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REPRESENTATIONS OF THE GROUP $GL(n,F)$ WHERE F IS A NON-ARCHIMEDEAN LOCAL FIELD

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This article is a survey of recent results in the theory of representations of reductive $\wp$-adic groups. For simplicity of presentation only the groups $GL(n)$ are treated. Chapter I provides general information on representations of locally compact zero-dimensional groups. Chapter II presents Harish–Chandra's method of studying the representations of $GL(n)$, which is based on reduction to cuspidal representations. Some finiteness theorems are proved by this method. In Chapter III we study another approach to the representations of $GL(n)$, due to Gel'fand and Kazhdan; it is based on restricting the representations from $GL(n)$ to a subgroup P_n. All theorems are presented with detailed proofs. No prior information is assumed on the part of the reader except the most elementary familiarity with the structure of non-Archimedean local fields.

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Introduction

This paper is a survey of the theory of representations of reductive $\wp$-adic groups by the example of the group $GL(n, F)$, where F is a locally compact non-Archimedean field. Before presenting the contents of the paper, we recount the history of the problem.

0.1. The theory of representations of reductive $\wp$-adic groups began to develop comparatively recently. The first paper in this direction was written by Mautner [28] in 1958, in which spherical functions on the groups $SL(2, F)$ and $PGL(2, F)$ were studied.

In 1961 Bruhat [4] defined the "principal series" of representations for the $\wp$-adic Chevalley groups by analogy with the real case. These are representations induced by a one-dimensional representation of a Borel subgroup B (in the case of $GL(n, F)$, B is the subgroup of upper triangular matrices). Using his apparatus of distributions on locally compact groups, he proved some sufficient conditions for these representations to be irreducible.

In 1963 Gel'fand and Graev showed that, in addition to the principal series, there exist several series of irreducible unitary representations of $SL(2, F)$ that occur in the regular representation of $SL(2, F)$ (the so-called representations of the discrete series). In their construction these representations correspond to the multiplicative characters of quadratic distributions of F. Gel'fand and Graev wrote down explicit formulae for these representations, computed their characters, and, in the case when the characteristic p of the residue field of F is different from 2, obtained the Plancherel formula for $SL(2, F)$ (see [11], [12]).

Later other, more transparent constructions of representations of discrete series of $SL(2, F)$ were obtained by Shalika [38] and Tanaka [32].

Relying on the results of Gel'fand and Graev, Kirillov proved that for $p \neq 2$ these representations, together with the representations obtained from the principal series, exhaust all the irreducible unitary representations of $SL(2, F)$ (see [12], Ch. 2, Appendix). At present a complete list of irreducible representations exists only for $SL(2, F)$ and the groups close to it (such as $GL(2, F)$, the group of quaternions over F; all for $p \neq 2$).

In papers of Gel'fand and Pyatetskii–Shapiro a connection was revealed between the theory of representations of $\wp$-adic groups (and, more generally, adele groups) and the theory of automorphic forms (see [15], [12]).

The next important step was taken by Harish–Chandra [33]. He investigated the general properties of irreducible representations of reductive $\wp$-adic groups. By analogy with the real case, he studied those irreducible representations whose matrix elements are finite modulo the centre of the group. We call such representations cuspidal.[1] Harish–Chandra showed that any irreducible unitary representation of a reductive group G can be induced from a cuspidal representation of some subgroup. Thus, the study of arbitrary representations reduces, in a certain sense, to that of cuspidal representations. He showed that any irreducible cuspidal representation of G is admissible; that is, for each open subgroup $N \subset G$ the space of N-invariant vectors is finite-dimensional. This enabled him to develop the theory of characters for cuspidal representations.

[1] Harish–Chandra himself used the term cuspidal for unitary representations whose matrix elements are square-integrable modulo the centre. He called absolutely cuspidal representations whose matrix coefficients are finite modulo the centre. However, recently there has been a tendency to call the latter cuspidal (see [18], [13]), and we follow it.

A great stimulus to the subsequent development of the theory of representations of $\wp$-adic groups was the book by Jacquet and Langlands *Automorphic Forms on* $GL(2)$ (see [23]) published in 1970. This book pointed out an astonishing connection between this theory and the theory of numbers: the so-called non-commutative reciprocity law (for more details, see 0.3). This book, in many respects, determines the present direction of research in the theory of representations.

In 1971 Gel'fand and Kazhdan [14] extended the main technical tool used by Jacquet and Langlands, Kirillov's model and Whittaker's model, from $GL(2, F)$ to $GL(n, F)$ for all n. A more precise form of the hypothetical reciprocity law for $GL(n, F)$ is also presented in this paper.

In 1970–1973 Jacquet (see [22], [19]) and Howe (see [35], [36]) obtained a number of important results on the representations of $GL(n, F)$. Their methods readily carry over to a wide class of groups (see [9], [34]). We present part of their results in our survey.

Throughout this entire period the theory of representations of reductive $\wp$-adic groups had its eye on the admissibility conjecture, which kept resisting theorem status. This conjecture is that all irreducible unitary representations of a reductive group G are admissible. For $GL(2, F)$ it was proved by different methods in [24] and [29].

Harish–Chandra in [33] reduced the proof of the conjecture to the assertion that for a cuspidal representation the dimension of the space of N-invariant vectors does not exceed a certain number depending only on N (where N is an open subgroup of G). With the help of this result the conjecture was proved by Howe [36] for $GL(n, F)$ and by Bernshtein [2] for the general case.

0.2. In our paper we present mainly the results obtained by Harish–Chandra, Jacquet, Howe, Gel'fand, Kazhdan, and Bernshtein (as they apply to $GL(n, F)$). No preliminary knowledge is required on the part of the reader, except a familiarity with the structure of locally compact non-Archimedean fields (it is sufficient to know Chapters I and II of Weil's *Basic Number Theory* [10]).

We also use some standard facts on modules over infinite-dimensional algebras; in all such cases we give precise references. An exception is the Appendix, for whose understanding it is necessary to know the chapter on algebraic geometry in [3]; however, the article can be read entirely independently from the Appendix.

We now briefly present the contents of this paper.

Chapter I is of a preliminary nature. In § 1 we study locally compact zero-dimensional spaces and groups (we call them l-spaces and l-groups). We construct (following Bruhat [4]) the theory of distributions on l-spaces. In this case it is considerably simpler than the corresponding theory for real varieties. In addition, we introduce the new concept of an l-sheaf, which turns out to be very useful in the study of representations, and we treat the distributions on these sheaves.

In §2 general information on representations of l-groups is presented. Here, as throughout the article, we work with algebraic representations and do not treat unitary representations. In fact, studying algebraic representations is significantly simpler, because it is not necessary to introduce a topology in the representation space, and so all investigations are of a purely algebraic nature. On the other hand, as is shown in 4.21, the study of irreducible unitary representations reduces to that of algebraic representations.

Jacquet and Langlands [23] were the first to investigate algebraic representations systematically; we reproduce many of their results. We also note that in writing Chapter I we have used Rodier's article [30].

Chapter II is devoted to the general theory of representations of the group $G_n = GL(n, F)$. In §3 Harish–Chandra's theory is presented. First we introduce the functors $i_{\alpha,\beta}$ and $r_{\beta,\alpha}$, which make it possible to reduce the study of all representations of G_n to that of certain special representations, which we call quasi-cuspidal. Next we prove that quasi-cuspidal representations are precisely those whose matrix elements are finite modulo the centre (Harish–Chandra's theorem). Using this theorem and the theory of finite representations developed in §2, we study the properties of quasi-cuspidal representations. As a corollary, we obtain the theorem that each irreducible representation of G_n is admissible.

In writing §3 we have used [19].

In §4 two fundamental theorems are proved: Howe's theorem that any admissible finitely generated representation of $GL(n, F)$ has finite length, and the admissibility theorem (for each open subgroup $N \subset G_n$ the dimension of the space of N-invariant vectors in any irreducible representation of G_n does not exceed a constant depending only on N).

Next we deduce some important corollaries from the admissibility theorem. In particular, in 4.21 we explain how the admissibility theorem for irreducible unitary representations follows from it.

The results of 4.1–4.8 are due to Howe [35], [36]. The proofs given here are due to Bernshtein and are based on the methods of §3.[1]

In Chapter III the Gel'fand–Kazhdan theory is presented. In comparison with the original paper [14] we have used the following technical improvements: a) l-sheaves are used systematically; b) the representations of the group P_n (see 5.1) are investigated in detail; this investigation is based on the inductive passage from the representations of P_{n-1} to those of P_n (the idea of this inductive passage is due to Kazhdan).

In §5 we classify the representations of P_n. Next, with the help of this, we construct the models of Whittaker and Kirillov for irreducible representations of G_n.

[1] We recommend the reader to become acquainted with Howe's very beautiful original papers.

At the end of §5 we derive an important criterion, due to Kazhdan, for a representation of G_n or P_n to have finite length. From this it follows, in particular, that an irreducible representation of G_n has finite length under its restriction to P_n. The results obtained generalize the Jacquet–Langlands theorem to the effect that in Kirillov's model for $GL(2, F)$ the space of finite functions has codimension not greater than 2. This is shown in 5.24.

In §7 the main technical theorems stated in §5 are proved. Here the general theorem on invariant distributions proved in §6 (Theorem 6.10) is used. It asserts that if an l-group G and an automorphism σ act on an l-sheaf $\mathcal{F}$, then under certain conditions any G-invariant distribution on $\mathcal{F}$ is invariant under σ. The idea of using this theorem is due to Gel'fand and Kazhdan, but the conditions we use are much simpler, because they are stated in topological terms and not in terms of algebraic varieties.

In the Appendix we prove a theorem from algebraic geometry, which greatly simplifies the verification of the conditions of Theorem 6.10 for algebraic groups. The proof is Bernshtein's.

0.3. We now dwell briefly on those questions that we do not touch upon in our survey.

1. The results and formulations of Chapter II of our survey carry over to an arbitrary reductive group G. The complications that arise are of a technical nature and are connected with the necessity of analyzing the geometric structure of G in detail (see [33], [8], [2]).

The main theorem of Gel'fand and Kazhdan, Theorem 5.16, can be proved for an arbitrary decomposable group with a connected centre (see [14], [30]). Moreover, the Gel-fand–Kazhdan theory is completely preserved for a finite field F.

Throughout the paper G denotes a reductive $\wp$-adic group, and π an irreducible admissible representation of G.

2. CONJECTURE. *The character* tr π *of a representation* π *is a locally summable function on* G *and is locally constant on the set* G' *of regular elements.*

This conjecture was proved by Harish-Chandra [33] for the case when π is cuspidal and char $K = 0$, and by Jacquet and Langlands [23] for $G = GL(2, F)$. See also [35], [36], [9].

3. CONJECTURE. A Haar measure can be chosen on G such that the formal dimensions of all cuspidal representations of G are integers.

This conjecture is discussed in [33] and is proved for $GL(n, F)$ in [36].

4. Macdonald's survey article [27] on spherical functions on $\wp$-adic groups is excellent.

5. The non-commutative reciprocity law. This is the name given to Langland's conjecture, which connects the representations of $GL(n, F)$ with the n-dimensional representations of the Weyl group W_F of F (W_F is a subgroup of the Galois group $\mathrm{Gal}(\bar{F}/F)$; it is defined in [10]). This conjecture is discussed in [23] for $GL(2, F)$ and is proved in [20] for $GL(2, \mathbf{Q}_p)$.

Generalizations of this conjecture are discussed in Langland's lectures [26].

A more precise statement of the conjecture for $GL(n, F)$ is given by Gel'fand and Kazhdan (see [13], [16]).

6. One of the most unexpected results of Jacquet and Langlands is the connection they discovered between the representations of $GL(2, F)$ and the group of quaternions over F (this connection is discussed in detail in §15 of their book [23]). Apparently, for any groups G and G' that are isomorphic over the closure of F some correspondence must exist between their representations. The results of [37] confirm this.

In conclusion, we wish to thank I. M. Gel'fand, who gave us the idea of writing this survey, and D. A. Kazhdan for much useful advice. We wish to stress the strong influence on our work of Weil's lectures on the theory of Jacquet and Godement, which he gave at the Moscow State University in May–June 1972. We are also grateful to A. G. Kamenskii for reading the manuscript and making several valuable comments.

CHAPTER I

PRELIMINARY INFORMATION

§1. Distributions on l-spaces, l-groups, and l-sheaves

l-SPACES AND l-GROUPS

1.1. We call a topological space X an *l-space* if it is Hausdorff, locally compact, and zero-dimensional; that is, each point has a fundamental system of open compact neighbourhoods. We call a topological group G an *l-group* if there is a fundamental system of neighbourhoods of the unit element e consisting of open compact subgroups.

It can be shown that a topological group is an l-group if and only if it is an l-space.

1.2. LEMMA. *Let X be an l-space and $Y \subset X$ a locally closed subset (that is, Y is the intersection of open and closed subsets of X, or, equivalently, Y is closed in a neighbourhood of any of its points). Then Y is an l-space in the induced topology.* ■

1.3. LEMMA. *Let K be a compact subset of an l-space X. Then any covering of K by open subsets of X has a finite refinement of pairwise disjoint open compact subsets of X.* ■

1.4. PROPOSITION. *Let G be an l-group and H a closed subgroup of G. We introduce the quotient topology on $H \setminus G$ (U is open in $H \setminus G \Longleftrightarrow p^{-1}(U)$ is open in G, where $p\colon G \to H \setminus G$ is the natural projection). Then $H \setminus G$ is an l-space in this topology, and p is a continuous open map.*

This will be proved in 6.5.

1.5. Throughout, when we say that "G acts on an l-space X", we always mean a continuous left action.

PROPOSITION. *Suppose that an l-group G is countable at infinity and acts on an l-space X: $(g, X) \mapsto \gamma(g)x)$. We assume that the number of G-orbits in X is finite. Then there is an open orbit $X_0 \subset X$, and for any point $x_0 \in X_0$ the mapping $G \to X_0$ defined by $g \mapsto \gamma(g)x_0$ is open. (G is countable at infinity means that G is the union of countably many compact sets.)*

PROOF. Let N be an open compact subgroup of G, $\{g_i\}$ $(i = 1, 2, \ldots)$ coset representatives of G/N, and $x_1, \ldots, x_n \in X$ orbit representatives. Then $G = \bigcup_{i=1}^{\infty} g_iN$ and hence,

$$X = \bigcup_{i,j} \gamma(g_iN)x_j$$

is the union of countably many compact sets. It is easy to verify that any l-space is a Baire space, in other words, is not representable as the union of countably many nowhere dense closed subsets. Therefore, one of the sets $\gamma(g_iN)x_j$ contains an interior point $\gamma(g_in)x_j$. But then $x_j = \gamma(g_in)^{-1}\gamma(g_in)x_j$ is an interior point of $\gamma(N)x_j = \gamma(g_in)^{-1}\gamma(g_iN)x_j$.

It is clear that for one of the points x_j this fact is true for arbitrarily small subgroups N. Therefore, the mapping $g \mapsto \gamma(g)x_j$ is open. The proposition now follows from this.

