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Range characterization of the cosine transform on higher Grassmannians

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Abstract

We characterize the range of the cosine transform on real Grassmannians in terms of the decomposition under the action of the special orthogonal group $SO(n)$. We also give a geometric interpretation of this image in terms of valuations. In addition, we discuss the non-Archimedean analogues.

Keywords: Grassmannians; Cosine transform; Radon transform

0. Introduction

The main results of this paper are Theorems 1.2, 1.3, and 2.1 below. In Section 1 we study the image of the cosine transform. The investigations of this problem were started by Matheron in 1974 [16]. It was investigated further by Goodey and Howard [6,7], and Goodey et al. [8]. For more details on the previous results see Section 1 of this paper. Theorem 1.2 describes the image of the cosine transform (defined in Section 1) acting from the space of functions on the Grassmannian of real j-dimensional subspaces in $\mathbb{R}^n$ to the space of functions on the Grassmannian of real i-dimensional subspaces. The description is given in terms of K-types, namely in terms of the decomposition into irreducible subspaces under the action of the special orthogonal group $SO(n)$. In Theorem 1.3 we consider separately the case $i=j$; this

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case is especially important since the study of other cases is reduced to this one using the results on the Radon transform due to Gelfand et al. [5] (see also [10]). It turns out that the cosine transform in this case can be interpreted as an intertwining operator of $GL(n,\mathbb{R})$-modules. This key observation allows us to use the representation theory of reductive groups. We prove that this intertwining operator has an irreducible image. This step uses connection of the cosine transform with the theory of valuations on convex sets (discussed in Section 1) and earlier results of one of the authors about representation theoretical properties of valuations [1,2]. Once we have this irreducibility result the description of the K-type structure is a direct consequence of the results of Howe and Lee [13].

In Section 2 we discuss an analogue of the cosine transform over non-Archimedean local fields.

The cosine transform appears naturally in convex, integral, and stochastic geometry (see [6–9,11,16,17,20]). For the basic notions of the representation theory we refer to [24].

Remark. (a) The results of this paper were applied further to the theory of valuations in [3].

(b) Recently, we have found a more general result about the cosine type transform of more general form (where the kernel $|\cos(E,F)|$ is replaced by $|\cos(E,F)|^{\alpha}$, $\alpha\in\mathbb{C}$; the notation is explained in Section 1). This will be discussed elsewhere.

1. Range of the cosine transform

First let us recall some notation. We will denote by $Gr_{k,n}$ the Grassmannian of real k-dimensional subspaces in the real n-dimensional space $\mathbb{R}^n$. Let us fix a Euclidean structure on $\mathbb{R}^n$. Let $E\in Gr_{i,n}$, $F\in Gr_{j,n}$. Assume that $i\leqslant j$. Let us call by *cosine of the angle* between E and F the following number:

$$|\cos(E,F)| := \frac{vol_i(Pr_F(A))}{vol_i(A)},$$

where A is any subset of E of non-zero volume, Pr_F denotes the orthogonal projection onto F, and vol_i is the i-dimensional measure induced by the Euclidean structure. (Note that this definition does not depend on the choice of a subset $A\subset E$). In the case $i\geqslant j$ we define the cosine of the angle between them as cosine of the angle between their orthogonal complements:

$$|\cos(E,F)| := |\cos(E^{\perp},F^{\perp})|.$$

(It is easy to see that if $i=j$ both definitions are equivalent.)

Let us call by *sine of the angle* between E and F the cosine between E and the orthogonal complement of F:

$$|\sin(E,F)| := |\cos(E,F^{\perp})|.$$

The following properties are well known (and rather trivial):

$$|\cos(E,F)| = |\cos(F,E)| = |\cos(E^{\perp},F^{\perp})|,$$

$$|\sin(E,F)| = |\sin(F,E)| = |\sin(E^{\perp},F^{\perp})|,$$

$$0 \leqslant |\cos(E,F)|,\ |\sin(E,F)| \leqslant 1.$$

For any $1 \leqslant i, j \leqslant n-1$ one defines the cosine transform

$$T_{j,i} : C(Gr_{i,n}) \to C(Gr_{j,n})$$

as follows:

$$(T_{j,i}f)(E) := \int_{Gr_{i,n}} |\cos(E,F)| f(F)\, dF,$$

where the integration is with respect to the Haar measure on the Grassmannian.

