Brownian motion — exercise sheet 1

Notation:

- $\{B(t) : t \in [0, \infty)\}$ is standard Brownian motion started at 0.
- For $a \in \mathbb{R}$ we denote $T_a = \inf\{t \geq 0 : B(t) = a\}$.

1. Set $T = \sup\{t \in [0, 1] : B(t) = 0\}$. Show that almost surely there exists times $t_n < s_n < T$ with $t_n \uparrow T$ such that
$$B(t_n) < 0 \quad \text{and} \quad B(s_n) > 0\,.$$
2. Show that for any $a \in \mathbb{R}$ almost surely $T_a < \infty$.
3. Define the (random) function $f : [0, \infty)$ defined by $f(a) = T_a$. Show that almost surely f is left-continuous and that the set of discontinuity points is countable and dense.
4. Let $\gamma_1, \ldots, \gamma_n$ be i.i.d. random variables distributed as T_1. Show that $n^{-2} \sum_{i=1}^{n} \gamma_i$ has the same distribution as T_1.
5. Let $\tau = \inf\{t \geq 0 : B(t) = \max_{s \in [0,1]} B(s)\}$. I.e., τ is the almost sure unique time in which the maximum of B in $[0, 1]$ is attained. Show that $\tau < 1$ almost surely and has distribution whose density is
$$g(t) = \frac{1}{\pi \sqrt{t(1-t)}} \mathbf{1}_{t \in [0,1]}\,.$$