

Weakly Saturated Hypergraphs and a Conjecture of Tuza

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Abstract

Given a fixed hypergraph H , let $\text{wsat}(n, H)$ denote the smallest number of edges in an n -vertex hypergraph G , with the property that one can sequentially add the edges missing from G , so that whenever an edge is added, a new copy of H is created. The study of $\text{wsat}(n, H)$ was introduced by Bollobás in 1968, and turned out to be one of the most influential topics in extremal combinatorics. While for most H very little is known regarding $\text{wsat}(n, H)$, Alon proved in 1985 that for every graph H there is a limiting constant C_H so that $\text{wsat}(n, H) = (C_H + o(1))n$. Tuza conjectured in 1992 that Alon's theorem can be (appropriately) extended to arbitrary r -uniform hypergraphs. In this paper we prove this conjecture.

1 Introduction

Typical problems in extremal combinatorics ask how large or small a discrete structure can be, assuming it possesses certain properties. For example, the Turán problem asks, for a fixed r -uniform hypergraph (r -graph for short) H , to determine the smallest integer $m = \text{ex}(n, H)$ so that every n -vertex r -graph with $m + 1$ edges has a copy of H . Another example is the Ramsey problem which asks to find the minimum integer $R = R(n)$ so that every 2-coloring of the edges of the complete graph on R vertices has a monochromatic clique of size n . While in many cases it seems hopeless to obtain full solutions to these problems, one would at least like to know that these extremal functions are “well behaved”. For example, it is natural to ask if the quantities $\text{ex}(n, H)/n^r$ and $R(n)^{1/n}$ tend to a limit. While it is easy to see that the first quantity indeed tends to a limit [23], it is a famous open problem of Erdős [12, 13, 16] to prove that the second one does so as well. Our aim in this paper is to prove that another well studied extremal function is well behaved.

For a set of vertices V we use $\binom{V}{r}$ to denote the complete r -graph on V . For a fixed r -graph H , an r -graph $G = (V, E)$ is called H -saturated if it does not contain a copy of H but for any edge $e \in \binom{V}{r} \setminus E(G)$ adding e to G creates a copy of H . We let $\text{sat}(n, H)$ denote the smallest number of edges in an H -saturated r -graph on n vertices. Let K_t^r denote the complete r -graph on t vertices; when $r = 2$ (i.e. when dealing with graphs) we use K_t instead of K_t^2 . The problem of determining $\text{sat}(n, K_t)$ was raised by Zykov [40] in the 1940's and studied in the 1960's by Erdős, Hajnal and Moon [15] who showed that $\text{sat}(n, K_t) = \binom{n}{2} - \binom{n-t+2}{2}$. Their result was later generalized

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by Bollobás [7] who showed that $\text{sat}(n, K_t^r) = \binom{n}{r} - \binom{n-t+r}{r}$. It is worth noting that the proof in [7] introduced the (equivalent and) highly influential *Two Families Theorem*, stating that if A_1, \dots, A_s and B_1, \dots, B_s are two families of sets, so that all $|A_i| = a$, all $|B_i| = b$, and $A_i \cap B_j = \emptyset$ if and only if $i = j$, then $s \leq \binom{a+b}{a}$.

We say that G is *weakly H -saturated* if the edges of $\binom{V}{r} \setminus E(G)$ admit an ordering e_1, \dots, e_k such that for each $i = 1, \dots, k$ the r -graph $G_i := G \cup \{e_1, \dots, e_i\}$ contains a copy of H containing the edge e_i . We refer to the sequence e_1, \dots, e_k as a *saturation process*. Define $\text{wsat}(n, H)$ to be the smallest number of edges in a weakly H -saturated r -graph on n vertices. Note that we may automatically assume that any G realizing $\text{wsat}(n, H)$ is H -free, as otherwise we could remove an edge from a copy of H in G to obtain a smaller weakly H -saturated r -graph. Hence weak saturation can be viewed as an extension of the notion of (ordinary) saturation.

The problem of determining $\text{wsat}(n, H)$ was first introduced in 1968 by Bollobás [8] who conjectured that $\text{wsat}(n, K_t) = \text{sat}(n, K_t)$. This was proved independently by Frankl [20] and Kalai [21, 22] using the skewed¹ variant of Bollobás's Two Families Theorem (a related statement for matroids was proven earlier by Lovász [24]) and further extended by Alon [1] and Blokhuis [6]. This result, which has several other equivalent formulations, is amongst the most classical and important results of extremal combinatorics. See e.g. the discussions in [2, 27, 32, 34].