Remark. One should notice that very often in the literature the cosine transform $T_{j,i}$ (resp. the Radon transform $R_{j,i}$) is denoted by $T_{i,j}$ (resp. $R_{i,j}$), i.e. with permutation of indexes. We prefer our notation since it is more convenient to write the composition formulas like $R_{j,i} = R_{j,k} \circ R_{k,i}$.

Clearly, the cosine transform commutes with the action of the orthogonal group $O(n)$, and hence its image is $O(n)$-invariant subspace of functions. Our first main result (Theorem 1.2) is the description of the image of the cosine transform in terms of the decomposition of it with respect to the action of $SO(n)$. Since the representation of $SO(n)$ in functions on the Grassmannian is multiplicity free, it is sufficient to list those irreducible representations of $SO(n)$ entering into the image of the cosine transform. Moreover, it is shown (Theorem 1.3) that for $i = j$ this image coincides with the image of even translation invariant i-homogeneous continuous valuations on convex sets under certain natural map (actually this fact is proved first and then used in the proof of the more explicit result).

Now let us recall standard facts on the representations of the special orthogonal group $SO(n)$ (see e.g. [25]).

Lemma 1.1. *The isomorphism classes of irreducible representations of* $SO(n)$, $n > 2$ *are parameterized by their highest weights, namely sequences of integers*

$(m_1, m_2, \ldots, m_{[n/2]})$ *which satisfy*:

(i) *if* n *is odd then* $m_1 \geqslant m_2 \geqslant \cdots \geqslant m_{[n/2]} \geqslant 0$:
(ii) *if* $n > 2$ *is even then* $m_1 \geqslant m_2 \geqslant \cdots \geqslant m_{n/2-1} \geqslant |m_{n/2}|$.

Recall also that for $n = 2$ the representations of $SO(2)$ are parameterized by a single integer m_1. We will use the following notation. Let us denote by Λ^+ the set of all highest weights of $SO(n)$, and by Λ_k^+ the set of all highest weights $\lambda = (m_1, m_2, \ldots, m_{[n/2]})$ with $m_i = 0$ for $i > k$ and all m_i are *even*.

Let us recall the decomposition of the space of functions on the Grassmannian $Gr_{k,n}$ under the action of $SO(n)$ referring for the proofs to [22,23]. Since $Gr_{k,n}$ is a symmetric space, each irreducible representation enters with multiplicity at most one. The representations which do appear have highest weights precisely from $\Lambda_k^+ \cap \Lambda_{n-k}^+$.

Now let us state our main result.

Theorem 1.2. *Let* $1 \leqslant i, j \leqslant n-1$. *Then the image of the cosine transform* $T_{j,i} : C(Gr_{i,n}) \to C(Gr_{j,n})$ *consists of irreducible representations of* $SO(n)$ *with highest weights* $\lambda = (m_1, \ldots, m_{[n/2]})$ *precisely with the following additional conditions*:

(i) $\lambda \in \Lambda_i^+ \cap \Lambda_{n-i}^+ \cap \Lambda_j^+ \cap \Lambda_{n-j}^+$;
(ii) $|m_2| \leqslant 2$;

Moreover the image of $T_{j,i}$ *is closed in the* C^∞ *topology.*

Note that as a corollary of this theorem we immediately get exactly the same characterization of the image of the sine transform. Theorem 1.2 was known for a long time for $i = j = 1$ (or equivalently for $i = j = n-1$), see [20] or [11]. The case $n = 4$, $i = j = 2$ was described completely in [8]; this paper contains also partial information on the general case.

