An Elementary Proof of a Theorem of Hardy and Ramanujan

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Abstract

Let $Q(n)$ denote the number of integers $1 \leq q \leq n$ whose prime factorization $q = \prod_{i=1}^{t} p_i^{a_i}$ satisfies $a_1 \geq a_2 \geq \ldots \geq a_t$. Hardy and Ramanujan proved that

$$\log Q(n) \sim \frac{2\pi}{\sqrt{3}} \sqrt{\frac{\log(n)}{\log\log(n)}} .$$

Before proving the above precise asymptotic formula, they studied in great detail what can be obtained concerning $Q(n)$ using purely elementary methods, and were only able to obtain much cruder lower and upper bounds using such methods.

In this paper we show that it is in fact possible to obtain a purely elementary (and much shorter) proof of the Hardy–Ramanujan Theorem. Towards this goal, we first give a simple combinatorial argument, showing that $Q(n)$ satisfies a (pseudo) recurrence relation. This enables us to replace almost all the hard analytic part of the original proof with a short inductive argument.

Keywords: Partition function, Additive number theory, Asymptotic number theory

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1 Introduction

Let $\ell_k = p_1 \cdot p_2 \cdots p_k$ denote the product of the first k prime numbers, and take $\mathcal{Q}$ to be the set of integers q which can be expressed as $q = \ell_1^{b_1} \cdot \ell_2^{b_2} \cdots \ell_t^{b_t}$, for some $t \geq 1$ and sequence of non-negative integers $b_1, \ldots, b_t$. Set[1] $Q(n) = |\mathcal{Q} \cap [n]|$, and note that this definition of $Q(n)$ is equivalent to the one given in the abstract. The problem of bounding $Q(n)$ was introduced by Hardy and Ramanujan [2]. As they explained, their motivation for studying this problem was its relation to highly composite numbers [7], its relation to variants of the partition function (see below), as well as the methods used in order to estimate $Q(n)$. The main result of [2] was the tight asymptotic bound

$$\log Q(n) \sim \frac{2\pi}{\sqrt{3}} \sqrt{\frac{\log(n)}{\log\log(n)}} . \tag{1}$$

In the first section of their paper, they studied what bounds can be obtained regarding $\log Q(n)$ using purely elementary methods. They were only able to use such methods in order to prove the much cruder bounds

$$C_1 \sqrt{\frac{\log(n)}{\log\log(n)}} \leq \log Q(n) \leq C_2 \sqrt{\log(n) \log\log(n)} .$$

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[1] Throughout the paper we use the standard notation $[n] := \{1, \ldots, n\}$.

They then used far more sophisticated methods (see below) in order to prove (1), not before noting that *"to obtain these requires the use of less elementary methods"*. We will show in this paper that one can in fact prove (1) via a completely elementary and short argument. But prior to discussing our new proof, let us first put it in perspective.

1.1 Historical perspective

The paper in which Hardy and Ramanjuan studied $Q(n)$ [2] was followed a year later by the celebrated paper [3] in which they obtained their famous asymptotic formula for the partition function $p(n)$. Much as they have done in [2], they devoted the first section of [3] to study what bounds can be obtained regarding $p(n)$ using purely elementary methods. They proved that $2\sqrt{n} \leq \log p(n) \leq \sqrt{8n}$, and remarked that they can prove the right asymptotic bound

$$\log p(n) \sim \pi\sqrt{2n/3} \tag{2}$$

using the methods they used in [2] in their proof of (1), but that this proof *"is of the difficult and delicate type"*. Indeed, in his book on Ramanujan's work from 1940 [1], Hardy remarked that *"It is actually true that* $\log p(n) \sim \pi\sqrt{2n/3}$*..., but we cannot prove this very simply"*. This shortcoming was resolved two years later by Erdős [4], who came up with an ingeniously simple proof of (2). His main idea was to take advantage of a certain recurrence relation satisfied by $p(n)$ in order to bound $p(n)$ by induction on n. See [5, 6] for further background and references.

With this perspective in mind, what we obtain in this paper can be considered an Erdős-type proof of (1). Unlike the case of $p(n)$, there is (to the best of our knowledge) no recurrence relation involving $Q(n)$. This suggests that Erdős's approach cannot be used to prove (1). However, as we explain below, there is an approximate such relation, which turns out to be sufficient for proving (1).