While the aforementioned results determine the exact value of $\text{wsat}(n, H)$ when $H = K_t^r$, our understanding of this function for general H is much more limited, despite decades of extensive study [1, 3, 4, 9, 14, 17, 25, 26, 29, 30, 35, 36, 38, 39]. Note that by the construction from [15], we know that every graph H we have

$$\text{wsat}(n, H) \leq \text{sat}(n, H) \leq \text{sat}(n, K_{|V(H)|}) = O_H(n). \quad (1.1)$$

As of now, the best known general bounds for $\text{wsat}(n, H)$ when H is a graph are due to Faudree, Gould and Jacobson [18] who showed that for graphs H of minimum degree $\delta = \delta(H)$ we have²

$$\left(\frac{\delta}{2} - \frac{1}{\delta+1} \right) \cdot n \leq \text{wsat}(n, H) \leq (\delta - 1) \cdot n + O(1).$$

At this point it is natural to ask if for every H there is a constant C_H so that

$$\text{wsat}(n, H) = (C_H + o(1))n. \quad (1.2)$$

Such a result was obtained in 1985 by Alon [1], who proved that for graphs the function $\text{wsat}(n, H)$ is (essentially) subadditive, implying that $\text{wsat}(n, H)/n$ tends to a limit, by Fekete's subadditivity lemma [19].

Much less was known when H is an r -graph with $r \geq 3$. Similarly to the case $r = 2$ above (1.1), Bollobás's construction from [7] gives a simple bound of

$$\text{wsat}(n, H) \leq \text{sat}(n, H) = O_H(n^{r-1}).$$

A more refined result was obtained by Tuza [39] who introduced the following key definition. The *sparseness* of an r -graph H , denoted $s(H)$, is the smallest size of a vertex set $W \subseteq V$ contained in

¹In the skewed version one assumes that $A_i \cap B_i = \emptyset$ as in Bollobás's theorem, but that $A_i \cap B_j \neq \emptyset$ only for $i < j$.

²The upper bound is known to be tight for many graphs, the cliques being one example. Concerning the lower bound, the authors of [18] give a construction of a graph H with $\text{wsat}(n, H) \leq (\delta/2 + 1/2 - 1/\delta)n$.

precisely one edge of H ; note that $1 \leq s(H) \leq r$ for every non-empty r -graph H . It was proved in [39] that for every r -graph H there are two positive reals c_H and C_H such that

$$c_H \cdot n^{s-1} \leq \text{wsat}(n, H) \leq C_H \cdot n^{s-1}. \quad (1.3)$$

It was further conjectured in [39] that the more refined bound $\text{wsat}(n, H) = C_H \cdot n^{s-1} + O(n^{s-2})$ holds for every r -graph of sparseness s . See also the recent survey [11] on saturation problems where this conjecture is further discussed. Since such a result is not known even for graphs (i.e. when $r = s = 2$), Tuza [39] asked if one can improve upon (1.3) by showing that for every r -graph we have $\text{wsat}(n, H) = C_H \cdot n^{s-1} + o(n^{s-1})$ where $s = s(H)$. Prior to this work, such a result was only known for $r = 2$ by Alon's result (1.2). In this paper we fully resolve Tuza's problem for all r -graphs.

Theorem 1.1. *For every r -graph H there is $C_H > 0$ such that*

$$\lim_{n \rightarrow \infty} \text{wsat}(n, H)/n^{s-1} = C_H,$$

where $s = s(H)$ is the sparseness of H . In particular³, for every r -graph H there is $C'_H \geq 0$ such that

$$\lim_{n \rightarrow \infty} \text{wsat}(n, H)/n^{r-1} = C'_H.$$

It is interesting to note that Tuza [37] (for graphs) and Pikhurko [28] (for arbitrary r -graphs) also conjectured that a theorem analogous to the second assertion of Theorem 1.1 should hold with respect to $\text{sat}(n, H)$. However, there are results suggesting that this analogous statement does not hold even for graphs, see [5, 10, 31] and the discussion in [11].

