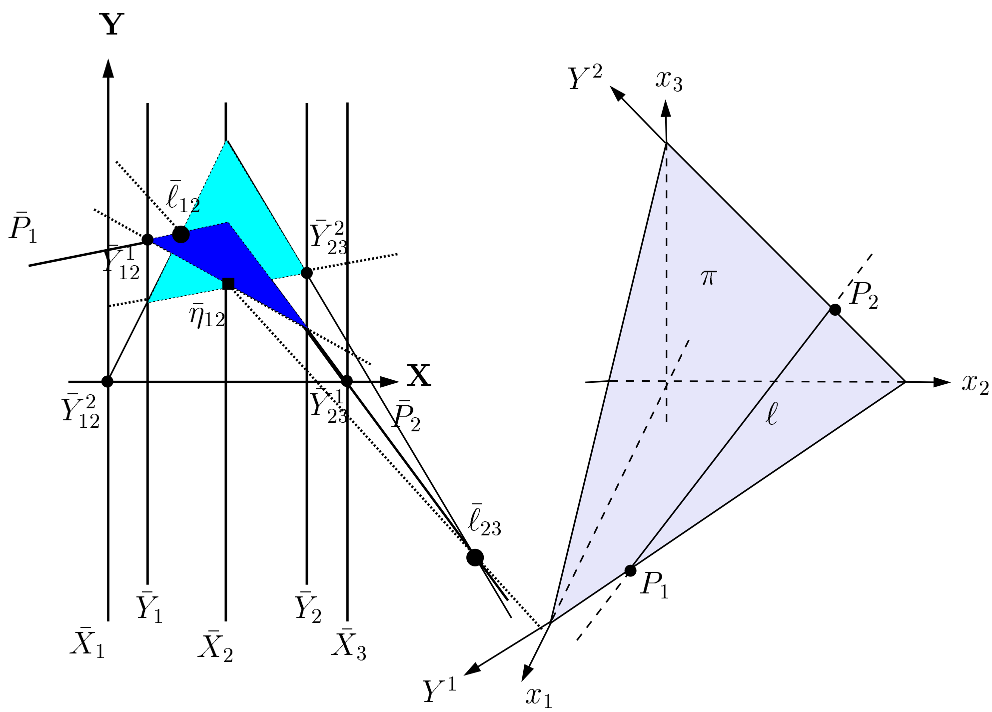


Figure 5: A line ℓ on a plane π is represented by one point $\bar{\eta}_{12}$ in terms of the coordinates (i.e. line in 2-D $\rightarrow$) point in $\bar{Y}_1$ and $\bar{Y}_2$ which is collinear with the two point $\bar{\ell}_{12}$ and $\bar{\ell}_{23}$. This is a consequence of Desargues projective geometry theorem.

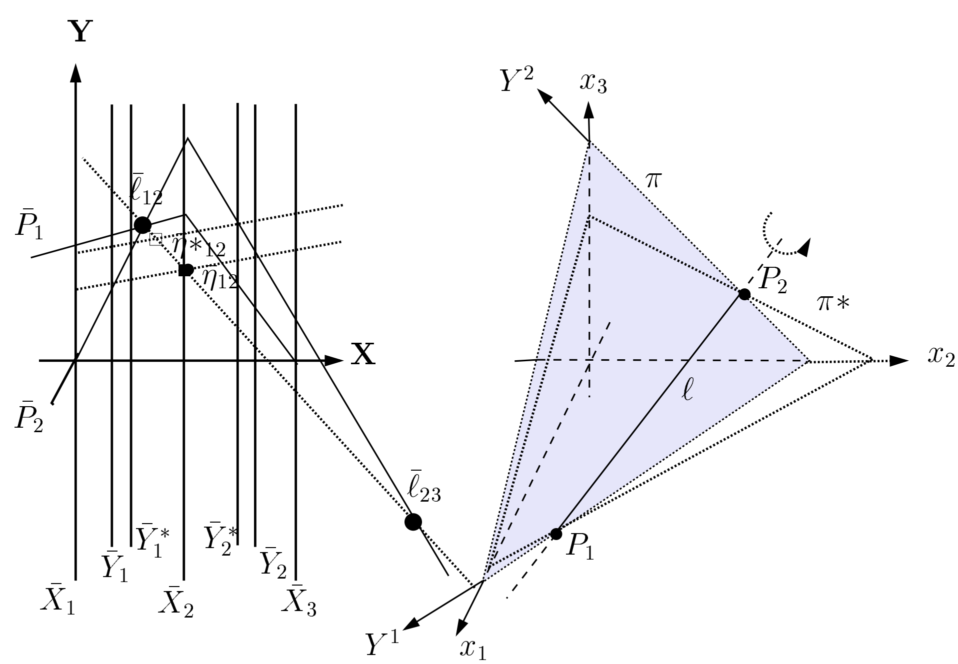


Figure 6: Rotation of a plane about a line $\leftrightarrow$ Translation of a point along a line.

The family or *pencil* of planes :

$$\pi : (x_3 - m_3 x_2 - b_3) + k(x_2 - m_2 x_1 - b_2) = 0 \; , \qquad (1)$$

is on the line:

$$\ell : \begin{cases} \ell_{12} & : \;\; x_2 = m_2 x_1 + b_2 \\ \ell_{23} & : \;\; x_3 = m_3 x_2 + b_3 \;\; . \end{cases} \qquad (2)$$

each value of k determines a(the rotated) plane and, in turn, the translated position $\bar{\eta}_{12}$:

$$\bar{\eta}_{12} = (\frac{m_3^2 - 2m_3 - k^2}{m_3^2 - m_3 + k^2(m_2 - 1)} \;\; , \;\; - \frac{b_2 k^2 + m_3 b_3}{m_3^2 - m_3 + k^2(m_2 - 1)}) \qquad (3)$$

The above generalize to $\mathbb{R}^N$ where a hyperplane being represented by $N-1$ vertical lines.

Representation by Indexed Points

The family of "Super-Planes" $\mathcal{E}$

We consider the set of points $P \in \mathbb{R}^N$ whose representation in $||$-coords collapses to a straight line. They form a 2-D subspace (2-flat) That is, $\bar{P} \;\; : \;\; y = mx + b$ and for each choice of (m, b) the corresponding point is :

$$P = (md_1 + b, \; md_2 + b, \ldots, md_N + b) = m(d_1, \; d_2, \ldots, d_N) + b(1, \ldots, 1) \; . \quad (4)$$

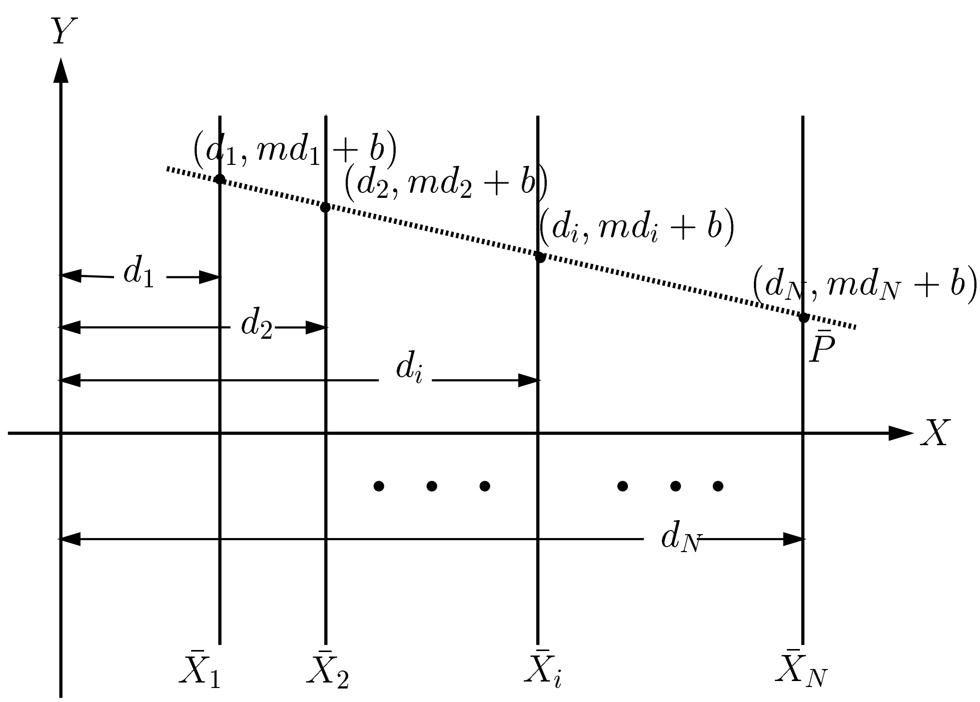


Figure 7: Points in $\mathbb{R}^N$ represented by lines.

Therefore, the super-planes (abbr.*sp*) are on the line u containing the points $(0,0,...,0),(1,1,...,1)$. They can be described in terms of the axes spacing and for $\mathbb{R}^3$ the *sp* are given by:

$$\pi^s : (d_3 - d_2)x_1 + (d_1 - d_3)x_2 + (d_2 - d_1)x_3 = 0 \tag{5}$$

For the standard axes spacing used so far, $d_1 = 0, d_2 = 2, d_3 = 2$ the corresponding, called the first, *sp* is :

$$\pi_1^s : x_1 - 2x_2 + x_3 = 0 \tag{6}$$

For a plane

$$\pi \ : \ c_1x_1 + c_2x_2 + c_3x_3 = c_o \ , \tag{7}$$

$$\ell_\pi = \pi \cap \pi_1^s \ : \ \begin{cases} \ell_{\pi_{12}} : \ x_2 = -\frac{c_1 - c_3}{c_2 + 2c_3}x_1 + \frac{c_o}{c_2 + 2c_3} \\ \\ \ell_{\pi_{23}} : \ x_3 = -\frac{2c_1 + c_2}{c_3 - c_1}x_2 + \frac{c_o}{c_3 - c_1}. \end{cases} , \tag{8}$$

These two points representing ℓ_π coincides since it is a line in a *sp*, and in homogeneous coordinates

$$\bar{\pi}_{123} = \bar{\ell}_{\pi_{12}} = \bar{\ell}_{\pi_{23}} = (c_2 + 2c_3, \ c_o, \ c_1 + c_2 + c_3) \ . \tag{9}$$

This is the first indexed point for π. To understand its significance follow the next two figures.

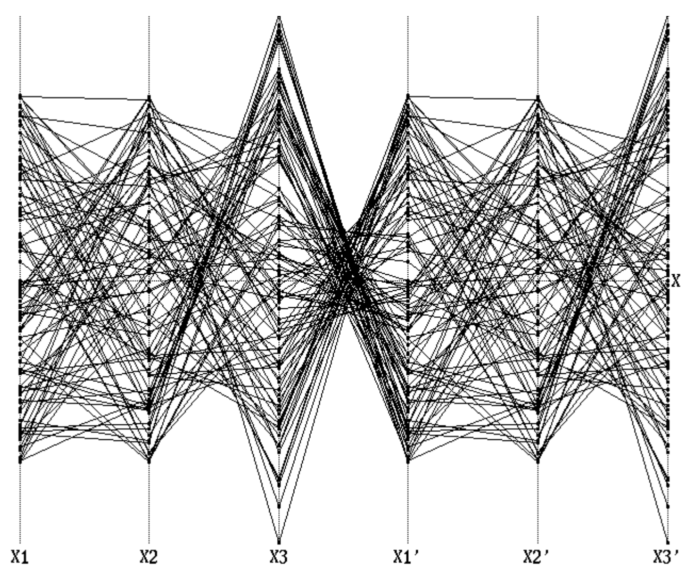


Figure 8: On the first 3 axes a set of polygonal lines representing a randomly sampled set of points on a plane $\pi \subset \mathbb{R}^3$.

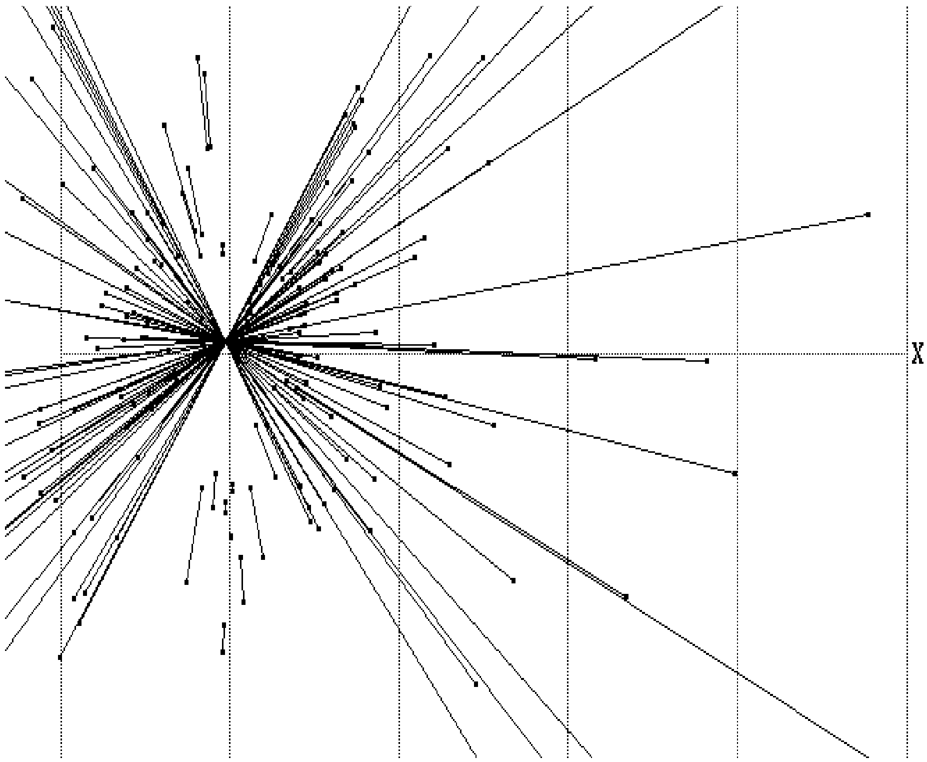


Figure 9: Coplanarity! In $\|$-coords joining the pairs of points representing lines on a plane forms a pencil of lines on a point. The point shown is $\bar{\pi}_{123}$ in eq. (9). Review also the 3-point-collinearity for multidimensional lines (previous chapter).