1.6. COROLLARY. *If an l-group G is countable at infinity and acts transitively on an l-space X, $x_0 \in X$, and H is the stability subgroup of x_0, then the natural map of $H \setminus G$ into X: $Hg \mapsto \gamma(g^{-1})x_0$ is a homeomorphism.*

DISTRIBUTIONS ON l-SPACES

1.7. DEFINITION. Let X be an l-space. We call locally constant complex-valued functions on X with compact support *Schwartz functions* on X. We denote the space of these functions by $S(X)$. We call linear functionals on $S(X)$ *distributions* on X. We denote the space of distributions on X by $S^*(X)$. Note that $S(X)$ and $S^*(X)$ are treated without any topology. If $f \in S(X)$ and $T \in S^*(X)$, then the value of T at f is denoted by $\langle T, f\rangle$ or $\int_X f(x)dT(x)$ (or briefly $\int fdT$). If $x_0 \in X$, then the Dirac distribution ε_{x_0} is defined by $\langle \varepsilon_{x_0}, f\rangle = f(x_0)$.

We denote by $C^\infty(X)$ the space of all locally constant complex-valued functions on X.

1.8. Let Y be an open and Z a closed subset of an l-space X. We define the mappings i_Y: $S(Y) \to S(X)$ and p_Z: $S(X) \to S(Z)$ as follows: $i_Y(f)$ is the continuation of f by zero outside Y, and $p_Z(f)$ is the restriction of f to Z.

PROPOSITION. *The sequence*

$$0 \to S(Y) \xrightarrow{i_Y} S(X) \xrightarrow{p_{X\setminus Y}} S(X\setminus Y) \to 0$$

is exact.

PROOF. The proof requires only the fact that $p_{X\setminus Y}$ is epimorphic, that is, that any Schwartz function on $X \setminus Y$ can be continued to a Schwartz function on X. This follows easily from 1.3.

1.9. COROLLARY. *The dual sequence is also exact*:

$$0 \to S^*(X\setminus Y) \xrightarrow{p^*_{X\setminus Y}} S^*(X) \xrightarrow{i^*_Y} S^*(Y) \to 0.$$

1.10. DEFINITION. We say that a distribution $T \in S^*(X)$ is equal to 0 on an open subset $Y \subset X$ if $i^*_Y(T) = 0$. The set of points $x \in X$ such that T is not equal to 0 in any neighbourhood of x is called the *support* supp T of T. It is clear that supp T is closed. A distribution T is called *finite* if supp T is compact. We denote the space of finite distributions by $S^*_c(X)$.

1.11. For each $T \in S^*_c(X)$ we construct a functional on $C^\infty(X)$ (see 1.7). If K = supp T, then by virtue of 1.9 and 1.10 there exists a unique $T_0 \in S^*(K)$ such that $p^*_K(T_0) = T$. Since K is compact, for any $f \in C^\infty(X)$ we have $p_K(f) \in S(K)$, and we set $\int\limits_X f\, dT = \int\limits_K p_K(f)\, dT_0$.

1.12. Let E be a vector space over $\mathbf{C}$. We denote by $C^\infty(X, E)$ the space of locally constant functions on X with values in E, and by $S(X, E)$ the subspace of $C^\infty(X, E)$ consisting of functions with compact support. Using 1.3, we can easily show that $S(X, E) \simeq S(X) \otimes E$. Therefore, to each $T \in S^*(E)$ there corresponds a mapping $S(X, E) \to E$ defined by $f\cdot\xi \mapsto \langle T, f\rangle \cdot \xi$, where $f \in S(X)$ and $\xi \in E$. For each $T \in S^*_c(X)$ a mapping $C^\infty(X, E) \to E$ is constructed just as in 1.11 (notation: $f \mapsto \int\limits_X f(x)\, dT(x)$, where $f \in S(X, E)$ or $f \in C^\infty(X, E)$).

l-SHEAVES AND DISTRIBUTIONS ON THEM

1.13. In what follows we find it necessary to deal with vector functions that take values in distinct spaces at distinct points. To do so, we now introduce the concept of an l-sheaf, which we shall use constantly throughout.

DEFINITION. Let X be an l-space. We say that an *l-sheaf* is defined on X if with each $x \in X$ there is associated a complex vector space $\mathcal{F}_x$ and there is defined a family $\mathcal{F}$ of cross-sections (that is, mappings φ defined on X such that $\varphi(x) \in \mathcal{F}_x$ for each $x \in X$) such that the following conditions hold:

(1) $\mathcal{F}$ is invariant under addition and multiplication by functions in $C^\infty(X)$.

(2) If φ is a cross-section that coincides with some cross-section in $\mathcal{F}$

in a neighbourhood of each point, then $\varphi \in \mathscr{F}$.

(3) If $\varphi \in \mathscr{F}$, $x \in X$, and $\varphi(x) = 0$, then $\varphi = 0$ in some neighbourhood of x.

(4) For any $x \in X$ and $\xi \in \mathscr{F}_x$ there exists a $\varphi \in \mathscr{F}$ such that $\varphi(x) = \xi$.

The l-sheaf itself is denoted by $(X, \mathscr{F})$ or simply $\mathscr{F}$. The spaces $\mathscr{F}_x$ are called *stalks*, and the elements of $\mathscr{F}$ *cross-sections of the sheaf.* We call the set $\operatorname{supp} \varphi = \{x \in X \mid \varphi(x) \neq 0\}$ the *support of the cross-section* $\varphi \in \mathscr{F}$. Condition (3) guarantees that $\operatorname{supp} \varphi$ is closed.

We call a cross-section $\varphi \in \mathscr{F}$ *finite* if $\operatorname{supp} \varphi$ is compact. We denote the space of finite cross-sections of $(X, \mathscr{F})$ by $\mathscr{F}_c$. It is clear that $\mathscr{F}_c$ is a module over $S(X)$, and that $S(X) \cdot \mathscr{F}_c = \mathscr{F}_c$. It turns out that this property can be taken as the basis for the definition of an l-sheaf.

1.14. PROPOSITION. *Let M be a module over $S(X)$ such that $S(X) \cdot M = M$. Then there exists one and up to isomorphism only one l-sheaf $(X, \mathscr{F})$ such that M is isomorphic as an $S(X)$-module to the space of finite cross-sections $\mathscr{F}_c$.*

PROOF. For each $x \in X$ we denote by $M(x)$ the linear subspace of M generated by elements of the form $f \cdot \xi$, where $f \in S(X)$, $\xi \in M$, and $f(x) = 0$. (An equivalent definition: $M(x) = \{\xi \in M \mid \text{for some } f \in S(X),\ f(x) \neq 0 \text{ and } f \cdot \xi = 0\}$.)

We define the stalk $\mathscr{F}_x$ of $\mathscr{F}$ at $x \in X$ by $\mathscr{F}_x = M/M(x)$. The elements of M become cross-sections in a natural way. We denote by $\mathscr{F}$ the set of cross-sections that coincide with some cross-section of M in a neighbourhood of each point. Conditions (1)–(4) in 1.13 are obvious, and it is easy to verify with the help of 1.3 that $\mathscr{F}_c = M$.

Thus, defining an l-sheaf on X is equivalent to defining an $S(X)$-module M such that $S(X) \cdot M = M$.

EXAMPLE. $M = S(X)$. In this case $\mathscr{F} = C^\infty(X)$ is the sheaf of locally constant functions.

EXAMPLE. Let $q\colon X \to Y$ be a continuous mapping of l-spaces. Then $S(X)$ can be turned into an $S(Y)$-module in a natural way, and this defines an l-sheaf $\mathscr{F}$ on Y. It is easy to verify that the stalk $\mathscr{F}_y$ of $\mathscr{F}$ at $y \in Y$ is isomorphic to $S(q^{-1}y)$.

1.15. DEFINITION. Let $(X, \mathscr{F})$ be an l-sheaf. We call linear functionals on the space $\mathscr{F}_c$ of finite cross-sections of $\mathscr{F}$ *distributions on* $\mathscr{F}$. We denote the space of distributions on $\mathscr{F}$ by $\mathscr{F}^*$. It is clear that $\mathscr{F}^*$ is a module over the ring $C^\infty(X)$ (see 1.7) with respect to the multiplication $\langle fT, \varphi \rangle = \langle T, f\varphi \rangle$ $(T \in \mathscr{F}^*, \varphi \in \mathscr{F}_c,\ f \in C^\infty(X))$.

1.16. Let Y be a locally closed subset of X. Let us define a sheaf $(Y, \mathscr{F}(Y))$, which we call the *restriction of $\mathscr{F}$ to Y*. The stalks of $(Y, \mathscr{F}(Y))$ coincide with the corresponding stalks of $\mathscr{F}$, and $\mathscr{F}(Y)$ consists of the cross-sections that coincide with the restriction of some cross-section of $\mathscr{F}$ in a neighbourhood of each $y \in Y$. Conditions (1)–(4) from 1.13 are obviously fulfilled.

Let Y be an open and Z a closed subset of X. The mappings $i_Y : \mathscr{F}_c(Y) \to \mathscr{F}_c(X)$ and $p_Z : \mathscr{F}_c(X) \to \mathscr{F}_c(Z)$ are defined exactly as in 1.8. With the help of 1.3, it is easy to verify that these mappings are well-defined and that the results of 1.8–1.11 generalize to the case in question in an obvious way.

1.17. DEFINITION. Let $\gamma: X \to Y$ be a homeomorphism of l-spaces. We define an isomorphism $S(X) \to S(Y)$, which we also denote by γ, by setting $(\gamma f)(y) = f(\gamma^{-1}y)$ $(f \in S(X), y \in Y)$.

Let $(X, \mathscr{F})$ and $(Y, \mathscr{E})$ be l-sheaves. By an isomorphism $\gamma: \mathscr{F} \to \mathscr{E}$ we mean a pair consisting of a homeomorphism $X \to Y$ and a linear isomorphism $\mathscr{F}_c \to \mathscr{E}_c$ (we denote both these mappings also by γ) such that $\gamma(f\varphi) = \gamma(f)\gamma(\varphi)$, where $f \in S(X)$ and $\varphi \in \mathscr{F}_c$. It is clear that γ induces an isomorphism $\gamma: \mathscr{F}_x \to \mathscr{E}_{\gamma(x)}$ for any $x \in X$.

If $\gamma: \mathscr{F} \to \mathscr{E}$ is an isomorphism, then we define $\gamma: \mathscr{F}^* \to \mathscr{E}^*$ by $\langle \gamma(T), \varphi \rangle = \langle T, \gamma^{-1}(\varphi)\rangle$ $(T \in \mathscr{F}^*, \varphi \in \mathscr{E}_c)$.

By the action of an l-group G on an l-sheaf $(X, \mathscr{F})$ we mean a system of isomorphisms $\gamma(g)$ $(g \in G)$ of $(X, \mathscr{F})$ having the natural group properties. Here we assume that the mapping $\gamma: G \times X \to X$ is continuous.

A distribution $T \in \mathscr{F}^*$ is called *G-invariant* if $\gamma(g)T = T$ for all $g \in G$.

HAAR MEASURE

1.18. Let us consider the action of an l-group G on itself by left translations $\gamma(g)g' = gg'$ and right translations $\delta(g)g' = g'g^{-1}$.

PROPOSITION. *There exists one and up to a factor only one left-invariant distribution $\mu_G \in S^*(G)$, that is, a distribution such that*

$\int_G f(g_0 g)\, d\mu_G(g) = \int_G f(g)\, d\mu_G(g)$ *for all $f \in S(G)$ and $g_0 \in G$. We can take μ_G to be a positive functional (that is, $\langle \mu_G, f\rangle > 0$ for any non-zero non-negative function $f \in S(X)$). We call such a distribution μ_G a (left-invariant) Haar measure.*

A right-invariant Haar measure ν_G is defined in a similar manner.

PROOF. Let $\{N_\alpha\}$, $\alpha \in \mathfrak{A}$, be a family of open compact subgroups of G that form a fundamental system of neighbourhoods of the unit element; suppose that there exists an index $\alpha_0 \in \mathfrak{A}$ such that $N_\alpha \subset N_{\alpha_0}$ for all $\alpha \in \mathfrak{A}$. We set $S_\alpha = \{f \in S(G) \mid \delta(G)f = f$ for any $g \in N_\alpha\}$. It is clear that $S_\alpha \subset S_\beta$ for $N_\alpha \supset N_\beta$ and that $S(G) = \bigcup_\alpha S_\alpha$. Obviously each space S_α is invariant under left translations and to define a left-invariant distribution on G comes to the same thing as to define a system of mutually compatible left-invariant functionals $\mu_\alpha \in S_\alpha^*$. It is clear that S_α is generated by the left translations of the characteristic function of N_α, hence that μ_G is unique. To prove the existence, we define μ_α by

$$\mu_\alpha(f) = |N_{\alpha_0}/N_\alpha|^{-1}\cdot\sum f(g), \text{ where } f\in S_\alpha,$$

and the sum is taken over $g \in G/N_\alpha$. Since supp f is compact, in this sum only finitely many summands are non-zero, and it can be verified immediately that the functionals μ_G are left-invariant, compatible, and positive.

The assertions pertaining to a right-invariant distribution are proved similarly.

1.19. We call a locally constant homomorphism of an l-group G into the multiplicative group of complex numbers a *character* of G.

PROPOSITION. a) *Under the conditions of* 1.18, *for each* $g \in G$ *there exists a unique number* $\Delta_G(g) > 0$ *such that* $\delta(g)\mu_G = \Delta_G(g)\mu_G$. *The function* Δ_G *is a character of* G.

b) *The restriction of* Δ_G *to any compact subgroup* $\Gamma \subset G$ *is trivial.*

c) *The distribution* $\Delta_G^{-1}\cdot\mu_G$ *is right-invariant. If* $T \to \check{T}$ *is the transformation in* $S^*(G)$ *corresponding to the homeomorphism* $g \mapsto g^{-1}$ (see 1.16), *then the distribution* $\check{\mu}_G$ *is right-invariant and coincides with* $\Delta_G^{-1}\mu_G$.

The character Δ_G is called the *modulus* of G. If $\Delta_G = 1$, then G is called *unimodular*. As follows from b), a group generated by compact subgroups is always unimodular.

PROOF. a) It is clear that the distribution $\delta(g)\mu_G$ is left-invariant and positive; that is, $\delta(g)\mu_G = \Delta_G(g)\mu_G$, where $\Delta_G(g) > 0$ and $\Delta_G(g_1)\cdot\Delta_G(g_2) = \Delta_G(g_1g_2)$ (see 1.18). If $f \in S(G)$ and $\langle\mu_G, f\rangle > 0$, then for elements g lying in some neighbourhood of the unit element we have $\delta(g)f = f$, so that $\Delta_G(g) = 1$. Hence Δ_G is a character.

b) $\Delta_G(\Gamma)$ is a finite subgroup of $(\mathbf{R}^+)^*$, that is, $\Delta_G(\Gamma) = 1$.

c) Since $\delta(g)\mu_G = \Delta_G(g)\mu_G$ and $\delta(g)\Delta_G = \Delta_G(g)\cdot\Delta_G$, the distribution $\Delta_G^{-1}\cdot\mu_G$ is right-invariant. It is clear that $\check{\mu}_G$ is also right-invariant, so that $\Delta_G^{-1}\cdot\mu_G$ and $\check{\mu}_G$ are proportional. Since their values on the characteristic function of any open compact subgroup coincide and are positive, they are equal.

