HYPERGEOMETRIC ORTHOGONAL POLYNOMIALS OF JACOBI TYPE

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Abstract. Motivated by the theory of hypergeometric orthogonal polynomials, we consider *quasi-orthogonal* polynomial families, *i.e.* those that are orthogonal with respect to a non-degenerate bilinear form defined by a linear functional, and in which the ratio of successive coefficients is given by a rational function $f(u, s)$ which is polynomial in u. We call this a family of Jacobi type.

Our main result is that there are precisely *five* families of Jacobi type. These are the classical families of Jacobi, Laguerre and Bessel polynomials, and two more one parameter families $E_n^{(c)}, F_n^{(c)}$. The last two families can be expressed through Lommel polynomials, and they are orthogonal with respect to a positive measure on $\mathbb{R}$ for $c > 0$ and $c > -1$ respectively. Each of the five families can be obtained as a suitable specialization of some hypergeometric series.

1. Introduction

1.1. **Orthogonal Polynomials.** Let $\mathcal{P} = (P_0, P_1, ..., P_n....)$ be a family of polynomials in $\mathbb{C}[z]$, where P_i is a monic polynomial of degree i.

We say that the family $\mathcal{P}$ is **quasi-orthogonal** if there exists a linear functional $M : \mathbb{C}[z] \to \mathbb{C}$ such that the matrix $Q_{ij} = M(P_i \cdot P_j)$ is a diagonal invertible matrix. In other words this means that $Q_{ij} = 0$ if $i \neq j$ and $Q_{ij} \neq 0$ if $i = j$.

The functional M is usually called a *moment functional*. It is easily seen that the moment functional M completely determines a quasi-orthogonal family $\mathcal{P}$. Conversely, the moment functional is uniquely determined by the family $\mathcal{P}$, and the normalization $M(1) = 1$.

We say that the family $\mathcal{P}$ is **orthogonal** if all polynomials P_i are defined over $\mathbb{R}$ and there exists a moment functional M defined over $\mathbb{R}$ such that all the diagonal entries Q_{ii} are real and strictly positive. It is known that in this case the moment functional M is defined by integration with some positive measure μ on $\mathbb{R}$ that has an infinite support, and $\mathcal{P}$ is the standard family of orthogonal polynomials defined by this measure (see *e.g.* [Chi78, Ch. II] or [Ism09, Ch. 2]).

There are many classical (quasi) orthogonal families of polynomials. These families play an important role in many areas of Mathematics, Physics and Engineering (see §1.2 below).

The characteristic property of these classical families is that they can be described in terms of hypergeometric functions. There is a general idea to axiomatize the notion of a hypergeometric (quasi)orthogonal family and try to classify all such families. One such attempt is presented by Askey-Wilson scheme (see §1.2 below). We will see that this scheme misses some natural families. The difficulty is that the notion of a hypergeometric family is not easy to define formally, so it is difficult to axiomatize.

In this paper we propose an axiomatic description and classification of some class of hypergeometric (quasi)orthogonal families of polynomials that we call the families of Jacobi type.

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Let us write our polynomials explicitly in terms of their coefficients $P_n = \sum c(n,k)z^k$. We extend the domain of definition of the coefficients $c(n,k)$ to $\mathbb{Z}^2$ by setting $c(n,k) = 0$ outside of the range $0 \le k \le n$. Roughly speaking, we say that the family $\mathcal{P}$ of polynomials is of *rational type* if there exists a rational function $f(u,s)$ such that $c(n,k+1) \equiv f(n,k)c(n,k)$. In this case our polynomials are (up to constants) hypergeometric functions of the variable z (with some parameters). We will say that the family $\mathcal{P}$ of polynomials is of Jacobi type if it is of rational type, and f depends polynomially on the first variable u. Since the function f can have poles we need to be more careful.

Definition 1.1. *Let $\mathcal{P}$ be a family of monic polynomials. We say that this family is of* ***Jacobi type*** *if it is quasi-orthogonal and there exist relatively prime polynomials $N(u,s)$ and $D(s)$ such that*

$$c(n,k+1)D(k) \equiv c(n,k)N(n,k). \tag{1}$$

Remark. *A priori the polynomials N and D could have a finite number of common zeroes. This would mean that we do not have conditions on a finite number of coefficients $c(n,k)$. However, in Lemma 3.1 below we show that the polynomials N and D determine the Jacobi type family uniquely.*

To formulate our main result, let us recall the definition of the hypergeometric series

$${}_pF_q(\underline{a};\underline{b};z) = \sum_{k\ge 0} \frac{(a_1)_k \cdots (a_p)_k}{(b_1)_k \cdots (b_q)_k k!} z^k, \quad (c)_k := c(c+1)\cdots(c+k-1). \tag{2}$$

and note that if we set $a_1 = -n$ then the infinite series (2) truncates to give a polynomial of degree $\le n$. If the n-th coefficient of this polynomial is not 0 then we can divide by it to obtain a monic polynomial of degree n. It turns out that all the Jacobi type families can be obtained in this manner.

Theorem A. *There are precisely* five *polynomial families of Jacobi type. Each can be obtained as above from the indicated hypergeometric series, and is orthogonal for the indicated parameter range:*

(1) Jacobi: ${}_2F_1(-n, n+a; b; z)$; $\quad a > b-1,\ a > 0,\ b > 0$
(2) Bessel: ${}_2F_0(-n, n+a; ; z)$; $\quad a > 0$
(3) Laguerre: ${}_1F_1(-n; b; z)$; $\quad b > 0$
(4) $E_n^{(c)}$: ${}_4F_1(-n, -n-c+1, n+c, n+1; 1/2; z)$; $\quad c > 0$
(5) $F_n^{(c)}$: ${}_4F_1(-n, -n-c+1, n+c+1, n+2; 3/2; z)$; $\quad c > -1$

The Jacobi, Bessel, and Laguerre families are, of course, classical. It turns out that $E_n^{(c)}$ and $F_n^{(c)}$ can be expressed through the Lommel polynomials $h_n^{(c)}$ ([Wat44, §9.6], [Ism09, §6.5]), as follows

$$E_n^{(c)}(-z^2) = \frac{(-1)^n}{2^{2n}c_{2n}} h_{2n}^{(c)}, \quad zF_n^{(c)}(-z^2) = \frac{(-1)^n}{2^{2n+1}c_{2n+1}} h_{2n+1}^{(c)}. \tag{3}$$

We recall that the Lommel polynomials $h_n^{(c)}$satisfy the following remarkably simple recursion:

$$h_{n+1}^{(c)} = 2(c+n)zh_n^{(c)} - h_{n-1}^{(c)}. \tag{4}$$

However, being alternately even and odd, they have many vanishing coefficients, and thus they are not a Jacobi type family. In this respect the $h_n^{(c)}$ are similar to the Hermite polynomials, which are not of Jacobi type, but which can be obtained as in (3) from two Laguerre families.

1.2. **Background, related results, future directions.** The theory of orthogonal polynomials on the real line is a classical subject with applications to many areas of mathematics. We refer the reader to [Wat44, WW21] for the classical theory and to [Ism09, KWKS10, KLS10, AAR99, GR04, Gro78] for more recent developments.