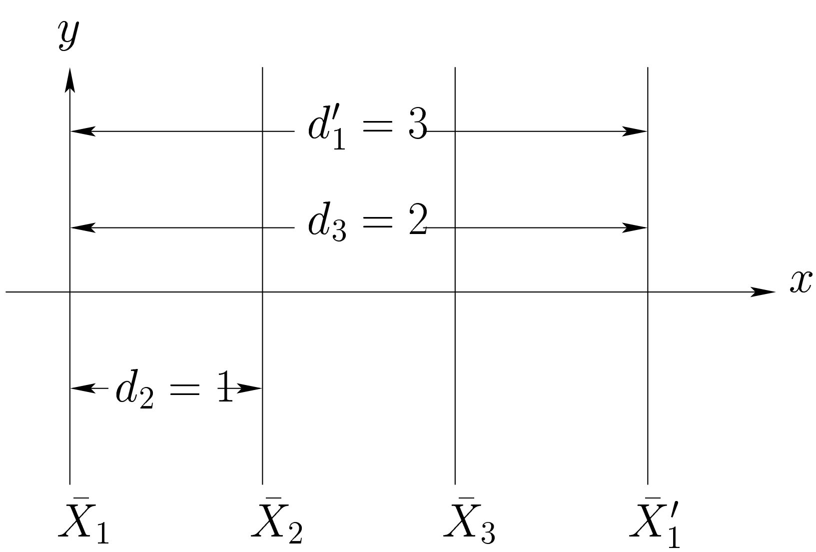


Figure 10: The axes spacing for the second super-plane $\pi^s_{1'}$.

Next the axis $\bar{X}_1$ is translated to the position $\bar{X}'_1$ one unit to the right of the $\bar{X}_3$ providing the new axes spacing $d_1 = 4, d_2 = 1, d_3 = 2$. The corresponding sp is

$$\pi^s_{1'} : x_1 + x_2 - 2x_3 = 0 \; . \tag{10}$$

The x_1 values of the coplanar points shown in Fig. 8 are transfered to the $\bar{X}_{1'}$ – see Fig.

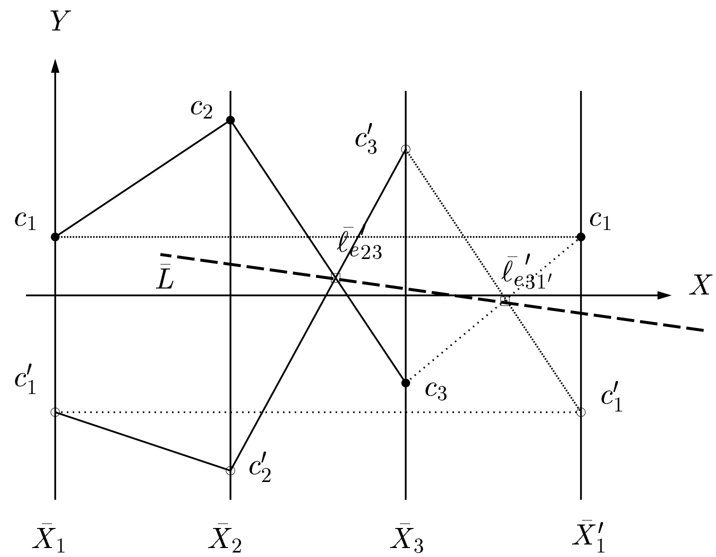


Figure 11: Transferring the values from the $\bar{X}_1$ to the $\bar{X}_{1'}$-axis.

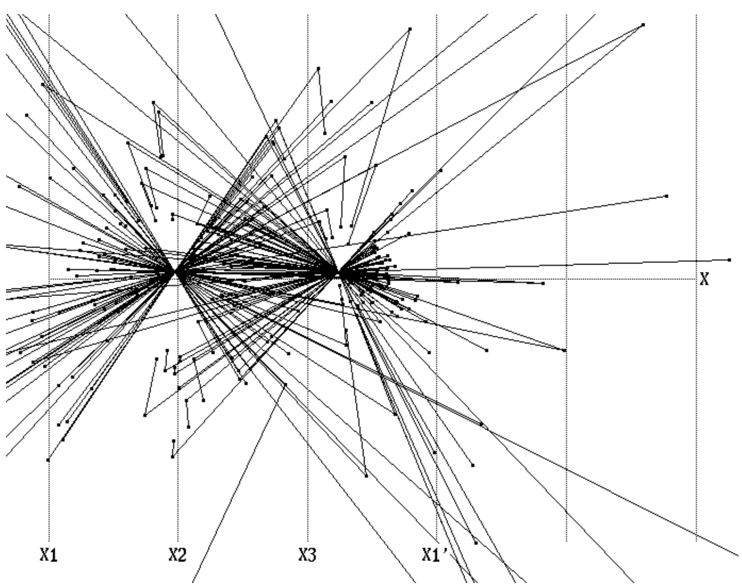


Figure 12: The plane π represented by two points

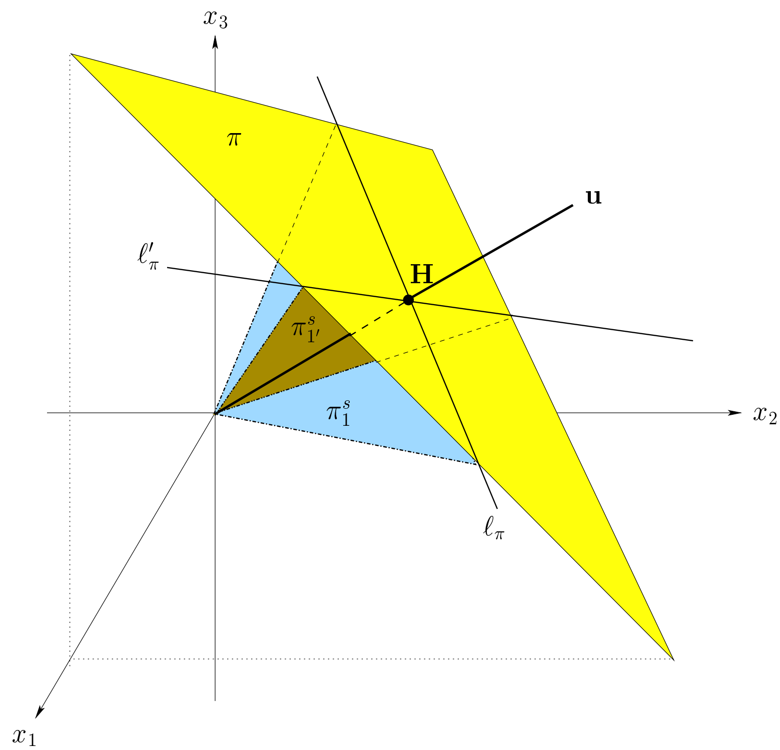


Figure 13: The intersections of a plane π with the two super-planes ${\pi^s}_1$ and $\pi^s_{1'}$ are two lines ℓ_π , ℓ'_π which specify the plane and provide its representation. This is the equivalent of the previous figure but in cartesian coordinates.

11– and the construction in Fig. 9 is repeated providing the second point

$$\bar{\pi}_{231'} = \bar{\ell'}_{\pi_{1'2}} = \bar{\ell'}_{\pi_{23}} = (3c_1 + c_2 + 2c_3,\ c_o,\ c_1 + c_2 + c_3). \tag{11}$$

shown Fig. 12. These two points represent the plane π since from their coordinates the coefficients of eq. (7). Geometrically, we have determined the plane π by the two lines ℓ_π, $\ell'_\pi \subset \pi$ shown in Fig. 13. A plane in $\mathbb{R}^3$ can be specified in terms of any two intersecting lines it contains. The reason for choosing the lines in the *sp* is that in $\|$-coords such lines are represented by **one** rather than two points and there are further advantages. Note that

$$\bar{\pi}_{231'} - \bar{\pi}_{123} = (3c_1, 0, 0)\ . \tag{12}$$

The four Indexed Points

The $\bar{X}_2$ and $\bar{X}_3$ axes are each translated to positions $\bar{X}'_2$ and $\bar{X}'_3$ 3 units to the right providing the third

$$\pi^s_{1'2'} : -2x_1 + x_2 + x_3 = 0\ , \tag{13}$$

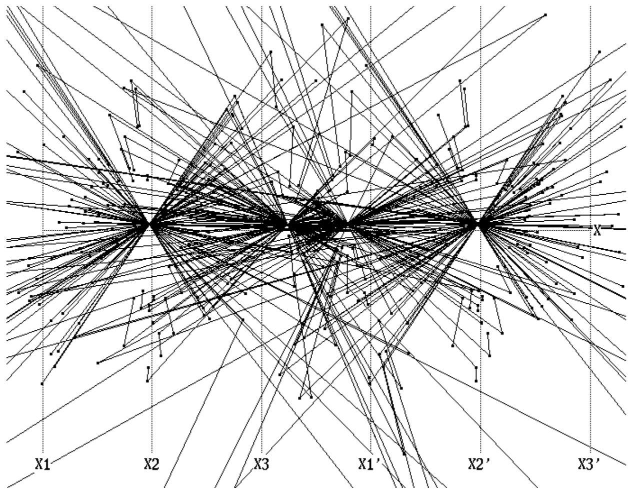


Figure 14: The plane π intersected with four super-planes. Each point represents one of the intersection lines.

and similarly the fourth sp $\pi^s_{1'2'3'}$. Two new points are constructed and shown in Fig. 14 As for the previous 2 points

$$\begin{cases} \bar{\pi}_{31'2'} - \bar{\pi}_{231'} = (3c_2, 0, 0) \\ \bar{\pi}_{31'2'} - \bar{\pi}_{1'2'3'} = (3c_3, 0, 0) \ . \end{cases} \tag{14}$$

It is easiy checked that the translations correspond to 120^o rotations of the sp π^s_1 about the line u on the points $(0,0,0),(1,1,1)$ with $\pi_{1'2'3'}$ coinciding with π^s_1. To simplify notation the index permutation is unimportant so that $\pi_{231'} = \pi_{1'23}$.

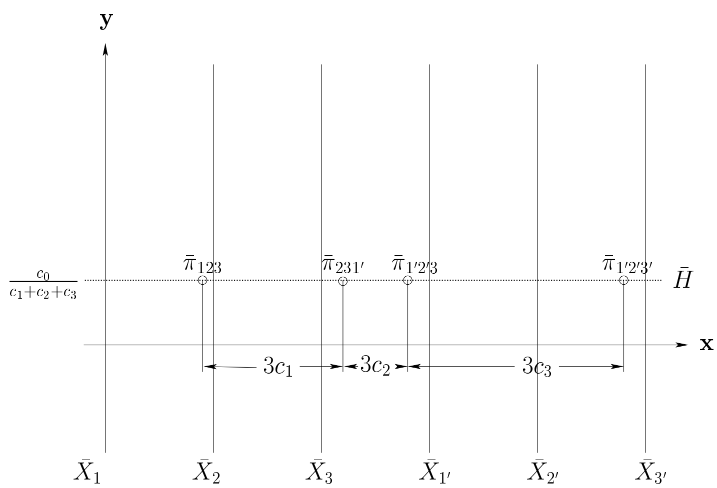


Figure 15: The distances between adjacent points are proportional to the coefficients of $\pi \ : \ c_1x_1 + c_2x_2 + c_3x_3 = c_0$. The proportionality constant is the dimensionality of the space. The plane's equation can be read from the picture!

Synthetic Constructions in $\mathbb{R}^3$

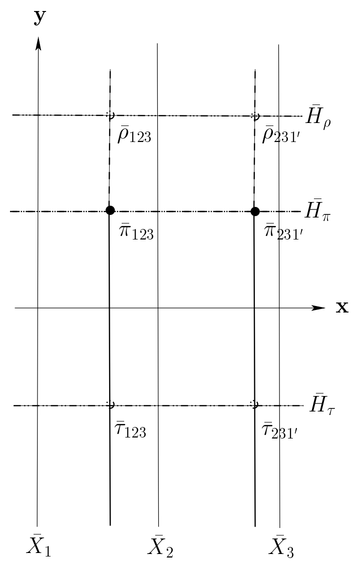


Figure 16: The parallel planes ρ, τ are above and below respectively the plane π whose upper half-space is marked by the two dashed half-lines.

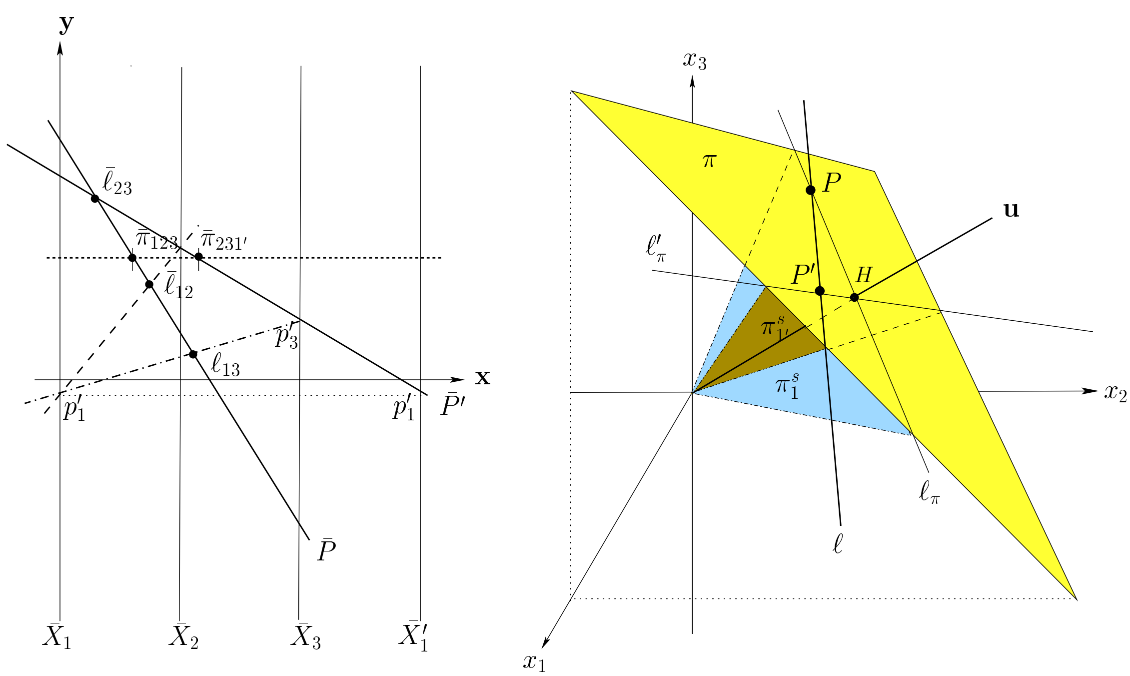


Figure 17: A line ℓ is contained in a plane $\pi \Leftrightarrow$ the points $\bar{\ell}_{12}, \bar{\ell}_{13}, \bar{\ell}_{23}, \bar{\pi}_{123}$ are on a line $\bar{P}$ and $\bar{\ell}_{1'2}, \bar{\ell}_{1'3}, \bar{\ell}_{23}, \bar{\pi}_{231'}$ are on a line $\bar{P}'$. Then $P = \ell_\pi \cap \pi$ and $P' = \ell_{\pi'} \cap \pi$.