HAAR MEASURE ON A FACTOR SPACE

1.20. Let H be a closed subgroup of an l-group G, and let $\Delta = \Delta_G/\Delta_H$ be a character of H. We consider functions f in $C^\infty(G)$ that satisfy the following conditions:

a) For any $h \in H$ and $g \in G$, $f(hg) = \Delta(h)f(g)$.

b) f is finite modulo H; that is, there is a compact set $K_f \subset G$ such that supp $f \subset H\cdot K_f$.

We denote the space of such functions by $S(G, \Delta)$. It is clear that it is invariant under the right action δ of G (see 1.18). If $\Delta = 1$, then $S(G, \Delta)$ is naturally isomorphic to $S(H\setminus G)$.

1.21. THEOREM. *There exists one and up to a factor only one functional $\nu_{H\backslash G}$ on $S(G, \Delta)$ that is invariant under the action δ of G. This functional can be chosen to be positive. We call it a Haar measure on $H\backslash G$ and write* $f \mapsto \langle \nu_{H\backslash G}, f\rangle = \int\limits_{H\backslash G} f(g)\, d\nu_{H\backslash G}(g), f \in S(G, \Delta)$.

PROOF. We define an operator p: $S(G) \to C^\infty(G)$ by

$$(pf)(g) = \int\limits_H f(hg)\, \Delta_G^{-1}(h)\, d\mu_H(h).$$

It clearly commutes with the right action of G. Moreover, if $h_0 \in H$, then

$$pf(h_0 g) = \int\limits_H f(hh_0 g)\, \Delta_G^{-1}(h)\, d\mu_H(h) = \int\limits_H f(hh_0 g)\, \Delta_G^{-1}(h) \cdot \Delta_H(h) \cdot d\nu_H(h) =$$

$$= \Delta(h_0) \cdot \int\limits_H f(hh_0 g)\, \Delta_G^{-1}(hh_0)\, \Delta_H(hh_0)\, d\nu_H(h) =$$

$$= \Delta(h_0) \cdot \int\limits_H f(h'g)\, \Delta_G^{-1}(h')\, \Delta_H(h')\, d\nu_H(h') = \Delta(h_0)\, (pf)(g)$$

(see 1.18 and 1.19). Hence $p(S(G)) \subset S(G, \Delta)$.

LEMMA. a) *The homomorphism* p: $S(G) \to S(G, \Delta)$ *is surjective.*

b) *If* $f \in S(G)$ *and* $pf = 0$, *then* $\langle \nu_G, f\rangle = 0$.

PROOF. Let N be an open compact subgroup of G, and let $g \in G$. We denote by $S(G)_g^N$ (respectively, $S(G, \Delta)_g^N$) the subspace of functions $f \in S(G)$ (respectively, $f \in S(G, \Delta)$) such that supp $f \subset HgN$ and $\delta(n)f = f$ for any $n \in N$. It is clear that $p(S(G)_g^N) \subset S(G, \Delta)_g^N$ and that it suffices to verify the assertions of the lemma for these subspaces.

a) Let $f \in S(G, \Delta)_g^N$. We set $f' = c^{-1} \cdot \chi_{gN} \cdot f$, where χ_{gN} is the characteristic function of gN and $c = \int\limits_H \chi_{gN}(hg)\, \Delta_H^{-1}(h) d\mu_H(h)$. It is clear that $f' \in S(G)_g^N$ and that $pf' = f$.

b) The space $S(G)_g^N$ can be identified with the space of finite functions on the discrete set HgN/N, on which H acts transitively on the left. Therefore, any two functionals T on $S(G)_g^N$ such that $T(\gamma(h)f) = \Delta_G^{-1}(h)T(f)$ for all $h \in H$, $f \in S(G)_g^N$ are proportional. It can be verified as above that the functionals $Tf = pf(g)$ and $T'f = \langle \nu_G, f\rangle = \langle \Delta_G^{-1}\mu_G, f\rangle$ satisfy these conditions, that is, are proportional; hence b) follows.

It follows from this lemma and 1.18 that $S(G, \Delta) \simeq S(G)/\text{Ker } p$ and that any functional on $S(G)$ that is invariant under the right action of G is equal to 0 on Ker p. Therefore, by 1.18, $\nu_{H\backslash G}$ exists and is unique. Since ν_G is positive, $\nu_{H\backslash G}$ is positive.

DISTRIBUTIONS ON l-GROUPS; CONVOLUTIONS

1.22. Let X_1 and X_2 be l-spaces. It is easy to see that $S(X_1) \times S(X_2)$, is isomorphic to the tensor product $S(X_1) \otimes S(X_2)$, where the isomorphism is defined by

$$(f_1 \otimes f_2)(x_1, x_2) = f_1(x_1)f_2(x_2)(f_i \in S(X_i),\ x_i \in X_i).$$

This isomorphism makes it possible to define the tensor product of distributions: if $T_i \in S^*(X_i)$ $(i = 1, 2)$, then we define $T_1 \otimes T_2 \in S^*(X_1 \times X_2)$ by

$$\int\limits_{X_1\times X_2} (f_1 \otimes f_2)\, d\,(T_1 \otimes T_2) = \int\limits_{X_1} f_1\, dT_1 \cdot \int\limits_{X_2} f_2\, dT_2.$$

It is easy to verify that Fubini's theorem is valid:

$$\int\limits_{X_1\times X_2} f(x_1, x_2)\, d\,(T_1 \otimes T_2)(x_1, x_2) = \int\limits_{X_1} \left\{ \int\limits_{X_2} f(x_1, x_2)\, dT_2(x_2) \right\} dT_1(x_1) =$$
$$= \int\limits_{X_2} \left\{ \int\limits_{X_1} f(x_1, x_2)\, dT_1(x_1) \right\} dT_2(x_2); \qquad (f \in S(X_1 \times X_2),\ T_i \in S^*(X_i)).$$

1.23. PROPOSITION. $\operatorname{supp}(T_1 \otimes T_2) = \operatorname{supp} T_1 \times \operatorname{supp} T_2$.

PROOF. If Y_i is open in $X_i (i = 1, 2)$, then it follows immediately from the definition that $i^*_{Y_1\times Y_2}(T_1 \otimes T_2) = i^*_{Y_1}(T_1) \otimes i^*_{Y_2}(T_2)$ (see 1.9). Our assertion follows from this immediately.

COROLLARY. *The tensor product of finite distributions is finite.*

1.24. DEFINITION. Let G be an l-group. By the *convolution* of two distributions $T_1, T_2 \in S^*_c(G)$ we mean the distribution $T_1 * T_2 \in S^*(G)$ defined by

$$\int\limits_G f(g)\, d\,(T_1 * T_2)(g) = \int\limits_{G\times G} f(g_1 g_2)\, d\,(T_1 \otimes T_2)(g_1, g_2) \quad (f \in S(G))$$

(the expression on the right-hand side makes sense; see 1.11 and 1.23).

If $T \in S^*_c(G)$ and $f \in S(G)$, then we define the convolution

$$T * f \in S(G) \text{ by } (T * f)(g_0) = \int f(g^{-1}g_0)\, dT(g).$$

PROPOSITION. a) $\operatorname{supp}(T_1 * T_2) \subset \operatorname{supp} T_1 \cdot \operatorname{supp} T_2$. *In particular,* $T_1 * T_2 \in S^*_c(G)$.

b) *If* $T_1, T_2 \in S^*_c(G)$ *and* $f \in S(G)$, *then* $T_1 * (T_2 * f) = (T_1 * T_2) * f$.

PROOF. a) follows from 1.23, and b) follows easily from Fubini's theorem.

1.25. COROLLARY. a) *The convolution operation* $*$ *determines on* $S^*_c(G)$ *the structure of an associative algebra. The Dirac distribution* ε_e (see 1.7) *is the unit element of this algebra.*

b) *If* $T \in S^*_c(G)$ *and* $g \in G$, *then* $\varepsilon_g * T = \gamma(g)T$ *and* $T * \varepsilon_g = \delta(g^{-1})T$. *In particular,* $\varepsilon_g * \varepsilon_{g'} = \varepsilon_{gg'}$.

1.26. Let Γ be a compact subgroup of an l-group G and μ_Γ a two-sided invariant Haar measure on Γ (see 1.18 and 1.19), normalized by the condition $\langle \mu_\Gamma, 1 \rangle = 1$. We define $\varepsilon_\Gamma \in S^*_c(G)$ by $\varepsilon_\Gamma = p^*_\Gamma(\mu_\Gamma)$ (see 1.9).

PROPOSITION. a) *If* $g \in \Gamma$, *then* $\varepsilon_g * \varepsilon_\Gamma = \varepsilon_\Gamma * \varepsilon_g = \varepsilon_\Gamma$.

b) *If* Γ_1 *and* Γ_2 *are compact subgroups of* Γ *and* $\Gamma = \Gamma_1 \cdot \Gamma_2$, *then* $\varepsilon_\Gamma = \varepsilon_{\Gamma_1} * \varepsilon_{\Gamma_2}$ *(and hence* $\varepsilon_\Gamma = \varepsilon_{\Gamma_2} * \varepsilon_{\Gamma_1}$, *since* $\Gamma = \Gamma^{-1} = \Gamma_2^{-1} \cdot \Gamma_1^{-1} = \Gamma_2 \cdot \Gamma_1$). *In particular,* $\varepsilon_\Gamma^2 = \varepsilon_\Gamma$.

c) *If* $g \in G$, *then* $\varepsilon_g * \varepsilon_\Gamma * \varepsilon_{g^{-1}} = \varepsilon_{g\Gamma g^{-1}}$.

a) can be verified directly; b) and c) follow from the next lemma.

1.27. LEMMA. *Let* Γ_1 *and* Γ_2 *be compact subgroups of an l-group* G, *and let* $g \in G$. *Then there exists a unique distribution* $T \in S_c^*(G)$ *such that* supp $T \subset \Gamma_1 g \Gamma_2$ *and* $\gamma(g_1)T = \delta(g_2)T = T$ *for all* $g_1 \in \Gamma_1$, $g_2 \in \Gamma_2$, *and* $\langle T, 1\rangle = 1$.

PROOF. We define the action γ' of $\Gamma_1 \times \Gamma_2$ on $\Gamma_1 g \Gamma_2$ by $\gamma'(g_1 g_2)x = g_1 x \cdot g_2^{-1}$ $(g_1 \in \Gamma_1,\ g_2 \in \Gamma_2,\ x \in \Gamma_1 g \Gamma_2)$. Since this action is transitive, it follows from 1.21 and 1.19 that there exists a unique γ'-invariant distribution $T_0 \in S^*(\Gamma_1 g \Gamma_2)$ such that $\langle T_0, 1\rangle = 1$. The lemma now follows from 1.9.

T can be described explicitly: it is equal to $\varepsilon_{\Gamma_1} * \varepsilon_g * \varepsilon_{\Gamma_2}$.

1.28. DEFINITION. We call a distribution $T \in S^*(G)$ *locally constant on the left* (respectively, *right*) if there exists an open subgroup $N \subset G$ such that $\gamma(g)T = T$ (respectively, $\delta(g)T = T$) for all $g \in N$. Locally constant functions in $C^\infty(G)$ on the left (or right) are defined similarly.

PROPOSITION. *Let* μ_G *be a left-invariant Haar measure on an l-group* G. *Then any* $T \in S^*(G)$ *that is locally constant on the left has the form* $T = f \cdot \mu_G$, *where* f *is a function on* G *that is locally constant on the left. The situation is similar for distributions that are locally constant on the right.*

PROOF. Let N be an open compact subgroup of G such that $\gamma(g)T = T$ for all $g \in N$. By 1.27, the restriction of T to any set of the form $N \cdot g (g \in G)$ is proportional to the restriction of μ_G, hence the required proposition follows.

1.29. COROLLARY. *The mapping* $f \mapsto f\mu_G$ *defines an isomorphism of* $S(G)$ *with the space of finite distributions that are locally constant on the left. In particular, these distributions are locally constant on the right. The space* $\mathcal{H}(G)$ *of locally constant finite distributions on* G *is a two-sided ideal in* $S_c^*(G)$.■

1.30. We shall often identify $S(G)$ with $\mathcal{H}(G)$ by the isomorphism $f \mapsto f\mu_G$, where μ_G is a fixed left-invariant Haar measure on G. Here a convolution operation arises on $S(G)$: $(f_1\mu_G) * (f_2\mu_G) = (f_1 * f_2)\mu_G$. An explicit formula can be written down for this operation.

LEMMA. a) *If* $T \in S_c^*(G)$ *and* $f \in S(G)$, *then* $T * (f\mu_G) = (T * f)\mu_G$.

b) *If* $f_1, f_2 \in S(G)$, *then* $(f_1 * f_2)(g_1) = \int f_1(g) f_2(g^{-1}g_1)\, d\mu_G(g)$.

§2. Representations of *l*-groups

In what follows, instead of the sentence, "Let π be a representation of an

l-group G in a complex vector space E'', we write $\pi = (\pi, G, E)$. We call E a *G-module*. All vector spaces that we encounter are treated without any topology. A G-module E is called *irreducible* if it is different from 0 and has no G-submodules except 0 and E.

ALGEBRAIC AND ADMISSIBLE REPRESENTATIONS

2.1. DEFINITION. a) A representation (π, G, E) is called *algebraic* if for any $\xi \in E$ the stabilizer stab $\xi = \{g \in G \mid \pi(g)\,\xi = \xi\}$ is open in G. For any representation (π, G, E) we set $E_\alpha = \{\xi \in E \mid \text{stab } \xi \text{ is open in } G\}$. It is clear that E_α is a G-submodule of E and that $\pi \mid E_\alpha$ is an algebraic representation, called the *algebraic part* of π.

b) A representation (π, G, E) is called *admissible* if it is algebraic and if for any open subgroup $N \subset G$ the subspace E^N consisting of the N-invariant vectors in E is finite-dimensional.

We denote by Alg G the category whose objects are the algebraic representations of G and whose morphisms are the usual morphisms of representations (intertwining operators).

2.2. EXAMPLE. Suppose that an action γ of an l-group G on an l-space X is defined. Then the representation $(\gamma, G\ S(X))$ is algebraic, but the representation $(\gamma, G, C^\infty(X))$ is not necessarily algebraic. For example, if γ is the action of G on itself by left translations (see 1.18), then $C^\infty(G)_a$ consists of the functions that are locally constant on the left (see 1.28).

2.3. Let (π, G, E) be an algebraic representation. For each $T \in S_c^*(G)$ we define an operator $\pi(T)$ in E; namely, if $\xi \in E$, then the function $g \mapsto \pi(g)\xi$ lies in $C^\infty(G, E)$, and we set $\pi(T)\xi = \int_G \pi(g)\xi\, dT(g)$ (see 1.2).

Using 1.24, we can easily verify that

$$\pi(T * T') = \pi(T) \circ \pi(T') \qquad (T, T' \in S_c^*(G)),$$
$$\pi(\varepsilon_g) = \pi(g), \qquad (g \in G).$$

Thus, the mapping $T \mapsto \pi(T)$ defines a representation of $S_c^*(G)$ (see 1.25), which we denote by the same letter π.