1.2 Our simplification

The proof of (1) in [2] began with an estimate for the function $\phi(\delta) = \sum_{k=1}^{\infty} \ell_k^{-\delta}$. Hardy and Ramanujan [2] then used their estimate for $\phi(\delta)$ in order to estimate the generating function of $Q(n)$ around zero. They then moved to the second and main step of their proof in which they proved a Tauberian theorem, which enabled them to translate their estimate for the generating function of $Q(n)$, to the estimate for $Q(n)$ in (1).

As in [2], our proof starts with a certain sum estimate. Set $\Phi(\delta) = \sum_{k=1}^{\infty} \log(\ell_k)\ell_k^{-\delta}$.

Lemma 1.1. *We have* $\Phi(\delta) \sim \delta^{-2}/\log(1/\delta)$ *as* $\delta \to 0$.

At this point our proof departs from that of [2] by doing away with its entire hard analytic part. This is achieved by the following (pseudo[2]) recurrence relation.

Lemma 1.2. *For every integer* n *set* $W(n) := \prod_{w \in \mathcal{Q} \cap [n]} w$. *Then the following relation holds*

$$W(n) = \prod_k \ell_k^{\sum_s Q(\lfloor n/\ell_k^s \rfloor)}. \tag{3}$$

Proof. Let $Q(n,k,s)$ and $Q^*(n,k,s)$ be the number of integers in $\mathcal{Q} \cap [n]$ for which $b_k = s$ and $b_k \geq s$ respectively. Since each integer has at most one representation as a product of ℓ_k, we have

$$W(n) = \prod_{s,k} \ell_k^{s \cdot Q(n,k,s)} = \prod_k \ell_k^{\sum_s sQ(n,k,s)} = \prod_k \ell_k^{\sum_s Q^*(n,k,s)} = \prod_k \ell_k^{\sum_s Q(\lfloor n/\ell_k^s \rfloor)}. \qquad \blacksquare$$

[2]One expects that most of the contribution to $Q(n)$ comes from integers q satisfying $\log(q) \approx \log(n)$. In this case we expect $W(n) \approx n^{Q(n)}$, which then turns equation (3) into a "genuine" recurrence relation.

What Lemma 1.2 gives us is the ability to prove (1) by induction on n (with the aid of Lemma 1.1) with (almost) the same ease with which Erdős [4] proved (2). Lemma 1.1 is proved in Section 2. The proof of the upper bound of (1) appears in Section 3, and the proof of the lower bound, which is almost identical to that of the upper bound, is given in Section 4. The appendix contains proofs of a few elementary inequalities.

2 Proof of Lemma 1.1

Let us mention that throughout the rest of the paper, we use $f(n) = \Theta(g(n))$ to denote that there are $c, C > 0$ such that $cg(n) \le f(n) \le Cg(n)$ for large enough n. We first recall Chebyshev's elementary bound for the n^{th} prime p_n (see [6]) which states that $p_n = \Theta(n \log n)$. Observe that this bound immediately gives the following

$$\log(\ell_n) = \sum_{k=1}^{n} \log(p_k) = \sum_{k=1}^{n} (\log(k \log k) + \Theta(1)) = n \log(n \log n) + \Theta(n) = (1 + o(1)) n \log n\ . \tag{4}$$

We now claim that the following three asymptotic estimates hold, which together imply Lemma 1.1.

$$\delta^{-2}/\log(1/\delta) \sim \int_1^\infty x \log x e^{-\delta x \log x} dx \sim \sum_{k \ge 2} k \log k e^{-\delta k \log k} \sim \Phi(\delta). \tag{5}$$

The first estimate is proved below. The second estimate follows from the monotonicity regimes of the function $f(x) = x \log x e^{-\delta x \log x}$ for small δ. Indeed, $f(x)$ is monotone increasing from $1/e$ up to w_δ and monotone decreasing from w_δ onwards, where w_δ is the unique maximum of $f(x)$ with $x > 1/e$. We can now estimate the sum of $f(x)$ by its integral with an error term of the order of $f(w_\delta) = 1/(e\delta)$. The last estimate in (5) follows by replacing[3] δ by $(1 + o(1))\delta$ and from (4), that is, from the fact that $\Phi(\delta)$ is bounded from above (resp. below) by $\sum_k k \log k e^{-\delta(1+o(1))k \log k}$.