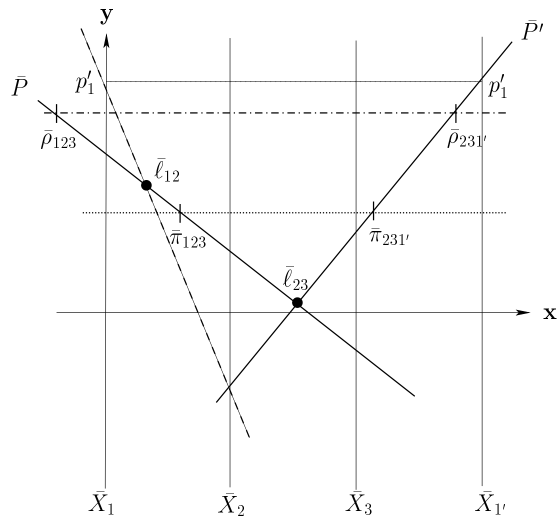


Figure 18: Two intersecting planes

Constructing the four indexed points

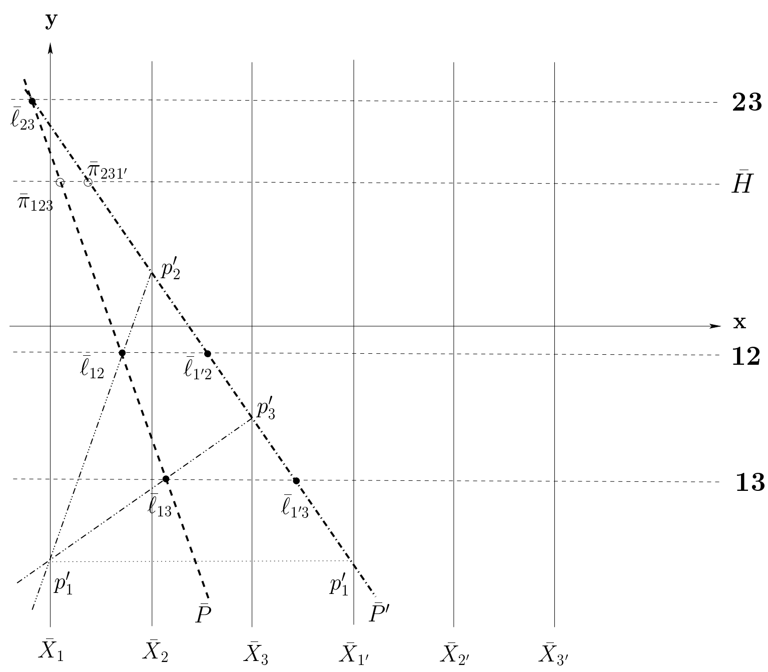


Figure 19: Representation of a plane $\pi \subset \mathbb{R}^3$ by two indexed points. First step in the construction of the points $\bar{\pi}_{32'1'}, \bar{\pi}_{1'2'3'}$ from $\bar{\pi}_{123}, \bar{\pi}_{231'}$. A (any) line $\ell \subset \pi$ is constructed as in Fig. 17. The points $\bar{\ell}_{12}, \bar{\ell}_{13}$ are constructed and the horizontal lines $\mathbf{12}, \mathbf{13}, \mathbf{23}$ are drawn on the $\bar{\ell}$ s with the corresponding indices.

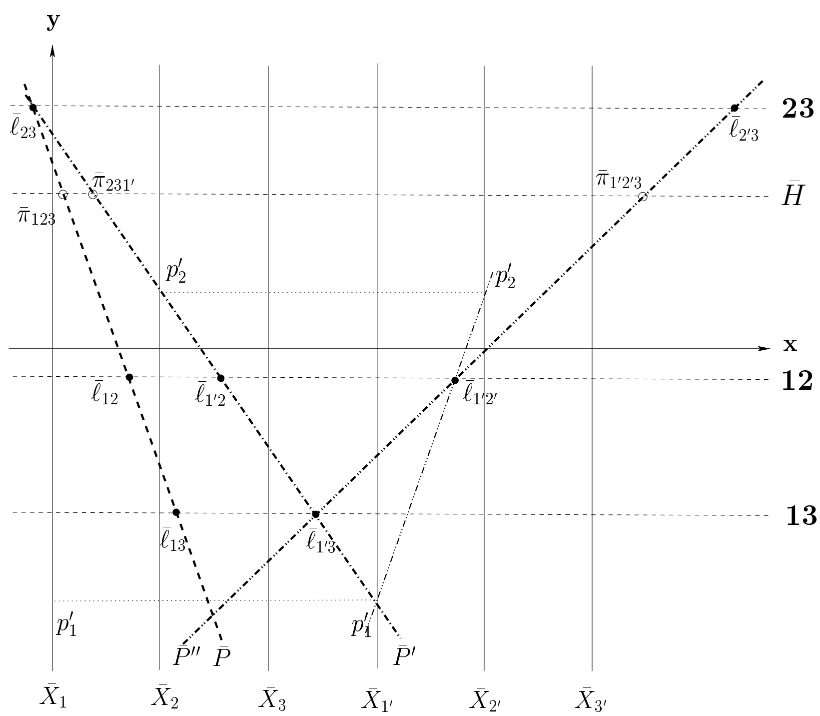


Figure 20: Construction of the third point $\bar{\pi}_{1'2'3}$.

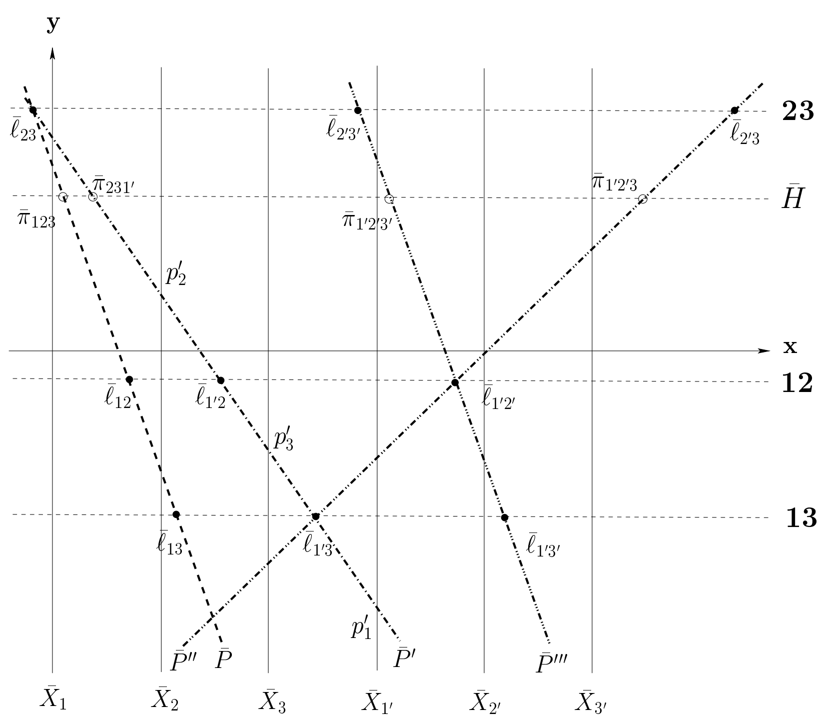


Figure 21: Construction of the fourth point $\bar{\pi}_{1'2'3'}$. Note that $\bar{P}'''$ on $\bar{\ell}_{1'2'}$ is parallel to $\bar{P}$ since P''' is the point $P = \pi \cap \ell_\pi$ in the $\bar{X}_{1'}, \bar{X}_{2'}, \bar{X}_{3'}$ axes.

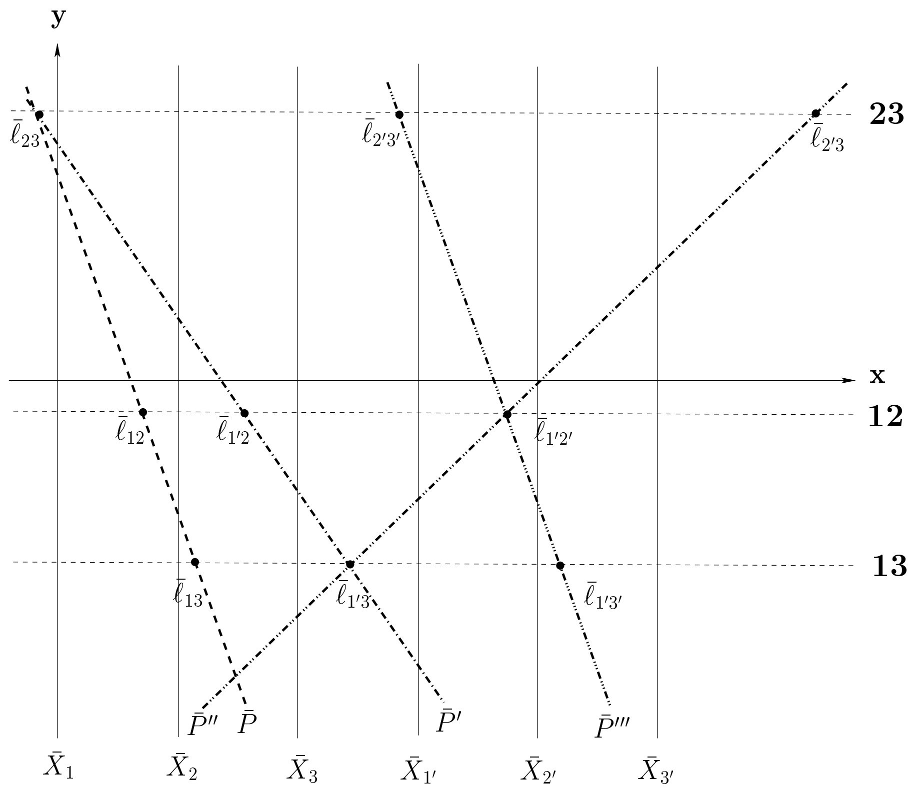


Figure 22: The collinear triples of points $\bar{\ell}$ corresponding to a line ℓ as they appear in each of the four coordinate systems $\bar{X}_1, \bar{X}_2, \bar{X}_3$ through $\bar{X}_{1'}, \bar{X}_{2'}, \bar{X}_{3'}$.

Special Planes

Principal 2-planes

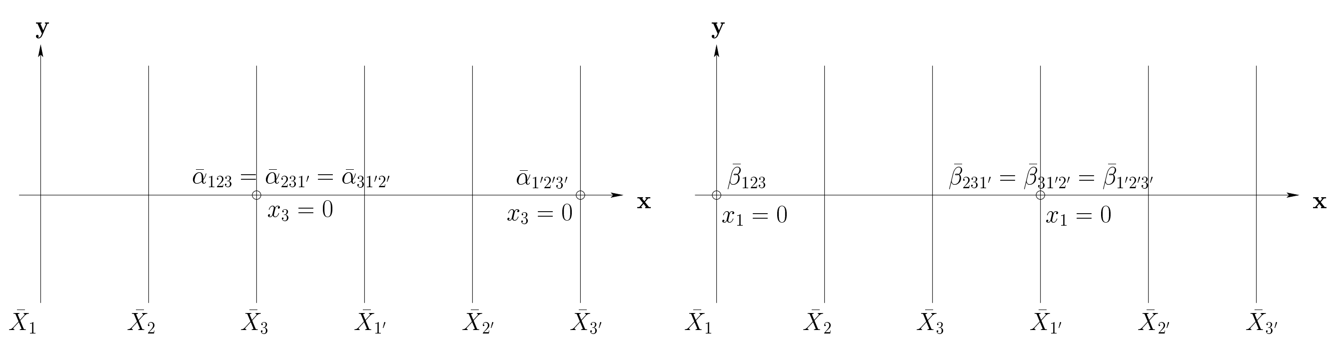


Figure 23: Indexed points corresponding to $\alpha : x_3 = 0$ the x_1x_2 principal 2-plane on the left and for the x_2x_3 principal 2-plane $\beta : x_1 = 0$ on the right.

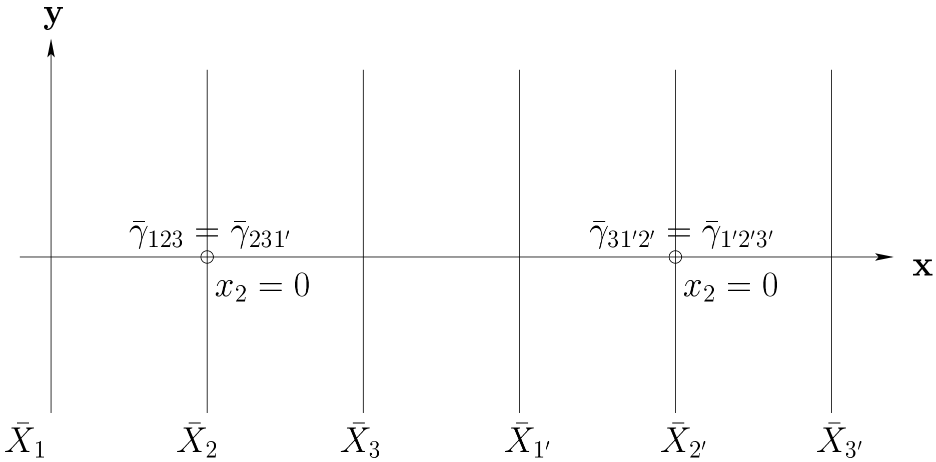
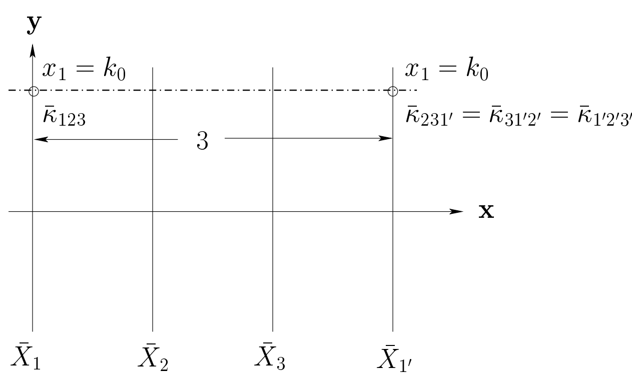


Figure 24: Indexed points corresponding to $\gamma : x_2 = 0$ the principal 2-plane x_1x_3.

The constant planes



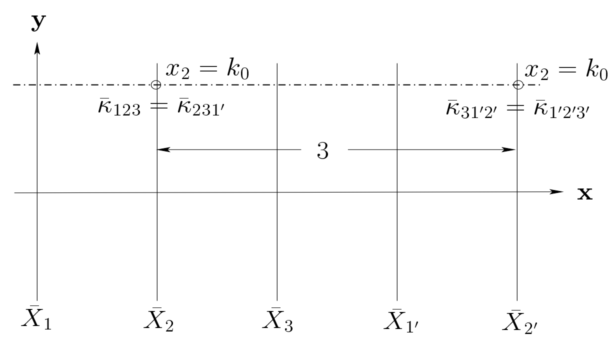


Figure 25: Indexed points representing the constant planes $\kappa : x_1 = k_0$ (left) and $\kappa : x_2 = k_0$ (right).