It is instructive to compare our result with that of Bochner [Boc29], who classified second order differential operators with polynomial eigenfunctions. The classic Sturm-Liouville theorem provides sufficient conditions for such eigenfunctions to constitute an orthogonal family, and in Bochner's case this yields the Jacobi, Laguerre, Bessel, and Hermite polynomials. Note however that the eigenfunction condition is a very strong requirement, which has the immediate effect of restricting the set of possible families to lie in a three dimensional parameter space, and which makes the classification problem quite a bit easier [Ism09]. In contrast, the classification result in Theorem A is obtained in the infinite dimensional parameter space corresponding to the coefficients of $f(u,s)$ in $\mathbb{C}(s)[u]$. Thus it is remarkable that there are only *two* additional families in this general setting, both of which are hypergeometric orthogonal polynomials.

There are other ways of obtaining orthogonal families from hypergeometric functions and from their q-analogs, the basic hypergeometric functions ${}_r\phi_s$. The well-known Askey scheme [KLS10] is a hierarchical organization of some 44 families which arise in this manner. Most of these families can be obtained by limiting procedures applied to the Askey-Wilson polynomial, which is a ${}_4\phi_3$ family that depends on four parameters a, b, c, d in addition to q. Ismail [Ism09] describes a generalization of Bochner's theorem to second order q-difference operators and shows orthogonal families arising in this way are limits of Askey-Wilson polynomials. More recently Verde-Star [V-S21] has shown that almost all of the families in the Askey scheme belong to a class of families characterized by three recursive sequences. Still more recently, Koornwinder [Koo22] has obtained abstract characterization theorems for the Askey scheme within the Verde-Star class.

Just as in the Bochner setting, the assumptions of Ismail and Verde-Star immediately restrict the possible families to lie in a finite dimensional parameter space – 6 dimensions for Ismail and 11 dimensions for Verde-Star. It seems to us however that one can hope for a classification of hypergeometric and q-hypergeometric orthogonal families in a much greater generality, and we plan to return to this question in subsequent work.

We note also that the Askey-Wilson polynomials and their various limits can be understood in terms of a certain representation of a double affine Hecke algebra (DAHA) of rank 1 [Sah99, Sah00]. Thus one may ask whether the new polynomials can be understood in terms of a generalization of the DAHA theory – either a more general DAHA or a more general representation of the same DAHA.

1.3. **Structure of the paper and ideas of the proofs.** Our proof of Theorem A is purely algebraic. The main point of our argument is that starting with a rational family $\mathcal{P}$ we construct some algebraic object - a module M - that encodes this family. This module has much more structure than the family $\mathcal{P}$, so it is much easier to analyze.

More precisely, consider the field $K = \mathbb{C}(u,s)$ of bivariate rational functions. Denote by A the subalgebra of $\mathrm{End}_{\mathbb{C}}(K)$ generated by multiplication operators and the shift operators $S^{\pm} : s \mapsto s \pm 1$, $U^{\pm} : u \mapsto u \pm 1$. Starting with a family $\mathcal{P}$, defined by a rational function $f \in K$, we construct an A-module $M(f)$. The condition that the family $\mathcal{P}$ is quasi-orthogonal can be written as 3-term relation. This gives very strong restrictions on the module $M(f)$.

For example, the first major result is that this module is one dimensional over the field K. After this we use a general classification of one dimensional modules to fix the shape of this module.

In §2 we give some necessary preliminaries, including a classification result of one-dimensional A-modules from [Ore30] (see Theorem 2.3 below), and Favard's theorem, which states that a family of monic polynomials is quasi-orthogonal if and only if there exist sequences $\alpha_0(n)$ and $\alpha_{-1}(n)$ of complex numbers such that

$$zP_n = P_{n+1} + \alpha_0(n)P_n + \alpha_{-1}(n)P_{n-1} \text{ for all } n \geq 1; \tag{5}$$

In §3 we construct a one-dimensional A-module from the coefficient function $c(n,k)$. First we express the $\alpha_0(n)$ and $\alpha_{-1}(n)$ through f, implying that they are rational functions of n. Then we consider the space of all $\mathbb{C}$-valued functions on $\mathbb{Z}^2$ as a module over $\mathbb{C}[u,s]$, and extend scalars to obtain a vector space over the field of rational functions $K = \mathbb{C}(u,s)$. This vector space has a natural structure of an module A-module. Then we consider the submodule M generated by the image of $c(n,k)$, that we denote by w. By (5), it is spanned by w and $U^{-1}w$, since Uw is linearly dependent on $w, U^{-1}w$, and $S^{-1}w$, but $S^{-1}w = (S^{-1}f)w$ by the construction of w. Thus $\dim_K M \leq 2$, but in Theorem 3.3 below we show that $\dim_K M = 1$.

To prove this theorem we write Uw as the span $Uw = \gamma U^{-1}w + \delta w$, and apply the shift operator S. Using the relation $US = SU$ we obtain that SUw can be expressed as two different linear combinations of w and $U^{-1}w$. This shows that w and $U^{-1}w$ are linearly dependent, and thus each of them spans M.

In §4 we use the explicit realization from Theorem 2.3 to almost classify all pairs M, v, where M is a one-dimensional A-module, and v is a vector satisfying a three-term relation. Namely, we realize v as a function

$$\Phi = \exp(cu + ds)g(s,u)\prod\Gamma(k_iu + l_is + c_i),$$

where $g \in K = \mathbb{C}(u,s)$, satisfying

$$S^{-1}\Phi = U\Phi + \alpha_0\Phi + \alpha_{-1}U^{-1}\Phi, \tag{6}$$

For every i, Φ has an infinite family of poles in the lines $k_iu + l_is + c_i = n$ for $n \in \mathbb{Z}_{\geq 0}$. Any of the poles of one of the terms in (6) has to be a pole of at least one other term. Thus for $a = k_iu + l_is + c_i$ we obtain that two of the functions $S^{-1}a, Ua, a, U^{-1}a$ coincide, and thus $k_i, l_i \in \{-1, 0, 1\}$. Further analyzing the poles we obtain that there exist polynomials $p, q, w \in \mathbb{C}[t]$ and $g \in \mathbb{C}[u,s]$ such that

$$f = S\Phi/\Phi = \frac{q(u+s)p(s-u)Sg}{w(s)g} \tag{7}$$

In the rest of the section we use elementary algebraic considerations to deduce from (6) strong restrictions on p, q, and w (see Theorem 4.3 and Corollary 4.8).

In §5 we prove Theorem A. For this purpose we show that the condition that f depends polynomially on u implies that in (7) we can assume that $g = 1$. Then we use Corollary 4.8 to classify all possible w, p, and q. Finally, we use hypergeometric functions to construct an example for each case, completing the proof of Theorem A. In §5.1 we explicitly compute the coefficients α_i in the 3-term relation for each of the cases.

In §6 we investigate the two new families $E_n^{(c)}$ and $F_n^{(c)}$, express them through Lommel polynomials, and find the discrete measures on the real line with respect to which they are orthogonal. We also give the 4th order differential equations that they satisfy.

In Appendix A we prove some technical lemmas from §3.

2. Preliminaries

2.1. **Favard's theorem.** Let $\{P_n\}_{n=0}^{\infty}$ be a family of monic polynomials.

Theorem 2.1 (see *e.g.*[Chi78, §4])**.**

(i) The family $\{P_n\}$ is quasi-orthogonal if and only if there exist (unique) sequences $\alpha_0(n)$ and $\alpha_{-1}(n)$ of complex numbers, and $c_0 \in \mathbb{C}$ such that

$$zP_n = P_{n+1} + \alpha_0(n)P_n + \alpha_{-1}(n)P_{n-1}\ \forall n \in \mathbb{Z}_{\geq 1}; \quad P_0 = 1,\ \textit{and}\ P_1 = z + c_0, \tag{8}$$

and $\alpha_{-1}(n) \neq 0$ for all $n \in \mathbb{Z}_{\geq 1}$.

