Complex analysis 2
Homework #2
Due date: Thursday in class, April 11th, 2019

$\mathbb{R}$ = real numbers; $\mathbb{C}$ = complex plane; $\mathbb{D} = \{z : |z| < 1\}$.

1. Suppose that $f : \mathbb{C} \to \mathbb{C}$ is entire and not a translation (i.e., it is not of the form $f(z) = az + b$ for fixed $a, b \in \mathbb{C}$). Show that $f \circ f$ has a fixed point. I.e., show that there exists $z_0 \in \mathbb{C}$ such that $f(f(z_0)) = z_0$.

2. Let $\Omega = \mathbb{D} \setminus \{0\}$. Show that there does not exists $h : \Omega \to \mathbb{R}$ such that h is harmonic in Ω and $\lim_{z\to 0} h(z) = 0$ and $\lim_{z\to\partial\mathbb{D}} h(z) = 1$.

3. Let $h : \Omega \to \mathbb{R}$ be a harmonic function. Assume that $|h(z)| \leq M$ for all $z \in \Omega$. Show that $|\nabla u(z)| \leq \frac{2M}{\operatorname{dist}(z,\partial\Omega)}$ where $\operatorname{dist}(z, \partial\Omega)$ is the distance of z from $\partial\Omega$ and $|\nabla u(z)|$ is the norm of the gradient of u at z (that is, $|\nabla u(z)| = \sqrt{u_x(z)^2 + u_y(z)^2}$).

4. Let $h : \partial\mathbb{D} \to \mathbb{R}$ be a continuous function and $u : \mathbb{D} \to \mathbb{R}$ defined by the Poisson formula
$$u(a) = \frac{1}{2\pi}\int_0^{2\pi} \frac{1-|a|^2}{|e^{it}-a|^2} h(e^{it})dt\,.$$
Show that $\lim_{a\to\xi} u(a) = h(\xi)$ for any $\xi \in \partial\mathbb{D}$.

5. Let $h : \partial\mathbb{D} \to \mathbb{R}$ be a continuous function and let $a \in \mathbb{D}$. For each point $e^{i\theta} \in \partial\mathbb{D}$, let $e^{i\theta^*}$ be the point in $\partial\mathbb{D}$ such that $a, e^{i\theta}, e^{i\theta*}$ lie on a straight line. Show that the harmonic extension $u : \mathbb{D} \to \mathbb{R}$ of h is given by
$$u(a) = \frac{1}{2\pi}\int_0^{2\pi} h(e^{i\theta^*})d\theta\,.$$

6. Suppose $u : \mathbb{D} \to \mathbb{R}$ is harmonic and $u(z) \geq 0$ for all $z \in \mathbb{D}$. Show that for any $z \in \mathbb{D}$
$$\frac{1-|z|}{1+|z|}u(0) \leq u(z) \leq \frac{1+|z|}{1-|z|}u(0)\,.$$

7. Suppose that $u : \mathbb{D} \setminus \{0\} \to \mathbb{R}$ is harmonic and that $\lim_{z\to 0} u(z)$ is either $+\infty$ or $-\infty$. Show that u can be written as
$$u(z) = \beta \ln|z| + u_1(z)\,,$$
where $\beta \neq 0$ is real, and $u_1 : \mathbb{D} \to \mathbb{R}$ is harmonic in $\mathbb{D}$. [Hint: show that the residue of $u_x - iu_y$ is real]