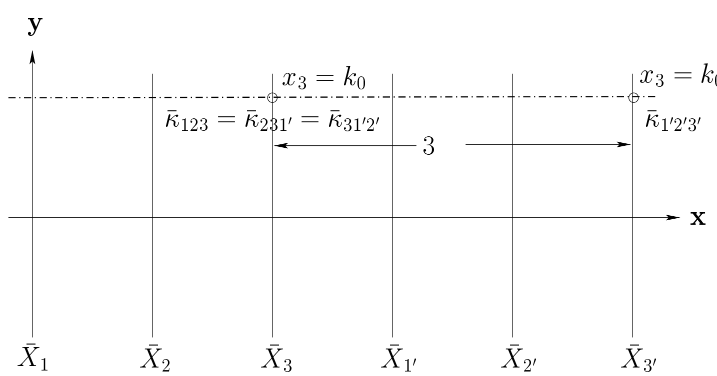


Figure 26: Representation of the plane $\kappa : x_3 = k_0$.

Projecting Planes and Lines

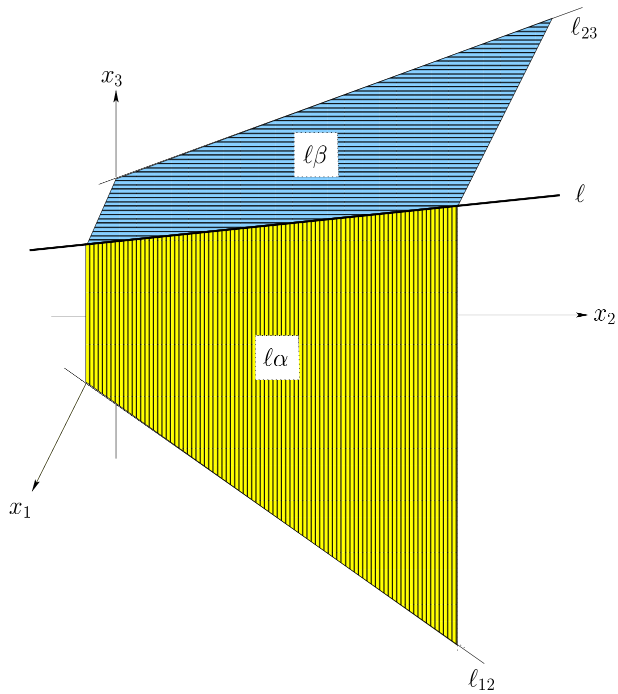


Figure 27: A line ℓ as the intersection of the projecting planes $\ell\alpha \perp \alpha$ and $\ell\beta \perp \beta$.

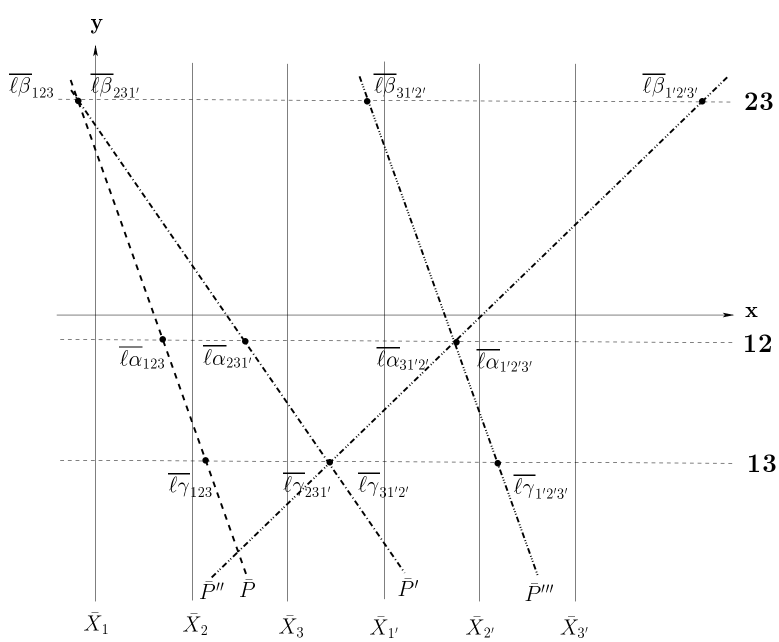


Figure 28: The projecting planes $\ell\alpha$, $\ell\beta$, $\ell\gamma$ the line ℓ whose $\bar{\ell}$ points are shown in Fig. 22.

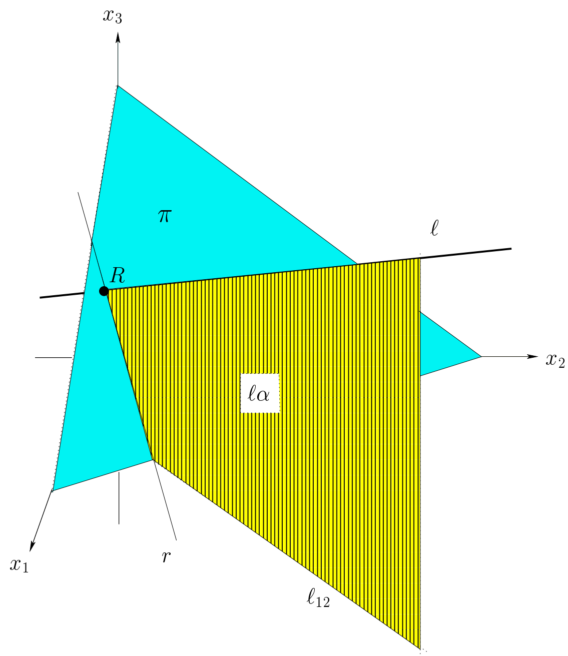


Figure 29: Finding $\ell \cap \pi$ using the projecting plane $\ell\alpha$.

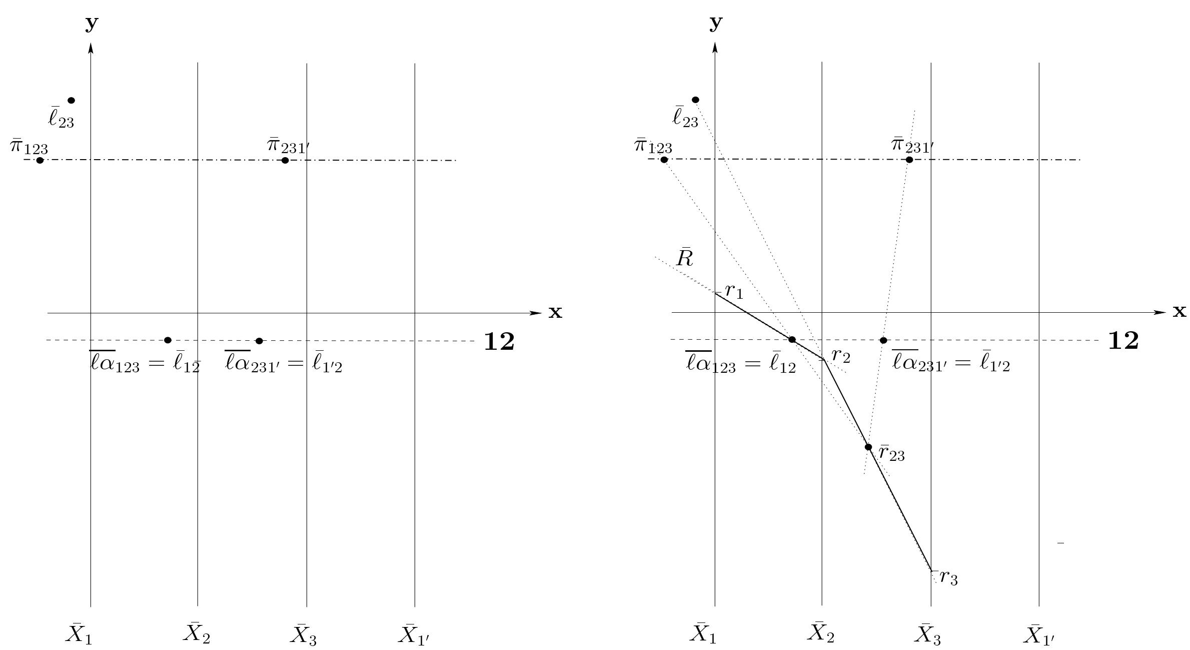


Figure 30: On the left are the initial data, are points specifying the plane π and line ℓ, for intersection construction on the right. First $r = \pi \cap \ell\alpha$ is constructed (need only $\bar{r}_{23}$ since $\bar{r}_{12} = \bar{\ell}_{12}$). Then $R = (r_1, r_2, r_3) = r \cap \ell = \pi \cap \ell$.

This is a geometrical/graphical solution of the system of linear equations

$$\left\{ \begin{array}{lll} \pi & : & c_1x_1 + c_2x_2 + c_3x_3 \quad = c_0 \\ \ell_{1,2} & : & c_{11}x_1 + c_{12}x_2 \qquad\quad = c_{10} \\ \ell_{2,3} & : & c_{22}x_2 + c_{23}x_3 \qquad\quad = c_{20} \end{array} \right. . \tag{15}$$

Separation in $\mathbb{R}^3$ – Points and Planes

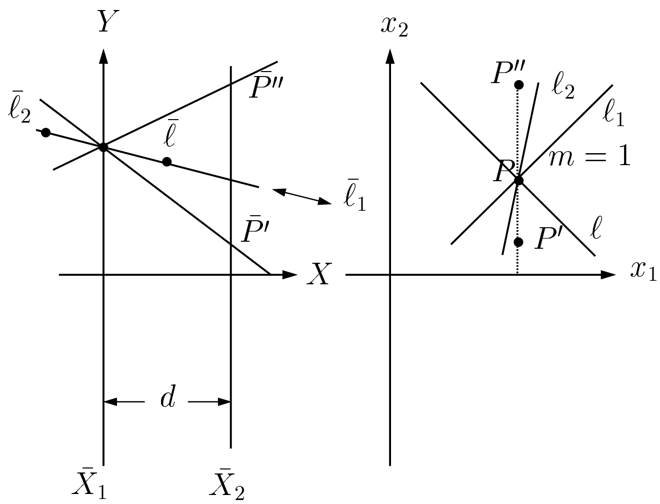


Figure 31: The "above" and "below" relations between points and lines switch at $m = 1$.

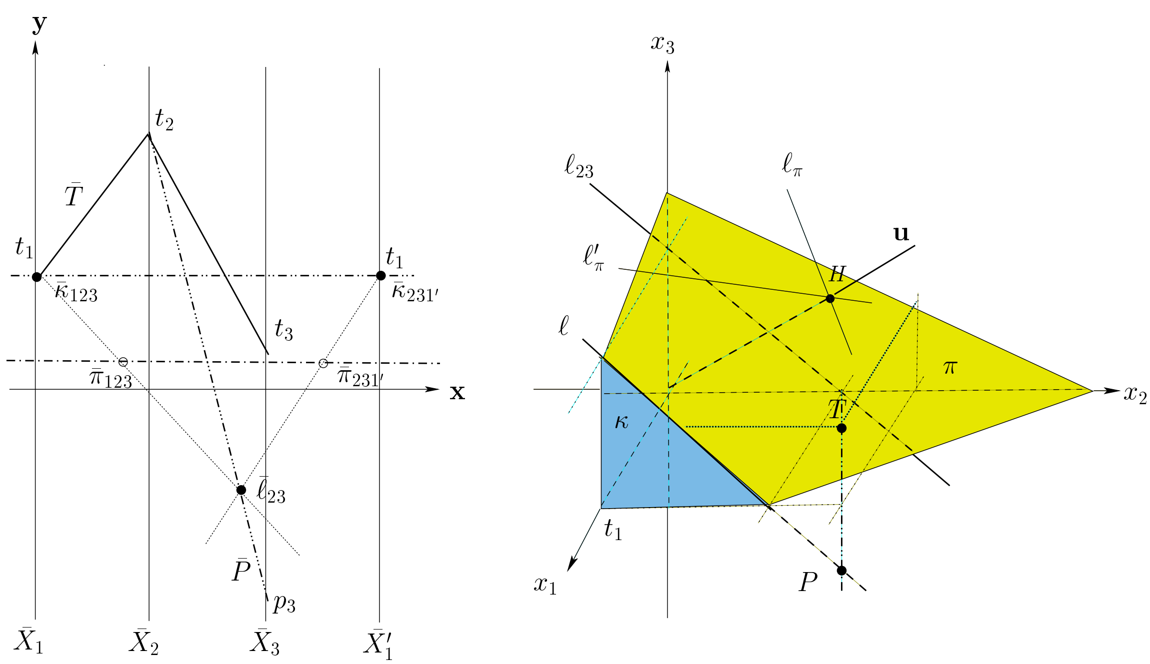


Figure 32: Point $T = (t_1, t_2, t_3)$ is above the plane π. Line $\ell = \pi \cap \kappa$ where $\kappa : x_1 = t_1$ with $P = (t_1, t_2, p_3) \in \ell \cap \pi$. With $\bar{\ell}_{23}$ between the $\bar{X}_2, \bar{X}_1$, i.e. the slope of ℓ_{23} is negative, and *below* the portion $\bar{T}_{23}$ of the polygonal line $\bar{T}$, T is *above* ℓ in the plane κ and also π. This is also clear from the picture since T, $P \in \kappa$ have the same x_1, x_2 coords and $p_3 < t_3$.

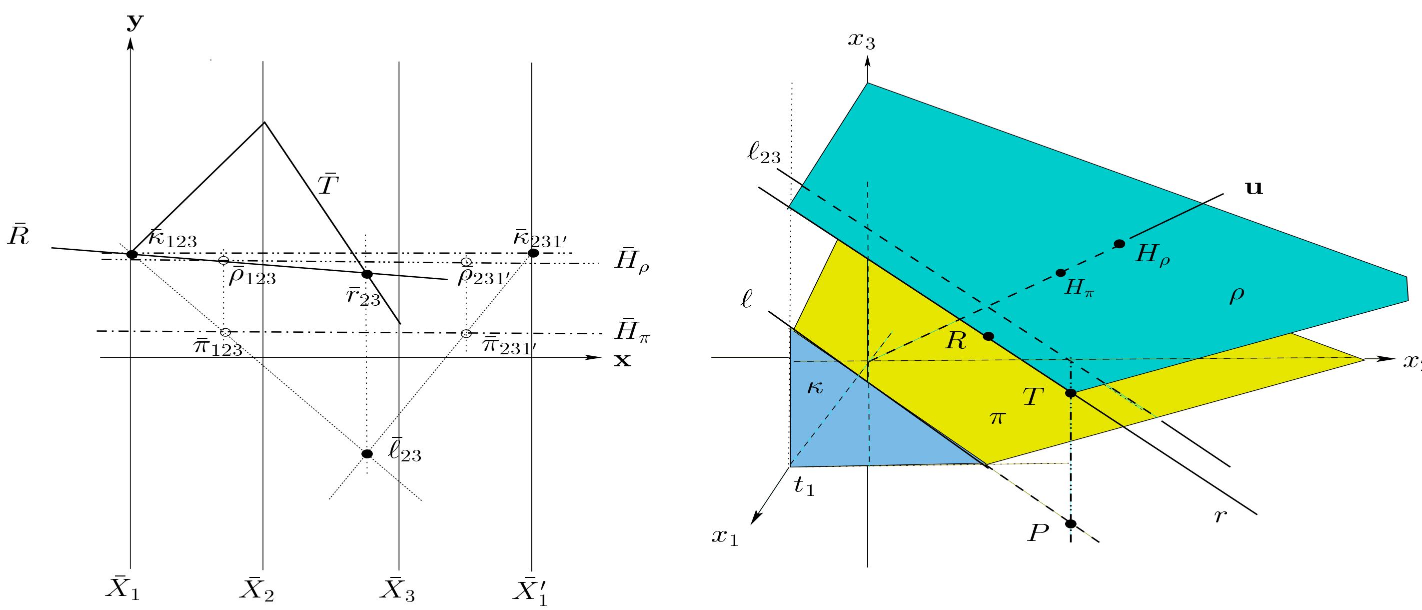


Figure 33: Point $T = (t_1, t_2, t_3)$ is above the plane π. Line $\ell = \pi \cap \kappa$ where $\kappa : x_1 = t_1$ with $P = (t_1, t_2, p_3) \in \ell \cap \pi$.