(ii) The family $\{P_n\}$ is orthogonal if and only if: it is quasi-orthogonal, all the coefficients of all the polynomials P_n are real numbers, and for all $n \in \mathbb{Z}_{\geq 1}$ we have $\alpha_0(n), \alpha_{-1}(n) \in \mathbb{R}$, and $\alpha_{-1}(n) > 0$.

2.2. **One-dimensional A-modules.** Recall the notation $K = \mathbb{C}(u, s)$ and $A :=$ the subalgebra of $\mathrm{End}_{\mathbb{C}}(K)$ generated by K and the shift operators $U^{\pm 1} : u \mapsto u \pm 1$; $S^{\pm 1} : s \mapsto s \pm 1$. The $S^{\pm 1}, U^{\pm 1}$ act on K by $(U^{\pm 1}g)(u, s) := g(u \pm 1, s)$ and $(S^{\pm 1}g)(u, s) := g(u, s \pm 1)$. We say that an A-module is *one-dimensional* if it is one-dimensional as a vector space over K.

For every one-dimensional module M and every non-zero vector $v \in M$ there exist unique $f, h \in K$ such that $Sv = fv$ and $Uv = hv$. It is easy to see that

$$Uf/f = Sh/h \tag{9}$$

and that if we replace v by gv for some non-zero $g \in K$, the pair (f, h) will change to $(fSg/g, hUg/g)$. We will call such pairs equivalent. Conversely, it is easy to see that every pair $(f, h) \in K^{\times} \times K^{\times}$ satisfying (9) defines a one-dimensional A-module. The set of isomorphism classes of one-dimensional A-modules forms a group under tensor product. This corresponds to elementwise products of pairs (f, h). This group was computed in [Ore30]. To formulate this result, note that the field of meromorphic functions in u, s has a natural A-module structure.

Definition 2.2. *A meromorphic function η in two variables u, s is of γ type if it is a product of the form*

$$\exp(au + bs)\prod_{i=1}^{n}\Gamma(k_iu + l_is + c_i), \quad a, b, c_i \in \mathbb{C},\ (k_i, l_i)\ \textit{coprime in}\ \mathbb{Z}^2, \tag{10}$$

where $\Re c_i \in [0, 1)\,\forall i$, and if $(k_i, l_i) = (-k_j, -l_j)$ then $c_i \neq 1 - c_j$.

It is easy to see that for every η of γ type, the space $K\eta$ spanned by it is invariant under $U^{\pm 1}$ and $S^{\pm 1}$, and thus is an A-module.

Theorem 2.3 ([Ore30], see also [Sab05, Proposition 1.7])**.** *The correspondence $\eta \mapsto K\eta$ is an isomorphism between the group of functions of γ type and the group of isomorphism classes of 1-dimensional A-modules.*

Remark 2.4. *The formulations of the theorem in* [Ore30] *and in* [Sab05, Proposition 1.7] *are slightly different from ours. For example, in* [Sab05, Proposition 1.7] *negative powers of $\Gamma(k_iu + l_is + c_i)$ are allowed. On the other hand,* [Sab05, Proposition 1.7] *reduces the set of allowed pairs (k_i, l_i) to include exactly one of $\{(k, l), (-k, -l)\}$ for any coprime $(k, l) \in \mathbb{Z}^2$. The equivalence of the two formulations follows from the Euler's reflection formula:*

$$\Gamma(x)\Gamma(1 - x) = \frac{\pi}{\sin(\pi x)} = \frac{2\pi i}{1 - \exp(-2\pi i x)}\exp(-i\pi x) = \phi(x)\exp(-i\pi x), \tag{11}$$

where ϕ is a periodic function with period 1. Thus multiplication by $\phi(ku+ls+c)\exp(-i\pi c)$ defines an isomorphism between $\langle\Gamma(ku+ls+c)^{-1})\rangle$ and $\langle\Gamma(-ku-ls+1-c)\exp(i\pi ku+i\pi ks)\rangle$.

3. From hypergeometric polynomials to 1-dimensional A-modules

Let $P_n(z)=\sum c(n,k)z^k$ be a family of polynomials of Jacobi type. In particular, there exist relatively prime polynomials $N(u,s)$ and $D(s)$ such that

$$c(n,k+1)D(k)\equiv c(n,k)N(n,k)\quad \forall n,k\in\mathbb{Z} \tag{12}$$

Recall that we extend the domain of definition of the coefficients $c(n,k)$ to $\mathbb{Z}^2$ by setting $c(n,k)=0$ outside of the range $0\leq k\leq n$. A central role in this paper is played by the rational function

$$f(u,s):=\frac{N(u,s)}{D(s)} \tag{13}$$

Lemma 3.1 (Appendix A). *(i) There exist polynomials $x\in\mathbb{C}[u,s]$ and $y\in\mathbb{C}[s]$ such that $N(u,s)=(u-s)x(u,s)$, $D(s)=(s+1)y(s)$, and $y(k)\neq 0$ for any $k\in\mathbb{Z}_{\geq 0}$.*
(ii) For all $n\geq k\geq 0$ we have $c(n,k)\neq 0$, and for all $n>k\geq 0$ we have $N(n,k)\neq 0$.
(iii) For every $n\geq k\geq 0$ we have $c(n,k)=\prod_{i=k}^{n-1}f(n,i)^{-1}$.

We postpone the proof of this technical lemma to Appendix A.

By Favard's theorem (Theorem 2.1 above), there exist (unique) sequences $\{\alpha_0(n)\}_{n=1}^{\infty}$ and $\{\alpha_{-1}(n)\}_{n=1}^{\infty}$ of complex numbers such that $\alpha_{-1}(n)\neq 0$ for all $n\in\mathbb{Z}_{\geq 1}$ and

$$zP_n=P_{n+1}+\alpha_0(n)P_n+\alpha_{-1}(n)P_{n-1}\quad \forall n\in\mathbb{Z}_{\geq 1} \tag{14}$$

Lemma 3.2. *The functions $\alpha_i(n)$ are rational functions. Moreover, we have*

$$\alpha_0(n)=f(n,n-1)^{-1}-f(n+1,n)^{-1} \tag{15}$$

and

$$\begin{aligned}\alpha_{-1}(n)=f(n,n-2)^{-1}f(n,n-1)^{-1}-f(n+1,n-1)^{-1}f(n+1,n)^{-1}-\\ -f(n,n-1)^{-2}+f(n,n-1)^{-1}f(n+1,n)^{-1}\end{aligned} \tag{16}$$

Proof. Looking at the coefficient of z^n in (14) we have

$$c(n,n-1)=c(n+1,n)+\alpha_0(n)$$

Thus we have

$$\alpha_0(n)=\frac{c(n,n-1)-c(n+1,n)}{c(n,n)}=f(n,n-1)^{-1}-f(n+1,n)^{-1}\in\mathbb{C}(n) \tag{17}$$

From the coefficient of z^{n-1} we have $c(n,n-2)=c(n+1,n-1)+\alpha_0(n)c(n,n-1)+\alpha_{-1}(n)$. This implies (16). □

Let L be the $\mathbb{C}[u,s]$-module of all $\mathbb{C}$-valued functions on $\mathbb{Z}^2$, with the action given by restriction of any polynomial in $\mathbb{C}[u,s]$ to $\mathbb{Z}^2$, by substitution of n for u and k for s. Let N be the extension of scalars $N:=L\otimes_{\mathbb{C}[u,s]}\mathbb{C}(u,s)$.