Proof and paper overview: It is natural to ask why Alon's [1] one-paragraph proof of Theorem 1.1 for $s = 2$ is hard to extend to $s > 2$.⁴ Perhaps the simplest reason is that one cannot hope to show that in these cases the function $\text{wsat}(n, H)$ is subadditive (and then apply Fekete's lemma) since a subadditive function is necessarily of order $O(n)$, while we know from (1.3) that when $s \geq 3$ the function $\text{wsat}(n, H)$ is of order at least n^2 . One can of course try to come up with more complicated recursive relations for $\text{wsat}(n, H)$ and combine them with variants of Fekete's lemma, but this seems to lead to a dead-end (we have certainly tried to go down that road). The main novelty in this paper is in finding a direct and efficient way to use an m -vertex r -graph witnessing the fact that $\text{wsat}(m, H)$ is small, in order to build arbitrarily large n -vertex r -graphs witnessing the fact that $\text{wsat}(n, H)$ is small. One of the main tools we use to construct such an example is Rödl's approximate designs theorem [33] which enables us to efficiently combine many examples of size m into one of size n . Rödl's result would only allow us to construct a saturation process generating part of the edges of K_n^r . To complete this saturation process we would also need another set of gadgets. In Section 2 we establish some general facts about weak saturation of r -graphs. The main proof of Theorem 1.1 is carried out in Section 3.

³Here we simply use the fact that for every r -graph H we have $1 \leq s(H) \leq r$.

⁴While formally [1] only deals with $r = 2$, the proof very similarly applies to $s = 2$ for arbitrary r .

2 Preliminaries

In this section we establish a few useful facts regarding $\text{wsat}(n, H)$. Perhaps counterintuitively, a graph G can be weakly H -saturated but not weakly H' -saturated for some subgraph $H' \subseteq H$. In fact, $\text{wsat}(n, H)$ is not even monotone with respect to H . For example, if H' is a triangle and H is a triangle with a pendant edge, then $\text{wsat}(n, H') = n - 1$ (with extremal examples being all n -vertex trees), while $\text{wsat}(n, H) = 3$ (the triangle being one extremal example). We now define a setting where one does have such a monotonicity.

Given $s \leq r \leq h$, let $T_{r,h,s}^-$ be the r -graph obtained from the complete h -vertex r -graph K_h^r by choosing a set Z of s vertices and deleting all edges containing Z as a subset. Define the *template r -graph* $T_{r,h,s}$ to be the (unique up to isomorphism) r -graph obtained from $T_{r,h,s}^-$ by adding a single missing edge f (on the same vertex set), we call f the *special edge*. To practise the definition, note that $T_{r,h,r}$ is simply the clique K_h^r . We say that an r -graph G is *$T_{r,h,s}$ -template saturated* if the edges in $\binom{V(G)}{r} \setminus E(G)$ admit an ordering e_1, \dots, e_k (the *$T_{r,h,s}$ -template saturation process*) such that for each $i = 1, \dots, k$ the r -graph $G_i := G \cup \{e_1, \dots, e_i\}$ contains a copy of $T_{r,h,s}$ in which the edge e_i plays the role of the special edge f . The next lemma shows that comparing $T_{r,h,s}$ -template saturation with weak H -saturation, for an r -graph H with $s(H) = s$, we do have monotonicity.

Lemma 2.1. *Suppose G and H are r -graphs with $|V(H)| = h$ and $s(H) = s \geq 2$. Suppose that G is $T_{r,h,s}$ -template saturated. Then G is weakly H -saturated.*

Proof. By the definition of sparseness, H contains a set S of s vertices contained in precisely one edge $e \in E(H)$. Deleting e from H gives the r -graph H^- of order h and in which no edge contains S as a subset. By the definition of $T_{r,h,s}$ we have that H^- is a subgraph of $T_{r,h,s}^-$. More importantly, H^- can be embedded into $T_{r,h,s}^-$ in a way that maps S bijectively on Z . Indeed, any map $\phi : V(H^-) \mapsto V(T_{r,h,s}^-)$ which sends the set S of H^- to the set Z of $T_{r,h,s}^-$ has this property.

Consider now a $T_{r,h,s}$ -template saturation process of G . By the above argument, at every step the newly created copy of $T_{r,h,s}$ (with the new edge playing the role of the special edge) gives rise to a new copy of H , where the new edge plays the role of e . Therefore, the same process certifies weak H -saturation of G . \square

We will frequently use the following simple observation stating that saturation processes are monotone with respect to the starting graph G .

