Planar maps, random walks and circle packing — exercise sheet 1

This assignment is due on Monday, November 27th.

1. Let $G_z(a, x)$ be the Green's function defined in class, that is,

$$G_z(a, x) = \mathbb{E}_a\big[\#\text{visits to } x \text{ before visiting } z\big] .$$

   Show that the function $h(x) = G_z(a, x)/\pi(x)$ is harmonic.

2. Show that the effective resistance satisfies the triangle inequality. That is, for any three vertices x, y, z we have $R_{\text{eff}}(x \leftrightarrow z) \leq R_{\text{eff}}(x \leftrightarrow y) + R_{\text{eff}}(y \leftrightarrow z)$.

3. Let a, z be two vertices of a finite network and let τ_a, τ_z be the first visit time to a and z, respectively, of the weighted random walk. Show that for any vertex x

$$\mathbf{P}_x(\tau_a < \tau_z) \leq \frac{R_{\text{eff}}(x \leftrightarrow \{a, z\})}{R_{\text{eff}}(x \leftrightarrow a)} .$$

4. Consider the following tree T. At height n it has 2^n vertices (the root is at height $n = 0$) and if $(v_1, \ldots, v_{2^n})$ are the vertices at level n we make it so that v_k has 1 child at level $n+1$ and if $1 \leq k \leq 2^{n-1}$ and v_k has 3 children at level $n+1$ for all other k.

   (a) Show that T is recurrent.

   (b) Show that for any disjoint edge cutsets Π_n we have that $\sum_n |\Pi_n|^{-1} < \infty$. (So, the Nash-Williams criterion for recurrence is not sharp)

5. (a) Let G be a finite planar graph with two distinct vertices $a \neq z$ such that a, z are on the outer face. Consider an embedding of G so that a is the left most point on the real axis and z is the right most point on the real axis. Split the outer face of G into two by adding the ray from a to $-\infty$ and the ray from z to $+\infty$. Consider the dual graph G^* of G and write a^* and z^* for the two vertices corresponding to the split outer face of G. Assume that all edge resistances are 1. Show that

$$R_{\text{eff}}(a \leftrightarrow z; G) = \frac{1}{R_{\text{eff}}(a^* \leftrightarrow z^*; G^*)} .$$

   (b) Show that the probability that a simple random walk on $\mathbb{Z}^2$ started at $(0, 0)$ has probability $1/2$ to visit $(0, 1)$ before returning to $(0, 0)$.