It thus remains to estimate the integral in (5). We start with the upper bound, that is, that for every $\varepsilon > 0$ and all small enough $\delta < \delta_0(\varepsilon)$, this integral is bounded from above by $(1+\varepsilon)\delta^{-2}/\log(1/\delta)$. Let $a_0 = a_0(\varepsilon)$ be such that $\log(x) \ge (1 - \frac{1}{8}\varepsilon) \log(x \log x)$ for every $x \ge a_0$. Then

$$\begin{aligned}
\int_1^\infty x \log x e^{-\delta x \log x} dx \sim \int_{a_0}^\infty x \log x e^{-\delta x \log x} dx &\le \frac{1}{(1 - \varepsilon/8)\delta^2} \int_{\delta a_0 \log a_0}^\infty \frac{z e^{-z}}{\log(z/\delta)} dz \\
&\le \frac{1 + \varepsilon/4}{\delta^2} \left(\int_{\delta a_0 \log a_0}^{1/\log^2(1/\delta)} \frac{z e^{-z}}{\log(z/\delta)} dz + \int_{1/\log^2(1/\delta)}^\infty \frac{z e^{-z}}{\log(z/\delta)} dz \right) \\
&\le \frac{1 + \varepsilon/4}{\delta^2} \left(\frac{C}{\log^2(1/\delta)} + \frac{1}{\log(1/\delta) - 2 \log\log(1/\delta)} \int_0^\infty z e^{-z} dz \right) \\
&\le \frac{\varepsilon/2}{\delta^2 \log(1/\delta)} + \frac{1 + \varepsilon/2}{\delta^2 \log(1/\delta)} \int_0^\infty z e^{-z} dz = \frac{1 + \varepsilon}{\delta^2 \log(1/\delta)}\ ,
\end{aligned}$$

where in the first inequality we used the substitution $z = \delta x \log x$, which, by the choice of a_0, guarantees that $dz = \delta(\log(x) + 1)dx \ge \delta(1 - \varepsilon/8) \log(z/\delta)dx$, and in the third line C is a constant that depends only on a_0. The lower bound is similar. Take any $\varepsilon > 0$ and let $b_0 = b_0(\varepsilon) \ge 1$ be such that $\log(x) + 1 \le (1 + \frac{1}{8}\varepsilon) \log(x \log x)$ for all $x \ge b_0$. Then for all small enough $\delta < \delta_0(\varepsilon)$ we have

$$\int_1^\infty x \log x e^{-\delta x \log x} dx \ge \int_{b_0}^\infty x \log x e^{-\delta x \log x} dx \ge \frac{1}{(1 + \varepsilon/8)\delta^2} \int_{\delta b_0 \log b_0}^\infty \frac{z e^{-z}}{\log(z/\delta)} dz$$

[3] This is allowed by the continuity properties of the left-hand side of (5).

$$\geq \frac{1-\varepsilon/4}{\delta^2}\int_{\delta b_0 \log b_0}^{\log(1/\delta)} \frac{ze^{-z}}{\log(z/\delta)}dz \geq \frac{1-\varepsilon/4}{\delta^2 \log(\log(1/\delta)/\delta)}\int_{\delta b_0 \log b_0}^{\log(1/\delta)} ze^{-z}dz$$
$$\geq \frac{1-\varepsilon}{\delta^2 \log(1/\delta)} \, .$$

3 An Upper Bound for $Q(n)$

Set $c = 2\pi/\sqrt{3}$ and fix any $0 < \varepsilon < 1/2$. We will prove that $Q(n) \leq Ke^{(1+\varepsilon)c\sqrt{\log(Cn)/\log\log(Cn)}}$ by induction on n, where $K = K(\varepsilon)$ and $C = e^{10}+1$. Note that by choosing K large enough, we can assume that the induction assumption holds for all $n \leq n_0(\varepsilon)$, allowing us to assume in what follows that $n \geq n_0(\varepsilon)$. We first split the contributions of $Q(n)$ as follows

$$\begin{aligned}
Q(n) &= Q(n^{1-\varepsilon/4}) + (Q(n) - Q(n^{1-\varepsilon/4})) \\
&\leq Ke^{(1+\varepsilon)c\sqrt{\log(Cn^{1-\varepsilon/4})/\log\log(Cn^{1-\varepsilon/4})}} + (Q(n) - Q(n^{1-\varepsilon/4})) \\
&\leq \frac{\varepsilon}{8}Ke^{(1+\varepsilon)c\sqrt{\log(Cn)/\log\log(Cn)}} + (Q(n) - Q(n^{1-\varepsilon/4})) \, .
\end{aligned}$$