Observation 2.2. *For any r -graphs G and H with $|V(G)| = n$, if G is weakly H -saturated then so is any intermediate r -graph $G \subseteq G' \subseteq K_n^r$. The analogous statement holds for $T_{r,h,s}$ -template saturation.*

As an immediate consequence we obtain

Lemma 2.3. *Suppose s' satisfies $r \geq s' \geq s \geq 2$, and let G be a supergraph of $T_{r,h,s'}^-$ on the same vertex set. Then G is $T_{r,h,s}$ -template saturated in K_h^r .*

Proof. The assertion is true for $G = T_{r,h,s}$: the missing edges can be added in any order. For arbitrary $s' \geq s$, the r -graph $T_{r,h,s'}^-$ and, by extension, every supergraph thereof, contain $T_{r,h,s}^-$ as a subgraph. Therefore, the assertion holds by Observation 2.2. \square

Our next goal is to obtain a certain “approximate continuity” of $\text{wsat}(n, H)$ with respect to n . We first need the following lemma.

Lemma 2.4. *Let $h \geq r \geq s \geq 2$, suppose $V = A \sqcup B$ is a set of vertices, where $|B| \leq |A|$, and let $E = \binom{A}{r}$ be the edges contained in A . Then there exists a set $E' \subseteq \binom{V}{r}$ of size at most $rh^r |A|^{s-2} |B|$ so that $G = (V, E \cup E')$ is $T_{r,h,s}$ -template saturated in $\binom{V}{r}$.*

Proof. Let $C \subseteq A$ be a fixed set of h vertices, and let

$$E' := \{f \in \binom{V}{r} \setminus E : |f \setminus C| \leq s - 1\}.$$

Note that every such f contains at least one vertex from B (as otherwise we would have $f \in E$). Since $|B| \leq |A|$ we have $|E'| \leq rh^r |A|^{s-2} |B|$. We claim that $G = (V, E \cup E')$ is $T_{r,h,s}$ -template saturated, as desired. To describe the corresponding saturation process, we consider a missing edge f and apply induction on $\lambda(f) := |f \setminus C|$. The base case of $\lambda(f) \leq s - 1$ is given by the fact that these edges are already in $E \cup E'$.

Suppose now that $\lambda \geq s$ is arbitrary, f is a missing edge with $\lambda(f) = \lambda$, and every edge e with $\lambda(e) < \lambda$ has already been added. Let $L := f \setminus C$ (so that $|L| = \lambda$), and let $P \subseteq C \setminus f$ be a set of $h - r$ vertices. By the induction hypothesis, all edges on the vertex set $P \cup f$ not containing L as a subset have already been added. Conversely, every currently missing edge must contain L as a subset, which means the currently present edges on $P \cup f$ form a supergraph of $T_{r,h,\lambda}^-$. Since $\lambda \geq s$, by Lemma 2.3 we can add all missing edges on the set $P \cup f$, including the edge f , via a $T_{r,h,s}$ -template saturation process. This completes the induction step. \square

In the following statement the reader should think of $k_2 = o(k_1)$. Since $\text{wsat}(k_1, H)$ is of order k_1^{s-1} (by (1.3)) this means that in this regime $\text{wsat}(k_1 + k_2, H) = (1 + o(1))\text{wsat}(k_1, H)$.

Corollary 2.5. *Let $h \geq r \geq s \geq 2$ and H be an r -graph with $|V(H)| = h$ and $s(H) = s$. Then for every $k_2 \leq k_1$ we have*

$$\text{wsat}(k_1 + k_2, H) \leq \text{wsat}(k_1, H) + rh^r \cdot k_1^{s-2} \cdot k_2.$$

Proof. Given a minimal weakly H -saturated r -graph $G^- = (A, E^-)$ on k_1 vertices, construct a weakly H -saturated r -graph $G = (V, E)$ on $k_1 + k_2$ vertices as follows. Let B be a set of k_2 vertices disjoint from A , let $V := A \sqcup B$ and $E := E^- \cup E'$ where E' is the edge set as described in Lemma 2.4. Then G is weakly H -saturated. Indeed, first run a saturation process inside A . Afterwards the remaining missing edges can be added by Lemma 2.4 and Lemma 2.1. Moreover, by Lemma 2.4 we have

$$|E| \leq |E^-| + |E'| \leq \text{wsat}(k_1, H) + rh^r \cdot k_1^{s-2} \cdot k_2.$$

\square

3 Proof of Theorem 1.1

As we mentioned at the end of Section 1, our approach to proving Theorem 1.1 is to use an m -vertex weakly H -saturated graph with few edges in order to build, for all large enough n , an n -vertex

weakly H -saturated graph with few edges. In the first step of the proof we will take ℓ disjoint vertex “clusters” (for some large ℓ) and cover them with copies of the m -vertex example. To do so efficiently, we shall need the following classical theorem of Rödl [33] (formerly, the Erdős-Hanani conjecture).