Recall the notation $K=\mathbb{C}(u,s)$ and $A:=$ the subalgebra of $\mathrm{End}_{\mathbb{C}}(K)$ generated by K and the shift operators $U^{\pm 1}:u\mapsto u\pm 1$; $S^{\pm 1}:s\mapsto s\pm 1$. The $S^{\pm 1},U^{\pm 1}$ act on K and on L by $(U^{\pm 1}g)(u,s):=g(u\pm 1,s)$ and $(S^{\pm 1}g)(u,s):=g(u,s\pm 1)$. Let $S^{\pm 1},U^{\pm 1}$ act on N by $S^{\pm 1}(l\otimes g):=S^{\pm 1}l\otimes S^{\pm 1}g$ and $U^{\pm 1}(l\otimes g):=U^{\pm 1}l\otimes U^{\pm 1}g$. This defines a structure of an A-module on N.

Let w denote the image of the function $c(n,k)$ in N and let M be the A-submodule of N generated by w.

Theorem 3.3. $\dim_K M = 1$.

We will use the following technical lemma, whose proof is postponed to Appendix A.

Lemma 3.4 (Appendix A)**.** *We have $w \neq 0$ and $Sw = fw$.*

Proof of Theorem 3.3. By the previous lemma, w is an eigenvector of S. Thus the three-term relation (14) implies that the module M is spanned by w and $U^{-1}w$. Indeed, from $Sw = fw$ we have $S^{-1}w = (S^{-1}f)^{-1}w$. Looking at the coefficient of z^k in (14) we have

$$(S^{-1}f)^{-1}c(n,k) = (S^{-1}c)(n,k) = Uc(n,k) + \alpha_0(n)c(n,k) + \alpha_{-1}U^{-1}c(n,k) \tag{18}$$

for every $n \geq 1$. This also trivially holds for every $n \leq -2$. Thus, if we multiply both sides of the equality by $n(n+1)$, it will hold for all n. Therefore, in the A-module $M \subset N = L \otimes_{\mathbb{C}[u,s]} K$ we have

$$(S^{-1}f)^{-1}w = S^{-1}w = Uw + \alpha_0 w + \alpha_{-1}U^{-1}w \tag{19}$$

Thus Uw is linearly dependent on $w, U^{-1}w$, and thus $w, U^{-1}w$ span M, and $\dim_K M \leq 2$. Suppose by way of contradiction that $\dim_K M = 2$. Let $v := U^{-1}w$. Since $\dim_K M = 2$, v and Uv span M and are linearly independent. From (19) we have

$$U^2 v = -\alpha_{-1}v + \delta U v, \tag{20}$$

where $\delta = S^{-1}f^{-1} - \alpha_0 \in K$. Since U and S commute, $Sv = SU^{-1}w = U^{-1}Sw = U^{-1}(fw) = (U^{-1}f)U^{-1}w = (U^{-1}f)v$. In other words $Sv = \beta v$, where $\beta = U^{-1}f \in K$. Applying S to (20) and using $US = SU$ we have

$$SU^2v = S(-\alpha_{-1}v) + S(\delta U v) = -S(\alpha_{-1})\beta v + S(\delta)U(\beta v) = -S(\alpha_{-1})\beta v + S(\delta)U(\beta)U(v) \tag{21}$$

We also have

$$SU^2v = U^2(Sv) = U^2(\beta v) = U^2\beta U^2 v = -U^2(\beta)\alpha_{-1}v + U^2(\beta)\delta U v \tag{22}$$

Thus

$$-S(\alpha_{-1})\beta v + S(\delta)U(\beta)U(v) = -U^2(\beta)\alpha_{-1}v + U^2(\beta)\delta U v \tag{23}$$

Since v, Uv are linearly independent, comparing their coefficients we have

$$-S(\alpha_{-1})\beta = -U^2(\beta)\alpha_{-1} \text{ and } S(\delta)U(\beta) = U^2(\beta)\delta \tag{24}$$

The first equation implies

$$U^2(\beta)/\beta = S(\alpha_{-1})/\alpha_{-1} = 1, \tag{25}$$

Thus β does not depend on u, and since $\beta = U^{-1}f$ neither does f. However, this contradicts Lemma 3.1, that says that f has the term $u - s$. ☐

Corollary 3.5. *There exists a (unique) rational function $h \in \mathbb{C}(u,s)$ such that for all $n, k \in \mathbb{Z}$ we have $h(n,n)f(n+1,n) = 1$ and*

$$c(n+1,k) = h(n,k)c(n,k). \tag{26}$$

Proof. Since $\dim_K M = 1$, there exists a (unique) rational function $h \in \mathbb{C}(u,s)$ such that $Sw = hw$. Then we have

$$fShw = SUw = USw = hUfw,$$

thus $fSh = hUf$. By Lemma 3.1 we have for every $n \geq k \geq 0$

$$\frac{c(n+1,k)}{c(n,k)} = f(n+1,n)^{-1}\prod_{i=k}^{n-1}\frac{f(n,i)}{f(n+1,i)} = f(n+1,n)^{-1}\prod_{i=k}^{n-1}\frac{h(n,i)}{h(n,i+1)} = \frac{h(n,k)}{h(n,n)f(n+1,n)} \tag{27}$$

By definition of $N = L \otimes_{\mathbb{C}[u,s]} K$, the equality $Uw = hw$ implies that there exists $r \neq 0 \in \mathbb{C}[u,s]$ such that (26) holds for every $n,k \in \mathbb{Z}$ with $r(n,k) \neq 0$. Combining this with (27) we get that for every $n > \deg r$ we have $h(n,n)f(n+1,n) = 1$. Thus $h(n,n)f(n+1,n) = 1$ for all $n \in \mathbb{Z}$. This also implies $h(n,n+1)f(n,n) = 1$, and thus $h(n,n+1) = \infty$. Thus (26) holds for $k = n+1$ (since $c(n,n) = 1$ and $c(n,n+1) = 0$). It also holds trivially for $k > n+1$ and for $k < 0$. Thus (26) holds for all $n \geq 0$ and all $k \in \mathbb{Z}$. For $n < -1$ it also holds trivially. For $n = -1$ and $k > 0$ it also holds trivially. For $k = 0$ we have $1 = c(0,0) = \infty \cdot 0 = h(-1,0)c(-1,0)$, that is (26) also holds. $\square$

4. Classification of pointed 1-dimensional A-modules satisfying the 3-term relation

Let $\Phi(u,s)$ be a meromorphic function of the form $g\phi$, where ϕ is of γ type and $g \in \mathbb{C}(u,s)$. Then $S\Phi = f\Phi$ for some $f \in \mathbb{C}(u,s)$. Suppose also that Φ satisfies the 3-term recurrence relation (19):

$$S^{-1}\Phi = U\Phi + \alpha_0\Phi + \alpha_{-1}U^{-1}\Phi, \tag{28}$$

where $\alpha_0, \alpha_{-1} \in \mathbb{C}(u)$ with $\alpha_{-1} \neq 0$. By Theorem 3.3, Φ generates a one-dimensional A-module. Thus $U\Phi = h\Phi$ for some $h \in \mathbb{C}(u,s)$. This h necessarily satisfies

$$Uf/f = Sh/h \tag{29}$$

Since $S\Phi = f\Phi$, we have $S^{-1}\Phi = S^{-1}(f)^{-1}\Phi$, and (28) is equivalent to

$$S^{-1}(f)^{-1} = h + \alpha_0 + \alpha_{-1}/U^{-1}(h), \tag{30}$$

We are looking for the space of joint solutions of (29) and (30).