For example, for $(\gamma, G, S(G))$ we have $\gamma(T)f = T * F$ (see 1.24).

2.4. Under the conditions of 2.3, let Γ be a compact subgroup of G. Then $\pi(\varepsilon_\Gamma)$ (see 1.26) is the projection operator $(\pi(\varepsilon_\Gamma)^2 = \pi(\varepsilon_\Gamma * \varepsilon_\Gamma) = \pi(\varepsilon_\Gamma)$ by virtue of 1.26).

The image $\pi(\varepsilon_\Gamma)E$ coincides with the subspace E^Γ of Γ-invariant vectors in E. It follows, in particular, that for any exact sequence of algebraic G-modules $0 \to E_1 \to E_2 \to E_3 \to 0$ the sequence $0 \to E_1^\Gamma \to E_2^\Gamma \to E_3^\Gamma \to 0$ is also exact.

The kernel of $\pi(\varepsilon_\Gamma)$ is the space $E(\Gamma)$ spanned by all vectors of the form $\pi(g)\xi - \xi$, $g \in \Gamma$, $\xi \in E$. For it is clear that $\pi(\varepsilon_\Gamma) \mid E(\Gamma) = 0$ and that Γ acts trivially on $E/E(\Gamma)$, so that $\pi(\varepsilon_\Gamma)$ is the identity on this space.

2.5. Let $(\pi, G, E) \in \mathrm{Alg}\, G$. By 2.3, a representation π of $\mathcal{H}(G)$ corresponds to it (see 1.29) and $E = \bigcup \pi(\varepsilon_N)E$, where the union is taken over all open compact subgroups $N \subset G$.

Conversely, let τ be a representation of $\mathcal{H}(G)$ in E, where $E = \bigcup_N \tau(\varepsilon_N)E$. If $T \in S_c^*(G)$ and $\xi \in E$, then we set $\tau(T)\xi = \tau(T * \varepsilon_N)\xi$, where N is an open compact subgroup of G such that $\tau(\varepsilon_N)\xi = \xi$. This is well-defined by virtue of 1.26b). We also set $\pi(g) = \tau(\varepsilon_g)$.

PROPOSITION. *The representation* (π, G, E) *is algebraic, and for any* $T \in S_c^*(G)$ *we have* $\pi(T) = \tau(T)$ (see 2.3).

Thus, Alg G is isomorphic to a complete subcategory of the category of modules over $\mathcal{H}(G)$. This subcategory consists of the modules E such that $E = \bigcup_N \varepsilon_N E$.

SOME INFORMATION ON MODULES

2.6. To facilitate the presentation, we recall some standard facts from general algebra.

LEMMA. *Let* $\mathcal{H}$ *be an algebra over* $\mathbf{C}$ *and* E *a non-zero* $\mathcal{H}$*-module.*

a) *If* E *is finitely generated, then* E *has an irreducible factor module.*

b) *In general,* E *has an irreducible subfactor module; that is, submodules* $E' \subset E'' \subset E$ *can be chosen in* E *such that* $E' \neq E''$ *and* E''/E' *is irreducible.*

a) follows easily from Zorn's lemma (see [25]); b) follows easily from a).

2.7. LEMMA-DEFINITION. *Let* $\mathcal{H}$ *be an algebra over* $\mathbf{C}$ *and* E *an* $\mathcal{H}$*-module. Then the following conditions are equivalent:*

a) E *is isomorphic to the direct sum of irreducible* $\mathcal{H}$*-modules.*

b) E *is generated by its irreducible* $\mathcal{H}$*-submodules.*

c) *Each* $\mathcal{H}$*-submodule* $E' \subset E$ *has an* $\mathcal{H}$*-invariant complement* E'' *such that* $E = E' \oplus E''$.

A proof can be found in [5] (Ch. VIII, §3, Proposition 7).

A module E that satisfies these conditions is called *completely reducible.* It is clear that any subfactor module of a completely reducible module is also completely reducible.

2.8. Let $\mathcal{H}$ be an algebra over $\mathbf{C}$ and E an $\mathcal{H}$-module. We call a sequence of submodules $0 \subsetneq E_1 \subsetneq \dots \subsetneq E_k = E$ a *chain length* k. We say that E has finite length if the lengths of all its chains of submodules are bounded. In this case each chain can be included in a maximal chain $0 \subsetneq E_1' \subsetneq \dots \subsetneq E_l' = E$, that is, such that all factor modules E_i'/E_{i-1}' are irreducible. The length l of a maximal chain and the collection of factor modules (up to rearrangement) depend only on E (see [1]).

The number l is called the *length* of E and is denoted by $l(E)$. For any subfactor module E' of E we have $l(E') \leqslant l(E)$.

RESTRICTION TO SUBGROUPS OF FINITE INDEX

2.9. LEMMA. *Let G be a group, H a normal subgroup of G of finite index, and (π, G, E) a representation.*

a) *If $\pi|_H$ is completely reducible, then π is also completely reducible.*

b) *If π is irreducible, then $\pi|_H = \pi_1 \oplus \ldots \oplus \pi_k$, where (π_i, H, E_i) is an irreducible representation and $k \leqslant |G/H|$.*

PROOF. Let $g_1, \ldots, g_l$ be coset representatives of G/H.

a) If E_1 is a G-submodule of E, then by 2.7, E_1 has a complementary H-submodule $E_1^\perp$. Let A be the projection of E onto E_1 along $E_1^\perp$. We set $A_0 = l^{-1} \cdot \sum_{i=1}^{l} \pi(g_i)A\pi(g_i)^{-1}$. It is easy to see that A_0 projects E onto E_1 and commutes with the action of G. Therefore, its kernel Ker A_0 is a G-submodule of E complementary to E_1.

b) Let $\xi \in E$, $\xi \neq 0$. Then the vectors $\pi(g_i)\xi$ generate E as an H-module. By 2.6, there exists an H-submodule E' of E such that the H-module E/E' is irreducible. We set $E_i' = \pi(g_i)E'$ and consider the H-equivariant mapping $A: E \to \oplus E/E_i'$ $(A\xi = (\xi \bmod E_1', \ldots, \xi \bmod E_l'))$. Since Ker A is G-invariant and not equal to E and since π is irreducible, Ker $A = 0$; that is, A is an embedding. Therefore, E, as an H-module, is isomorphic to a submodule of $\bigoplus_{i=1}^{l} E/E_i'$. The lemma now follows from 2.7 and 2.8

THE REPRESENTATIONS π_N

2.10. Let N be an open compact subgroup of an l-group G. We set $\mathscr{H}_N = \varepsilon_N * S_c^*(G) * \varepsilon_N$. This is an algebra with a unit element, whose role is played by ε_N. For each representation $(\pi, G, E) \in$ Alg G we denote by π_N the natural representation of $\mathscr{H}_N$ in E^N.

PROPOSITION. a) *π is irreducible if and only if $\pi \neq 0$ and for all subgroups N either $\pi_N = 0$ or π_N is irreducible.*

b) *Suppose that $(\pi_i, G, E_i) \in$ Alg G is irreducible and that N is such that $E_i^N \neq 0$ $(i = 1, 2)$. Then π_1 is equivalent to π_2 if and only if $(\pi_1)_N$ is equivalent to $(\pi_2)_N$.*

c) *For each irreducible representation τ of $\mathscr{H}_N$ there exists an irreducible representation $\pi \in$ Alg G such that $\tau \simeq \pi_N$.*

PROOF. Let $(\pi, G, E) \in$ Alg G and let V be an $\mathscr{H}_N$-invariant subspace of E^N. We consider the G-submodule $E' \subset E$ generated by V. Then $E'^N = \pi(\varepsilon_N)E'$ is generated by the vectors $\pi(\varepsilon_N)\pi(T)v = \pi(\varepsilon_N * T * \varepsilon_N)v$ $(T \in S_c^*(G),\ v \in V)$; that is, $E'^N = V$.

a) Suppose that $(\pi, G, E) \in \mathrm{Alg}\, G$ is irreducible. If E^N contained an $\mathcal{H}_N$-invariant subspace V other than 0 and E^N, then, as follows from our arguments, it would generate a G-module E' other than from 0 and E. Hence π_N is irreducible. Conversely, if there exists a non-trivial G-submodule E' of E, then $0 \subsetneq E'^N \subsetneq E^N$ for some group N; that is, π_N is reducible.

b) Suppose that $(\pi_1)_N$ is equivalent to $(\pi_2)_N$ and that $j\colon E_1^N \to E_2^N$ is the operator that defines this equivalence. Then the space $V = \{(x, jx)\} \subset E_1^N \oplus E_2^N$ is $\mathcal{H}_N$-invariant. We consider the G-submodule E' of $E_1 \oplus E_2$ generated by V. As shown above, $E'^N = V$, so that E' does not contain, and is not contained in, E_1 or E_2. Since E_1 and E_2 are irreducible G-modules, the projections of E' onto E_1 and E_2 are isomorphisms, so that $E_1 \simeq E' \simeq E_2$. The converse in b) is obvious.

c) Let L be the space of τ. Since L is an irreducible $\mathcal{H}_N$-module, it is isomorphic to $\mathcal{H}_N/V$, where V is a left ideal in $\mathcal{H}_N$. We regard $\mathcal{H}(G)$ as a left $\mathcal{H}(G)$-module and denote by E_1 and E_2 the submodules of $\mathcal{H}(G)$ generated by the subspaces $\mathcal{H}_N \subset \mathcal{H}(G)$ and $V \subset \mathcal{H}_N \subset \mathcal{H}(G)$. Then $E_1^N = \mathcal{H}_N$ and $E_2^N = V$. If we set $E_3 = E_1/E_2$, then $E_3^N = \mathcal{H}_N/V \simeq L$, where E_3 is generated by E_3^N.

Since L is an irreducible module, for any G-submodule $E' \subset E_3$ either $E'^N = 0$, hence $(E_3/E')^N \simeq L$, or $E'^N = E_3^N$, hence $E' = E_3$. Therefore, taking an irreducible factor module E of E_3 (which exists by 2.6, because E_3 is generated by any $\xi \in E_3^N$, $\xi \neq 0$), we obtain $E^N \simeq L$, as required.

SCHUR'S LEMMA

2.11. PROPOSITION. *Let G be an l-group that is countable at infinity and suppose that $(\pi, G, E) \in \mathrm{Alg}\, G$ is irreducible. Then the space* $\mathrm{Hom}_G(\pi, \pi)$ *consists of scalar operators.*

PROOF. It is clear that any non-zero operator $A \in \mathrm{Hom}_G(\pi, \pi)$ is invertible. We assume that $A \neq \lambda \cdot 1$ ($1 = 1_E$ is the identity operator $E \to E$). Then the operator $R_\lambda = (A - \lambda)^{-1}$ is defined for each $\lambda \in \mathbf{C}$. We choose a non-zero vector $\xi \in E$. Then the vectors $R_\lambda \xi$, $\lambda \in \mathbf{C}$, are linearly independent (if $\Sigma c_i R_{\lambda_i} \xi = 0$, then by writing $\Sigma c_i R_{\lambda_i} = c \Pi R_{\lambda_i} \cdot \Pi(A - \mu_j)$, where $c, \mu_j \in \mathbf{C}$ and $c \neq 0$, we obtain $\Pi R_{\lambda_i} \cdot \Pi(A - \mu_j)\xi = 0$, which contradicts the fact that R_{λ_i} and $(A - \mu_j)$ are invertible). On the other hand, it is clear that E is of at most countable dimension: it is generated by the set $\pi(G)\xi$, which is countable, because the open subgroup stab ξ has countable index in G. So we have a contradiction to the fact that $\mathbf{C}$ is uncountable.

REMARK. In case π is admissible, the lemma is true even without the assumption that G is countable at infinity.

COMPLETENESS OF THE SYSTEM OF IRREDUCIBLE REPRESENTATIONS

2.12. PROPOSITION. *Let G be a l-group, countable at infinity, and let $T \in S_c^*(G)$, $T \neq 0$. Then there is an irreducible representation $\pi \in$ Alg G such that $\pi(T) \neq 0$.*

PROOF. For some open compact subgroup $N \subset G$ we have $h = \varepsilon_N * T * \varepsilon_N \neq 0$, so that we may assume that $T = h \in \mathscr{H}_N$. We claim that there exists an $h^* \in \mathscr{H}_N$ such that h^*h is not nilpotent.

We fix a left-invariant Haar measure μ_G and use it to identify $S(G)$ and $\mathscr{H}(G)$ (see 1.30). Here $(f_1 * f_2)(g') = \int f_1(g)f_2(g^{-1}g')d\mu_G(g)$ (see 1.30). We define an involution $f \mapsto f^*$ in $S(G)$ by $f^*(g) = \overline{f(g^{-1})}$, where the bar denotes complex conjugation. It is easy to verify that:

1) $(f^*)^* = f$ and $(f_1 * f_2)^* = f_2^* * f_1^*$ $(f, f_1, f_2 \in S(G))$.

2) If $f \in S(G)$, then $(f^* * f)(e) = \int |f(g)|^2 d\mu_G(g)$; that is, if $f \neq 0$, then $(f^* * f)(e) \neq 0$ (e is the unit element of G).

Now if $h = f\mu_G$, then we set $h^* = f^*\mu_G$. We have $f_0 = f^* * f \neq 0$ and $f_0^* = f_0$. Therefore, $f_0^2 = f_0^* * f_0 \neq 0$, $f_0^4 = (f_0^2)^* * f_0^2 \neq 0$, etc.; that is, $h^*h \in \mathscr{H}_N$ is not nilpotent.

The assertion we need now follows from 2.10c and the following lemma:

LEMMA. Let $\mathscr{H}$ be a countable-dimensional algebra over **C**, and suppose that $h \in \mathscr{H}$ is not a nilpotent element. Then there is an irreducible representation τ of $\mathscr{H}$ such that $\tau(h) \neq 0$.

A proof of this lemma can be found in [21], Ch.I, §10, Theorem 2.

THE CONTRAGRADIENT REPRESENTATION

2.13. DEFINITION. Let $(\pi, G, E) \in$ Alg G.

a) Let E^* be the space of all linear functionals on E. We define the representation $\pi^* = (\pi^*, G, E^*)$ conjugate to π by $\langle \pi^*(g)\xi^*, \xi \rangle = \langle \xi^*, \pi(g^{-1})\xi \rangle$, where $\xi^* \in E^*$, $\xi \in E$, and $\langle \xi^*, \xi \rangle$ is the value of ξ^* at ξ.

b) The algebraic part of (π^*, G, E^*) (see 2.1) is called the *contragradient representation* to π, and is denoted by $(\widetilde{\pi}, G, \widetilde{E})$.