Theorem 3.1 (Rödl [33]). *For every $k > t > 1$ and $\delta > 0$ for all $N > N_0(k, t, \delta)$ the following holds. There exists a collection $\mathcal{F} \subseteq \binom{[N]}{k}$ of size at most $(1 + \delta) \binom{N}{t} / \binom{k}{t}$ such that every $A \in \binom{[N]}{t}$ is contained in some $F_A \in \mathcal{F}$.*

The outcome of applying Rödl’s theorem will be a graph (denoted G'_n in the proof of Theorem 1.1) that has an H -saturation process generating part of the edges of K_n^r , namely the edges containing vertices from at most $s - 1$ of the ℓ clusters. To generate the remaining edges, we will add to G'_n another collection of gadgets (the edge set E_2 in the proof of Theorem 1.1). These are described in the next two lemmas. We note that the bound guaranteed by Lemma 3.3 is crucial for establishing that $|E_2| = o(n^{s-1})$, thus making sure that these extra edges have a negligible effect on the total number of edges of the graphs we construct.

Lemma 3.2. *Suppose $G = (V, E)$ is an r -graph such that $V = \bigsqcup_{i=1}^s V_i$ with $|V_i| \geq h$ for all i and E contains all r -tuples in V missing at least one of the sets V_i . For each $i \in [s]$ let $R_i \subseteq V_i$ be a set of h vertices. Let E' be the set of all edges containing at least $r - s + 2$ vertices from $R := \bigcup_i R_i$. Then $E \cup E'$ is $T_{r,h,s}$ -template saturated in $\binom{V}{r}$.*

Proof. For each $i \in [s]$ let $L_i := V_i \setminus R_i$ and let $L := \bigcup_i L_i$. Let the vertices of R and L be called *rigid* and *loose*, respectively. Our aim is to define a $T_{r,h,s}$ -template saturation process. Note that by assumption the edges in $\binom{V}{r}$ containing at most $s - 2$ loose vertices are already present.

Consider first the missing edges $C \in \binom{V}{r} \setminus (E \cup E')$ containing exactly $s - 1$ loose vertices. By pigeonhole, for every such edge there is an index $j \in [s]$ such that no vertex in $C_j := C \cap V_j$ is loose. Let

$$\rho(C) := \min\{|C_j| : C_j \subseteq R\}.$$

We now apply induction on ρ in order to construct a $T_{r,h,s}$ -template saturation process adding successively the edges with $\rho = 0, 1, 2, \dots$. For the base case $\rho = 0$, note that such edges necessarily do not contain any vertex from (at least) one of the sets V_1, \dots, V_s , and therefore are already in E .

For the induction step let $\rho(C) \geq 1$ be arbitrary, and suppose that the edges with a smaller value of ρ are already present. Let $j \in [s]$ satisfy $C_j \subseteq R$ and $|C_j| = \rho$, let $i \in [s] \setminus \{j\}$ be another index and let $D \subseteq R_i \setminus C_i$ be a set of size $h - r$. Observe now that inside the set $D \cup C$ the only edges not yet present are the ones containing $(C \cap L) \cup C_j$ as a subset. Indeed, since $(D \cup C) \cap L = C \cap L$ every edge in $D \cup C$ missing a vertex from $C \cap L$, contains at most $s - 2$ loose vertices, and is thus in E' . Furthermore, every edge in $D \cup C$ missing a vertex from C_j contains fewer than $\rho(C)$ from R_j (and no vertex from L_j). Therefore, it is already present by the induction hypothesis. Thus the currently present edges on $D \cup C$ induce a supergraph of $T_{r,h,s'}^-$, where $s' = |(C \cap L) \cup C_j|$. Since $s' = s - 1 + \rho(C) \geq s$, by Lemma 2.3 we can add all the missing edges on $D \cup C$, including C , via a $T_{r,h,s}$ -template saturation process.