Rotation of a Plane about a Line and the Dual Translation

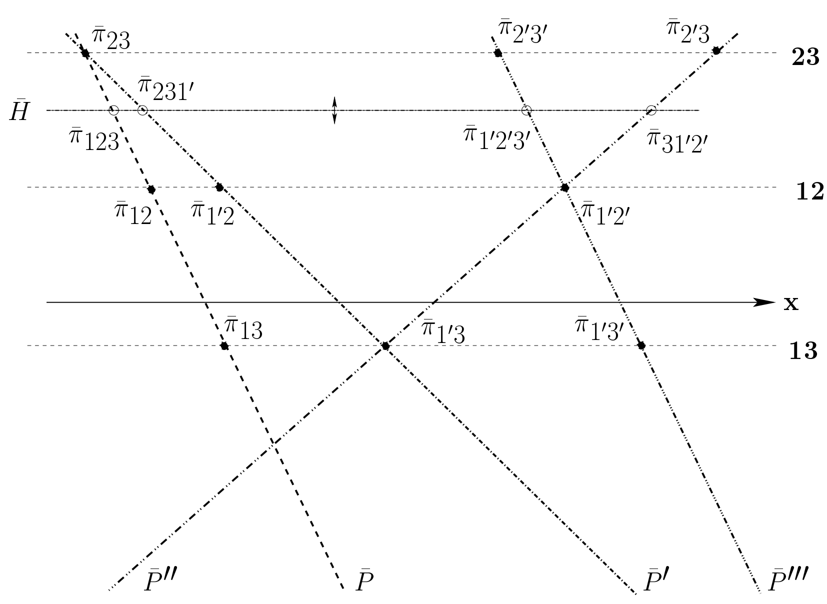


Figure 34: Rotation of a 2-flat (plane) about a 1-flat(line) in $\mathbb{R}^3$ corresponds to a translation of the points with 3 indices on the horizontal line $\bar{H}$ along the lines $\bar{L}$, $\bar{L}'$, $\bar{L}''$, $\bar{L}'''$ joining the points with 2 indices.

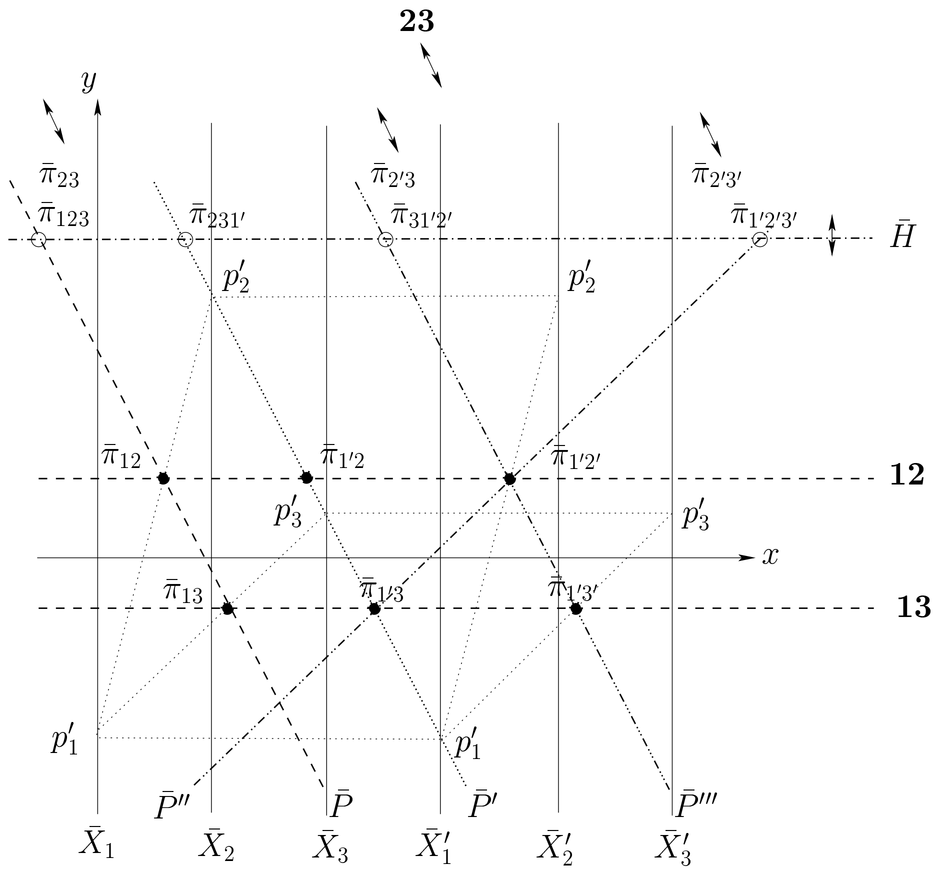


Figure 35: Rotation of a plane π^2 about a line π^1 such that c_1 remains constant.

Hyperplanes and p-flats in $\mathbb{R}^N$

Designate the axes spacing by $\mathbf{d} = (d_1, d_2, ..., d_i, ..., d_N)$ and recall that the super-planes *sp* are on the line

$$u : x_2 = x_1 \ , \ x_3 = x_2 \ , \ . \ . \ . \ , \ x_i = x_{i-1} \ , \ . \ . \ . \ , \ x_N = x_{N-1} \ . \tag{16}$$

Being 2-flats the *sp* are described by $N-2$ linearly independent equations which after some matrix manipulations can be written with 2 variables each as :

$$\pi^{Ns} : \begin{cases} \pi^{Ns}_{123} & : \quad (x_2 - x_1) + k_1(x_3 - x_2) = 0 \ , \\ \pi^{Ns}_{234} & : \quad (x_3 - x_2) + k_2(x_4 - x_3) = 0 \ , \\ & \ \ \ldots \\ \pi^{Ns}_{i(i+1)(i+2)} & : \quad (x_{i+i} - x_i) + k_i(x_{i+2} - x_{i+1}) = 0 \ , \\ & \ \ \ldots \\ \pi^{Ns}_{(N-2)(N-1)N} & : \quad (x_{N-i} - x_{N-i-1}) + k_{N-1}(x_N - x_{N-1}) = 0 \ . \end{cases} \tag{17}$$

The form of the equations stems from the fact that in the 3-dimensional subspace $x_{i-2}x_{i-1}x_i$, $\pi^{Ns}_{(i-2)(i-1)i}$ the equation for $\pi^{Ns}_{i(i+1)(i+2)}$ describes the pencil of 2-flats on the 1-flat (line) $x_{i-1} =$

$x_{i-2},\ x_i = x_{i-1}$. Pointing out that each such 2-flat contains the points $(1,1,1),\ (d_{i-2}, d_{i-1}, d_i)$ enables the elimination of the parameter k_i in the equations π^{Ns} and rewriting them in terms of the axes spacing as :

$$\begin{array}{rcl}
\pi^{Ns}(d_1,d_2,d_3) & : & (d_3-d_2)x_1+(d_1-d_3)x_2+(d_2-d_1)x_3=0\ , \\
\pi^{Ns}(d_2,d_3,d_4) & : & (d_4-d_3)x_2+(d_2-d_4)x_3+(d_3-d_2)x_4=0\ , \\
 & \cdots & \\
\pi^{Ns}(d_i,d_{i+1},d_{i+2}) & : & (d_{i+2}-d_{i+1})x_i+(d_i-d_{i+2})x_{i+1}+(d_{i+3}-d_i)x_{i+2}=0\ , \\
 & \cdots & \\
\pi^{Ns}(d_{N-2},d_{N-1},d_N) & : & (d_N-d_{N-1})x_{N-2}+(d_{N-2}-d_N)x_{N-1}+(d_{N-1}-d_{N-2})x_N=0\ .
\end{array} \tag{18}$$

The actual axes spacing used can be stated explicitly, as above rather, than subscripts when it is clear from the context. To get a better feel let's play around a bit in $\mathbb{R}^4$ with

$$\pi^{4s}:\left\{\begin{array}{ll} \pi^{4s}_{123} & :(d_3-d_2)x_1+(d_1-d_3)x_2+(d_2-d_1)x_3=0 \\ \pi^{4s}_{234} & :(d_4-d_3)x_2+(d_2-d_4)x_3+(d_3-d_2)x_4=0 \end{array}\right. . \tag{19}$$

Substituting $x_i = d_i, x_j = d_j$ yields the remaining two d_k, d_s for $i \neq j \neq k \neq s$ confirming that this is the correct 2-flat.

This is a good time to make some notational conventions[1] by defining the axes spacing recursively. Initially it is $\mathbf{d^0_N} = (0,1,2,\cdots,N)$ and after i successive translations, where the $\bar{X}_i$ axis is in the $\bar{X}_{i'}$ position,

$$\mathbf{d^i_N} = \mathbf{d^0_N} + (\overbrace{N,N,\cdots,N}^{i-1},0,\cdots,0) = (d_{i1}, d_{i2},\ldots,d_{ik},\ldots,d_{iN}). \tag{20}$$

When the dimensionality is clear from the context the subscript N can be omitted. For a flat π^p expressed in terms of the $\mathbf{d^i_N}$ spacing points $\bar{\pi}^p_{1',\cdots,i',i+1,\cdot,N}$ of its representation are denoted compactly by $\bar{\pi}^p_{i'}$ and it is consistent to write $\pi^p = \pi^p_0$ which we may do on occasion.

For the standard axis spacing $\mathbf{d} = (0,1,2,3)$ the *sp* is

$$\pi^{4s}_{1234}:\left\{\begin{array}{ll} \pi^{s}_{123} & :x_1-2x_2+x_3=0 \\ \pi^{s}_{234} & :x_2-2x_3+x_4=0 \end{array}\right. . \tag{21}$$

To emphasize the notational equivalence note that $\pi^{4s}(0,1,2) = \pi^{4s}_{123}$. The axes are translated, as in $\mathbb{R}^3$, to generate different *sp* corresponding to rotations about u which are summarized in Fig 36. First, the axis $\bar{X}_1$ is translated to position $\bar{X}_{1'}$ one unit to the right of $\bar{X}_4$ with the resulting the axes spacing $\mathbf{d} = (4,1,2,3)$ yielding

$$\pi^{4s}_{1'234}:\left\{\begin{array}{ll} \pi^{s}_{1'23} & :x_1+2x_2-3x_3=0 \\ \pi^{s}_{234} & :x_2-2x_3+x_4=0 \end{array}\right. . \tag{22}$$

The angle of rotation between π^{4s}_{123} and $\pi^{4s}_{1'23}$ computed via eq. (**??**) is $cos^{-1}\ (-\sqrt{3/7}) = 180^o - \phi\ ,\quad \phi = cos^{-1}\ (\sqrt{3/7}) \approx 49.1^o$. Note that since π^{s}_{234} remains unchanged this is not

[1] I am indebted to Liat Cohen for providing this notation.

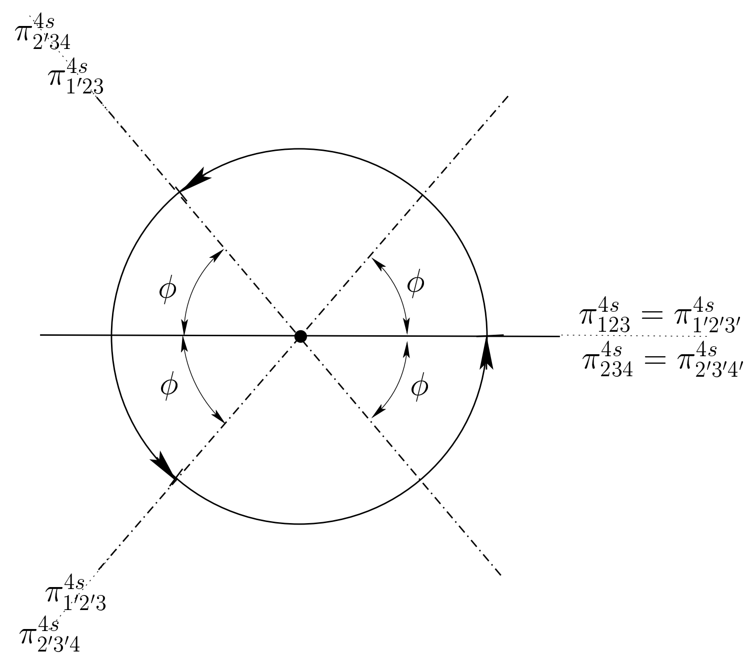


Figure 36: The rotations of $\pi^4 s$ about the line u. This is a a projection on a plane perpendicular to u and the 3-flats π^{4s}_{123} , π^{4s}_{234} whose projections are lines.

a complete rotation of π^{4s}_{1234} about the line u. Proceeding, with the translation of $\bar{X}_2$ to position $\bar{X}'_2$ one unit to the right of $\bar{X}_{1'}$ provides the axes spacing $\mathbf{d} = (4, 5, 2, 3)$ and the *sp*

$$\pi^s_{1'2'34} : \begin{cases} \pi^s_{1'2'3} & : 3x_1 - 2x_2 - x_3 = 0 \\ \pi^s_{2'34} & : x_2 + 2x_3 - 3x_4 = 0 \end{cases} . \tag{23}$$