The first inequality holds by induction and the second holds provided n is large enough. Hence, proving that $Q(n) - Q(n^{1-\varepsilon/4}) \leq (1-\frac{\varepsilon}{8})Ke^{(1+\varepsilon)c\sqrt{\log(Cn)/\log\log(Cn)}}$ would complete the proof. Since

$$\log W(n) \geq (1-\varepsilon/4)\log(n)(Q(n) - Q(n^{1-\varepsilon/4}))$$

we just need to establish that for large enough n

$$\log W(n) \leq (1-\varepsilon/2)\log(n)Ke^{(1+\varepsilon)c\sqrt{\log(Cn)/\log\log(Cn)}} \, . \tag{6}$$

To simplify the presentation, we set $m = \log(Cn)$. Further, set $f(x) = \sqrt{x/\log(x)}$ and denote its derivative by $f'(x) = \frac{1-1/\log(x)}{2\sqrt{x\log(x)}}$. It follows immediately from Lagrange's remainder theorem (see the appendix for a detailed proof) that for every $r > 10$ and $0 \leq t \leq r-10$

$$f(r-t) \leq f(r) - tf'(r) \, . \tag{7}$$

Using Lemma 1.2 and then applying induction, we can bound $\log W(n)$ as follows

$$\begin{aligned}
\log W(n) &= \sum_k \sum_s \log(\ell_k) Q(\lfloor n/\ell_k^s \rfloor) \leq \sum_s \sum_k \log(\ell_k) Ke^{(1+\varepsilon)cf(\log(Cn/\ell_k^s))} \\
&\leq Ke^{(1+\varepsilon)cf(m)} \sum_{s=1}^{\infty}\sum_{k=1}^{\infty} \log(\ell_k)\ell_k^{-(1+\varepsilon)csf'(m)} \, ,
\end{aligned} \tag{8}$$

where the second inequality ($f'(m)$ means substituting $x = m = \log(Cn)$ in $f'(x)$) holds by (7) with $r = m$ and $t = \log(\ell_k^s)$ noting that $m = \log(Cn) \geq 10$ and assuming that in the first line we only consider indices k, s such that $\log(\ell_k^s) \leq \log(n) \leq m - 10$ (as otherwise $Q(\lfloor n/\ell_k^s \rfloor) = 0$).

Hence, to complete the proof of (6) it remains to establish that the double sum in (8) is bounded from above by $(1-\varepsilon/2)\log(n)$. Since $c \geq 1$ and $\alpha\lceil k\log(k+1)\rceil \leq \log(\ell_k) \leq \beta\lceil k\log(k+1)\rceil$ for some absolute $\alpha, \beta > 0$ for all $k \geq 1$ (by (4)), this double sum is clearly (as we only increase the number of summands which are positive) bounded from above by $S_1 + S_2$ where

$$S_1 = \sum_{s=1}^{(\log\log(n))^2}\sum_{k=1}^{\infty}\log(\ell_k)\ell_k^{-(1+\varepsilon)csf'(m)} \quad \text{and} \quad S_2 = \sum_{s=(\log\log(n))^2}^{\infty}\sum_{k=1}^{\infty}\beta k e^{-\alpha sf'(m)k} \, .$$

Bounding S_1: By Lemma 1.1 there is a $\delta_0 = \delta_0(\varepsilon)$ such that $\Phi(\delta) \le (1+\varepsilon/12)\delta^{-2}/\log(1/\delta)$ holds for every $\delta < \delta_0$. Assume n is large enough such that $(1+\varepsilon)c(\log\log(n))^2 f'(m) \le \delta_0$. Then,[4]

$$S_1 \le \sum_{s=1}^{(\log\log(n))^2} \frac{(1+\varepsilon/12)f'(m)^{-2}}{(1+\varepsilon)^2c^2\log\left(\frac{1}{(1+\varepsilon)csf'(m)}\right)s^2} \le \frac{8(1+\varepsilon/12)^2\log(n)}{(1+\varepsilon)^2c^2}\sum_{s=1}^{\infty}\frac{1}{s^2} \le (1-\varepsilon)\log(n)\ .$$

The first inequality uses Lemma 1.1 with $\delta = (1+\varepsilon)csf'(m)$ $(\le \delta_0)$, and the second inequality holds as $\frac{f'(m)^{-2}}{-\log((1+\varepsilon)csf'(m))} \le 8(1+\varepsilon/12)\log(n)$ for all $1 \le s \le (\log\log(n))^2$ and large n.