The main case in the proof of this theorem is $i = j$. The cosine transform for $i \neq j$ can be written as a composition of the Radon transform between different Grassmannians and the cosine transform for $i = j$. Thus to deduce the general case we use the characterization of the image of the Radon transform [5], see also [10]. In order to treat the case $i = j$ we first interpret the cosine transform as an intertwining operator between certain representations on the (non-compact) group $GL(n, \mathbb{R})$ induced from maximal parabolic subgroups. Next, we prove that the image of this operator is an irreducible $GL(n, \mathbb{R})$-module . In order to do it we first show (see Theorem 1.3) that the image is contained in the subspace corresponding to even translation invariant valuations (see the definitions below). But the last space is an irreducible $GL(n, \mathbb{R})$-module by the main result in [2]. Hence it coincides with the image of the cosine transform. The decomposition of the space of even valuations with respect to the action of $SO(n)$ was described in [2] as an easy corollary of the irreducibility and the computations due to Howe and Lee [13].

Now let us describe the construction of the intertwining operator. Let us denote by L the line bundle over the Grassmannian $Gr_{i,n}$ whose fiber over a subspace $E \in Gr_{i,n}$ is the space of Lebesgue measures on E (which is denoted by $|\wedge^i E^*|$). Clearly, L is $GL(n,\mathbb{R})$-equivariant line bundle over $Gr_{i,n}$. Moreover if we fix the Euclidean structure on $\mathbb{R}^n$ we can identify L with the trivial bundle in a way compatible with the action of $SO(n)$. Let M denote the line bundle over the Grassmannian $Gr_{n-i,n}$ whose fiber over $F \in Gr_{n-i,n}$ is the space of Lebesgue measures on the quotient space $\mathbb{R}^n/F$ denoted by $|\wedge^i(\mathbb{R}^n/F)^*|$. Let $|\omega|$ denote the line bundle of densities over $Gr_{n-i,n}$. Let $N := M \otimes |\omega|$. Define an intertwining operator T from the space of continuous sections of N to the space of continuous sections of L

$$T : \Gamma(Gr_{n-i,n}, N) \to \Gamma(Gr_{i,n}, L)$$

as follows. For $E \in Gr_{i,n}$ and $f \in \Gamma(Gr_{n-i,n}, N)$ set

$$(Tf)(E) = \int_{F \in Gr_{n-i,n}} pr^*_{E,F}(f(F)),$$

where $pr_{E,F}$ denotes the natural map $E \to \mathbb{R}^n/F$ and $pr^*_{E,F}$ is the induced map $|\wedge^i(\mathbb{R}^n/F)^*| \to |\wedge^i E^*|$. Clearly T is a non-trivial operator commuting with the action of $GL(n,\mathbb{R})$.

Theorem 1.3. *The image of the operator T is an irreducible $GL(n,\mathbb{R})$-module. Moreover if we identify L with the trivial bundle in an $SO(n)$-equivariant way then the image of T coincides with the image of the cosine transform $T_{i,i}$.*

Note that the second statement of the theorem easily follows from the definitions. In order to prove the irreducibility of the image we will need one more construction which we are going to describe.

Let $\mathcal{K}^n$ denote the family of all convex compact subsets in $\mathbb{R}^n$.

Definition 1.4. (1) A function $\phi : \mathcal{K}^n \to \mathbb{C}$ is called a valuation if for any $K_1, K_2 \in \mathcal{K}^n$ such that their union is also convex one has

$$\phi(K_1 \cup K_2) = \phi(K_1) + \phi(K_2) - \phi(K_1 \cap K_2).$$

(2) A valuation ϕ is called continuous if it is continuous with respect the Hausdorff metric on $\mathcal{K}^n$.

(3) A valuation ϕ is called translation invariant if $\phi(K + x) = \phi(K)$ for every $x \in \mathbb{R}^n$ and every K.

(4) A valuation ϕ is called even if $\phi(-K) = \phi(K)$ for every $K \in \mathcal{K}^n$.

(5) A valuation ϕ is called homogeneous of degree k (or k-homogeneous) if for every $K \in \mathcal{K}^n$ and every scalar $\lambda \geqslant 0$ $\phi(\lambda \cdot K) = \lambda^k \phi(K)$.

We refer to further details on valuations to the surveys [18,19]. We will need only few facts about them.

Lemma 1.5 (Hadwiger [12]). *Every translation invariant n-homogeneous continuous valuation on $\mathcal{K}^n$ is proportional to the Lebesgue measure on $\mathbb{R}^n$.*

Let us denote by $Val_{n,i}^{\mathrm{ev}}$ the linear space of translation invariant i-homogeneous even continuous valuations. It is a Frechet space with respect to the topology of uniform convergence on compact subsets of $\mathcal{K}^n$.