The angle between $\pi^s_{1'23}$ and $\pi^s_{1'2'3}$ is $cos^{-1}(-1/7) = 2\phi$ while the angle between π^s_{234} and $\pi^s_{2'34}$ is $180^o - \phi$. Continuing by translating $\bar{X}_3$ to position $\bar{X}_{3'}$ one unit to the right of $\bar{X}'_2$ yields $\mathbf{d} = (4, 5, 6, 3)$ and

$$\pi^s_{1'2'3'4} : \begin{cases} \pi^s_{1'2'3'} & : x_1 - 2x_2 + x_3 = 0 \\ \pi^s_{2'3'4} & : 3x_2 - 2x_3 - x_4 = 0 \end{cases} . \tag{24}$$

returning $\pi^s_{1'2'3'}$ to its original position π^s_{123} while bringing $\pi^s_{2'3'4}$ at an angle 2ϕ from $\pi^s_{2'34}$. Again this is not a complete rotation of the whole *sp* about the line u. The final translation of $\bar{X}_4$ to $\bar{X}_{4'}$ one unit to the right of $\bar{X}_3$ provides $\mathbf{d} = (4, 5, 6, 7)$ and

$$\pi^s_{1'2'3'4'} : \begin{cases} \pi^s_{1'2'3'} & : x_1 - 2x_2 + x_3 = 0 \\ \pi^s_{2'3'4'} & : x_2 - 2x_3 + x_4 = 0 \end{cases} . \tag{25}$$

which is identical to π^s_{1234}. Unlike $\mathbb{R}^3$ the rotations angles are not all equal though the sum is a full circle. The "anomaly" suggest that $\phi(N)$ is a function of the dimensionality N with $\phi(3) = 60^o$ and $\phi(4) \approx 49.1^o$. It is interesting to investigate that. For $\mathbb{R}^N$ we look at π^{Ns}_{123} with $\mathbf{u} = (0, 1, 2, ..., N-2, N-1)$ and after the translation of $\bar{X}_1$ to position $\bar{X}_{1'}$ one unit to the right of $\bar{X}_N$ the axes spacing is $\mathbf{u} = (N, 1, 2, ..., N-2, N-1)$ yielding respectively the two corresponding $\pi^{Ns}(d_1, d_2, d_3)$:

$$\begin{cases} \pi^{Ns}_{123} & : x_1 - 2x_2 + x_3 = 0 \;, \\ \pi^{Ns}_{1'23} & : x_1 + (N-2)x_2 + (1-N)x_3 = 0 \;. \end{cases} \tag{26}$$

The angle function is

$$\phi(N) = cos^{-1}\left(\frac{\sqrt{3}(N-2)}{2\sqrt{3-3N+N^2}} \right) \;, \tag{27}$$

some of values are $\phi(5) \approx 47.88^o$, $\phi(6) = 45^o$ and the $\lim_{N\to\infty}\phi(N) = 30^o$. Next let us compute the one-flat (line) intersection $\ell_\pi = \pi^{4s} \cap \pi$ where the plane $\pi \subset \mathbb{R}^4$. These will provide the index point representation for π representing the one-flats :

$$\begin{cases} \pi & : c_1x_1 + c_2x_2 + c_3x_3 + c_4x_4 = c_o \;, \\ \pi^{4s}_{123} & : (d_3 - d_2)x_1 + (d_1 - d_3)x_2 + (d_2 - d_1)x_3 = 0 \\ \pi^{4s}_{234} & : (d_4 - d_3)x_2 + (d_2 - d_4)x_3 + (d_3 - d_2)x_4 = 0. \end{cases} \tag{28}$$

$$\begin{cases} a = c_1(d_2 - d_1) + c_3(d_2 - d_3) + c_4(d_2 - d_4) \;, \\ b = c_2(d_2 - d_1) + c_3(d_3 - d_1) + c_4(d_4 - d_1) \end{cases} \tag{29}$$

$$\ell_\pi : x_2 = -\frac{a}{b}x_1 + \frac{(d_2 - d_1)}{b}c_o \;. \tag{30}$$

Since $\ell_\pi \subset \pi^{4s}_{1234}$, $\bar{\ell}_{\pi_{ij}} = \bar{\ell}_{\pi_{kr}}$ for distinct indices $i, j, k, r \in (1, 2, 3, 4)$ and in homogeneous coordinate

$$\bar{\ell}_{\pi_{12}} = \left(\; (d_2 - d_1)b + d_1 \textstyle\sum_{i=1}^4 c_i \;, (d_2 - d_1)c_o \;, \sum_{i=1}^4 c_i) \; \right) = (\displaystyle\sum_{i=1}^{4} c_i d_i \;, c_o) \;, \sum_{i=1}^{4} c_i) \;. \tag{31}$$

From these the "generic" indexed points which provide the representation for a $\pi \subset \mathbb{R}^4$ for the axes spacing $\mathbf{d}$ we write $\bar{\pi}(\mathbf{d})$ are obtained. After performing the standard translations $d_i \to d_i + N$ we have and take the differences we obtain :

$$\begin{cases} \bar{\pi}_{1'234} - \bar{\pi}_{1234} & = (4c_1 \;, 0 \;, 0) \;, \\ \bar{\pi}_{1'2'34} - \bar{\pi}_{1'234} & = (4c_2 \;, 0 \;, 0) \;, \\ \bar{\pi}_{1'2'34} - \bar{\pi}_{1'23'4} & = (4c_3 \;, 0 \;, 0) \;, \\ \bar{\pi}_{1'2'3'4} - \bar{\pi}_{1'2'3'4} & = (4c_4 \;, 0 \;, 0) \;. \end{cases} \tag{32}$$

Indexed Points in $\mathbb{R}^N$

$\mathcal{R}$emarkably, the collinearity property (as in 2-D with the 3-pt-collinearity and 3-D with the collinearity of two pts with 2-indices and one of the pts representing a plane on the line see Fig. 9) generalizes to higher dimensions enabling the recursive (on the dimensionality) construction of the representation of p-flats for $2 \leq p \leq N-1$. To achieve this some intermediate steps are needed. The ss corresponding to the axes spacing $\mathbf{d^i_N}$ (i.e. obtained from the translation of the axes $\bar{X}_1, \ldots, \bar{X}_i$ to the positions $\bar{X}_{1'}, \ldots, \bar{X}_{i'}$ see 20) is denoted by $\pi_{i'}^{Ns}$.

Theorem: The 1-flat $\pi \cap \pi_{i'}^{Ns}$, where

$$\pi : \sum_{k=1}^{N} c_k x_k = c_0 \ , \tag{33}$$

is a hyperplane in $\mathbb{R}^N$ an $(N-1)$-flat, is represented by the point :

$$\bar{\pi}_{i'} = (\sum_{k=1}^{N} d_{ik} c_k \ , \ c_0 \ , \ \sum_{k=1}^{N} c_k) \tag{34}$$

where the d_{ik} are the inter-axes distances for the spacing $\mathbf{d_N^i}$ as given in eq. (20).

Corollary [Hyperplane Representation] The hyperplane π given by eq. (33) is represented by the $N-1$ points $\bar{\pi}_{i'}$, given by eq. (34), for $i = 0, 1, 2, \ldots, (N-2)$.

A p-flat in $\mathbb{R}^N$ is specified by $N-p$ linearly independent linear equations which, without loss of generality, can be of the form:

$$\pi^p : \begin{cases} \pi^p_{12\cdots(p+1)} : c_{11}x_1 & + \ldots + \quad c_{(p-1)1}x_p + c_{p1}x_{p+1} = c_{10} \\ \pi^p_{23\cdots(p+2)} : c_{22}x_2 & + \ldots + \quad c_{p2}x_{p+1} + c_{(p+1)2}x_{p+2} = c_{20} \\ & \ldots \\ \pi^p_{j\cdots(p+j)} \ : c_{jj}x_j & + \ldots + \quad c_{(p+j-1)j}x_{p+j-1} + c_{(p+j)j}x_{p+j} = c_{j0} \\ & \ldots \\ \pi^p_{(N-p)\cdots N} : c_{(N-p)(N-p)}x_{N-p} & + \ldots + \quad c_{(N-1)(N-p)}x_{N-1} + c_{N(N-p)}x_N = c_{(N-p)0} \end{cases}$$

and is rewritten compactly as

$$\pi^p \ : \ \{\pi^p_{j\ldots(p+j)} \ : \ \sum_{k=i}^{p+j} c_{jk}x_k = c_{j0} \ \ , \ \ j = 1, 2, \ldots, (N-p)\}. \tag{35}$$

A p-flat $\pi^p \subset \mathbb{R}^N$ is the intersection of $N-p$ hyperplanes and eq. 35 is the analogue of the "adjacent-variable" description for lines in Chapter **??** the indexing being a direct extension. Unless otherwise specified, a p-flat is described by eq. (35) with the standard spacing $\mathbf{d_N^0}$.

Theorem A p-flat in $\mathbb{R}^N$ given by eq. (35) is represented by the $(N-p)p$ points :

$$\bar{\pi}^p_{\{j\ldots(p+j)\}_{i'}} = (\sum_{k=1}^{p+1} d_{ik}c_{jk}, \ c_{j0}, \ \sum_{k=1}^{p+1} c_{jk}), \tag{36}$$

where $j = 1, 2, \ldots, N-p, \quad i = 1, 2, \ldots, p$ and the d_{ik} are the distances specified by the axes spacing $\mathbf{d_N^i}$.

To clarify, a hyperplane π in $\mathbb{R}^4$ π (i.e. 3-flat) can be represented by the three points

$$\bar{\pi}_{1234}, \ \bar{\pi}_{1'234}, \ \bar{\pi}_{1'2'34} \ . \tag{37}$$

For a 2-flat π^2, $p = 2$, $N = 4$, $p(N-p) = 4$ and it can be represented by the four points :

$$\pi^2_{123} : \bar{\pi}^2_{123},\ \bar{\pi}^2_{1'23}\ ; \quad \pi^2_{234} : \bar{\pi}^2_{234},\ \bar{\pi}^2_{2'34}. \tag{38}$$

Similarly in $\mathbb{R}^5$, a hyperplane π is represented by the four, a 3-flat π^3 and a 2-flat π^2 by six points each i.e.

$$\begin{cases} \pi \quad : \bar{\pi}_{12345}\ ,\ \bar{\pi}_{1'2345}\ ,\ \bar{\pi}_{1'2'345}\ ,\ \bar{\pi}_{1'2'3'45} \\ \\ \pi^3 \quad : \begin{cases} \pi^3_{1234} & : \bar{\pi}^3_{1234}\ ,\ \bar{\pi}^3_{1'234}\ ,\ \bar{\pi}^3_{1'2'34} \\ \pi^3_{2345} & : \bar{\pi}^3_{2345}\ ,\ \bar{\pi}^3_{2'345}\ ,\ \bar{\pi}^3_{2'3'45} \end{cases} \\ \\ \pi^2 \quad : \begin{cases} \pi^2_{123} & : \bar{\pi}^2_{123}\ ,\ \bar{\pi}^2_{1'23} \\ \pi^2_{234} & : \bar{\pi}^2_{234}\ ,\ \bar{\pi}^2_{2'34} \\ \pi^2_{345} & : \bar{\pi}^2_{345}\ ,\ \bar{\pi}^2_{3'45}\ . \end{cases} \end{cases} \tag{39}$$

In many instances it is possible to use simplified notation by just retaining the subscripts so that the three points in eq. (37) are referred by $1234, 1'234, 1'2'34$. The dimensionality is one less than the number of indices. Continuing, the points representing pi^3 in eq. (39) denoted simply by $1234, 1'234, 1'2'34$; $\quad 2345, 2'345, 2'3'45$; since there 2 sets of 3 points with 5 different indices altogether we can conclude that this is a 2-flat in $\mathbb{R}^5$. The simplified and the more formal notation are used interchangeably. This theorem unifies all previous results for p-flats π^p where $0 \leq p < N$. Recall that in Chapter on lines the representation of a point $P \in \mathbb{R}^N$, a 0-flat π^0, is also be given in terms of N points each with one index.

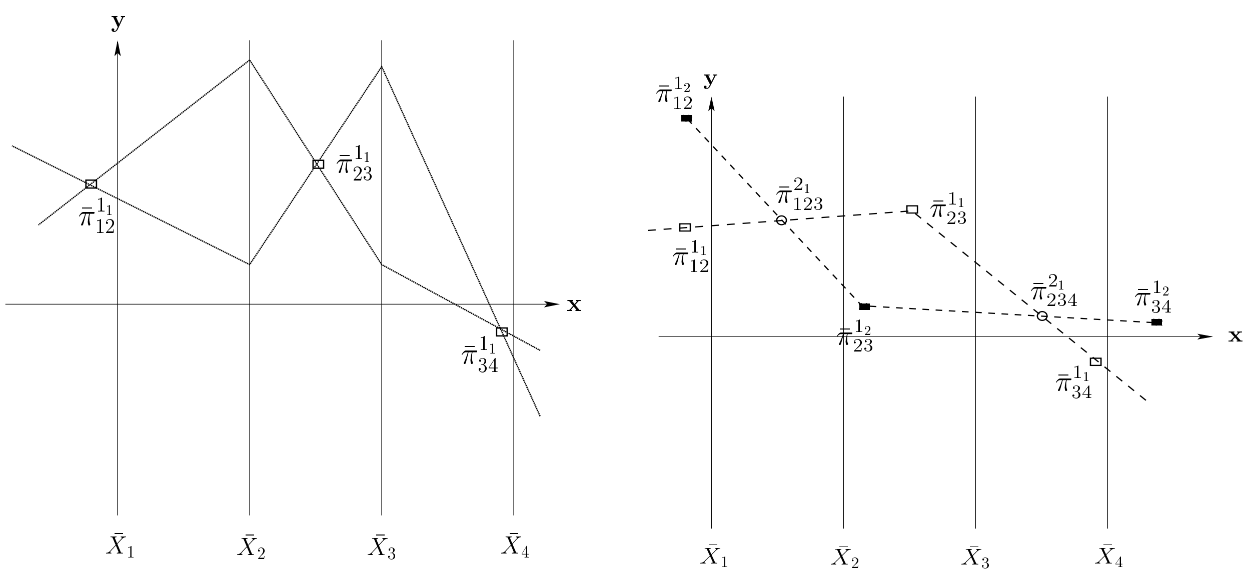


Figure 37: Recursive Construction in $\mathbb{R}^4$. A pair of points (polygonal lines) determines a line (1-flat) π^{1_1} represented by the 3 constructed points $\bar{\pi}^{1_1}_{i,i-1}$, $\quad i = 1, 2, 3, 4$ (left). The 1-flat π^{1_1} and another π^{1_2} (right),] represented by the 3 black points, determine a 2-flat (plane) π^{2_1} represented by the two points $\bar{\pi}^{2_1}_{123}$, $\bar{\pi}^{2_1}_{234}$. These points are the intersections of the two polygonal lines joining the points previously obtained representing the 1-flat π^{1_1}.

These points are simply the values of the coordinates p_i, $i \in [1, \ldots, N]$ of P. They can also be thought of as N equations:

$$x_i = p_i \ .$$

Let N_r be the number of points and n_r the number of indices appearing in the representation of a flat in $\mathbb{R}^N$ then for p-flat, $N_r = (N-p)p$, $n_r = p+1$ and

$$N_r + n_r = (N-p)p + (p+1) = -p^2 + p(N+1) + 1 \ . \tag{40}$$

Then in $\mathbb{R}^N$ for

1. $p = 0$ — points π^0 : $N_r + n_r = N + 1$,
2. $p = 1$ — lines π^1 : $N_r + n_r = (N-1) + 2 = N + 1$,
3. $p = 2$ — 2-flats (2-planes) π^2 : $N_r + n_r = (N-2)2 + 3 = 2N - 1$,
4. $p = N-1$ — hyperplanes $N_r = N-1$, $n_r = N$: $N_r + n_r = N - 1 + N = 2N - 1$.