Bounding S_2: Since $S_1 \le (1-\varepsilon)\log(n)$ it remains to prove that $S_2 \le \frac{1}{2}\varepsilon\log(n)$. Indeed

$$S_2 = \sum_{s=(\log\log(n))^2}^{\infty} \frac{\beta e^{-\alpha sf'(m)}}{\left(1-e^{-\alpha sf'(m)}\right)^2} \le \frac{\beta}{\alpha^2 f'(m)^2}\sum_{s=(\log\log(n))^2}^{\infty}\frac{1}{s^2} \le \frac{9\beta m\log(m)}{\alpha^2((\log\log(n))^2-1)} \le \frac{1}{2}\varepsilon\log(n)\ .$$

The equality is $\sum_{a=1}^{\infty} at^a = \frac{t}{(1-t)^2}$, the first inequality uses the elementary inequality $\frac{e^{-z}}{(1-e^{-z})^2} \le \frac{1}{z^2}$ which follows directly from the Taylor expansion of e^x, and the second inequality holds as for all t we have $\sum_{s=t}^{\infty}\frac{1}{s^2} \le \int_{t-1}^{\infty}\frac{1}{x^2}dx = \frac{1}{t-1}$.

4 A Lower Bound for $Q(n)$

The proof is almost identical to the proof in Section 3. The only difference is that instead of (7) we now use (9), hence we need to account for f''. Set $c = 2\pi/\sqrt{3}$ and fix any $0 < \varepsilon < 1/2$. We will prove that $Q(n) \ge \frac{1}{K}e^{(1-\varepsilon)c\sqrt{\log(Cn)/\log\log(Cn)}}$ by induction on n, where $K = K(\varepsilon)$, and $C = e^{10^3}+1$. Note that by choosing K large enough, we can assume that the induction assumption holds for all $n \le n_0(\varepsilon)$, allowing us to assume in what follows that $n \ge n_0(\varepsilon)$.

To simplify the presentation we set $m = \log(Cn)$. Further, set $f(x) = \sqrt{x/\log(x)}$, denote its derivative by $f'(x) = \frac{1-1/\log(x)}{2\sqrt{x\log(x)}}$, and its second derivative by $f''(x) = \frac{-1+3/\log^2(x)}{4x\sqrt{x\log(x)}}$. As with (7), Lagrange's remainder theorem implies (see the appendix for a detailed proof) that for $r \ge 10^3$ and any $0 \le t \le r/10$

$$f(r-t) \ge f(r) - tf'(r) + t^2 f''(r)\ . \qquad (9)$$

Setting $s' = \log\log(n)$ and $k' = \log^{3/4}(n)$ we have $\ell_{k'}^{s'} \le n^{1/10}$ for large enough n, and hence

$$\begin{aligned}
\log(n)Q(n) \ge \log\left(W(n)\right) &= \sum_k\sum_s \log(\ell_k)Q(\lfloor n/\ell_k^s\rfloor) \\
&\ge \sum_{s\le s'}\sum_{k\le k'}\log(\ell_k)\frac{1}{K}e^{(1-\varepsilon)cf(\log(Cn)-(1+\varepsilon/4)\log(\ell_k^s))} \\
&\ge \frac{1}{K}e^{(1-\varepsilon)cf(m)}\sum_{s\le s'}\sum_{k\le k'}\log(\ell_k)e^{-(1-\varepsilon/2)c\log(\ell_k^s)f'(m)}\cdot e^{2c\log^2(\ell_k^s)f''(m)} \\
&\ge \frac{1}{K}e^{(1-\varepsilon)cf(m)}\sum_{s\le s'}\sum_{k\le k'}\log(\ell_k)\ell_k^{-(1-\varepsilon/2)csf'(m)}\left(1+2c\log^2(\ell_k^s)f''(m)\right)\ , \qquad (10)
\end{aligned}$$