There is a natural map

$$\gamma : Val_{n,i}^{\mathrm{ev}} \to \Gamma(Gr_{i,n}, L),$$

where L is the line bundle defined above. To define it, fix any $\phi \in Val_{n,i}^{\mathrm{ev}}$. Take any $E \in Gr_{i,n}$. Consider the restriction of ϕ to the class of convex compact subsets of E. It is again translation invariant i-homogeneous continuous valuation on E. Hence by Lemma 1.5 it is proportional to the Lebesgue measure on E. Hence ϕ defines a continuous section $\gamma(\phi)$ of L. This map was used in [1,2] in the proof of McMullen's conjecture. This map was also independently considered by Klain [15]. Note that this map turns out to be injective by a theorem of Klain [14]. The main fact on $Val_{n,i}^{\mathrm{ev}}$ we use is the following result proved in [2].

Lemma 1.6. *For every integer i, $1 \leqslant i \leqslant n-1$ the space $Val_{n,i}^{\mathrm{ev}}$ is an irreducible $GL(n, \mathbb{R}^n)$-module. Hence its image in $\Gamma(Gr_{i,n}, L)$ is an irreducible submodule.*

Proof of Theorem 1.3. By Lemma 1.6 it remains to show that the image of T is contained in $\gamma(Val_{n,i}^{\mathrm{ev}})$. Fix any $f \in \Gamma(Gr_{n-i,n}, N)$. Let us define a valuation ϕ as follows. For every $K \in \mathcal{K}^n$ set

$$\phi(K) := \int_{F \in Gr_{n-i,n}} f(pr_{\mathbb{R}^n/F}(K)),$$

where $pr_{\mathbb{R}^n/F} : \mathbb{R}^n \to \mathbb{R}^n/F$ is the canonical map. It is easy to see that $\phi \in Val_{n,i}^{\mathrm{ev}}$; and moreover $\gamma(\phi) = T(f)$. Thus Theorem 1.3 is proved. □

Proof of Theorem 1.2. First consider the case $i = j$. By Theorem 1.3 and its proof the image of the cosine transform coincides with the image under the map γ of translation invariant i-homogeneous continuous valuations. The explicit decomposition of the last space under the action of $SO(n)$ was given in [2] (it was heavily based on [13]). Thus Theorem 1.2 is proved for $i = j$. Now consider the case $i \neq j$. Clearly we may assume that $j < i$. One has the Radon transform

$$R_{j,i} : C(Gr_{i,n}) \to C(Gr_{j,n})$$

defined by $(R_{j,i}f)(H) = \int_{F \supset H} f(F)\, dF$. □

The next lemma is well known, but we will present a proof for convenience of the reader.

Lemma 1.7. *Let $1 \leqslant j < i \leqslant n$. Then*

$$T_{j,i} = cT_{j,j}R_{j,i},$$

here c is a constant depending on n, i, j.

Proof. Fix a subspace $E \in Gr_{j,n}$. Let A be any convex compact subset of E of positive measure. Let f be a continuous function on $Gr_{i,n}$. Then by definition

$$(T_{j,i}f)(E) = \int_{F \in Gr_{i,n}} \frac{vol_j(Pr_F(A))}{vol_j(A)} f(F)\, dF. \tag{1}$$

Let us fix $F \in Gr_{i,n}$ for a moment. Let $B := Pr_F(A)$. By the Cauchy–Kubota formula (see e.g. [21])

$$vol_j(B) = c \cdot \int_{H \in Gr_j(F)} vol_j(Pr_H(B))\, dH,$$

where $Gr_j(F)$ denotes the Grassmannian of j-dimensional subspaces of F, and c is a non-zero normalizing constant depending on i, j, and n only. Substituting this into (1) and using the standard change of integration (normalizing all the Haar measures to be probability measures) one gets

$$\begin{aligned}(T_{j,i}f)(E) &= \frac{c}{vol_j(A)} \int_{F \in Gr_{i,n}} dF \cdot f(F) \left(\int_{H \subset F} dH \cdot vol_j(Pr_F(A)) \right) \\ &= \frac{c}{vol_j(A)} \int_{H \in Gr_{j,n}} dH \cdot vol_j(Pr_H(A)) \left(\int_{F \supset H} dF \cdot f(F) \right).\end{aligned}$$

This identity clearly proves the lemma. $\square$

To finish the proof of Theorem 1.2 we need the following fact proved in [5] (see also [10]).