Note that eq. (40) does not cover the case points ($p = 0$). In summary, the point representation reveals that the object being represented is a flat whose dimensionality is one less than the number of indices used. The dimensionality of the space where the flat resides is, of course, equal to the number of parallel axes.

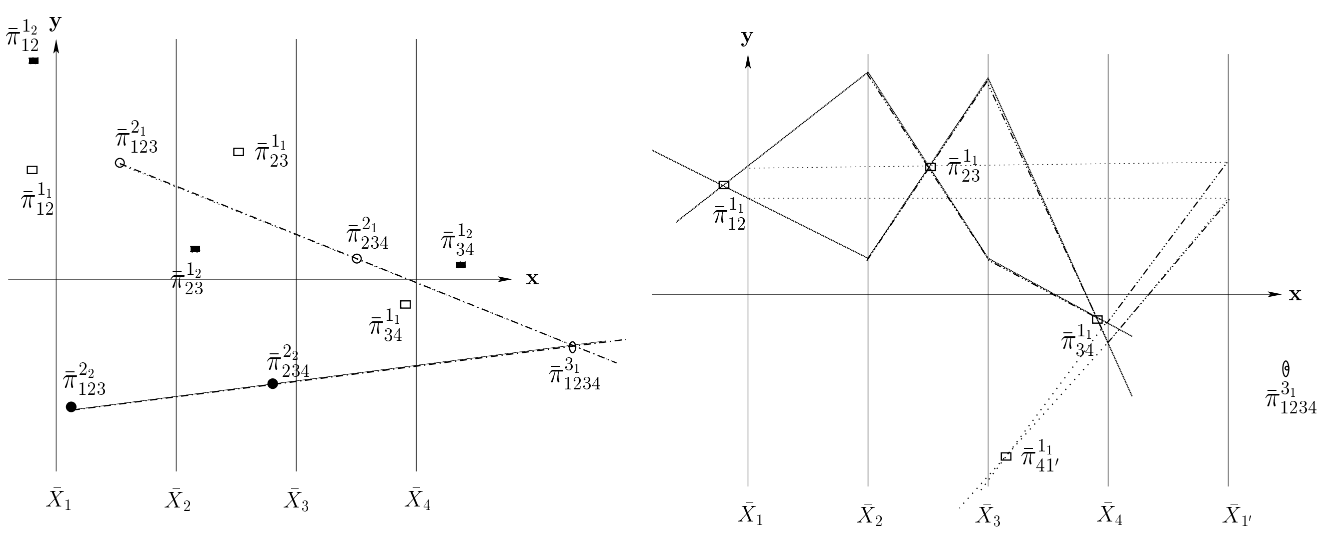


Figure 38: Recursive Construction in $\mathbb{R}^4$. Two 2-flats, π^{2_1} constructed previously and another π^{2_2} represented by the 2 black points (left), determine a 3-flat π^{3_1}. Pairs of points representing the same 2-flat are joined and their intersection is the point $\bar{\pi}^{3_1}_{1234}$. This is one of the 3 points representing the 3-flat. The "debris" from the previous constructions, points with fewer than 4 indices, can now be discarded.A new axis (right) $\bar{X}_{1'}$ is placed one unit to the right of $\bar{X}_3$ and the x_1 values are transfered to it from the $\bar{X}_1$ axis. Points are now represented by new polygonal lines between the $\bar{X}_2$ and $\bar{X}_{1'}$ axes and one of the points $\bar{\pi}^{1_1}_{41'}$, representing the 1-flat π^{1_1} on the new triple of $\|$-coords axes, is constructed as in 1st step.

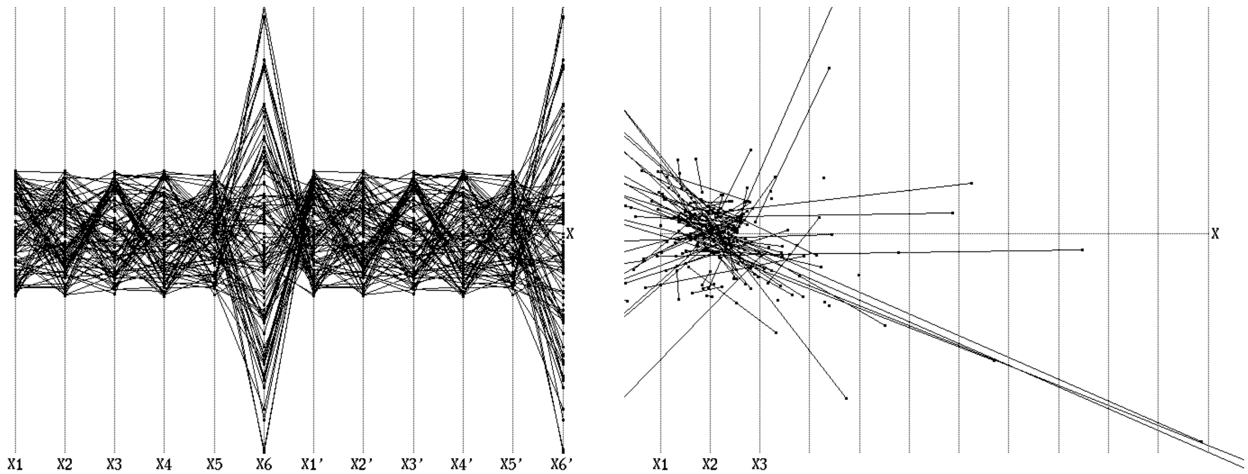


Figure 39: Polygonal lines (left) on the $\bar{X}_1 \ldots \bar{X}_6$ axes representing randomly selected points on a 5-flat $\pi^5 \subset \mathbb{R}^6$. ted points on a 5-flat $\pi^5 \subset \mathbb{R}^6$ .The $\bar{\pi}_{12}^{1_i}$, $\bar{\pi}_{23}^{1_i}$ portions of the 1-flats $\subset \pi^5$ constructed (right) from the polygonal lines lines. No pattern is evident.

Collinearity Property

The underpining of the construction algorithm for the point representation of a 2-flat $\pi^2 \subset \mathbb{R}^3$, as we saw, is the collinearity property. Namely for *any* $\pi^1 \subset \pi^2$ the points $\bar{\pi}_{12}^1$, $\bar{\pi}_{13}^1$, $\bar{\pi}_{23}^1$

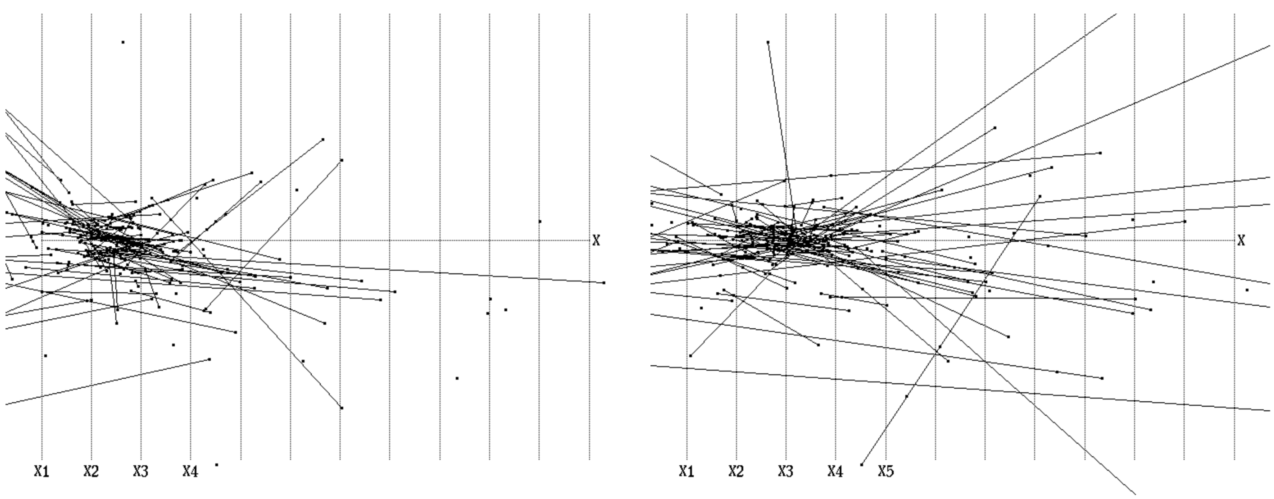


Figure 40: The $\bar{\pi}_{123}^{2_i}$, $\bar{\pi}_{234}^{2_i}$ (left) portions of the 2-flats $\subset \pi^5$ constructed from the polygonal lines joining $\bar{\pi}_{12}^{1_i}, \bar{\pi}_{23}^{1_i}$, $\bar{\pi}_{34}^{1_i}$. The $\bar{\pi}_{123}^{2_i}$, $\bar{\pi}_{234}^{2_i}$ (right) portions of the 2-flats $\subset \pi^5$ constructed from the polygonal lines joining $\bar{\pi}_{12}^{1_i}, \bar{\pi}_{23}^{1_i}$, $\bar{\pi}_{34}^{1_i}$.

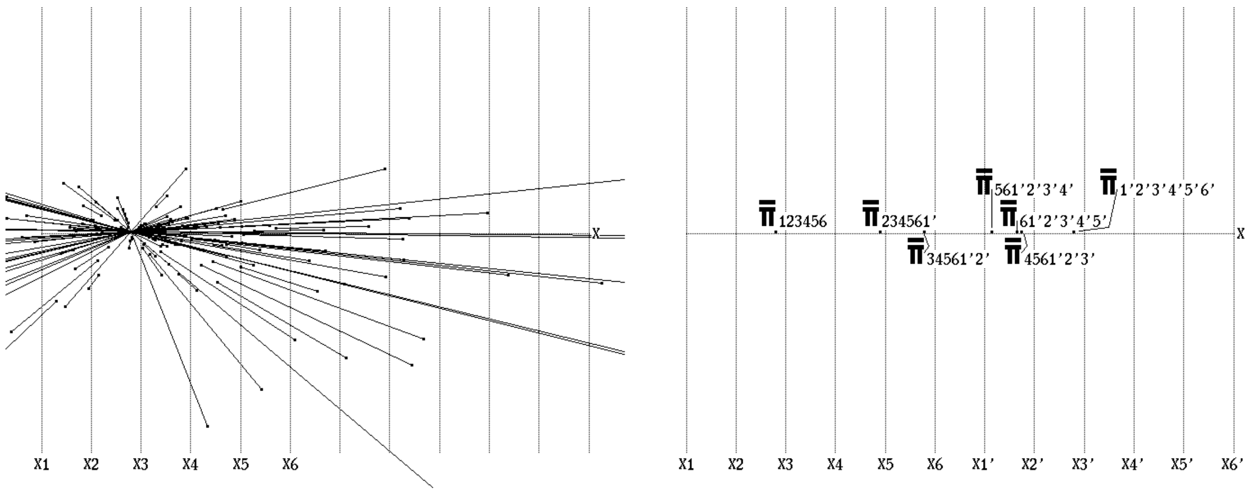


Figure 41: This is it (left)! The $\bar{\pi}^{4_i}_{12345}$, $\bar{\pi}^{4_i}_{23456}$ of the 4-flats $\subset \pi^5$ constructed from the polygonal lines joining $\bar{\pi}^{3_i}_{1234}$, $\bar{\pi}^{3_i}_{2345}$, $\bar{\pi}^{3_i}_{3456}$. This shows that the original points whose representation is in Fig. 39(left) are on a 5-flat in $\mathbb{R}^6$. The remaining points of the representation are obtained in the same way and all 7 points of the representation of π^5 are seen on the right. The coefficients of its equation are equal to 6 times the distance between sequentially indexed points for in Fig. **??** for $\mathbb{R}^3$.

are collinear with $\bar{\pi}_{123}$. For the generalization to p-flats let

$$\bar{L}^{p_k}_j = \bar{\pi}^{p_k}_{j\cdots(p+j)} \bullet \bar{\pi}^{p_k}_{(j+1)\cdots(p+j+1)} \tag{41}$$

denote the line $\bar{L}^{p_k}_j$ on the indicated two points. The gist of this section is the proof prove that $\pi^{(p-1)_1}$, $\pi^{(p-1)_2} \subset \pi^p \subset \mathbb{R}^N$

$$\bar{\pi}^p_{j\cdots(p+j+1)} = \bar{L}^{(p-1)_1}_j \cap \bar{L}^{(p-1)_2}_j \ . \tag{42}$$

An example is for $j = 1, p = 2, N = 3$ recasts our old friend from section **??** as :

$$\bar{L}^{\pi^{1_k}}_1 = \bar{\pi}^{1_k}_{12} \bullet \bar{\pi}^{1_k}_{23}, \quad k = 1, 2, \quad \bar{\pi}^2_{123} = \bar{L}^{\pi^{1_1}}_1 \cap \bar{L}^{\pi^{1_2}}_1 \ .$$

The pair (41) and (42) state the basic recursive construction implied in the *Representation Mapping* stated formally below. The recursion is on the dimensionality, increased by one at each stage, of the flat whose representative points are constructed. Though the notation may seem intimidating the idea is straight forwar, and to clarify it we illustrate it for $N = 4, p = 3$ in Figs. 37 and 38. Starting with the polygonal lines on a 3-flat π^{3_1}, first the points $\bar{\pi}^{1_i}_{12}$, , $\bar{\pi}^{1_i}_{23}$, $\bar{\pi}^{1_i}_{34}$, representing 1-flats (lines) on π^3, are constructed and joined to form polygonal lines having 3 vertices (the points) joined by **two** lines. From the intersection of these new polygonal lines the points $\bar{\pi}^{2_j}_{123}$, , $\bar{\pi}^{2_j}_{234}$, representing 2-flats on π^{3_1}, are constructed. At any stage a point representing $\bar{\pi}^r$, where the superscript is the flat's dimension, is obtained by *any pair* of lines joining points representing flats of dimension $r - 1$ and contained in π^r.