[4] It is worth noting that this is the main term we need to bound. In particular, this is the place where the constant $c = 2\pi/\sqrt{3}$ emerges.

where the equality holds by Lemma 1.2, the second inequality holds by induction, the fact that for all $x > 2$ we have $\log(\lfloor x \rfloor) \geq \log(x) - 2/x$, and by assuming n is large enough such that $\ell_k^s/n \leq \frac{1}{8}\varepsilon \log(\ell_k^s)$ for all $s \leq s'$ and $k \leq k'$; the third inequality holds[5] by (9) with $r = m$, $t = (1+\varepsilon/4)\log(\ell_k^s)$ noting that $m = \log(Cn) \geq 10^3$ and $\log(\ell_{k'}^{s'}) \leq \log(n^{1/10}) \leq m/10$, and the last inequality holds as for all x we have $1 + x \leq e^x$.

Hence, to complete the proof it remains to establish that the double sum in (10) is bounded from below by $\log(n)$. Since $(1-\varepsilon/2)c > 1$ and since $\alpha\lceil k\log(k+1)\rceil \leq \log(\ell_k) \leq \beta\lceil k\log(k+1)\rceil$ for all $k \geq 1$ (by (4)), the double sum in (10) is at least $S_1 - S_2 + S_3$ where (note that $f''(m) < 0$)

$$S_1 = \sum_{s=1}^{s'}\sum_{k=1}^{\infty} \log(\ell_k)\ell_k^{-(1-\varepsilon/2)csf'(m)} \quad , \quad S_2 = \sum_{s=1}^{s'}\sum_{k=k'}^{\infty} \beta k e^{-\alpha s f'(m) k} ,$$

$$S_3 = \sum_{s=1}^{s'}\sum_{k=1}^{\infty} 2\beta^3 c s^2 k^3 f''(m) e^{-\alpha s f'(m) k} .$$

Bounding S_1: By Lemma 1.1 there is a $\delta_0 = \delta_0(\varepsilon)$ such that $\Phi(\delta) \geq (1-\varepsilon/4)\delta^{-2}/\log(1/\delta)$ holds for every $\delta < \delta_0$. Assume n is large enough such that $(1-\varepsilon/2)c\log\log(n)f'(m) \leq \delta_0$. Then,

$$S_1 \geq \sum_{s=1}^{\log\log(n)} \frac{(1-\frac{\varepsilon}{4})f'(m)^{-2}}{(1-\frac{\varepsilon}{2})^2 c^2 \log\left(\frac{1}{(1-\frac{\varepsilon}{2})csf'(m)}\right) s^2} \geq \frac{8(1-\frac{\varepsilon}{4})^{3/2}\log(n)}{(1-\frac{\varepsilon}{2})^2c^2} \sum_{s=1}^{\log\log(n)} \frac{1}{s^2} \geq \left(1 + \frac{\varepsilon}{2}\right)\log(n) ,$$

where the first inequality holds by Lemma 1.1 applied with $\delta = (1-\varepsilon/2)csf'(m)$ $(\leq \delta_0)$, the second inequality holds as $\frac{f'(m)^{-2}}{-\log((1-\varepsilon/2)csf'(m))} \geq 8\sqrt{1-\varepsilon/4}\log(n)$ for all $1 \leq s \leq \log\log(n)$ and large n. The last inequality holds provided n is large enough such that $\sum_{s=1}^{\log\log(n)} \frac{1}{s^2} \geq \sqrt{1-\varepsilon/4}\cdot \pi^2/6$.

Bounding S_2: Observe that the following holds for large enough n:

$$S_2 \leq \beta\log\log(n)(\log(n))^{3/4}\cdot\frac{e^{-\alpha f'(m)\log^{3/4}(n)}}{\left(1-e^{-\alpha f'(m)}\right)^2} \leq \frac{\beta\log\log(n)(\log(n))^{3/4}\cdot e^{-\alpha f'(m)\log^{3/4}(n)}}{\alpha^2\left(f'(m)-f'(m)^2\right)^2} \leq \frac{1}{4}\varepsilon\log(n) ,$$

where the first inequality holds as for all $M \geq 1$ and $0 < x < 1$ we have $\sum_{k=M}^{\infty} kx^k \leq \frac{Mx^M}{(1-x)^2}$, the second inequality holds as for all $0 < x < 1$ we have $(1-e^{-x})^2 > (x-x^2)^2$.