Proposition 1.8. *For $j < i$ the Radon transform $R_{j,i} : C(Gr_{i,n}) \to C(Gr_{j,n})$ is injective iff $i + j \geqslant n$ and has a dense image iff $i + j \leqslant n$.*

This proposition, description of the decomposition of the space of functions on the Grassmannians under the action of $SO(n)$, and the characterization of the image of the cosine transform for $i = j$ imply the first part of Theorem 1.2.

It remains to prove that the image of $T_{j,i}$ is closed in C^∞-topology. We may assume that $j < i$. By Lemma 1.7 $T_{j,i} = cT_{j,j}R_{j,i}$. We will need the following fact due to Casselman and Wallach [4].

Proposition 1.9. *Let G be a real reductive group. Let K be its maximal compact subgroup. Let $\xi : X \to Y$ be a morphism of two admissible Banach G-modules of finite* *length which has a dense image. Then ξ induces an epimorphism on the spaces of smooth vectors.*

In our situation we will need the following more precise form of Proposition 1.9. In fact, it can be proved in a much more general context, but we do not need it and do not have precise reference.

Lemma 1.10. *Let $G = GL(n, \mathbb{R})$. Let $K = O(n)$ be the maximal compact subgroup. Let X and Y be G-modules of continuous sections of some finite-dimensional G-equivariant vector bundles over the Grassmannians (or any other partial flag manifolds). Let $\xi : X \to Y$ be a morphism of these G-modules. Then if $f \in Y$ is a smooth vector then there exists a smooth vector $g \in X$ such that $\xi(g) = f$ and the K-types entering into the decomposition of g are the same as those of f.*

We postpone the proof of this lemma till the end of the section. Now let us continue the proof of Theorem 1.2. We have given an interpretation of $T_{j,j}$ as an intertwining operator of two $GL(n, \mathbb{R})$-modules; they satisfy the conditions of Proposition 1.10, since they are induced from characters of parabolic subgroups (see [24]). Hence by Proposition 1.10 there exists a C^{∞}-smooth function g on the Grassmannian $Gr_{j,n}$ such that $f = T_{j,j}(g)$ and with the same K-types as f. Next, there exists an interpretation of the Radon transform as an intertwining operator of some admissible $GL(n, \mathbb{R})$-modules of finite length (it was given in [5]). Hence Lemma 1.10 implies the statement. $\square$

Proof of Lemma 1.10. Let $f \in Y$ be as in the statement of Lemma 1.10. By Proposition 1.9 we can choose a smooth vector $h \in X$ such that $\xi(h) = f$. Then h is just a smooth section of the corresponding vector bundle over the Grassmannian (or the partial flags manifold). Let us consider the image g under the orthogonal projection of h to the closure with respect to the L_2-metric of the span in X of those K-irreducible subspaces which have the same K-types as those entering into the decomposition of f. Clearly $\xi(g) = \xi(h) = f$. To finish the proof it remains to show that g is also a smooth section. Indeed, since the orthogonal projection commutes with the action of K, K-smooth vectors go to K-smooth, hence g is K-smooth vector. But since K acts transitively on the Grassmannians (and in fact on all partial flag manifolds), every K-smooth section of every K-equivariant bundle must be smooth in the usual sense. $\square$

2. Non-Archimedean analogue of the cosine transform

In this section we study non-Archimedean analogue of the cosine transform. More precisely, we study a non-Archimedean analogue of the intertwining operator $T_{i,i}$ (in the notation of the previous section). We show that it has an irreducible image.