Theorem 4.1. *Suppose that $U\Phi = h\Phi$, $S\Phi = f\Phi$, and $\Phi \neq 0$ satisfies* (28). *Then there exists $g \in \mathbb{C}[u,s]$ and $p,q,w \in \mathbb{C}[t]$ and $\sigma \in \mathbb{C}(u)$ such that*

$$f = \frac{q(u+s)p(s-u)Sg}{w(s)g}, \quad h = \frac{\sigma(u)q(u+s)Ug}{p(s-u-1)g}, \tag{31}$$

and g has no common factors with $w(s), q(u+s), p(s-u)$.

To prove the theorem will us the following proposition.

Proposition 4.2. *Let $a \in \mathbb{C}(u)[s]$ be an irreducible monic non-constant polynomial in s with coefficients in $\mathbb{C}(u)$. If $S^kU^la = a$ and $(k,l) \neq (0,0)$ then $l \neq 0$ and $a = s - (k/l)u + c$ with $c \in \mathbb{C}$.*

Proof. First of all $l \neq 0$ since otherwise $S^ka = a$, thus $a \in \mathbb{C}(u)$, and since a is monic $a = 1$.

Let $d := gcd(k,l), k' := k/d, l' := l/d$. Let m,n be such that $mk' + nl' = 1$. Now, perform a linear change of variables: $x := l's - k'u$, $y := ms + nu$. Under this change, the shift operator in y becomes $S^{k'}U^{l'}$.

The condition $S^kU^la = a$ implies that a is invariant to shifts $y \mapsto y + d$. Thus a depends only on x. Thus a is a polynomial in the expression $ls - ku$ with complex coefficients. To be monic in s and irreducible it must be $s - (k/l)u + c$. □

Proof of Theorem 4.1. By definition, we can present our Φ in the form

$$\Phi = \exp(cu + ds)g(s, u) \prod \Gamma(k_iu + l_is + c_i),$$

with $k_i, l_i \in \mathbb{Z}$ and (k_i, l_i) coprime for every i, $\Re c_i \in [0, 1)$ and if $(k_i, l_i) = (-k_j, l_j)$ then $c_i \neq 1 - c_j$.

Now, fix i and look at all the poles of Φ along the lines of the form $k_iu + l_is + c_i + n = 0$ with $n \in \mathbb{Z}$.

Assume first $g = 1$. Then these lines will be exactly those with $n \geq 0$. For $S^{-1}\Phi$ these will be the lines with $n \geq l_i$, for $U\Phi$ it will be $n \geq -l_i$, and for $U^{-1}\Phi$ we have $n \geq l_i$.

We see that unless $(k_i, l_i) \in \{(\pm1, \pm1), (\pm1, 0), (0, \pm1)\}$, one of those lines "sticks out", and cannot be covered by the other 3 terms in the 3-term relation (28). Since $\alpha_{-1} \neq 0$, this line is not a pole of $U\Phi$ or of $U^{-1}\Phi$.

Now, if $g \neq 1$, and g has a term that can cancel exactly the top line of $\exp(cu+ds) \prod \Gamma(k_iu+l_is+c_i)$ then we can get rid of this term by replacing c_i by $c_i \pm 1$ (relaxing the condition $\Re c_i \in [0, 1)$).

Thus we see that in any case $(k_i, l_i) \in \{(\pm1, \pm1), (\pm1, 0), (0, \pm1)\}$. This form of Φ gives the desired form of f, except that p, q, w and g are allowed to be rational functions. From (29) we obtain that h has the desired form as well.

Let $\gamma, \beta_1, \beta_0, \beta_{-1} \in \mathbb{C}[u]$ be polynomials that have no overall common factor and satisfy

$$\sigma = \beta_1/\gamma, \quad \alpha_0 = \beta_0/\gamma, \quad \alpha_{-1}/U^{-1}\sigma = \beta_{-1}/\gamma. \tag{32}$$

By (31), equation (30) becomes

$$\begin{aligned} &\gamma(u)w(s-1)S^{-1}(g) = \\ &\beta_1(u)q(u+s)q(u+s-1)Ug + \beta_0(u)q(s+u-1)p(s-u-1)g + \beta_{-1}(u)p(s-u)p(s-u-1)U^{-1}g, \end{aligned} \tag{33}$$

Further analyzing the poles let us show that the rational functions p, q, w and g are polynomials. Suppose first, by way of contradiction that g is not a polynomial.

Let a be an irreducible factor of the denominator of g. Then it is also a factor in the denominator of $S^{-1}f^{-1}$. Thus at least one of the terms in the right-hand side of (33) has a in denominator. Thus $S^{-1}a = U^ib$, where $i \in \{-1, 0, 1\}$ and b is another term in the denominator of g. Then $S^{-1}b$ is also a factor in the denominator of $S^{-1}f^{-1}$. Since the number of factors is finite, continuing in this way we obtain that $S^{-k}U^la = a$, where $k, l \in \mathbb{Z}$ and $k > 0$. Thus, by Proposition 4.2, $a = d(ls + ku + c)$ for some $c, d \in \mathbb{C}$. Without loss of generality we assume $d = 1$, and that among all the factors of the form $ls + ku + e$ with the same k, l, a has minimal $\Re e$. As we said, $S^{-1}a \in \{Ub, b, U^{-1}b\}$. If $S^{-1}a = b$ then $b = Sa = a - k$, which contradicts the choice of a. If $S^{-1}a = Ub$ then $b = a - l - k$, and thus $l \leq -k < 0$, thus the RHS of (33) also has b or $U^{-1}b$ in the denominator of one of the terms. Thus $S^{-1}a' \in \{b, U^{-1}b\}$ for some a'. But then $a' \in \{a - k, a - k + l\}$. Since $-k, -k + l < 0$, this contradicts the choice of a.

Now, if w is not a polynomial then Φ has a term of the form $\Gamma(s + c)$. Then $S^{-1}\Phi$ has a pole along the line $s + c = 1$, while the other terms in (28) do not have such a pole, which is again a contradiction. In the same way we show that p and q are polynomials as well. □

Theorem 4.3. *Let p, q, w, γ, β_i be polynomials in one variable, and $g \in \mathbb{C}[u, s]$ such that* (33) *holds, and not all β_i are identically 0. Then one of the following holds:*

(a) $\deg p, \deg q \leq 1$
(b) $\deg p = \deg w = 0$ *and* $w = p$.
(c) $\deg q = \deg w = 0$ *and* $w = q$.
(d) $\deg p = \deg q = \deg w = 2$.

Proof. Without loss of generality assume that p, q, w are monic. Denote $d_1 := \deg p$, $d_2 := \deg q, d_3 := \deg w$. Substituting $s = ku$ in (33) and taking the leading coefficient in u we get the relation

$$c_1(k-1)^{2d_1} + c_2(k-1)^{d_1}(k+1)^{d_2} + c_3(k+1)^{2d_2} + c_4k^{d_3} = 0 \tag{34}$$

with $c_i \in \mathbb{C}$ and $c_j \neq 0$ for some j. This is since g,Sg,Ug, and $U^{-1}g$ have the same leading term. In this equation some c_i may vanish even if the corresponding β_i or γ do not vanish.

Lemma 4.4. *One of the following holds:*
(a) $d_1, d_2 \leq 1$
(b) $|d_1 - d_2| = 1$
(c) $d_1 = d_3 = 0$*,* $w = p$*, and* $\beta_0 = \beta_{-1} = 0$
(d) $d_2 = d_3 = 0$*,* $w = q$*, and* $\beta_0 = \beta_1 = 0$.