2.14. LEMMA. a) *Let Γ be a compact subgroup of G. Then for any Γ-invariant vector $\xi^* \in E^*$ we have $\langle \xi^*, \xi \rangle = \langle \xi^*, \pi(\varepsilon_\Gamma)\xi \rangle$ for all $\xi \in E$. In particular, $(E^*)^\Gamma = (E^\Gamma)^*$, and if Γ is open then $\widetilde{E}^\Gamma = (E^*)^\Gamma = (E^\Gamma)^*$.*

b) *Let $\check{T} \in S_c^*(G)$ and let T be the distribution obtained from T by means of the homeomorphism $g \mapsto g^{-1}$ of G. Then for all $\xi \in E$ and $\widetilde{\xi} \in \widetilde{E}$ we have $\langle \pi(T)\widetilde{\xi}, \xi \rangle = \langle \xi, \pi(\check{T})\xi \rangle$. In particular, $\langle \widetilde{\pi}(\varepsilon_\Gamma)\widetilde{\xi}, \xi \rangle = \langle \widetilde{\xi}, \pi(\varepsilon_\Gamma)\xi \rangle$ for any compact subgroup $\Gamma \subset G$.*

PROOF. a) follows from 2.4.

b) We have $\langle \widetilde{\pi}(T)\widetilde{\xi}, \xi\rangle = \int \langle \widetilde{\pi}(g)\widetilde{\xi}, \xi\rangle\, dT(g) = \int \langle \widetilde{\xi}, \pi(g^{-1})\xi\rangle\, dT(g) =$
$= \int \langle \widetilde{\xi}, \pi(g)\xi\rangle\, d\check{T}(g) = \langle \widetilde{\xi}, \pi(\check{T})\xi\rangle$.

2.15. PROPOSITION. *Suppose that* $(\pi, G, E) \in \mathrm{Alg}\, G$ *is admissible. Then*:
a) $\widetilde{\pi}$ *is also admissible.*
b) *The natural embedding* $E \to \widetilde{\widetilde{E}}$ *is an isomorphism, that is,* $\widetilde{\widetilde{\pi}} = \pi$.
c) π *is irreducible if and only if* $\widetilde{\pi}$ *is irreducible.*

PROOF. a) and b) follow from the fact that $(\widetilde{E})^N = (E^N)^*$ and $(\widetilde{\widetilde{E}})^N = (\widetilde{E}^N)^*$, that is, $\dim \widetilde{\widetilde{E}}^N = \dim \widetilde{E}^N = \dim E^N < \infty$.

c) If E_1 is a non-trivial G-submodule of E, then its orthogonal complement $E_1^\perp = \{\widetilde{\xi} \in \widetilde{E} \mid \langle \widetilde{\xi}, \xi\rangle = 0 \text{ for all } \xi \in E_1\}$ is a non-trivial G-submodule of $\widetilde{E}$; that is, if π is reducible, then $\widetilde{\pi}$ is also reducible. If $\widetilde{\pi}$ is reducible, $\widetilde{\widetilde{\pi}} = \pi$ is also reducible.

THE TENSOR PRODUCT OF REPRESENTATIONS

2.16. Let G_i be an l-group ($i = 1, 2, \ldots, r$). We set $G = \prod_{i=1}^{r} G_i$. This is also an l-group. Let $(\pi_i, G_i, E_i) \in \mathrm{Alg}\, G_i$. We define their tensor product $(\pi = \otimes\, \pi_i,\ G,\ E = \otimes\, E_i)$ by $\pi(g_1, \ldots, g_r)\,(\bigotimes_{i=1}^{r} \xi_i) = \bigotimes_{i=1}^{r} \pi(g_i)\xi_i$.

PROPOSITION. *Suppose that all the* π_i *are admissible. Then*:
a) π *is admissible.*
b) $\widetilde{\pi} = \otimes\, \widetilde{\pi}_i$.
c) *If all the* π_i *are irreducible, then* π *is also irreducible.*
d) *Conversely, any irreducible admissible representation* π *of* G *has the form* $\pi = \otimes\, \pi_i$, *where the* π_i *are irreducible and admissible. The* π_i *are uniquely determined by* π.

PROOF. It suffices to examine the case $r = 2$.

Let $N_1 \subset G_1$ and $N_2 \subset G_2$ be open compact subgroups and let $N = N_1 \times N_2 \subset G$. It is easy to see that $\mathscr{H}_N = \mathscr{H}_{N_1} \otimes \mathscr{H}_{N_2}$ and that $E^N = E_1^{N_1} \otimes E_2^{N_2}$ (see 2.10). Therefore, $\dim E^N = \dim E_1^{N_1} \cdot \dim E_2^{N_2} < \infty$, and the natural mapping $\widetilde{E}_1^{N_1} \otimes \widetilde{E}_2^{N_2} = (E_1^{N_1})^* \otimes (E_2^{N_2})^* \to (E^N)^* = \widetilde{E}^N$ is an isomorphism. This proves a) and b). c) and d) follow from 2.10 and the following lemma on finite-dimensional representations.

LEMMA. *Let* τ_i *be an irreducible finite-dimensional representation of an algebra with unit element* $\mathscr{H}_i$ ($i = 1, 2$). *Then the representation* $\tau_1 \otimes \tau_2$ *of* $\mathscr{H}_1 \otimes \mathscr{H}_2$ *is irreducible. Conversely, each irreducible finite-dimensional representation* τ *of* $\mathscr{H}_1 \otimes \mathscr{H}_2$ *has the form* $\tau = \tau_1 \otimes \tau_2$, *where* τ_i *is an irreducible representation of* $\mathscr{H}_i$ ($i = 1, 2$); τ_1 *and* τ_2 *are uniquely determined by* τ (see [5], Ch. VIII, §7, Proposition 8).

THE CHARACTER OF A REPRESENTATION

2.17. Let (π, G, E) be an admissible representation. Then for any $T \in \mathscr{H}(G)$ the operator $\pi(T)$ has finite rank; consequently, the trace $\operatorname{tr} \pi(T)$ is defined. We choose a left-invariant Haar measure μ_G on G and define the distribution $\operatorname{tr} \pi \in S^*(G)$ by

$$\langle \operatorname{tr} \pi, f\rangle = \operatorname{tr} \pi(f\mu_G), \quad f \in S(G).$$

This distribution is called the character of π.

2.18. PROPOSITION. *Let $g_0 \in G$ and let A_{g_0} be the corresponding automorphism of G ($A_{g_0}(g) = g_0 g g_0^{-1}$). Then $A_{g_0}(\operatorname{tr} \pi) = \Delta_G(g_0) \cdot \operatorname{tr} \pi$* (see 1.16 and 1.19).

PROOF. If $T \in \mathscr{H}(G)$, then $A_{g_0}(T) = \varepsilon_{g_0} * T * \varepsilon_{g_0^{-1}}$ (see 1.25b), so that $\operatorname{tr} \pi(A_{g_0}(T)) = \operatorname{tr} [\pi(g_0)\pi(T)\pi(g_0^{-1})] = \operatorname{tr} \pi(T)$. Therefore, $\langle A_{g_0}(\operatorname{tr} \pi), f\rangle =$ $= \langle \operatorname{tr} \pi, A_{g_0}^{-1}(f)\rangle = \operatorname{tr} \pi(A_{g_0}^{-1}(f) \cdot \mu_G) = \Delta_G(g_0) \operatorname{tr} \pi(A_{g_0}^{-1}(f\mu_G))$ (see 1.19) $=$ $= \Delta_G(g_0) \operatorname{tr} \pi(f\mu_G) = \Delta_G(g_0)\langle \operatorname{tr} \pi, f\rangle$.

2.19. PROPOSITION. *If $\pi_1, \ldots, \pi_r$ are pairwise inequivalent, irreducible, admissible representations of an l-group G, then their characters* $\operatorname{tr} \pi_1, \ldots, \operatorname{tr} \pi_r$ *are linearly independent.*

PROOF. We choose an open compact subgroup $N \subset G$ such that all the $(\pi_i)_N$ are non-zero. By 2.10, they are irreducible and pairwise inequivalent. It remains to quote a standard fact about finite-dimensional representations.

LEMMA. *Let $\tau_1, \ldots, \tau_r$ be irreducible, finite-dimensional, pairwise inequivalent representations of $\mathscr{H}$. Then the linear forms* $\operatorname{tr} \tau_i$ *on $\mathscr{H}$ are linearly independent* (see [5], Ch. VIII, §13, Proposition 2).

2.20. COROLLARY. *Irreducible admissible representations π_1 and π_2 of an l-group G are equivalent if and only if* $\operatorname{tr} \pi_1 = \operatorname{tr} \pi_2$. ■

INDUCED REPRESENTATIONS

2.21. Let H be a closed subgroup of an l-group G, and let $(\rho, H, E) \in \operatorname{Alg} H$. We denote by $L(G, \rho)$ (or simply $L(\rho)$) the space of functions $f\colon G \to E$ satisfying the following conditions:

1) $f(hg) = \rho(h)f(g)$ for all $h \in H$, $g \in G$.

2) There exists an open compact subgroup $N_f \subset G$ such that $f(gg_0) = f(g)$ for all $g \in G$, $g_0 \in N_f$.

DEFINITION. The representation $(\pi, G, L(G, \rho))$ acting according to the formula $(\pi(g_0)f)(g) = f(gg_0)$ is said to be *induced by* ρ (notation: $\pi = \operatorname{Ind}(G, H, \rho)$). Condition 2) guarantees that π is algebraic.

2.22. Under the conditions of 2.21, we denote by $S(G, \rho)$ (or simply $S(\rho)$) the subspace of $L(G, \rho)$ consisting of the functions f that satisfy the following additional condition:

3) f is finite modulo H (see 1.20).

DEFINITION. The representation $(\pi, G, S(G, \rho))$ acting according to the formula $(\pi(g_0)f)(g) = f(gg_0)$ is said to be *finitely induced by* ρ (notation: π = ind (G, H, ρ)).

REMARK. Instead of 1) in 2.21 we often take the condition $f(hg) = (\Delta_G(h)/\Delta_H(h))^{1/2} \cdot \rho(h)f(g)$. This simplifies many formulae (see, for example, 2.25c)). But in the Frobenius duality theorem it is more convenient to use 1).

2.23. We define an action of G on $H \setminus G$ by $g_0[Hg] = Hgg_0^{-1}$. We also define for each $(\rho, H, E) \in \text{Alg}\, H$ an $S(H\setminus G)$-module structure on $S(G, \rho)$ as follows: if $f \in S(H \setminus G)$ and $\varphi \in S(G, \rho)$, then $(f\varphi)(g) = f(Hg) \cdot \varphi(g)$.

PROPOSITION. a) *There is one and up to isomorphism only one l-sheaf* $\mathcal{F}^\rho$ *on* $H \setminus G$ *such that* $S(G, \rho)$ *is isomorphic to* $\mathcal{F}^\rho_c$ *as an* $S(H \setminus G)$*-module. The action of* G *on* $S(G, \rho)$, *together with the action on* $H \setminus G$ *described above, defines an action of* G *on* $\mathcal{F}^\rho$ (see 1.16 and 1.17). *The resulting representation of* H *in the stalk over* $\bar{e} = He \in H \setminus G$ *is equivalent to* ρ.

b) Conversely, suppose that an l-sheaf $(H \setminus G, \mathcal{F})$ is defined on $H \setminus G$ and that the action of G on $H \setminus G$ is continued to an action γ of G on $\mathcal{F}$. We assume that the action of G in $\mathcal{F}_c$ is algebraic. If the resulting representation of H in the stalk E over $\bar{e} \in H \setminus G$ is denoted by ρ, then $(\gamma, G, \mathcal{F}_c)$ is isomorphic to ind (G, H, ρ), and the algebraic part of $(\gamma, G, \mathcal{F})$ (see 2.1) is isomorphic to Ind (G, H, ρ).

PROOF. a) follows easily from 1.14. Let us prove b). Since γ is algebraic, it follows immediately that ρ is algebraic. We define mappings α: $\mathcal{F}_c \to S(G, \rho)$ and β: $S(G, \rho) \to \mathcal{F}_c$ by

$$\alpha(\varphi)(g) = (\gamma(g)\varphi)(\bar{e}) \in E \qquad (\varphi \in \mathcal{F}_c),$$

$$\beta(f)(\gamma(g)\bar{e}) = \gamma(g)\,(f(g^{-1})) \in E_{\gamma(g)\bar{e}}\ (f \in S(G, \rho)).$$

It is easy to verify that α and β are well-defined and specify the equivalence of $(\gamma, G, \mathcal{F}_c)$ and ind (G, H, ρ). Similarly it is proved that $(\gamma, G, \mathcal{F}_a)$ and Ind (G, H, ρ) are equivalent.

2.24. We now list several properties of induced representations. First we state a simple auxiliary lemma.

LEMMA. *Under the conditions of* 2.21, *let* $N \subset G$ *be an open compact subgroup and* $\Omega \subset G$ *a system of coset representatives of* $H \setminus G/N$. *For each* $g \in \Omega$ *we consider the open compact subgroup* $N_g = H \cap gNg^{-1}$ *of* H. *Then the restriction of functions from* G *to* Ω *defines an isomorphism of* $L(G, \rho)^N$ *with the space of functions* f: $\Omega \to E$ *for which* $f(g) \in E^{N_g}$ *for all* $g \in \Omega$. *Here* $S(G, \rho)^N$ *is mapped isomorphically onto the subspace of functions having only a finite number of non-zero values.*■

2.25. PROPOSITION. a) *The mappings* $\rho \mapsto \text{Ind}(G, H, \rho)$ *and* $\rho \mapsto \text{ind}(G, H, \rho)$ *define functors from* Alg H *into* Alg G. *These functors are exact; that is, for any exact sequence of H-modules* $0 \to E_1 \to E_2 \to E_3 \to 0$ *the sequence* $0 \to L(G, \rho_1) \to L(G, \rho_2) \to L(G, \rho_3) \to 0$ *and the analogous sequence with* $S(G, \rho_i)$ *are also exact (here* $(\rho_i, H, E_i) \in \text{Alg}\, H$ $(i = 1, 2, 3)$).

b) *If* F *is a closed subgroup of* H *and* $\tau \in \operatorname{Alg} F$, *then* $\operatorname{ind}(G, H, \operatorname{ind}(H, F, \tau)) = \operatorname{ind}(G, F, \tau)$ *and* $\operatorname{Ind}(G, H, \operatorname{Ind}(H, F, \tau)) = \operatorname{Ind}(G, F, \tau)$.

c) *If* $(\rho, H, E) \in \operatorname{Alg} H$ *and* θ *is a character of* H, *then we denote by* θ_ρ *the representation* (θ_ρ, H, E) *defined by* $(\theta_\rho)(h) = \theta(h) \cdot \rho(h)$.

With this notation we have $\widetilde{\operatorname{ind}(G, H, \rho)} = \operatorname{Ind}(G, H, \Delta_G/\Delta_H \widetilde{\rho})$ *for any* $(\rho, H, E) \in \operatorname{Alg} H$.

PROOF. a) It is obvious that ind and Ind are functorial, and it follows from 2.24 and 2.4 that they are exact.

b) Immediate verification.

c) If $f \in S(G, \rho)$ and $\widetilde{f} \in L(G, (\Delta_G/\Delta_H)\widetilde{\rho})$, then the function $\{\widetilde{f}, f\}(g) = \langle \widetilde{f}(g), f(g) \rangle$ on G lies in $S(G, \Delta)$ (see 1.20). Using 1.21, we can define a pairing $\langle \widetilde{f}, f \rangle$ by

$$\langle \widetilde{f}, f \rangle = \int_{H \setminus G} \{\widetilde{f}, f\}(g)\, d\nu_{H\setminus G}(g).$$

This defines a mapping α: $L(G, (\Delta_G/\Delta_H)\widetilde{\rho}) \to \widetilde{S(G, \rho)}$, which, as is easy to see, is an embedding and commutes with the action of G. It follows from 2.24 and 2.14a) that α is epimorphic.