Now let us introduce the necessary notation. Let F be a non-Archimedean local field. In this section we will denote by $Gr_{i,n}$ the Grassmannian of i-dimensional subspaces in F^n. Denote by L the line bundle over the Grassmannian $Gr_{i,n}$ whose fiber over a subspace $E \in Gr_{i,n}$ is the space of Lebesgue measures on E (which is denoted by $|\wedge^i E^*|$). Clearly L is $GL(n,F)$-equivariant line bundle over $Gr_{i,n}$. Let M denote the line bundle over the Grassmannian $Gr_{n-i,n}$ whose fiber over $H \in Gr_{n-i,n}$ is the space of Lebesgue measures on the quotient space F^n/H denoted by $|\wedge^i (F^n/H)^*|$. Let $|\omega|$ denote the line bundle of densities over $Gr_{n-i,n}$. Let $N := M \otimes |\omega|$. Define an intertwining operator T from the space of continuous sections of N to the space of continuous sections of L

$$T : \Gamma(Gr_{n-i,n}, N) \to \Gamma(Gr_{i,n}, L)$$

as follows. For $E \in Gr_{i,n}$ and $f \in \Gamma(Gr_{n-i,n}, N)$ set

$$(Tf)(E) = \int_{H \in Gr_{n-i,n}} pr^*_{E,H}(f(H)),$$

where $pr_{E,H}$ denotes the natural map $E \to F^n/H$ and $pr^*_{E,H}$ is the induced map $|\wedge^i (F^n/H)^*| \to |\wedge^i E^*|$. Clearly, T is a non-trivial operator commuting with the action of $GL(n,F)$. Recall that an irreducible $GL(n,F)$-module is called *unramified* if it has a non-zero vector invariant with respect to maximal compact subgroup of $GL(n,F)$.

Theorem 2.1. *The operator T has an irreducible image. Moreover its image is an unramified $GL(n,F)$-module.*

Remark 2.2. It can be shown that the representation of $GL(n,F)$ in $\Gamma(Gr_{i,n}, L)$ is irreducible for $i = 0, 1, n-1, n$, and for $2 \leqslant i \leqslant n-2$ it has length two. This follows from the results of Zelevinsky [26] (see also the discussion below).

The proof of this theorem is an application of the results of the paper by Zelevinsky [26]. Let us remind the necessary facts following the notation of [26].

We will denote for brevity the group $GL(n,F)$ by G_n. By $Irr(G_n)$ we will denote the set of isomorphism classes of irreducible representations of G_n, and by $Rep(G_n)$ we will denote the set of isomorphism classes of all smooth representations of G_n. Let $\alpha = (n_1, \dots, n_r)$ be an ordered partition of n. Let G_α be the subgroup $G_{n_1} \times \cdots \times G_{n_r}$ of G_n consisting of block-diagonal matrices. Let P_α denote the subgroup of G_n consisting of block-upper-diagonal matrices. Then P_α is a parabolic subgroup with the Levi factor isomorphic to G_α. For $\rho_i \in Rep(G_{n_i})$, $i = 1, \dots, r$, let $\rho_1 \otimes \cdots \otimes \rho_r \in Rep(G_\alpha)$ be their (exterior) tensor product. This representation can be extended (trivially) to the representation of P_α. Define

$$\rho_1 \times \cdots \times \rho_r := Ind_{P_\alpha}^{G_n}(\rho_1 \otimes \cdots \otimes \rho_r),$$

where the induction is normalized.

Let $\mathscr{C}$ be the set of equivalence classes of irreducible cuspidal representations of G_n, $n = 1, 2, \ldots$. Note that a (one-dimensional) character of F^* can be considered as a cuspidal representation of $G_1 = F^*$. Let us call a segment in $\mathscr{C}$ any subset of $\mathscr{C}$ of the form $\Delta = \{\rho, \nu\rho, \ldots, \nu^k\rho = \rho'\}$, where ν is the character $\nu(g) = |det(g)|$. We will write it also as $\Delta = [\rho, \rho']$. To each segment Δ one can associate the irreducible representation $\langle \Delta \rangle$ which can be defined as the unique irreducible submodule of $\rho \times \nu\rho \times \cdots \times \nu^k\rho$.