Proof. Replacing k by $-k$ we can assume $d_1 \geq d_2$. If $d_1 \geq d_2 + 2$ then the polynomial $(k-1)^{2d_1}$ has, among others, monomials of degree d_1 and $d_1 - 1$. No term of $(k+1)^{2d_2}$ can cancel any of these, and the term k^{d_3} can only cancel one of them. Thus we must have $c_1 = 0$. But then, the polynomial $(k-1)^{d_1}(k+1)^{d_2}$ has terms of degrees $d_1 + d_2$ and $d_1 + d_2 - 1$ and again k^{d_3} can only cancel one of them. Thus we have $c_1 = c_2 = 0$ and thus $c_3, c_4 \neq 0$. If $d_2 > 0$ then $(k+1)^{2d_2}$ has at least three terms, and they cannot be cancelled by $c_4k^{d_3}$. Thus we must have $d_2 = d_3 = 0$. Then looking at the degree of s in (33), we obtain $\beta_1 = \beta_0 = 0$. This also implies $w = q$. $\square$

Lemma 4.5. *Suppose that* $d_1, d_2 \geq 1$*. Then* $d_1 = d_2 \in \{1, 2\}$.

Proof. Suppose $c_4 = 0$ in (34). Then setting $k = \pm 1$ we deduce $c_1, c_3 = 0$ and hence $c_2 = 0$ also. Thus we may assume $c_4 \neq 0$. Now setting $k = \pm 1$ we get $c_1, c_3 \neq 0$.

Let β be a root of the quadratic polynomial $c_1x^2 + c_2x + c_3$, and let α be a root of the polynomial $(k-1)^{d_1} - \beta(k+1)^{d_2}$. Then setting $k = \alpha$ in (34) the sum of the first three terms vanish and we get $c_4\alpha^{d_3} = 0$, which implies $\alpha = 0$ (and $d_3 \geq 1$). Thus 0 is the only root of $(k+1)^{d_1} - \beta(k-1)^{d_2}$, which means this polynomial is a monomial:

$$(k-1)^{d_1} - \beta(k+1)^{d_2} = \gamma k^{d_4} \quad \text{for some } \gamma, d_4 \tag{35}$$

If $d_4 = 0$ then $d_1 = d_2 = 1$, so we may assume $d_4 > 0$. This also implies $\beta = (-1)^{d_1}$ and $c_1 + (-1)^{d_1}c_2 + c_3 = 0$.

Now setting $k = 0, 1, -1$ we get

$$\beta = (-1)^{d_1},\ \gamma = (-1)^{d_1+1}2^{d_2},\ \gamma = (-1)^{d_1+d_4}2^{d_1}.$$

This forces $d_1 = d_2 =: d$, and (35) becomes

$$(k-1)^d - (-1)^d(k+1)^d = (-1)^{d+d_4}2^dk^{d_4} \tag{36}$$

The coefficient of k on the left is $2d(-1)^{d-1}$ which implies $d_4 = 1$ and $2d = 2^d$. Thus $d \in \{1, 2\}$. $\square$

Lemma 4.6. *If* $d_1 = d_2 = 2$ *then* $\deg w \leq 2$ *and* $\beta_1 + \beta_0 + \beta_{-1} = 0$.

Proof. Shifting u and s we can assume $p(t) = t^2 + c$, $q(t) = t^2 + d$ for some c, d. Using the substitution $s = ku$ as before, we see that $\deg w \leq 2$. This means that $\beta_1 + \beta_0 + \beta_{-1} = 0$. $\square$

□

From the proof of the last lemma we obtain the following one.

Lemma 4.7. *If $d_1 = d_2 = 2$ and $g = 1$ then $\deg w = 2$. If moreover $w(-1) = 0$, and w is monic then one of the following holds.*

(a) $w(s) = (s + 1/2)(s + 1)$, $p(t) = q(-1 - t)$
(b) $w(s) = (s + 1)(s + 3/2)$, $p(t) = q(-2 - t)$

Proof. Shifting u and s we can assume $p(t) = t^2 + c$, $q(t) = t^2 + d$ for some c, d. Since $\deg w \leq 2$, the term of s^3 in the RHS of (33) must vanish. But this term is $4(\beta_1 - \beta_{-1})u - 2(\beta_1 + 2\beta_0 + \beta_{-1})$. Since $\beta_1 + \beta_0 + \beta_{-1} = 0$, we obtain that $\beta_0 = 2(\beta_1 - \beta_{-1})u$, and thus $\beta_1(2u + 1) = \beta_{-1}(2u - 1)$. Thus we can assume without loss of generality that $\beta_{-1} = 2u + 1$ and $\beta_1 = 2u - 1$. This implies $\beta_0 = -4u$.

Then the coefficient of s^2 in the RHS of (33) is $32u^3 - 8u + 2(d - c)$. This does not vanish identically, thus w must have degree 2. Then we can assume w is monic and $\gamma(u) = 32u^3 - 8u + 2(c - d)$.

The coefficient of s in the RHS of (33) is $2(d - c) + 12u - 16u^2(c - d) - 48u^3$. Since this has to be divisible by γ, we have $c = d$. In this case $w(s - 1) = (s - 1)(s - 1/2)$, $\gamma = 8u(4u^2 - 1)$. Thus $w(s) = s(s + 1/2)$.

Shifting back in s we get the same shift for p and q, while shifting in u we obtain opposite shifts. Since $w(-1) = 0$ after the shift, we can shift s either by $1/2$ or by 1. If we shift by $1/2$ we get case (a), and if we shift it by 1 we get case (b). □

This lemma, together with Theorem 4.3, implies the following corollary.

Corollary 4.8. *Let p, q, w, γ, β_i be polynomials in one variable such that (33) holds, and not all β_i are identically 0. Suppose that $g = 1$, $w(-1) = 0$, and w is monic. Then one of the following holds.*

(a) $\deg p, \deg q \leq 1$
(b) $w(s) = (s + 1/2)(s + 1)$, $p(t) = q(-1 - t)$
(c) $w(s) = (s + 1)(s + 3/2)$, $p(t) = q(-2 - t)$

5. Proof of Theorem A

Lemma 5.1 (Direct computation)**.** *Let*

$$Q_u(z) := \frac{(-1)^n \prod_{k=1}^{j} (\beta_k)_{(n)}}{\prod_{l=1}^{i} (\gamma_l)_{(n)}} \cdot {}_iF_j \left(\begin{matrix} -u & & \gamma \\ & \beta & \end{matrix} ; z \right) \tag{37}$$

Then $SQ_u = fQ_u$, where

$$f(u, s) = \frac{s - u}{s + 1} \frac{\prod_{l=1}^{i} (s + \gamma_l)}{\prod_{k=1}^{j} (s + \beta_k)} \tag{38}$$

Conversely, if $f \in \mathbb{C}(u, s)$ is a rational function given by the formula (38), and $P_n(z)$ is a monic polynomial family of rational type given by f, then $P_n(z) = Q_n(z)$.

The 5 hypergeometric polynomial families in Theorem A are $Q_n(z)$, with the parameters β and γ given by the following table.