2.26. LEMMA. *Under the conditions of* 2.21, *suppose that* G *is compact modulo* H, *that is, there exists a compact set* $K \subset G$ *such that* $G = H \cdot K$. *Let* (ρ, H, E) *be an admissible representation. Then* $\pi = \operatorname{Ind}(G, H, \rho) = \operatorname{ind}(G, H\ \rho)$ *is also admissible.*

This lemma follows immediately from 2.24.

2.27. Let $h \to {}^\sigma h$ be an automorphism of an l-group H, and let $(\rho, H, E) \in \operatorname{Alg} H$. We define (ρ^σ, H, E) by $\rho^\sigma(h) = \rho({}^\sigma h)$ $(h \in H)$.

LEMMA. *Let* H *be a closed subgroup of an* l*-group* G, *and suppose that* $g \in G$ *normalizes* H. *We define an automorphism of* H *by* $h \to {}^g h = ghg^{-1}$. *Then* $\operatorname{Ind}(G, H\ \rho^g)$ *is equivalent to* $\operatorname{Ind}(G, H\ \rho)$ *for any* $\rho \in \operatorname{Alg} H$, *and similarly* $\operatorname{ind}(G, H, \rho^g) \simeq \operatorname{ind}(G, H, \rho)$.

PROOF. The equivalence determines an operator $A: L(G, \rho) \to L(G, \rho^g)$ defined by $(Af)(g') = f(gg')(f \in L(G, \rho))$.

FROBENIUS DUALITY

2.28. THEOREM. *Let* H *be a closed subgroup of an* l*-group* G, *and let* $(\rho, H, V) \in \operatorname{Alg} H$ *and* $(\pi, G, E) \in \operatorname{Alg} G$. *Then* $\operatorname{Hom}_G(\pi, \operatorname{Ind}(G, H, \rho)) \simeq \operatorname{Hom}_H(\pi|_H, \rho)$. *This isomorphism depends functorially on* π *and* ρ.

PROOF. Let $A \in \operatorname{Hom}_G(\pi, \operatorname{Ind}(G, H, \rho))$ and $B \in \operatorname{Hom}_H(\pi|_H, \rho)$. We define homomorphisms $\alpha(A) \in \operatorname{Hom}_H(\pi|_H, \rho)$ and $\beta(B) \in \operatorname{Hom}_G(\pi, \operatorname{Ind}(G, H, \rho))$ by $\alpha(A)\xi = A(\xi)(e)$ and $[\beta(B)\xi]\,(g) = B(\pi(g)\xi)$ (where $\xi \in E$ and e is the unit element of G). It can be verified immediately that α and β define the required isomorphism.

2.29. PROPOSITION. *Under the conditions of* 2.28,

$$\operatorname{Hom}_G(\operatorname{ind}(G, H, \rho), \tilde{\pi}) \simeq \operatorname{Hom}_H((\Delta_H/\Delta_G)\rho, (\widetilde{\pi|_H})).$$

This isomorphism depends functorially on π and ρ.

PROOF. If $\pi_1, \pi_2 \in \operatorname{Alg} G$, then $\operatorname{Hom}_G(\pi_1, \pi_2) \cong \operatorname{Hom}_G(\pi_2, \tilde{\pi}_1)$, because both of these spaces can be identified with the space of (π_1, π_2)-invariant bilinear forms. Therefore, our required assertion follows from 2.25c) and 2.28.

Finally, we write down α: $\operatorname{Hom}_H((\Delta_H/\Delta_G) \cdot \rho, (\widetilde{\pi|_H})) \to \operatorname{Hom}_G(\operatorname{ind}(G, H, \rho), \tilde{\pi})$ explicitly.

If $A \in \operatorname{Hom}_H((\Delta_H/\Delta_G) \cdot \rho, (\widetilde{\pi|_H}))$ and $f \in S(G, \rho)$, then

$$\langle \alpha(A) f, \xi \rangle = \int_{H\setminus G} \langle A[f(g)], \pi(g)\xi \rangle \, d\nu_{H\setminus G}(g) \quad (\xi \in E)$$

(see 1.21).

THE FUNCTOR $E \to E_{H,\theta}$

2.30. DEFINITION. Let H be an l-group and θ a character of H, that is, a one-dimensional algebraic representation of H. For each representation (π, H, E) we denote by $E(H, \theta)$ the subspace of E spanned by vectors of the form $\pi(h)\xi - \theta(h)\xi$ ($h \in H$, $\xi \in E$). We set $E_{H,\theta} = E/E(H, \theta)$. If $\theta = 1$, then we write $E(H)$ instead of $E(H, 1)$ and E_H instead of $E_{H,1}$.

$E^*_{H,\theta}$ is embedded in E^* and consists of the functionals $\xi^* \in E^*$ such that $\pi^*(h)\xi^* = \theta^{-1}(h)\xi^*$ for all $h \in H$.

EXAMPLE. Theorem 1.18 on the existence and uniqueness of a Haar measure is equivalent to the assertion that $\dim S(G)_G = 1$.

2.31. REMARKS. a) We consider the representation $(\theta^{-1}\pi, H, E)$ (see 2.25c)). It is then clear that $E_{\theta^{-1}\pi}(H) = E_\pi(H, \theta)$ and $(E_{\theta^{-1}\pi})_H = (E_\pi)_{H,\theta}$. Thus, the proof of the many assertions about $E(H, \theta)$ and $E_{H,\theta}$ can be reduced to the case $\theta = 1$.

b) Let H be a closed subgroup of an l-group G. We set $\operatorname{Norm}_G(H, \theta) = \{g \in G \mid ghg^{-1} \in H \text{ and } \theta(ghg^{-1}) = \theta(h) \text{ for all } h \in H\}$. It is clear that for any (π, G, E) the group $\operatorname{Norm}_G(H, \theta)$ preserves $E(H, \theta)$ and hence acts on $E_{H,\theta}$.

The correspondence $E \mapsto E_{H,\theta}$ defines a functor from $\operatorname{Alg} G$ into $\operatorname{Alg} \operatorname{Norm}_G(H, \theta)$.

2.32. LEMMA. *Under the conditions of* 2.30, *let H_1 and H_2 be subgroups of H, where $H_1H_2 = H$ and H_1 normalizes H_2. Then* $(E_{H_2,\theta/H_2})_{H_1,\theta/H_1} \cong E_{H,\theta}$.

PROOF. The lemma is an immediate consequence of the equality $E(H, \theta) = E(H_1, \theta/H_1) + E(H_2, \theta \mid H_2)$, which follows from the formula

$$\pi(h_1h_2)\xi - \theta(h_1h_2)\xi = [\pi(h_1)\pi(h_2)\xi - \theta(h_1)\pi(h_2)\xi] + [\pi(h_2)\theta(h_1)\xi - \theta(h_2)\theta(h_1)\xi].$$

2.33. We assume that an l-group H is exhausted by its compact subgroups; that is, any compact set in H lies in some compact subgroup. Let θ be a character of H, and let $(\pi, H, E) \in \mathrm{Alg}\, H$. Then there is a very convenient way of describing $E(H, \theta)$.

LEMMA. (Jacquet and Langlands). *A vector $\xi \in H$ lies in $E(H, \theta)$ if and only if there exists a compact subgroup $N \subset H$ such that*

$$\int_N \theta^{-1}(h)\,\pi(h)\,\xi\,d\mu_N(h) = \pi(\theta^{-1}\cdot\varepsilon_N)\,\xi = 0 \qquad \text{(see 2.3)}.$$

PROOF. Using Remark 2.31a), we may assume that $\theta = 1$. For this case the lemma follows from 2.4.

2.34. COROLLARY. *Under the conditions of* 2.33, *if E' is an H-submodule of E, then $E'(H, \theta) = E' \cap E(H, \theta)$.*

2.35. PROPOSITION. *Let H be an l-group, θ a character of H, and $0 \to E' \to E \to E'' \to 0$ an exact sequence of H-modules.*

a) *The sequence $E'_{H,\theta} \to E_{H,\theta} \to E''_{H,\theta} \to 0$ so arising is exact.*

b) *If H is exhausted by its compact subgroups, then the following sequence is exact:*

$$0 \to E'_{H,\theta} \to E_{H,\theta} \to E''_{H,\theta} \to 0.$$

PROOF. a) can be verified directly. The assertion that the mapping $E'_{H,\theta} \to E_{H,\theta}$ in b) is an embedding follows from 2.34.

REPRESENTATIONS IN CROSS-SECTIONS OF l-SHEAVES

2.36. Suppose that an action γ of an l-group G on an l-sheaf $(X, \mathscr{F})$ and a character θ of G are given. We are interested in the structure of $(\mathscr{F}_c)_{G,\theta}$.

We assume that there exists a continuous mapping of l-spaces $q\colon X \to Y$ such that $q(\gamma(g)x) = q(x)$ for all $g \in G$ and $x \in X$. Then $\mathscr{F}_c$ is naturally equipped with the structure of an $S(Y)$-module: if $f \in S(Y)$ and $\varphi \in \mathscr{F}_c$, then $f\varphi(x) = f(q(x)) \cdot \varphi(x)$. Here $\mathscr{F}_c(G, \theta)$ is a submodule of $\mathscr{F}_c$, so that $(\mathscr{F}_c)_{G,\theta}$ is an $S(Y)$-module. By virtue of 1.14, an l-sheaf on Y, which we denote by $\mathscr{F}'$, corresponds to $(\mathscr{F}_c)_{G,\theta}$. By definition, $\mathscr{F}'_c$ and $(\mathscr{F}_c)_{G,\theta}$ are isomorphic as $S(Y)$-modules.

PROPOSITION. *The stalk $\mathscr{F}'_y$ of $\mathscr{F}'$ at $y \in Y$ is naturally isomorphic to $\mathscr{F}_c(q^{-1}(y))_{G,\theta}$* (see 1.16).

PROOF. We may assume that $\theta = 1$ (see 2.31a)). We set $Z = q^{-1}(y)$. Then Z is a closed subset of X, and the restriction $p_Z\colon \mathscr{F}_c \to \mathscr{F}_c(Z)$ is an epimorphism (see 1.16). We claim that the kernel of p_Z is the subspace $L \subset \mathscr{F}_c$ generated by cross-sections of the form $f \cdot \varphi$, where $f \in S(Y)$, $f(y) = 0$, and $\varphi \in \mathscr{F}_c$. For it is clear that $L \subset \mathrm{Ker}\, p_Z$. Conversely, let $\varphi \in \mathrm{Ker}\, p_Z$. Then $q(\mathrm{supp}\, \varphi)$ is a compact set in Y not containing y. By virtue of 1.3, we can cover $q(\text{supp } \varphi)$ with an open compact subset $U \subset Y$ not containing y. If f is the characteristic function of U, then $f \in S(Y)$, $f(y) = 0$, and $\varphi = f \cdot \varphi$, that is, $\varphi \in L$.

Thus, $\mathscr{F}_c(Z) = \mathscr{F}_c/L$. Hence $(\mathscr{F}_c(Z))_G = \mathscr{F}_c/L'$, where L' is the subspace of $\mathscr{F}_c$ generated by L and $\mathscr{F}_c(G)$. But, as follows from the explicit description of a stalk of $\mathscr{F}'$ (see 1.14), $\mathscr{F}'_y$ also coincides with $\mathscr{F}_c/L'$, as required.

2.37. COROLLARY. *Under the conditions of* 2.36, *if there are no* G*-invariant* $\mathscr{F}$*-distributions on any stalk* $q^{-1}(y)$ *of* q, *then there are no* G*-invariant* $\mathscr{F}$*-distributions on* X. ∎

FINITE REPRESENTATIONS

2.38. We now introduce and study a certain class of representations, which plays a fundamental role in the study of the representations of $GL(n, F)$.

2.39. DEFINITION. Let $(\pi, G, E) \in \text{Alg } G$. For each $\xi \in E$ and $\tilde{\xi} \in \tilde{E}$ (see 2.13) we define $\varphi_{\xi,\tilde{\xi}} \in C^\infty(G)$ by $\varphi_{\xi,\tilde{\xi}}(g) = \langle \tilde{\xi}, \pi(g^{-1})\xi \rangle$. This function is called a *matrix element of* π. It is easy to verify that for any $T \in S_c^*(G)$ we have $\varphi_{\pi(T)\xi,\tilde{\xi}} = \gamma(T)\varphi_{\xi,\tilde{\xi}}$ (see 2.3).

2.40. LEMMA-DEFINITION. *Let* $(\pi, G, E) \in \text{Alg } G$. *Then the following conditions are equivalent*:

a) $\varphi_{\xi,\tilde{\xi}}$ *is a finite function for all* $\xi \in E$ *and* $\tilde{\xi} \in \tilde{E}$.

b) *For each* $\xi \in E$ *and each open compact subgroup* $N \subset G$ *the set* $K_{\xi,N} = \{g \in G \mid \pi(\varepsilon_N)\pi(g^{-1})\xi \neq 0\}$ *is compact.*

We call representations that satisfy these conditions *finite*. It is clear that any subfactor representation of a finite representation is finite.

PROOF. b) ⇒ a). If $\tilde{\xi} \in \tilde{E}^N$, then $\text{supp } \varphi_{\xi,\tilde{\xi}} \subset K_{\xi,N}$ for all $\xi \in E$ (see 2.14).

a) ⇒ b). For each $g \in G$ we set $\xi_g = \pi(\varepsilon_N) \cdot \pi(g^{-1})\xi$ and denote by E_ξ the linear span of the vectors $\xi_g (g \in G)$. Let us prove that $\dim E_\xi < \infty$. For if this is not the case, then there is a sequence g_i ($i = 1, 2, \dots$) such that the ξ_{g_i} are linearly independent. We complete them to a basis of E^N by vectors η_α and define a functional $\tilde{\xi}$ on E by $\langle \tilde{\xi}, \xi' \rangle = \langle \tilde{\xi}, \pi(\varepsilon_N)\xi' \rangle$ for all $\xi' \in E$, $\langle \tilde{\xi}, \xi_{g_i} \rangle = i$, and $\langle \tilde{\xi}, \eta_\alpha \rangle = 0$. Then $\tilde{\xi} \in \tilde{E}^N$ and $\varphi_{\xi,\tilde{\xi}}(g_i) = i$, which contradicts the fact that $\varphi_{\xi,\tilde{\xi}}$ is finite.

Thus, $\dim E_\xi < \infty$. We choose functionals $\tilde{\xi}_1, \dots, \xi_k \in \tilde{E}^N = (E^N)^*$ that separate points in E_ξ. Then $K_{\xi,N} \subset \bigcap_{i=1}^{k} \text{supp } \varphi_{\xi,\tilde{\xi}_i}$ is a compact set.

