Random walks on random fractals — exercise sheet 1

1. Consider a finite network G with distinct vertices $a \neq z$. Consider the weighted random walk starting at a and stopped when it reaches z. For a directed edge $e = (x, y)$ write $\theta(e) = \mathbb{E}[\text{net crossings of e}]$ (where by net crossings of e we mean we the number of crossing of (x, y) minus the number of crossings of (y, x)). Show that θ is the unit current flow.

2. Show that in the box $[0, n] \times [0, n] \subset \mathbb{Z}^2$ with unit resistances the effective resistance between $(0, 0)$ and (n, n) is $\Theta(\log n)$.

3. Let $G_z(a, x)$ be the Green's function defined in class, that is,

$$G_z(a, x) = \mathbb{E}_a\left[\#\text{visits to } x \text{ before visiting } z\right].$$

Show that the function $h(x) = G_z(a, x)/\pi(x)$ is harmonic.

4. Show that the effective resistance satisfies the triangle inequality. That is, for any three vertices x, y, z we have $R_{\text{eff}}(x \leftrightarrow z) \leq R_{\text{eff}}(x \leftrightarrow y) + R_{\text{eff}}(y \leftrightarrow z)$.

5. Let a, z be two vertices of a finite network and let τ_a, τ_z be the first visit time to a and z, respectively, of the weighted random walk. Show that for any vertex x

$$\mathbf{P}_x(\tau_a < \tau_z) \leq \frac{R_{\text{eff}}(x \leftrightarrow \{a, z\})}{R_{\text{eff}}(x \leftrightarrow a)}.$$

6. (**Extremal length**) Let $G = (V, E)$ be a finite network with edge weights $\{c(e)\}_{e \in E}$. Given an assignment of non-negative edge lengths $\ell : E \to [0, \infty)$ the *distance* between two vertices x and y, denoted $\text{dist}_\ell(x, y)$ is the minimum over all x to y paths of the sum of the edge lengths over the path. Prove that

$$R_{\text{eff}}(a \leftrightarrow z) = \max_\ell \left\{ \frac{\text{dist}_\ell^2(a, z)}{\sum_e c(e)\ell(e)^2} \right\}.$$

7. Consider the following tree T. At height n it has 2^n vertices (the root is at height $n = 0$) and if $(v_1, \ldots, v_{2^n})$ are the vertices at level n we make it so that v_k has 1 child at level $n + 1$ and if $1 \leq k \leq 2^{n-1}$ and v_k has 3 children at level $n + 1$ for all other k.

   (a) Show that T is recurrent.

   (b) Show that for any disjoint edge cutsets Π_n we have that $\sum_n |\Pi_n|^{-1} < \infty$. (So, the Nash-Williams criterion for recurrence is not sharp)

8. (a) Let G be a finite planar graph with two distinct vertices $a \neq z$ such that a, z are on the outer face. Consider an embedding of G so that a is the left most point on the real axis and z is the right most point on the real axis. Split the outer face of G into two by adding the ray from a to $-\infty$ and the ray from z to $+\infty$. Consider the dual graph G^* of G and write a^* and z^* for the two vertices corresponding to the split outer face of G. Assume that all edge resistances are 1. Show that

$$R_{\text{eff}}(a \leftrightarrow z; G) = \frac{1}{R_{\text{eff}}(a^* \leftrightarrow z^*; G^*)}.$$

   (b) Show that the probability that a simple random walk on $\mathbb{Z}^2$ started at $(0, 0)$ has probability $1/2$ to visit $(0, 1)$ before returning to $(0, 0)$.

9. Let $G = (V, E)$ be a graph so that $V = \mathbb{Z}$ and the edge set $E = \cup_{k \geq 0} E_k$ where $E_0 = \{(i, i+1) : i \in \mathbb{Z}\}$ and for $k > 0$

$$E_k = \Big\{ \big(2^k(n - 1/2), 2^k(n + 1/2)\big) : n \in \mathbb{Z} \Big\} .$$

Is G recurrent or transient?

10. Consider a finite network and let τ_u be the first time the random walker visits u. Show that for any three vertices a, x, z of the network we have

$$\mathbf{P}_x(\tau_z < \tau_a) = \frac{R_{\text{eff}}(a \leftrightarrow x) - R_{\text{eff}}(x \leftrightarrow z) + R_{\text{eff}}(a \leftrightarrow z)}{2R_{\text{eff}}(a \leftrightarrow z)} .$$

11. Show that in every finite network

$$\mathbb{E}_a[\tau_z] = \frac{1}{2} \sum_{x \in V} \pi(x)[R_{\text{eff}}(a \leftrightarrow z) + R_{\text{eff}}(z \leftrightarrow x) - R_{\text{eff}}(x \leftrightarrow a)] .$$