(39)

N	Name	Not-n	γ	β
1	Jacobi	$j_n^{b-1,a-b}$	$n+a$	b
2	Laguerre	$l_n^{(b-1)}$	$\emptyset$	b
3	Bessel	$BP_n^{(a+1)}$	$n+a$	$\emptyset$
4		$E_n^{(c)}$	$(1-n-c, n+1, n+c)$	$1/2$
5		$F_n^{(c)}$	$(1-n-c, n+2, n+c+1)$	$3/2$

The area of definition is $a \notin \mathbb{Z}_{<0}$; $b \notin \mathbb{Z}_{\leq 0}$; $c \notin \mathbb{Z}_{<0}$, and in addition for E_n: $c \neq 0$ and for $j_n^{b-1,a-b}$: $b-a \notin \mathbb{Z}_{>0}$.

Proof of Theorem A. Let M be the A-module defined by P_n in §3. By Theorem 3.3, it is one-dimensional. By abuse of notation, we will denote the element of M defined by P_n also as P_n. Let $f, h \in \mathbb{C}(u,s)$ be such that $SP_n = fP_n$ and $UP_n = hP_n$. By Theorems 2.3 and 4.1 we have

$$f = \frac{q(u+s)p(s-u)Sg}{w(s)g}, \quad h = \frac{\sigma(u)q(u+s)Ug}{p(s-u-1)g}, \tag{40}$$

By the assumptions on f in the definition of Jacobi type, g does not depend on u. Thus $Ug = U^{-1}g = g$. Without loss of generality, we can assume that Sg has no common factors with $w(s)$, and thus g has no common factors with $w(s-1)$. Let us show that g is constant. Suppose by way of contradiction that g is not constant, and let c be a root of g that is not a root of $S^{-1}g$. Substitute $s := c$ into the 3-term relation (33). Then the right-hand side vanishes, while the left-hand side does not. This is a contradiction, thus g must be constant, and thus we can assume $g = 1$.

By Corollary 3.5 we have $h(u,u)f(u+1,u) = 1$ for every n, and thus

$$\sigma(u)\frac{q(2u)}{p(-1)}\frac{q(2u+1)p(-1)}{w(u)} = 1$$

and

$$\sigma(u) = \frac{w(u)}{q(2u+1)q(2u)} \tag{41}$$

Altogether we obtain

$$f = \frac{q(u+s)p(s-u)}{w(s)}, \quad h = \frac{w(u)q(u+s)}{q(2u)q(2u+1)p(s-u-1)} \tag{42}$$

The polynomial family P_n is uniquely defined by f, which in turn, by (42), is defined by p, q, and w. By Lemma 3.1 we have $p(0) = 0$ and $w(-1) = 0$. Thus $\deg p, \deg w \geq 1$.

Assume first $\deg p = 1$, thus $p(t) = t$. By Theorem 4.3, in this case $\deg q \leq 1$.

Case 1. $\deg q = 1$. Since q is monic, we have $p(t) = t$, $q(t) = t + a$. From the 3-term relation we have $\deg w \leq 2$. Since $w(-1) = 0$, we have either $w(s) = s+1$ or $w(s) = (s+1)(s+b)$ for some $b \in \mathbb{C}$. If $w(s) = s+1$ this is the case of Bessel polynomials, and if $w(s) = (s+1)(s+b)$ this is the case of Jacobi polynomials. In both cases, we see that β and γ given in Table (39) give the same f that we found, and thus we have $P_n(z) = Q_n(z)$, where $Q_n(z)$ is defined in (37).

Case 2. $\deg q = 0$, $\deg w > 1$. From the 3-term relation in this case $\deg w = 2$, and thus $w = (s+1)(s+b)$. This is the case of Laguerre polynomials.

Case 3. $\deg q = 0$, $\deg w = 1$. Then $w(s) = s + 1$, and $q = 1$. In this case $\alpha_{-1} \equiv 0$, and the family is not of Jacobi type.

Now assume $\deg p > 1$. By Theorem 4.3, in this case $\deg p = 2$, thus $p = t(t + 1 - c)$ for some $c \in \mathbb{C}$. By Lemma 4.7, P_n is either $E_n^{(c)}$ or $F_n^{(c)}$. □

Summarizing the proof, we have the following table

(43)

family	$w(s)$	$p(t)$	$q(t)$
$j_n^{b-1,a-b}$	$(s+1)(s+b)$	t	$t+a$
$l_n^{(b-1)}$	$(s+1)(s+b)$	t	1
$BP_n^{(a+1)}$	$s+1$	t	$t+a$
$E_n^{(c)}$	$(s+1)(s+1/2)$	$t(t+1-c)$	$(t+1)(t+c)$
$F_n^{(c)}$	$(s+1)(s+3/2)$	$t(t+1-c)$	$(t+2)(t+c+1)$

Remark 5.2. *The assumption that α_{-1} does not vanish on $\mathbb{Z}_{>0}$ was never used in full strength. We only used that $\alpha_{-1} \not\equiv 0$, and only in two places: in Case 3 in the proof of Theorem A, and the proof of Theorem 4.1. In the proof of Theorem 4.1 this use can be easily avoided.*

Thus, dropping this assumption we obtain only one extra family - the one given by $f = \frac{s-u}{s+1}$, as in Case 3. By the Newton binomial formula this family is $P_n(z) = (1 - z)^n$. This family can also be obtained from Jacobi polynomials by setting $b := -a$, and sending a to infinity.

5.1. **Further details.** Combining Lemma 3.2 with (42) we obtain that the α_0 and α_{-1} for each family are given by the following table

(44)

family	α_0	α_{-1}
$j_n^{b-1,a-b}$	$\frac{2u^2+2au+b(a-1)}{(2u+a-1)(2u+a+1)}$	$\frac{u(u+a-1)(u+b-1)(u+a-b)}{(2u+a)(2u+a-1)^2(2u+a-2)}$
$l_n^{(b-1)}$	$2u+b$	$u(u+b-1)$
$BP_n^{(a+1)}$	$\frac{a-1}{(2u+a-1)(2u+a+1)}$	$\frac{u(u+a-1)}{(2u+a)(2u+a-1)^2(2u+a-2)}$
$E_n^{(c)}$	$-\frac{1}{2(2u+c-1)(2u+c+1)}$	$\frac{1}{16(2u+c)(2u+c-1)^2(2u+c-2)}$
$F_n^{(c)}$	$-\frac{1}{2(2u+c)(2u+c+2)}$	$\frac{1}{16(2u+c+1)(2u+c)^2(2u+c-1)}$

Each family is determined by the functions α_0 and α_{-1} and by the scalar $c_0 = P_1(0)$. For all the families except E_n we have $c_0 = -\alpha_0(0)$. Equivalently, for these families instead the initial condition $P_1 = z + c_0$ we could have used the condition $P_{-1} = 0$, and extend the function α_0 to 0. However, for the family $E_n^{(c)}$ we have $c_0 = 1/(4c(c + 1)) \neq 1/(2(c - 1)(c + 1)) = -\alpha_0(0)$. If we set c_0 to be $-\alpha_0(0)$, keeping the functions α_0 and α_{-1} of $E_n^{(c)}$ we obtain the family $F_n^{(c-1)}$.