Let $\Delta_1 = [\rho_1, {\rho_1}']$ and $\Delta_2 = [\rho_2, {\rho_2}']$ be two segments in $\mathscr{C}$. The segments Δ_1 and Δ_2 are called *linked* if $\Delta_1 \cup \Delta_2$ is also a segment and $\Delta_1 \not\subset \Delta_2$, $\Delta_2 \not\subset \Delta_1$. The segments Δ_1 and Δ_2 are called *juxtaposed* if they are linked and $\Delta_1 \cap \Delta_2 = \emptyset$. We say that Δ_1 *precedes* Δ_2 if they are linked and $\rho_2 = \nu^k\rho_1$ for $k > 0$.

The following theorem is one of the main results of [26] (Theorem 6.1).

Proposition 2.3. (a) *Let $\Delta_1, \ldots, \Delta_r$ be segments in $\mathscr{C}$. Suppose that for each pair of indices $i < j$, Δ_i does not precede Δ_j. Then the representation $\langle \Delta_1 \rangle \times \cdots \times \langle \Delta_r \rangle$ has a unique irreducible submodule; denote it by $\langle \Delta_1, \ldots, \Delta_r \rangle$.*

(b) *The representations $\langle \Delta_1, \ldots, \Delta_r \rangle$ and $\langle {\Delta'}_1, \ldots, {\Delta'}_s \rangle$ are isomorphic iff the sequences $\langle \Delta_1 \rangle, \ldots, \langle \Delta_r \rangle$ and $\langle {\Delta'}_1 \rangle, \ldots, \langle {\Delta'}_s \rangle$ are equal up to rearrangement.*

(c) *Any irreducible representation of G_n is isomorphic to some representation of the form $\langle \Delta_1, \ldots, \Delta_r \rangle$.*

We will also need the following fact [26, Theorem 4.2].

Proposition 2.4. *Let $\Delta_1, \ldots, \Delta_r$ be segments in $\mathscr{C}$. Then the representation $\langle \Delta_1 \rangle \times \cdots \times \langle \Delta_r \rangle$ is irreducible iff for all $i \neq j$ the segments Δ_i and Δ_j are not linked.*

Now let us introduce few more notation. Let Δ and Δ' be linked segments. Set

$$\Delta^{\cup} = \Delta \cup \Delta', \ \Delta^{\cap} = \Delta \cap \Delta'.$$

By definition $\Delta^{\cup}$ is a segment. $\Delta^{\cap}$ is a segment iff Δ and Δ' are not juxtaposed; otherwise $\Delta^{\cap} = \emptyset$. It was shown in [26, Section 4] that $\omega := \langle \Delta^{\cup} \rangle \times \langle \Delta^{\cap} \rangle$ is irreducible (and hence by Proposition 2.3 it is isomorphic to $\langle \Delta^{\cup}, \Delta^{\cap} \rangle$). (Here the term $\Delta^{\cap}$ should be ignored if it is empty.) The next result will be also used [26, Proposition 4.6].

Proposition 2.5. *Suppose Δ' precedes Δ* and set $\pi = \langle \Delta \rangle \times \langle \Delta' \rangle$. *Then π has a unique irreducible submodule $\omega_0 = \langle \Delta, \Delta' \rangle$. Moreover $\pi/\omega_0 \simeq \omega = \langle \Delta^{\cup} \rangle \times \langle \Delta^{\cap} \rangle$.*

Let us describe in terms of segments the representation dual to the given one. For each segment Δ in $\mathscr{C}$ let $\Delta^* := \{\rho^* | \rho \in \Delta\}$, where ρ^* is the representation dual to ρ. Clearly Δ^* is also a segment. The next result was proved in [26, Theorem 7.10].

Proposition 2.6. *For each segments $\Delta_1, \ldots, \Delta_r$ in $\mathscr{C}$*

$$\langle \Delta_1, \ldots, \Delta_r \rangle^* = \langle \Delta_1^*, \ldots, \Delta_r^* \rangle.$$

Now let us return to our situation.