Now consider the missing edges C having at least s loose vertices and apply induction on $\lambda(C) := |C \cap L|$; we can view the case $\lambda(C) = s - 1$ treated above as the base case. For the induction step, suppose that $\lambda(C) \geq s$ is arbitrary and that all the edges with a smaller value of λ are already

present. Let $D \subseteq R \setminus C$ be an arbitrary set of $h - r$ vertices. Then, by the induction hypothesis, all edges on $D \cup C$ not already present contain $C \cap L$ as a subset (for otherwise they would have fewer than $\lambda(C)$ loose vertices). Hence, the currently present edges within $D \cup C$ induce a supergraph of $T_{r,h,\lambda(C)}^-$. Since $|C \cap L| = \lambda(C) \geq s$, by Lemma 2.3 we can add all of the missing edges on $D \cup C$, including C , applying a $T_{r,h,s}$ -template saturation process.

Having reached $\lambda = r$, we have covered all edges in $\binom{V}{r}$. \square

Lemma 3.3. *Suppose $V = \bigsqcup_{i=1}^{\ell} V_i$ for some $\ell \geq s$, with $|V_i| \geq h$ for all i . Suppose further that for each $i \in [\ell]$ there is a designated subset $R_i \subseteq V_i$ with $|R_i| = h$. Let $G = (V, E)$ be an r -graph with $E = E_1 \cup E_2$ where E_1 contains all edges hitting at most $s - 1$ different V_i and*

$$E_2 := \bigcup_{Q \in \binom{[\ell-1]}{s-1}} E_2(Q),$$

where $E_2(Q)$ is a copy of E' as in Lemma 3.2 on $V_Q := V_\ell \sqcup \bigsqcup_{q \in Q} V_q$. Then G is $T_{r,h,s}$ -template saturated in $\binom{V}{r}$. Moreover, if $|V_i| = t$ for all i , then we have

$$|E_2| \leq rh^{r-s+2} \binom{\ell-1}{s-1} t^{s-2}.$$

Proof. First, for each $Q \in \binom{[\ell-1]}{s-1}$ consider the induced subgraph $G[V_Q]$. Note that with the partition $V_Q = V_\ell \sqcup \bigsqcup_{q \in Q} V_q$ this r -graph contains all the edges in the statement of Lemma 3.2. Hence, by Observation 2.2 and Lemma 3.2 we can apply a $T_{r,h,s}$ -template saturation process in order to add all missing edges inside V_Q . Thus we may assume from here on that the edges inside all sets V_Q are present.

For an edge $e \in \binom{V}{r}$ let $J(e) = e \setminus V_\ell$ and $j(e) = |J(e)|$. By the above, every edge e with $j(e) \leq s - 1$ has already been added and, conversely, every missing edge $e \in \binom{V}{r}$ satisfies $j(e) \geq s$. We construct a $T_{r,h,s}$ -template saturation process for the missing edges by adding them successively: first the edges with $j(e) = s$, followed by $j(e) = s + 1, \dots, j(e) = r$. To do so we apply induction on $j = j(e)$, where $j \leq s - 1$ can be viewed as the base case.

For the induction step, fix j and suppose that all edges e' with $j(e') < j$ have already been added. Let e be an arbitrary edge with $J(e) =: J$ and $j(e) = j$, and consider the set $T = e \cup P$ where $P \subseteq V_\ell \setminus e$ is an arbitrary set of $h - r$ vertices disjoint from e ; clearly, we have $|T| = h$. Notice now that every potential edge $f \subseteq T$ satisfies either $f \supseteq J$ or $|f \cap J| < j$. In the latter case, $j(f) < j$, so by the induction hypothesis, f has already been added. Thus, the only edges missing from T are the ones containing J as a subset. In other words, the edges currently present induce on T a supergraph of $T_{r,h,j}^-$. However, since $j \geq s$, by Lemma 2.3 we can add all the remaining edges of $\binom{T}{r}$, including e , via a $T_{r,h,s}$ -template saturation process. Since e was arbitrary subject to $j(e) = j$, this proves the induction step.