2.41. COROLLARY OF THE PROOF. *Each finitely generated* (*and, in particular, each irreducible*) *finite representation is admissible.* ∎

2.42. We wish to show that finite representations can be split off from the rest; more precisely, that any irreducible finite subfactor module is a direct summand. To do so, we use the following theorem:

THEOREM. *Let G be a unimodular l-group that is countable at infinity, and let $(\omega, G, V) \in$ Alg G be an irreducible finite representation.*

a) *For each open compact subgroup $N \subset G$ there exists a distribution $\varepsilon_N^\omega \in \mathscr{H}(G)$, such that $\omega(\varepsilon_N^\omega) = \omega(\varepsilon_N)$ and $\pi(\varepsilon_N^\omega) = 0$ for any irreducible representation $\pi \in$ Alg G not isomorphic to ω. This distribution is unique.*

b) *If $N' \subset N$ is an open compact subgroup, then*

$$\varepsilon_{N'}^\omega * \varepsilon_N^\omega = \varepsilon_N * \varepsilon_{N'}^\omega = \varepsilon_{N'}^\omega * \varepsilon_N = \varepsilon_N^\omega.$$

If $g \in G$, then $\varepsilon_g * \varepsilon_N^\omega * \varepsilon_{g^{-1}} = \varepsilon_{gNg^{-1}}^\omega$.

PROOF. (1) It follows from 2.12 that ε_N^ω is unique. b) follows easily from the uniqueness, Therefore, we only have to prove that ε_N^ω exists.

(2) We define a representation ω' of $G \times G$ in End V by $\omega'((g_1, g_2))A = \omega(g_1)A\omega(g_2^{-1})$ $(A \in \text{End } V)$ and consider its algebraic part $(\omega', G \times G, L)$. Let $(\omega \otimes \tilde{\omega}, G \times G, V \otimes \tilde{V})$ be the tensor product of ω and $\tilde{\omega}$ (see 2.16), and let τ: $V \otimes \tilde{V} \to L$ be the mapping defined by $\tau(\xi \otimes \tilde{\xi})(\eta) = \langle \tilde{\xi}, \eta \rangle \xi$. It is clear that τ $G \times G$ is equivariant. Since ω is admissible (see 2.41),

$$\dim (V \otimes \tilde{V})^{N \times N} = (\dim V^N)^2 = \dim L^{N \times N} < \infty,$$

so that τ is an isomorphism. In particular, L is an irreducible $G \times G$-module and, when restricted to $G = G \times \{e\}$, is the direct sum of modules isomorphic to V.

(3) We define φ: $L \to C^\infty(G)$ by $\varphi(A)(g) = \text{tr}\,(\omega(g^{-1})A)$. It is clear that $\varphi(\tau(\xi \otimes \tilde{\xi})) = \varphi_{\xi, \tilde{\xi}}$ $(\xi \in V, \tilde{\xi} \in \tilde{V})$. Since ω is finite and $\tau(V \otimes \tilde{V}) = L$, φ maps L into $S(G)$. We identify $S(G)$ and $\mathscr{H}(G)$ by means of the Haar measure μ_G (see 1.30) and define the "inverse" mapping ω: $[S(G) \cong \mathscr{H}(G)] \to L$ by $h \mapsto \omega(h) \in L$. We consider the representation $(\gamma', G \times G, S(G))$, where $[\gamma'((g_1, g_2))f]\,(g) = f(g_1^{-1} g g_2)$; it is easy to verify that φ and ω are $G \times G$-equivariant mappings. In particular, φ is an embedding.

(4) Let $(\pi, G, E) \in$ Alg G and $\xi \in E$. Then the mapping π_ξ: $[S(G) \simeq \mathscr{H}(G)] \to E$ defined by $h \mapsto \pi(h)\xi$ is G-equivariant. Since L splits into a direct sum of modules isomorphic to V, $\pi_\xi(\varphi(L))$ also splits into such a sum (see 2.7). In particular, if π is irreducible and not equivalent to ω, then $\pi(\varphi(L)) = 0$.

(5) We consider the mapping $\omega\varphi$: $L \to L$.

By virtue of Schur's lemma (see 2.11), $\omega\varphi$ is a scalar operator. If $A \in L$ and $A \neq 0$, then $\varphi(A) \neq 0$, and by 2.12, there is an irreducible representation $\pi \in$ Alg G such that $\pi(\varphi(A)) \neq 0$. It follows from (4) that $\omega\varphi(A) \neq 0$, that is, $\omega\varphi = c \cdot \text{Id}_L$, where $c \neq 0$.

(6) It is clear from what we have said that $\varepsilon_N^\omega = c^{-1}\varphi\omega(\varepsilon_N)$ satisfies the condition of the theorem.

2.43. Let $(\pi, G, E) \in \mathrm{Alg}\, G$. As follows from 2.42b), for each $\xi \in E$ the vectors $\pi(\varepsilon_N^\omega)\xi$ coincide for sufficiently small subgroups N. We denote this common vector by $\pi(\varepsilon^\omega)\xi$ (symbolically, $\pi(\varepsilon^\omega)\xi = \lim\limits_{N\to\{e\}} \pi(\varepsilon_N^\omega)\xi$). Using 2.42b), we can easily verify the following properties of $\pi(\varepsilon^\omega)$:

a) $\pi(\varepsilon^\omega)$ is a projection; that is $\pi(\varepsilon^\omega)^2 = \pi(\varepsilon^\omega)$.

b) $\pi(\varepsilon^\omega)$ commutes with the action of G.

c) If $\pi' \in \mathrm{Alg}\, G$ and $A \in \mathrm{Hom}_G(\pi, \pi')$, then $A\pi(\varepsilon^\omega) = \pi'(\varepsilon^\omega)A$. In addition, as shown in the proof of 2.42, $\mathrm{Im}\, \pi(\varepsilon^\omega)$ splits into a direct sum of G-submodules isomorphic to (ω, G, V).

2.44. THEOREM. a) *Let* $(\omega, G, V) \in \mathrm{Alg}\, G$ *be a finite irreducible representation, and let* $(\pi, G, E) \in \mathrm{Alg}\, G$. *Then* E *can be decomposed into a direct sum of submodules* $E = E_\omega \oplus E_\omega^\perp$, *where* E_ω *is a direct sum of submodules isomorphic to* V, *and* $E_\omega^\perp$ *contains no subfactor modules isomorphic to* V.

b) *The submodule* E_f *of* E *generated by the modules* E_ω *for all finite irreducible representations* ω *is completely reducible and finite, and* E/E_f *has no non-zero finite subfactor modules. In particular, any finite module is completely reducible.*

PROOF. a) We have to take $E_\omega = \pi(\varepsilon^\omega)$ and $E_\omega^\perp = \mathrm{Ker}\, \pi(\varepsilon^\omega)$. If $E_\omega^\perp$ contained a subfactor module isomorphic to V, then by 2.43c), we would have $\pi(\varepsilon^\omega)|_{E_\omega^\perp} \neq 0$, which is false.

b) follows from a), 2.7, and 2.6b).

CHAPTER II

THE GENERAL THEORY OF REPRESENTATIONS OF THE GROUP $GL(n, F)$

§3. Induced and quasi-cuspidal representations

THE STRUCTURE OF THE GROUP G_n

3.1. In what follows, F denotes a locally compact non-discrete field with a non-Archimedean valuation, R the ring of integers of F and $\wp$ a generator of the maximal ideal in R (for the definitions and main properties of F, R, and $\wp$ see [10], Ch. I). We shall study the representations of the group $G = G_n = GL(n, F)$. We set $\Gamma = \Gamma_n = GL(n, R)$ (this is a maximal compact subgroup of G) and $N_i = 1 + \wp^i.\ M(n, R)$ $(i = 1, 2, \ldots)$, where $M(n, R)$ is the ring of $m \times n$ matrices with coefficients in R.

The family $N_1 \supset N_2 \supset \ldots$ forms a fundamental system of neighbourhoods of the unit element of G consisting of open compact subgroups; that is, G is an l-group. We call the N_i congruence subgroups. It is clear that Γ normalizes all the congruence subgroups.

It is standard knowledge that G is unimodular (see [7]).

We denote by $Z = Z_n$ the centre of G; $Z = \{\lambda \cdot 1, \lambda \in F, \lambda \neq 0\}$.

3.2. We shall use three decompositions of G.

1. THE BRUHAT DECOMPOSITION. Let $B = B_n$ be the subgroup of upper triangular matrices in G, $U = U_n$ the subgroup of B consisting of matrices with 1's along the diagonal, and $D = D_n$ the subgroup of diagonal matrices. With each permutation ω of $1, \ldots, n$ we associate a matrix $w = (w_{ij})$, where $w_{ij} = \delta_{i,\omega(j)}$ and δ_{ij} is the Kronecker symbol. Let $W = W_n$ be the group of these matrices, which is isomorphic to the symmetric group S_n.

The following decomposition holds: $G = BWB = UDWU$. If $g = b_1 w b_2 = u_1 d w u_2$, then d and w are uniquely determined by g (see [6], Ch. 4, §2).

2. THE IWASAWA DECOMPOSITION. $G = B\Gamma$ (see [10], Ch. II, §2, Theorem 1).

3. THE CARTAN DECOMPOSITION. We consider the subgroup $\Delta = \Delta_n$ of G consisting of the diagonal matrices $d = (d_{ij})$ for which $d_{ii} = \wp^{m_i}$, where $m_1 \leqslant \ldots \leqslant m_n$ are integers (notation: $d = \operatorname{diag}(\wp^{m_1}, \ldots, \wp^{m_n})$). Then $G = \Gamma \cdot \Delta \cdot \Gamma$, where δ in the expression $g = \gamma_1 \delta \gamma_2 (\gamma_1, \gamma_2 \in \Gamma, \delta \in \Delta)$ is uniquely determined by g (see [10], Ch. II, §2, Theorem 2).

In particular, G is countable at infinity.

PARTITIONS AND THE GROUPS CONNECTED WITH THEM

3.3. We find it convenient to study the representations not only of G_n, but also of the groups $G_{n_1} \times G_{n_2} \times \ldots \times G_{n_r}$, which makes inductive transition possible.

DEFINITION. We denote by $\mathfrak{A}_n$ the index set $\{1, 2, \ldots, n\}$. A *segment* in $\mathfrak{A}_n$ is a subset of $\mathfrak{A}_n$ consisting of several consecutive numbers. By a *partition* α (of n) we mean a partition of $\mathfrak{A}_n$ into disjoint segments. Thus, if $n_1, \ldots, n_r$ are the lengths of the segments of a partition α (that is, the corresponding segments are equal to $I_1 = \{1, \ldots, n_1\}$, $I_2 = \{n_1 + 1, \ldots, n_1 + n_2\}, \ldots, I_r = \{n_1 + \ldots + n_{r-1} + 1, \ldots, n_1 + \ldots + n_r = n\}$), then we write $\alpha = (n_1, \ldots, n_r)$. We write $\beta < \alpha$ if each segment of α is a union of segments of β.

3.4. DEFINITION. Let $\alpha = (n_1, \ldots, n_r)$ be a partition of n. We set $G = \prod_{i=1}^{r} G_{n_i}$. For $\beta < \alpha$ we always assume that G_ρ is embedded in G_α in the natural way. In particular, G_α is embedded in $G_{(n)} = G_n$.

The decompositions 1, 2, and 3 in 3.2 are obviously valid for $G = G_\alpha = \Pi G_{n_i}$ with $\Gamma = \Gamma_\alpha = \Pi \Gamma_{n_i}$, $B = B_\alpha = \Pi B_{n_i}$, $W = W_\alpha = \Pi W_{n_i}$, $D = D_\alpha = \Pi D_{n_i}$, and $\Delta = \Delta_\alpha = \Pi \Delta_{n_i}$. The centre $Z = Z_\alpha$ of G_α is equal to ΠZ_{n_i}. We call subgroups of the form $N_i(\alpha) = N_i \cap G_\alpha$, where N_i is a congruence subgroup of G_n, *congruence subgroups* of G_α.

3.5. DEFINITION. Let β be a partition of n. We denote by U_β the subgroup of U_n (see 3.2) consisting of the matrices $u = (u_{ij})$ for which $u_{ij} = 0$ if $i \neq j$ and i, j lie in the same segment of β (note that $U_n = U_{(1,1,\ldots,1)}$). We set $P_\beta = G_\beta \cdot U_\beta \subset G_n$.

For $\beta < \alpha$ we set $U_\beta(\alpha) = U_\beta \cap G_\alpha$ and $P_\beta(\alpha) = P_\beta \cap G_\alpha$. It is clear that $U_\beta(\alpha)$ is a normal subgroup of $P_\beta(\alpha)$ and that $P_\beta(\alpha) = G_\beta \cdot U_\beta(\alpha)$ (semidirect product).

We call the $U_\beta(\alpha)$ *horospherical* subgroups of G_α and the $P_\beta(\alpha)$ *parabolic* subgroups.

We denote by $\bar{U}_\beta(\alpha)$ the transpose of $U_\beta(\alpha)$.

3.6. LEMMA. *Let $G = G_\alpha$ and let U be an horospherical and P a parabolic subgroup of G.*

a) *U is exhausted by its compact subgroups* (see 2.33).

b) *$G = P \cdot \Gamma$. In particular, G is compact modulo P* (see 2.26).

PROOF. Clearly it suffices to carry out the proof for $G = G_n$, $U = U_n$, and $P = B_n$.

a) We set $U^{(k)} = \{u = (u_{ij}) \in U_n \mid u_{ij} \in \wp^{k(i-j)} \cdot R\}$. It is easy to verify that $U^{(1)} \subset U^{(2)} \subset \ldots$ is a chain of open compact subgroups of U and that $U = \bigcup_k U^{(k)}$, hence a) follows.

b) follows from the Iwasawa decomposition.

AN INFORMAL DESCRIPTION OF THE METHOD

3.7. In this subsection we briefly describe the method used to study the representations of G_n. It is based on the study of the asymptotic behaviour of the matrix elements $\varphi_{\xi,\tilde{\xi}}(g) = \langle \tilde{\xi}, \pi(g^{-1})\xi \rangle$ as g "tends to infinity" (here $(\pi, G_n, E) \in \mathrm{Alg}\, G$, $\xi \in E$, and $\tilde{\xi} \in \tilde{E}$). Since each $g \in G_n$ can be represented in the form $g = \gamma_1 \delta \gamma_2 (\gamma_1, \gamma_2 \in \Gamma, \delta \in \Delta)$ and the sets $\pi(\Gamma)\xi$ and $\tilde{\pi}(\Gamma)\tilde{\xi}$ are finite, it suffices to study the behaviour of $\varphi_{\xi,\tilde{\xi}}(g)$ for $g = \delta \in \Delta$.

If N is a sufficiently small congruence subgroup of G_n, then $\xi \in E^N$ and $\tilde{\xi} \in \tilde{E}^N$, and we have

$$\langle \tilde{\xi}, \pi(\delta^{-1})\xi \rangle = \langle \tilde{\pi}(\varepsilon_N)\tilde{\xi}, \pi(\delta^{-1})\xi \rangle = \langle \tilde{\xi}, \pi(\delta^{-1})\pi(\varepsilon_{\delta N \delta^{-1}})\xi \rangle.$$

In this connection it is important for us to clarify how $\delta N \delta^{-1}$ behaves, as δ "tends to infinity".