Proof of Theorem 2.1. We use the notation of the beginning of this section. Recall that we study the intertwining operator

$$T : \Gamma(Gr_{n-i,n}, N) \to \Gamma(Gr_{i,n}, L).$$

For brevity, we will denote by κ the number $\frac{n-1}{2}$. It is easy to see that the representation of $GL(n,F)$ is isomorphic to $\Delta_1 \times \Delta_1{}'$, where

$$\Delta_1 = \underbrace{(\kappa - i, \kappa - (i-1), \ldots, \kappa - 1)}_{i \text{ times}},$$

$$\Delta_1{}' = \underbrace{(-\kappa, -\kappa + 1, \ldots, \kappa - i)}_{n-i \text{ times}}.$$

We see that if $i = 1$ or $n-1$ then the segments Δ_1 and $\Delta_1{}'$ are not linked (since one of them is contained in the other). Hence by Proposition 2.4 the representation $\Delta_1 \times \Delta_1{}'$ is irreducible. Hence in the cases $i = 1$, $n-1$ Theorem 2.1 is proved.

Now assume that $2 \leqslant i \leqslant n-2$. Then the segments Δ_1 and $\Delta_1{}'$ are linked and $\Delta_1{}'$ precedes Δ_1. Hence by Proposition 2.5 $\langle \Delta_1 \rangle \times \langle \Delta_1{}' \rangle$ has a unique irreducible submodule isomorphic to $\langle \Delta_1, \Delta_1{}' \rangle$, and the quotient module is irreducible and isomorphic to $\langle \Delta_1^{\cup}, \Delta_1^{\cap} \rangle$, where

$$\Delta_1^{\cup} = \underbrace{(-\kappa, -\kappa + 1, \ldots, \kappa - 1)}_{n-1 \text{ times}},$$

$$\Delta_1^{\cap} = (\kappa - i).$$

Now let us describe $\Gamma(Gr_{n-i,n}, N)$. Since $N = M \otimes |\omega|$ then $\Gamma(Gr_{n-i,n}, N)^* = \Gamma(Gr_{n-i,n}, M^*)$. It is easy to see that the representation of $GL(n,F)$ in $\Gamma(Gr_{n-i,n}, M^*)$ is equal to $\langle \Delta_2 \rangle \times \langle \Delta_2{}' \rangle$, where

$$\Delta_2 = \underbrace{(-\kappa + i, -\kappa + i + 1, \ldots, \kappa)}_{n-i \text{ times}},$$

$$\Delta_2{}' = \underbrace{(-\kappa + 1, -\kappa + 2, \ldots, -\kappa + i)}_{i \text{ times}}.$$

Clearly Δ_2' precedes Δ_2. Hence by Proposition 2.5 the unique irreducible submodule of $\langle \Delta_2 \rangle \times \langle \Delta_2' \rangle$ is isomorphic to $\langle \Delta_2, \Delta_2' \rangle$, and the quotient module is irreducible and isomorphic to $\langle \Delta_2^{\cup}, \Delta_2^{\cap} \rangle$, where

$$\Delta_2^{\cup} = \underbrace{(-\kappa+1, -\kappa+2, \dots, \kappa)}_{n-1 \text{ times}},$$

$$\Delta_2^{\cap} = (-\kappa+i).$$

Dualizing and using Proposition 2.6 we get that the $GL(n,F)$-module $\Gamma(Gr_{n-i,n}, N)$ has a unique irreducible submodule isomorphic to $\langle \Delta_3^{\cup}, \Delta_3^{\cap} \rangle$ with

$$\Delta_3^{\cup} = \underbrace{(-\kappa, -\kappa+1, \dots, \kappa-1)}_{n-1 \text{ times}},$$

$$\Delta_3^{\cap} = (\kappa - i);$$

and the quotient module is irreducible and is isomorphic to $\langle \Delta_3, \Delta_3' \rangle$ with

$$\Delta_3 = \underbrace{(-\kappa, -\kappa+1, \dots, \kappa-i)}_{n-i \text{ times}} = \Delta_1',$$

$$\Delta_3' = \underbrace{(\kappa-i, \kappa-i+1, \dots, \kappa-1)}_{i \text{ times}} = \Delta_1.$$

Comparing these computations with computations for $\Gamma(Gr_{i,n}, L)$ and using Proposition 2.3(b) we conclude that the image of any non-zero intertwining operator from $\Gamma(Gr_{n-i,n}, N)$ to $\Gamma(Gr_{i,n}, L)$ must have irreducible image.

It is easy to see that this image is unramified. $\square$

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