Bounding S_3: Since $S_1 - S_2 \geq (1+\varepsilon/4)\log(n)$, it is enough to prove that $S_3 \geq -\frac{1}{4}\varepsilon\log(n)$. Indeed,

$$S_3 \geq 2\beta^3 cf''(m)\sum_{s=1}^{\log\log(n)} s^2\frac{6e^{-\alpha s f'(m)}}{\left(1-e^{-\alpha s f'(m)}\right)^4} \geq \frac{36\beta^3 cf''(m)}{\alpha^4 f'(m)^4}\sum_{s=1}^{\infty}\frac{1}{s^2} \geq -\frac{1}{4}\varepsilon\log(n) .$$

The first and second inequality hold as for all $0 < z < 1$ we have $\sum_{k=0}^{\infty} k^3 z^k \leq \frac{6z}{(1-z)^4}$ (simply differentiate the identity $\sum_{k=0}^{\infty} z^k = \frac{1}{1-z}$ three times with respect to z) and $e^{-z}/(1-e^{-z})^4 \leq 3/z^4$ (this follows directly from the Taylor expansion of e^x). The last inequality holds for large enough n.

[5] As in Section 3, $f'(m)$ and $f''(m)$ mean plugging $x = m = \log(Cn)$ into the first/second derivatives of f.

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A Proofs of (7) and (9)

We first prove (7). By Lagrange's remainder theorem, for every $r > 1$ and t such that $0 \le t \le r$ there is $r - t \le c \le r$ such that

$$f(r-t) = f(r) - tf'(r) + t^2 f''(c)/2 \,.$$

Noting that $f''(z) = \frac{3-\log^2(z)}{4z^{3/2}\log^{5/2}(z)}$ is negative for all $z > e^{\sqrt{3}}$ we obtain (7) for $r > e^{\sqrt{3}}$ and $t > 0$ with $r - t > e^{\sqrt{3}}$.

We now prove (9). For every r and $t \le r$ we let

$$g_r(t) \coloneqq f(r-t) - f(r) + tf'(r) - t^2 f''(r) \,.$$

Note that $g_r(0) = g_r'(0) = 0$. Hence, proving that for all $r \ge 10^3$ and $0 \le t \le r/10$ we have $g_r''(t) > 0$ implies (9). Indeed, let $r \ge 10^3$ and let $t = \varepsilon r$ with $0 \le \varepsilon \le 1/10$. We have

$$\begin{aligned} g_r''(\varepsilon r) &= \frac{3 - \log^2((1-\varepsilon)r)}{4(1-\varepsilon)^{3/2} r^{3/2}\log^{5/2}((1-\varepsilon)r)} - \frac{3-\log^2(r)}{2r^{3/2}\log^{5/2}(r)} \\ &\ge \frac{1}{4r^{3/2}}\left(\frac{1}{(1-\varepsilon)^{3/2}}\cdot\frac{3-\log^2(9r/10)}{\log^{5/2}(9r/10)} - 2\cdot\frac{3-\log^2(r)}{\log^{5/2}(r)}\right) \\ &\ge \frac{1}{4r^{3/2}}\cdot\frac{2(1-1/10) - (1-\varepsilon)^{-3/2}(1+1/10)}{\sqrt{\log(r)}} > 0 \end{aligned}$$

where the first inequality holds as $\frac{3-\log^2(x)}{\log^{5/2}(x)}$ is monotone increasing for $x > e^{\sqrt{15}}$ and as $(1-\varepsilon)r \ge 900 \ge e^{\sqrt{15}}$, and the second inequality holds as $r > 10^3$ which implies both $\frac{3-\log^2(9r/10)}{\log^{5/2}(9r/10)} \ge \frac{-(1+1/10)}{\sqrt{\log(r)}}$ and $\frac{3-\log^2 r}{\log^{5/2}(r)} \le \frac{-(1-1/10)}{\sqrt{\log(r)}}$, and the last inequality holds as $\varepsilon \le 1/10$.