For the last assertion of the lemma, simply notice that, by construction in Lemma 3.2, each $E_2(Q)$ is of size at most $rh^{r-s+2}t^{s-2}$. \square

Proof of Theorem 1.1. Let H be an r -graph with $|V(H)| = h$. Suppose first that $s(H) = 1$, and observe that in this case $\text{wsat}(n, H) \leq \binom{h}{r}$ holds for every $n \geq h$. Indeed, take a set of n vertices

and put a copy of K_h^r on h of the vertices. Pick any other vertex v not in the copy of K_h^r , and note that since $s(H) = 1$ adding an edge containing v and $r - 1$ of the vertices of K_h^r is guaranteed to form a copy of H . Hence there is an H -saturation process that starts with the initial K_h^r and ends with K_{h+1}^r . We can then turn the K_{h+1}^r into K_{h+2}^r etc, until we obtain a complete r -graph on the n vertices. We can thus define $C_H := \min\{\text{wsat}(n, H) : n \geq h\}$, and let $n_1 \geq h$ satisfy $\text{wsat}(n_1, H) = C_H$. By the same reasoning as above, we also have $\text{wsat}(n, H) \leq \text{wsat}(n_1, H)$ for every $n \geq n_1$ (we first obtain $K_{n_1}^r$ and then complete it to K_n^r). By minimality of C_H we must have $\text{wsat}(n, H) = \text{wsat}(n_1, H)$. Therefore, $\lim_{n \rightarrow \infty} \text{wsat}(n, H)/n^{s-1} = C_H$.

Hence, from now on let us assume that $s(H) = s \geq 2$. Let

$$C_H := \liminf_{n \rightarrow \infty} \text{wsat}(n, H)/n^{s-1}.$$

For brevity we shall write C for C_H . Recall that by Tuza's theorem (1.3), we know that for every large enough n we have $c_1 n^{s-1} \leq \text{wsat}(n, H) \leq c_2 n^{s-1}$ for some positive constants $c_2(H) \geq c_1(H) > 0$, implying that $C > 0$. We now claim that C satisfies the assertion of Theorem 1.1. To this end we prove that for every $\varepsilon > 0$ we have $\text{wsat}(n, H) \leq (C + 8\varepsilon)n^{s-1}$ for all large enough n .

Let $\varepsilon > 0$ satisfy $\varepsilon < \varepsilon_0(H)$ where ε_0 is chosen so as to satisfy the inequalities required in the proof, and let m_1 satisfy (i) $\text{wsat}(m_1, H) \leq (C + \varepsilon)m_1^{s-1}$ and (ii) $m_1 \geq m_0(\varepsilon, H)$ so as to satisfy the various inequalities we require in the proof below. Note that by our choice of C there are infinitely many values of m_1 satisfying condition (i) hence we can always find m_1 satisfying condition (ii) as well. Let $m = \lceil m_1^{1/(s-1)} \rceil^{s-1}$ be the next largest perfect $(s - 1)$ -st power. Since

$$m = m_1 + O(m_1^{(s-2)/(s-1)}),$$

we can deduce from Corollary 2.5 (with $k_1 = m_1$ and $k_2 = m - m_1$) that

$$\text{wsat}(m, H) \leq \text{wsat}(m_1, H) + O(m_1^{s-2} m_1^{(s-2)/(s-1)}) \leq (C + \varepsilon)m_1^{s-1} + \varepsilon m_1^{s-1} = (C + 2\varepsilon)m^{s-1}, \quad (3.1)$$

where the second inequality uses the fact that $m_1 \geq m_0(\varepsilon, H)$. We now claim that for all sufficiently large $n \geq n_0(m_1, \varepsilon, h)$ we have $\text{wsat}(n, H) \leq (C + 8\varepsilon)n^{s-1}$. To this end, it suffices to show that for every large enough n which is a multiple of $m^{1/(s-1)}$, we have

$$\text{wsat}(n, H) \leq (C + 7\varepsilon)n^{s-1}. \quad (3.2)$$

Indeed, assuming this, let n be arbitrary and set $n_1 = m^{1/(s-1)} \cdot \lfloor n/m^{1/(s-1)} \rfloor$. By Corollary 2.5 (with $k_1 = n_1$ and $k_2 = n - n_1 = O(m^{1/(s-1)})$) and (3.2) we would get that

$$\text{wsat}(n, H) \leq \text{wsat}(n_1, H) + O(n_1^{s-2} m^{1/(s-1)}) \leq (C + 7\varepsilon)n_1^{s-1} + \varepsilon n_1^{s-1} = (C + 8\varepsilon)n^{s-1},$$

where the second inequality uses the fact that $n \geq n_0(m, \varepsilon, h)$.

