**Planar maps, random walks and circle packing — exercise sheet 2**

This assignment is due on Monday, December 18th.

1. Let $G$ be a finite triangulation. Without using the circle packing theorem, show that if $G$ is simple (that is, has no loops or parallel edges), then removing any two vertices from $G$ (and all the edges touching them) results in a connected graph.

2. Let $G$ be a finite simple planar map such that all of its faces have 3 edges except for the outer face which is a simple cycle. Show that there exists a circle packing of $G$ such that all the circles are inside the unit disc $\{z : |z| \leq 1\}$ and all the circles corresponding to the vertices of the outer face are tangent to the unit circle $\{z : |z| = 1\}$.

3. Let $G$ be a plane triangulation such that all vertex degrees are 6. Prove that $G$ is recurrent.

4. Let $G$ be a plane triangulation that can be circle packed in the unit disc $\{z : |z| < 1\}$. Show that $G$ is transient. (Note that $G$ may have *unbounded* degrees)

5. Let $P$ be a circle packing of a finite simple planar map with degree bounded by $D$ such that all of its faces are triangles except for the outerface. Assume that the carrier of $P$ is contained in $[-11, 11]^2$, contains $[-10, 10]^2$ and that all circles have radius at most 1. Let $h$ be a function taking the value 1 on all vertices with centers left of the line $\{-10\} \times \mathbb{R}$, taking the value 0 on all vertices with centers right of the line $\{10\} \times \mathbb{R}$, and is harmonic anywhere else. Assume $x$ and $y$ are two vertices such that their centers are contained in $[-1, 1]^2$ and that the Euclidean distance between these centers is at most $\epsilon > 0$. Show that

$$|h(x) - h(y)| \leq \frac{C}{\log(1/\epsilon)} ,$$

for some constant $C = C(D) > 0$ independent of $\epsilon$. [Hint: assume $h(x) < h(y)$ and consider the sets $A = \{v : h(v) \leq h(x)\}$ and $B = \{v : h(v) \geq h(y)\}$].