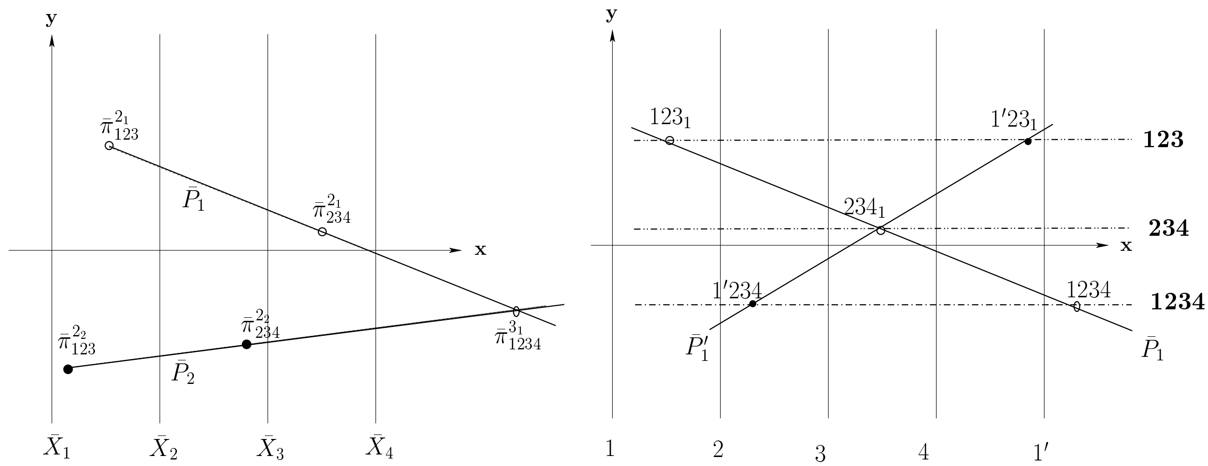


Figure 42: Continuing the construction of indexed points for a 3-flat π^3 in 4D. On the left is the point $\bar{\pi}_{1234}$ previously constructed in Fig. 38. Using simplified notation the construction is continued on the right to obtain the point $\bar{\pi}_{1'234}$ – marked by $1'234$. The $123_1, 1'23_1, 234_1$ are points of the representation of a 2-flat π^2_1 contained in π^3. The lines $\bar{P}_1, \bar{P}'_1$ on 1234 and $1'234$ share the indices 234 and necessarily intersect at the 234_1 point.

Theorem [Collinearity Construction Algorithm] : For any $\pi^{(p-2)} \subset \pi^{(p-1)} \subset \mathbb{R}^N$, the points $\bar{\pi}^{(p-2)}_{1\ldots(p-1)}$, $\bar{\pi}^{(p-2)}_{2\ldots(p-1)p}$, $\bar{\pi}^{(p-1)}_{1\ldots(p-1)p}$ are collinear.

Corollary For any $\pi^{(p-2)} \subset \pi^{(p-1)} \subset \mathbb{R}^N$, the points $\bar{\pi}^{(p-2)}_{\{j\ldots(p+j-2)\}_{i'}}$, $\bar{\pi}^{(p-2)}_{\{(j+1)\ldots(p+j-1)\}_{i'}}$, $\bar{\pi}^{(p-1)}_{\{(j\ldots(p+j-1)\}_{i'}}$ are collinear.

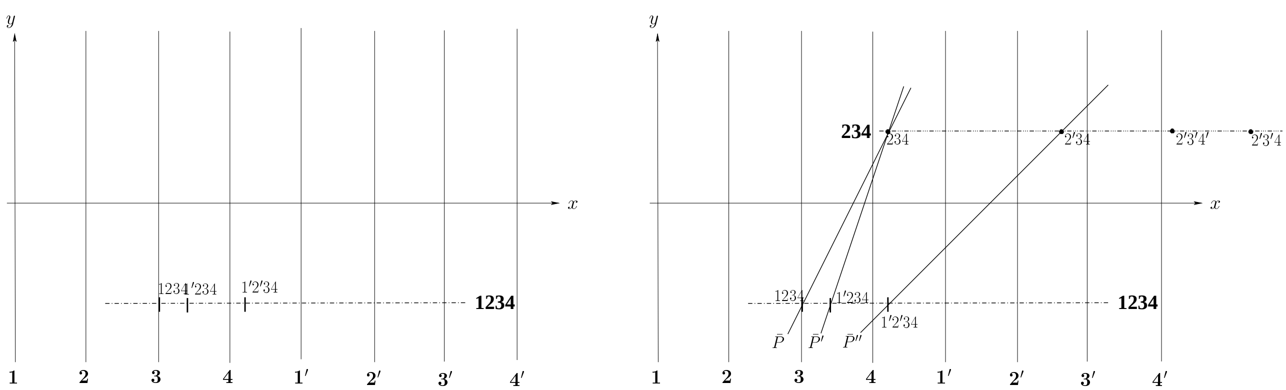


Figure 43: On the left are the initial data for a 3-flat π^3 in $\mathbb{R}^4$. Three points (right) $P \in \pi^3 \cap \pi^{4s}_1, P' \in \pi^3 \cap \pi^{4s}_1, P'' \in \pi^3 \cap \pi^{4s}$ are chosen. In the $x_2x_3x_4$ subspace of $\mathbb{R}^4$ they determine a 2-flat π^2_{234}.

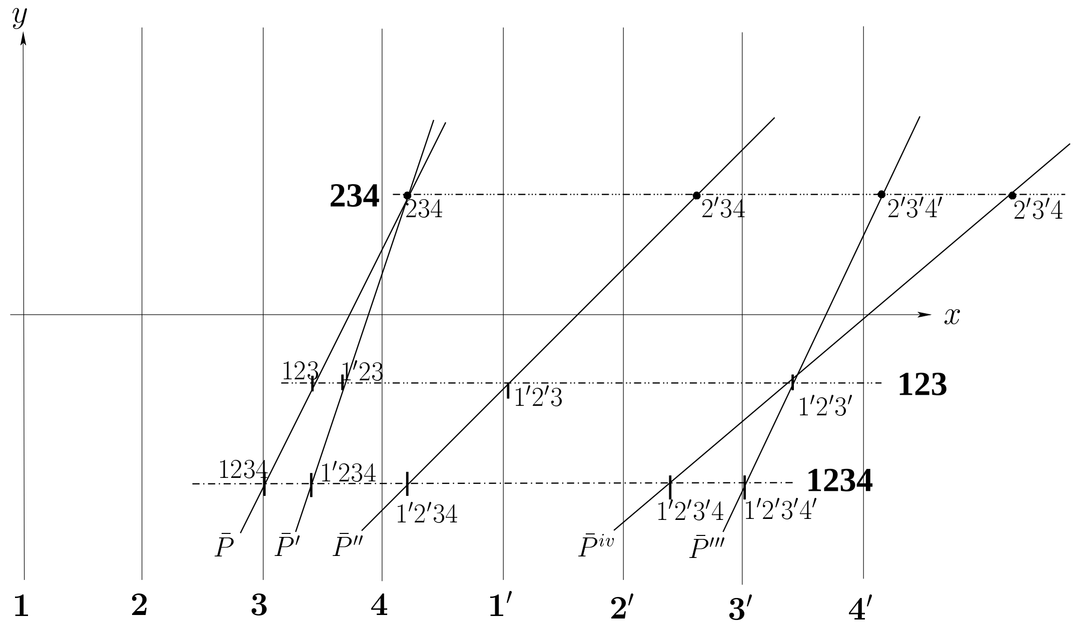


Figure 44: Completing the construction of the 5 points for a hyperplane's representation in $\mathbb{R}^4$.

Construction Algorithms in $\mathbb{R}^4$

The generalization of the 3-D construction algorithms to 4-D is direct and is an opportune time to introduce simplified notation which is used when the context is clear.

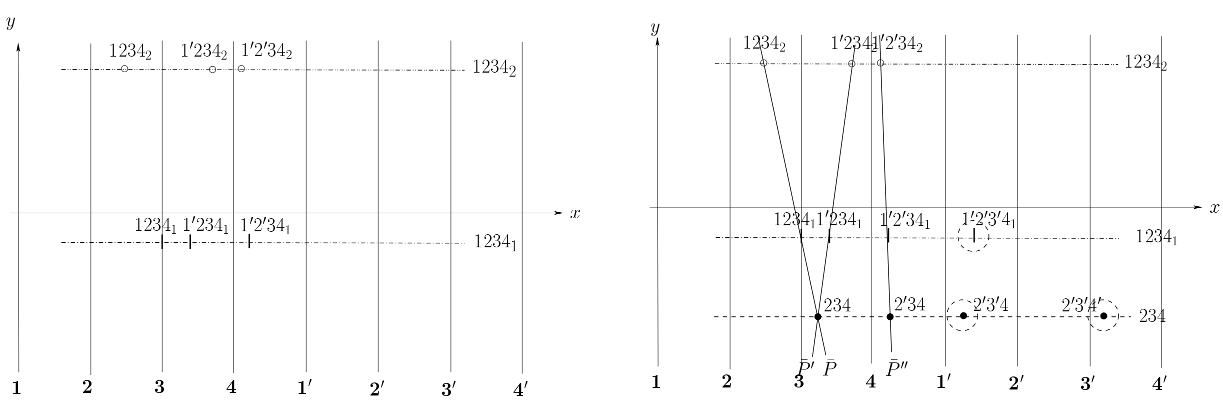


Figure 45: On the left are the initial data specifying two 3-flats π^{3_1}, π^{3_2} in $\mathbb{R}^4$. On the right the lines $\bar{P}, \bar{P}', \bar{P}''$ are drawn on the pair of points $1234, 1'234, 1'2'34$ respectively providing the 234 and $2'34$ points for the 2-flat $\pi^2 = \pi^{3_1} \cap \pi^{3_2}$. Then the points $2'3'4, 2'3'4'$ for π^2_{234} and $1'2'3'4_1$, shown within the dotted circles, are constructed.

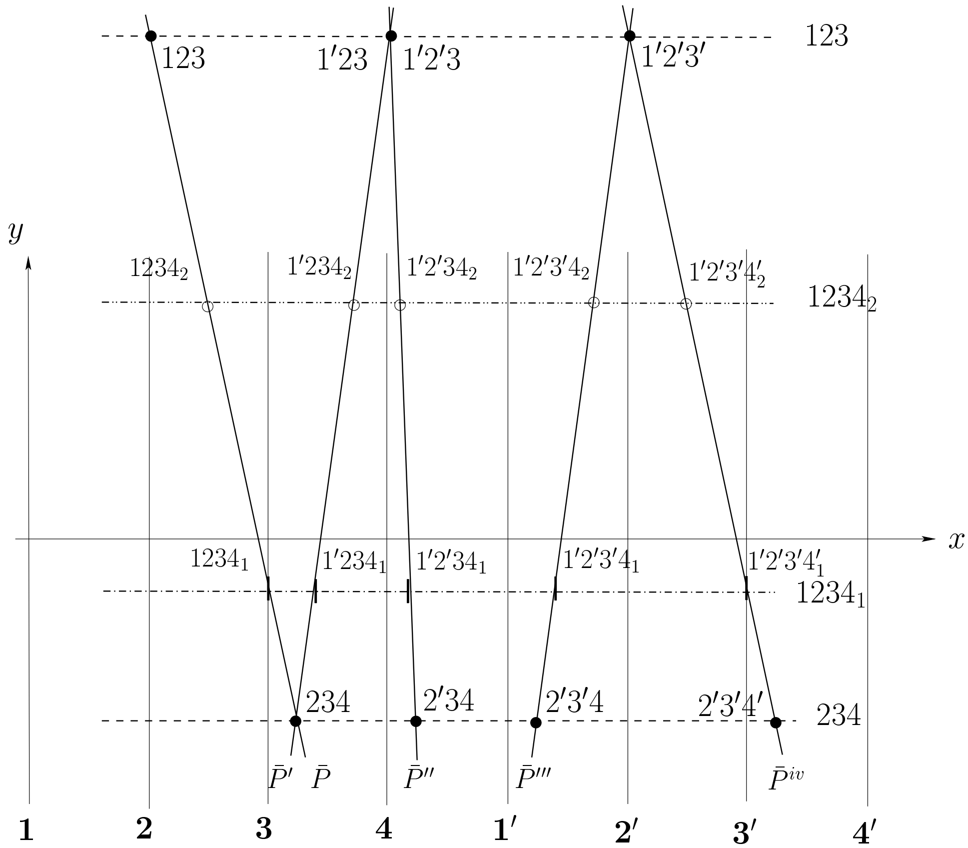


Figure 46: Intersection of two 3-flats in 4-D. The result is the 2-flat π^2 given by the points $123, 1'2'3'$ etc. representing π^2_{123} and $234, 2'34$ etc for π^2_{234}.

Detecting Near Coplanarity

The coplanarity of a set of points $S \subset \pi$ can be visually verified. What if the points are perturbed staying close to by no longer being on the plane π, can "near-coplanarity" still

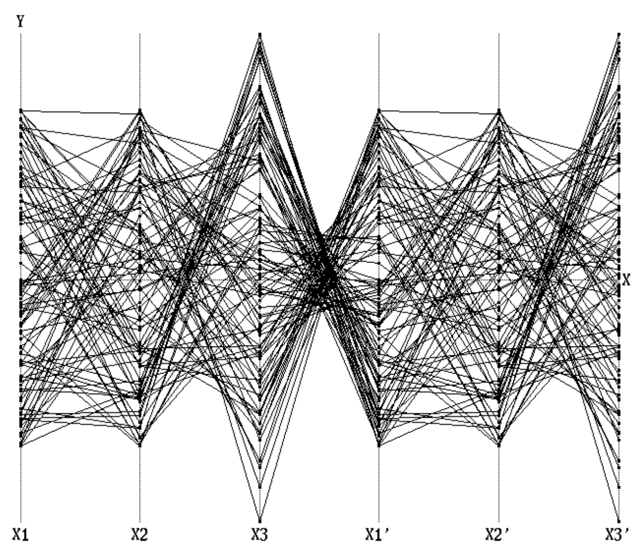


Figure 47: Polygonal lines representing a randomly selected set of "nearly" coplanar points.

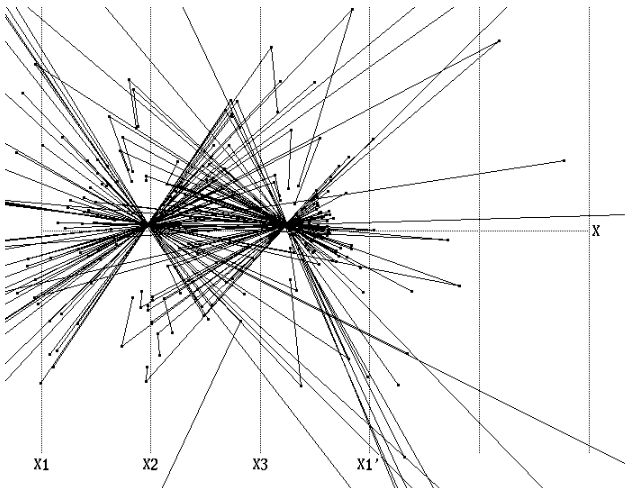


Figure 48: The "near-coplanarity" pattern (left) is very similar to that obtained for coplanarity with the points of intersection forming two clusters (right).

detected? Let us formulate this question more specifically by perturbing the the coefficients c_i of a plane's equation by a small amount ϵ_i. This generates a family of "proximate" planes forming a surface resembling a "slab" with nearly parallel sides. Now the experiment is performed by selecting a random set of points from such a "slab", and repeating the construction for the representation of planes. As shown in Fig. 47, 48 the there is a remarkable resemblance to the coplanarity pattern. The construction also works for any N. It is also possible to obtain error-bounds measuring the "near coplanarity" [3]. This topic is covered in a subsequent Chapter.

Experiments on points selected from several slabs simultaneously and performing similar construction showed that it is possible to determine the actual slabs from which the points were obtained or conversely can be fitted to. All this has important and interesting applications (USA patent # 5,631,982).

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