To prove (3.2) let m and n be as above, let V' be a set of $n/m^{1/(s-1)}$ vertices and let V be a set of n vertices, obtained by replacing each $v \in V'$ by a cluster S_v of $m^{1/(s-1)}$ vertices.

For all large enough $n \geq n_0(m, \varepsilon, h)$ by Rödl's theorem (Theorem 3.1, applied with $N = n/m^{1/(s-1)}$, $k = m^{1-1/(s-1)}$, $t = s - 1$ and $\delta = \varepsilon/C$) there is a collection \mathcal{D} of at most

$$(1 + \delta) \frac{\binom{n/m^{1/(s-1)}}{s-1}}{\binom{m^{1-1/(s-1)}}{s-1}} \leq (1 + 3\delta) \frac{n^{s-1}}{m^{s-1}}$$

subsets of V' of size $m^{1-1/(s-1)}$, so that each $(s-1)$ -tuple of vertices $\{v_1, \dots, v_{s-1}\} \subseteq V'$ belongs to at least one $D \in \mathcal{D}$. The inequality holds assuming⁵ $m \geq m_0(\varepsilon, H)$.

Define an r -graph G'_n as follows: go over all $D \in \mathcal{D}$ one by one in any order and apply the following procedure. Suppose $D = \{v_1, \dots, v_t\}$, where $t = m^{1-1/(s-1)}$ and let $S_D = S_{v_1} \cup \dots \cup S_{v_t}$ be the corresponding m vertices in V . By (3.1) there is a weakly saturated r -graph on m vertices with at most $(C + 2\varepsilon)m^{s-1}$ edges, denoted G_m ; put a copy of G_m on S_D . Let G'_n be the union over all S_D . Then, since $\delta = \varepsilon/C$ and assuming $\varepsilon < \varepsilon_0(H)$ we have

$$|E(G'_n)| \leq |\mathcal{D}||E(G_m)| \leq (1 + 3\delta) \frac{n^{s-1}}{m^{s-1}} (C + 2\varepsilon)m^{s-1} \leq (C + 6\varepsilon)n^{s-1}. \quad (3.3)$$

To complete the definition of G_n , we take $E(G_n) = E(G'_n) \cup E_2$, where E_2 is as in Lemma 3.3, with the parameters $\ell = n/m^{1/(s-1)}$, $t = m^{1/(s-1)}$ and the clusters $\{S_v : v \in V'\}$ playing the role of V_1, \dots, V_ℓ . By Lemma 3.3 we have

$$|E_2| \leq rh^{r-s+2} \binom{\ell-1}{s-1} t^{s-2} = rh^{r-s+2} \left(\frac{n}{m^{1/(s-1)}} - 1 \right) m^{\frac{s-2}{s-1}} = O\left(\frac{n^{s-1}}{m^{1/(s-1)}} \right) \leq \varepsilon n^{s-1},$$

where the last inequality assumes $m \geq m_0(\varepsilon, H)$. Combining this with (3.3) we have

$$|E(G_n)| \leq (C + 7\varepsilon)n^{s-1}.$$

Hence, to complete the proof of (3.2), it remains to describe an H -saturation process for G_n . Note by definition of G'_n , for each $D \in \mathcal{D}$ there is an H -saturation process for completing all hyperedges in S_D (namely, the H -saturation process of G_m , or of a supergraph of it). Since the sets in \mathcal{D} cover all $(s-1)$ -tuples $\{u_1, \dots, u_{s-1}\} \subseteq V'$, once all these processes are complete, we have all hyperedges $\{v_1, \dots, v_r\} \subseteq V$, hitting at most $s-1$ different sets S_u . Then, by Observation 2.2 and Lemma 3.3, our r -graph G_n is $T_{r,h,s}$ -template saturated, which by Lemma 2.1 implies it is weakly H -saturated. This completes the H -saturation process of G_n in K_n^r . \square

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⁵Indeed, denote $p = m^{1-1/(s-1)}$. If m is large enough so that $p - s \geq (1 - \frac{\delta}{2s})p$, then $\binom{n/m^{1/(s-1)}}{s-1} / \binom{p}{s-1} \leq (n^{s-1}/m) / \prod_{i=0}^{s-2} (p-i) \leq (n^{s-1}/m) / (1 - \delta/2s)^{s-1} p^{s-1} \leq (1 + \delta)n^{s-1}/m^{s-1}$.

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