Brownian motion — exercise sheet 3

Due Wednesday, June 16th. Notation:

- $\{B(t) : t \in [0, \infty)\}$ is standard Brownian motion started at 0, in $\mathbb{R}$ unless stated otherwise (as in questions 5 and 7).
- $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$.

1. Let $h : [0, \infty) \to \mathbb{R}$ be a fixed continuous function. Show that $\int_0^t h(s)dB_s$ has normal distribution with mean 0 and variance $\int_0^t h(s)^2 ds$.

2. Let $t > 0$ be fixed. Find the distribution of $\int_0^t B(s)ds$.

3. Let $T \in (0, \infty)$ be fixed. Show that almost surely

$$\lim_{\beta\to\infty} \sup_{t\in[0,T]} e^{-\beta t} \int_0^t e^{\beta s} dB_s = 0\,.$$

4. Assume $A(t)$ is a progressively measurable process (with respect to the Brownian filtration) in L^1, that is, $\mathbb{E}\int_0^T |A_t|dt < \infty$ for some fixed $T > 0$. Show that the process $\int_0^t A(s)ds$ $(t \leq T)$ is a martingale if and only if $A(t) = 0$ almost surely, for Lebesgue almost every $t \in [0, T]$.

5. Suppose $f : \mathbb{D} \to f(\mathbb{D})$ is a conformal map with $f(0) = 0$ and Taylor expansion $f(z) = \sum_{k=1}^{\infty} a_k z^k$. Let B_t be standard Brownian motion in $\mathbb{C}$ and $\tau = \inf\{t \geq 0 : B_t \notin f(\mathbb{D})\}$. Show that

$$\mathbb{E}_0\tau = \frac{1}{2}\sum_{k=1}^{\infty} |a_k|^2\,.$$

6. (a) Let $\Omega = \left\{x + iy \in \mathbb{C} : -\frac{\pi}{4} < x < \frac{\pi}{4}\right\}$. Show that the map

$$f(z) = \tan^{-1}(z) = z - \frac{z^3}{3} + \frac{z^5}{5} - \frac{z^7}{7}\dots$$

   conformally maps $\mathbb{D}$ onto Ω. [Hint: first map Ω into a half space using an exponential map, then into the disc using a Möbius map.]

   Remark: You will get full marks for this exercsise it even if you don't answer.

   (b) Derive that $\frac{\pi^2}{8} = 1 + 3^{-2} + 5^{-2} + \cdots$.

   (c) Conclude that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}\,.$$

7. Let Ω be a simply connected domain of area π and such that $0 \in \Omega$. Let $\tau = \inf\{t \geq 0 : B_t \notin \Omega\}$ where B_t is standard Brownian motion in $\mathbb{C}$. Show that $\mathbb{E}_0\tau$ is maximized among all possible Ω's at $\mathbb{D}$. (You may use Riemann's mapping theorem: there is a conformal map from $\mathbb{D}$ onto Ω mapping 0 to 0).