6. Investigation of the two new families $E_n^{(c)}, F_n^{(c)}$

6.1. **Preliminaries on Lommel polynomials.** Let us give some preliminaries on Lommel polynomials from [Wat44, §9] and [Ism09, §6.5]. They are defined by the recursive relation

$$2z(n+c)h_n^{(c)} = h_{n+1}^{(c)}(z) + h_{n-1}^{(c)}(z), \qquad h_{-1}^{(c)}(z) \equiv 0,\ h_0^{(c)} \equiv 1 \tag{45}$$

Explicitly, we have[1]

$$h_{2n}^{(c)}(z) = (-1)^n \sum_{k=0}^{n} \binom{n+k}{2k} (n+c-k)_{(2k)} (-2z)^{2k}, \tag{46}$$
$$h_{2n+1}^{(c)}(z) = (-1)^n \sum_{k=0}^{n} \binom{n+k+1}{2k+1} (n+c-k)_{(2k+1)} (-2z)^{2k+1}$$

To describe the discrete measure on the real line for which these polynomials are orthogonal we will need some notation. Let $J_v(z)$ denote the modified Bessel function [Wat44]:

$$J_v(z) = \sum_{n=0}^{\infty} \frac{(-1)^n (z/2)^{c+2n}}{\Gamma(n+\nu+1)n!} \tag{47}$$

Let $\{j_{\nu,k}\}_{k=1}^{\infty}$ denote the increasing sequence of positive zeroes of $J_{\nu,k}$ on $\mathbb{R}$. Denote $a_{\nu,k} := j_{\nu,k}^{-1}$.

Theorem 6.1 ([Ism09, (6.5.17)]). *For every $c \in \mathbb{R}_{>0}$ and every $n, m \in \mathbb{Z}_{\geq 0}$ we have*

$$\sum_{k=1}^{\infty} a_{c-1,k}^2 (h_{n,c}(a_{c-1,k}) h_{m,c}(a_{c-1,k}) + h_{n,c}(-a_{c-1,k}) h_{m,c}(-a_{c-1,k})) = \frac{\delta_{m,n}}{2(n+c)} \tag{48}$$

6.2. **Discrete measure for which $E_n^{(c)}(z)$ and $F_n^{(c)}(z)$ form orthogonal families.** Spelling out the definition of E_n and F_n as hypergeometric functions, we have

$$E_n^{(c)}(z) = \frac{1}{2^{2n} c_{(2n)}} \sum_{k=0}^{n} \binom{n+k}{2k} (n+c-k)_{(2k)} 2^{2k} z^k, \tag{49}$$
$$F_n^{(c)}(z) = \frac{1}{2^{2n} c_{(2n+1)}} \sum_{k=0}^{n} \binom{n+k+1}{2k+1} (n+c-k)_{(2k+1)} 2^{2k} z^k$$

Combining (49) and (46) we have for any $n \in \mathbb{Z}_{\geq 0}$:

$$E_n^{(c)}(-z^2) = \frac{(-1)^n}{2^{2n} c_{(2n)}} h_{2n}^{(c)}(z), \quad z F_n^{(c)}(-z^2) = \frac{(-1)^n}{2^{2n+1} c_{(2n+1)}} h_{2n+1}^{(c)}(z). \tag{50}$$

Theorem 6.1 and (50) imply the following corollary.

Corollary 6.2. *For every $c \in \mathbb{R}_{>0}$ and every $n, m \in \mathbb{Z}_{\geq 0}$ we have*

$$4c \sum_{k=1}^{\infty} a_{c-1,k}^2 E_n^{(c)}(-a_{c-1,k}^2) E_m^{(c)}(-a_{c-1,k}^2) = \frac{c}{2^{4n} c_{(2n)}^2 (2n+c)} \delta_{m,n} \tag{51}$$

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[1] M. Ismail kindly informed us of a typo in [Ism09, (6.5.8)]: $z/2$ should be $2z$.

and

$$16c^2(c+1)\sum_{k=1}^{\infty} a_{c-1,k}^4 F_n^{(c)}(-a_{c-1,k}^2)F_m^{(c)}(-a_{c-1,k}^2) = \frac{c+1}{2^{4n}(c+1)_{(2n)}^2(2n+1+c)}\delta_{m,n} \tag{52}$$

The arguments of [Ism09, §6.5] show that (52) continues to hold for every $c > -1$.

6.3. **Differential equations.** Being hypergeometric functions of type ${}_4F_1$, the $E_n^{(c)}$ and $F_n^{(c)}$ satisfy the following 4th order differential equations.

$$(D(D-1/2) - z(D-n)(D-n-c+1)(D+n+c)(D+n+1))E_n^{(c)} = 0 \tag{53}$$

and

$$(D(D+1/2) - z(D-n)(D-n-c+1)(D+n+c+1)(D+n+2))F_n^{(c)} = 0, \tag{54}$$

where $D = z\frac{\partial}{\partial_z}$ is the Euler operator.

Appendix A. Proof of the rationality of α_i and other technical lemmas

Proof of Lemma 3.1. Recall formula (12):

$$c(n,k+1)D(k) \equiv c(n,k)N(n,k), \tag{55}$$

(i) Setting $n = 0, k = -1$ we obtain $D(-1) = 0$. Thus we have $D(s) = (s+1)y(s)$ for some $y \in \mathbb{C}[s]$. Since P_n is monic of degree n, we have $c(n,n) = 1$ and $c(n,n+1) = 0$ for any $n \in \mathbb{Z}_{\geq 0}$. Thus substituting $k = n$ in (55) we have $N(n,n) = 0$. Thus the restriction of N to the diagonal $u = s$ vanishes on $\mathbb{Z}_{\geq 0}$, and thus vanishes identically. Thus $N(u,s)$ is divisible by $u - s$, that is $N(u,s) = (u-s)x(u,s)$ for some $x \in \mathbb{C}[u,s]$.

It is left to show that y has no zeros in $\mathbb{Z}_{\geq 0}$. Suppose the contrary, and let $k_0 \in \mathbb{Z}_{\geq 0}$ be the largest integer zero. Let $n > k_0 \in \mathbb{Z}_{\geq 0}$ be such that $x(n,i) \neq 0$ for any $0 \leq i \leq k_0$. Let us show that for any such n we have $c(n,k_0) \neq 0$. Indeed, for any integer $i \in [k_0, n]$ we have $c(n,i) \neq 0$. This follows by descending induction on i from $c(n,n) = 1$, $D(i+1) \neq 0$ and (55). Substituting n and k_0 into (55) we obtain $N(n,k_0) = 0$. Since this holds for infinitely many n, we obtain that $N(u,k_0) \equiv 0$, thus $N(u,s)$ is divisible by $s - k_0$, contradicting being coprime with $D(s)$.

(ii) follows similarly from (55) by descending induction on k, using $c(n,n) = 1$ and $D(k) \neq 0$.

(iii) follows from (55) by descending induction on $k \in [0,n]$, the base case being $c(n,n) = 1$. □

Proof of Lemma 3.4. Let us first show that $w \neq 0$. Equivalently, we have to show that c is not torsion. In other words, we have to show that the only polynomial $p \in \mathbb{C}[u,s]$ satisfying $p(n,k)c(n,k) = 0$ for all $n,k \in \mathbb{Z}^2$ is $p = 0$. Fix $k_0 \in \mathbb{Z}_{\geq 0}$. Then for any integer $n \geq k_0$, $c(n,k_0) \neq 0$ and thus if $p(n,k_0)c(n,k_0) = 0$ then $p(n,k_0) \equiv 0$, thus $(s-k_0)|p$. This holds for infinitely many values k_0, and thus $p \equiv 0$.

Let us now show that $Sw = fw$. By (55) and Lemma 3.1 we have

$$(k+1)y(k)c(n,k+1) = (k-n)x(n,k)c(n,k) \tag{56}$$

Thus in the $\mathbb{C}[u,s]$-module L we have the equality $(s+1)y(s)Sc = (s-u)x(u,s)c$, which translates to the equality $(s+1)y(s)Sw = (s-u)x(u,s)w$ in M. This implies $Sw = fw$. □

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