It turns out that $\delta \in \Delta$ can "go to infinity" along various paths and that the directions of its departure correspond to the partitions of n. Namely, let β be a partition of n. We say that the sequence $\delta_1, \delta_2, \ldots$, where

$$\delta_k = \mathrm{diag}(\wp^{m_1(k)}, \ldots, \wp^{m_n(k)}) \in \Delta_n,$$

tends to infinity in the complement of β if for any i and j in distinct segments of β

$$\lim_{k\to\infty}(m_i(k) - m_j(k)) = \infty.$$

We claim that if $\delta_1, \delta_2, \ldots$ tends to infinity in the complement of β, then $\bigcup_k \delta_k N \delta_k^{-1} \supset U_\beta$. It follows that if $\varphi_{\xi,\tilde{\xi}}(\delta_k) \neq 0$ for all k, then $\xi \notin E(U_\beta)$ (see 2.30 and 2.33). In this case we can extract information about (π, G_n, E) by investigating the representation of the "smaller" group G_β in $E/E(U_\beta)$. (The appropriate apparatus will be developed in 3.12–3.19.)

Otherwise, we can show that the matrix elements of π are finite modulo the centre Z of G_n, and we can apply the theory of finite representations developed in §2. (This will be done in 3.20–3.30.)

SEVERAL GEOMETRIC LEMMAS

3.8. We make a number of geometric assertions on the structure of the subgroups of $G = G_\alpha$. The motivation for these assertions is clear from 3.7.

DEFINITION. Let α and β, $\beta < \alpha$, be partitions of n, and let $\delta = \operatorname{diag}(\wp^{m_1}, \ldots, \wp^{m_n}) \in \Delta_\alpha$. We set

$$t_\beta^\alpha(\delta) = \min |m_i - m_j|,$$

where the minimum is taken over all index pairs i, j in distinct segments of β, but in one segment of α. For $\beta = \alpha$ we set $t_\beta^\beta(\delta) = \infty$. We usually drop α and simply write $t_\beta(\delta)$.

If $\delta_1, \delta_2, \ldots$ is a sequence of elements in Δ_n, then the condition for it to tend to infinity in the complement of β (see 3.7) is equivalent to $\lim_{k\to\infty} t_\beta(\delta_k) = \infty$.

3.9. LEMMA. *Let α and β, $\beta < \alpha$, be partitions of n. Then for any compact set $K \subset U_\beta(\alpha)$ and any neighbourhood V of the unit element in $U_\beta(\alpha)$ there exists a number $t = t(K, V)$ such that $\delta^{-1}K\delta \subset V$ for all $\delta \in \Delta_\alpha$ for which $t_\beta(\delta) > t$. Similarly, if $\bar{K}$ is a compact set and $\bar{V}$ is a neighbourhood of the unit element in $\overline{U_\beta(\alpha)}$, then for large $t_\beta(\delta)$ we have $\delta\bar{K}\delta^{-1} \subset \bar{V}$.*

PROOF. If $\delta = \operatorname{diag}(\wp^{m_1}, \ldots, \wp^{m_n})$, and $g = (g_{ij})$, then $(\delta^{-1}g\delta)_{ij} = \wp^{m_j - m_i}\cdot g_{ij}$. All the assertions of the lemma follow from this.

3.10. LEMMA. *Let $\Omega \subset \Delta = \Delta_\alpha$. Then the following conditions are equivalent*:

1. *Ω is compact modulo the centre Z of G.*
2. *There is a number t such that for all subpartitions $\gamma \lneq \alpha$ and all $\delta \in \Omega$ we have $t_\gamma(\delta) \leqslant t$.*

PROOF. Since Δ is a discrete set, it follows from condition 1. that Ω

is finite modulo Z. Therefore, 2. follows from 1. Now suppose that Ω is non-compact modulo Z. Then there are indices i and j in one segment of α such that $|m_i - m_j|$ is not bounded for $\delta = \mathrm{diag}(\wp^{m_1}, \ldots, \wp^{m_n}) \in \Omega$. Without loss of generality we may assume that $j = i + 1$. If α is subpartitioned between i and $i + 1$, then for the resulting partition $\gamma \lneqq \alpha$ the number $t_\gamma(\delta)$ is unbounded for $\delta \in \Omega$. Therefore, 1. follows from 2.

3.11. Let α and β, $\beta < \alpha$, be partitions of n. For each congruence subgroup $N \subset G_\alpha$ we set $N^- = N \cap \bar{U}_\beta(\alpha)$, $N^0 = N \cap G_\beta$, and $N^+ = N \cap U_\beta(\alpha)$.

LEMMA. a) $N = N^-N^0N^+ = N^+N^0N^-$. *These are called the decompositions of N associated with β.*

b) $\varepsilon_N = \varepsilon_{N^-} * \varepsilon_{N^0} * \varepsilon_{N^+} = \varepsilon_{N^+} * \varepsilon_{N^0} * \varepsilon_{N^-}$.

PROOF. It suffices to treat the case $\alpha = (n)$. Let $N = 1 + \wp^k \cdot M(n, R)$ (see 3.1). We perform the following "elementary" transformations on the matrices in N: if $i \leqslant j$, then the j-th row (column) multiplied by any $\alpha \in \wp^k \cdot R$ can be added to the i-th row (column). It is clear that the inverse transformations are also elementary. Since elementary transformations involve multiplications on the left by elements of N^+ or N^0 or multiplications on the right by elements of N^- or N^0, the set $N^+N^0N^-$ is invariant under these transformations. But it is clear that any matrix in N can be reduced to the unit matrix by means of these transformations, so that $N = N^+N^0N^-$. Passing to the inverse elements, we find that $N = N^-N^0N^+$.

b) Let $N_P = N \cap P_\beta(\alpha)$. It is then clear that $N_P = N^0N^+$ and that, by a), $N = N^-N_P$. Now b) follows from 1.26b).

THE FUNCTORS $i_{\alpha,\beta}$ AND $r_{\beta,\alpha}$

3.12. We shall study the representations of G_α by connecting them with the representations of G_β for $\beta < \alpha$. To detect this connection, we introduce functors $i_{\alpha,\beta}$: $\mathrm{Alg}\, G_\beta \to \mathrm{Alg}\, G_\alpha$ and $r_{\beta,\alpha}$: $\mathrm{Alg}\, G_\alpha \to \mathrm{Alg}\, G_\beta$. We set $P = P_\beta(\alpha)$ and $U = U_\beta(\alpha)$.

DEFINITION. a) Let $(\rho, G_\beta, V) \in \mathrm{Alg}\, G_\beta$. We denote by the same letter ρ the representation of P in V defined by $\rho(gu) = \rho(g)$ $(g \in G_\beta, u \in U)$. We set $i_{\alpha,\beta}(\rho) = \mathrm{ind}(G_\alpha, P, \rho) \in \mathrm{Alg}\, G_\alpha$.

b) Let $(\pi, G_\alpha, E) \in \mathrm{Alg}\, G_\alpha$. We define $r_{\beta,\alpha}(\pi)$ as the natural representation of G_β in E_U (see 2.30). We have $r_{\beta,\alpha}(\pi) \in \mathrm{Alg}\, G_\beta$.

3.13. PROPOSITION. a) *The mappings $i_{\alpha,\beta}$ and $r_{\beta,\alpha}$ define functors $i_{\alpha,\beta}$: $\mathrm{Alg}\, G_\beta \to \mathrm{Alg}\, G_\alpha$ and $r_{\beta,\alpha}$: $\mathrm{Alg}\, G_\alpha \to \mathrm{Alg}\, G_\beta$. These functors are exact.*

b) *If $\pi \in \mathrm{Alg}\, G_\alpha$ and $\rho \in \mathrm{Alg}\, G_\beta$, then there exists a natural isomorphism $\mathrm{Hom}_{G_\beta}(r_{\beta,\alpha}(\pi), \rho) \simeq \mathrm{Hom}_{G_\alpha}(\pi, i_{\alpha,\beta}(\rho))$. This isomorphism depends functorially on π and ρ.*

c) *If $\gamma < \beta < \alpha$, then $i_{\alpha,\beta} \circ i_{\beta,\gamma} = i_{\alpha,\gamma}$ and $r_{\gamma,\beta} \circ r_{\beta,\alpha} = r_{\gamma,\alpha}$.*

d) *If* $\rho \in \mathrm{Alg}\, G_\beta$ *is admissible, then* $i_{\alpha,\beta}(\rho)$ *is also admissible.*

e) *If* $(\pi, G_\alpha, E) \in \mathrm{Alg}\, G_\alpha$ *is finitely generated, then* $r_{\beta,\alpha}(\pi)$ *is also finitely generated.*

f) $\widetilde{i_{\alpha,\beta}(\rho)} = i_{\alpha,\beta}(\Delta \cdot \widetilde{\rho})$, *where* $\Delta = \Delta_{G_\alpha}/\Delta_P$.

PROOF. a) It is obvious that $i_{\alpha,\beta}$ and $r_{\beta,\alpha}$ are functorial. It follows from 2.25a) that $i_{\alpha,\beta}$ is exact, and from 2.35b) and 3.6 that $r_{\beta,\alpha}$ is exact.

b) By 3.6, $i_{\alpha,\beta}(\rho) = \mathrm{Ind}(G_\alpha, P, \rho)$. Therefore, it follows from the Frobenius duality (see 2.28) that $\mathrm{Hom}_{G_\alpha}(\pi, i_{\alpha,\beta}(\rho)) = \mathrm{Hom}_P(\pi, \rho)$. Since $\rho|_U = 1$, we see that $\mathrm{Hom}_{G_\alpha}(r_{\beta,\alpha}(\pi), \rho)$ stands on the right.

c) follows from 2.25b) and 2.32, and d) from 3.6 and 2.26.

e) Let us prove that E is a finitely generated P-module (hence it follows, of course, that its factor module E_U is finitely generated). Let Ω be a finite set of generators of E. Then E is equal to the linear span of $\pi(G_\alpha)\Omega = \pi(P)\pi(\Gamma)\Omega$ (see 3.6). Since π is algebraic, $\pi(\Gamma)\Omega$ is finite, as required.

Finally, f) follows from 2.25c) and 3.6.

3.14. THEOREM (Jacquet). *If* $(\pi, G_\alpha, E) \in \mathrm{Alg}\, G_\alpha$ *is admissible, then* $r_{\beta,\alpha}(\pi)$ *is also admissible.*

The proof will be given in 3.16–3.17.

3.15. REMARK. It can be proved that $i_{\alpha,\beta}$ carries finitely generated representations into finitely generated representations.

PROOF OF JACQUET'S THEOREM

3.16. Let $G = G_\alpha$, $\beta < \alpha$, $P = P_\beta(\alpha)$, and $U = U_\beta(\alpha)$. Let $(\pi, G, E) \in \mathrm{Alg}\, G$. We set $(\rho, G_\beta, V) = r_{\beta,\alpha}(\pi)$ and denote by A the natural projection $A\colon E \to V = E_U$. We fix a congruence subgroup N in G and consider its decomposition $N = N^+N^0N^-$ associated with β (see 3.11).

LEMMA. a) $A(E^N) \subset V^{N^0}$. b) *Let* $\eta \in V^{N^0}$. *Then there exists a* $t = t(\eta)$ *such that* $\rho(\delta^{-1})\eta \in A(E^N)$ *for all* $\delta \in \Delta_\alpha \cap Z_\beta$ *for which* $t_\beta(\delta) > t$.

PROOF. a) $A \in \mathrm{Hom}_P(\pi, \rho)$. b) Let $\eta = A\xi$, $\xi \in E$. If $\delta \in Z_\beta \cap \Delta_\alpha$, then $\rho(\delta^{-1})\eta = \rho(\delta^{-1})\rho(\varepsilon_{N^0})\eta = A(\pi(\varepsilon_{N^0})\pi(\delta^{-1})\xi)$. If $t_\beta(\delta)$ is large, then $\delta N^-\delta^{-1} \subset \mathrm{stab}\,\xi$ (see 3.9), and hence $\pi(\delta^{-1})\xi = \pi(\varepsilon_{N^-})\pi(\delta^{-1})\xi$. In addition, $A = \rho(\varepsilon_{N^+})A = A \cdot \pi(\varepsilon_{N^+})$. Therefore, if $\delta \in Z_\beta \cap \Delta_\alpha$, then for large $t_\beta(\delta)$ we have

$$\rho(\delta^{-1})\eta = A(\pi(\varepsilon_{N^+})\pi(\varepsilon_{N^0})\pi(\varepsilon_{N^-})\pi(\delta^{-1})\xi) =$$
$$= A(\pi(\varepsilon_N)\pi(\delta^{-1})\xi) \in A(E^N) \quad \text{(see 3.11).}$$

3.17. Under the conditions of 3.16, suppose that π is admissible. Let us prove that in this case $V^{N^0} = A(E^N)$, hence Jacquet's theorem follows.

We set $k = \dim A(E^N) < \infty$. By 3.16a), it suffices to show that $\dim V^{N_0} \leqslant k$. Let $\eta_1, \ldots, \eta_l$ be linearly independent vectors in V^{N^0}.

Using 3.16b), we can find a $\delta \in Z_\beta \cap \Delta_\alpha$ such that $\rho(\delta^{-1})\eta_i \in A(E^N)$ for all i. Since $\rho(\delta^{-1})$ is invertible, the vectors $\rho(\delta^{-1})\eta_i$ are linearly independent. Therefore, $l \leqslant k$, as required.

QUASI-CUSPIDAL AND CUSPIDAL REPRESENTATIONS

3.18. We now describe those representations that cannot be reduced to "smaller" groups by means of the functors $r_{\beta,\alpha}$.

DEFINITION. A representation $\pi \in \mathrm{Alg}\, G_\alpha$ is called *quasi-cuspidal* if $r_{\beta,\alpha}(\pi) = 0$ for all $\beta \lneqq \alpha$. An admissible quasi-cuspidal representation is called *cuspidal.*

π is quasi-cuspidal if and only if $\mathrm{Hom}_{G_\alpha}(\pi, i_{\alpha,\beta}(\rho)) = 0$ for all $\beta \lneqq \alpha$ and $\rho \in \mathrm{Alg}\, G_\beta$ (see 3.13b)).

It follows from 3.13a) that any subfactor module of a quasi-cuspidal module is also quasi-cuspidal.

3.19. PROPOSITION. *Let $\pi \in \mathrm{Alg}\, G_\alpha$ be irreducible. Then there exist a partition $\beta < \alpha$ and an irreducible quasi-cuspidal representation $\rho \in \mathrm{Alg}\, G_\beta$ such that π is embedded in $i_{\alpha,\beta}(\rho)$.*

PROOF. We consider a partition $\beta < \alpha$ such that $r_{\beta,\alpha}(\pi) \neq 0$ and $r_{\gamma,\alpha}(\pi) = 0$ for all $\gamma \lneqq \beta$. Then $r_{\beta,\alpha}(\pi)$ is quasi-cuspidal (see 3.13c)). It follows from 3.13e) that $r_{\beta,\alpha}(\pi)$ is finitely generated. Therefore, $r_{\beta,\alpha}(\pi)$ has an irreducible (quasi-cuspidal) factor representation ρ (see 2.6). Since $\mathrm{Hom}_{G_\alpha}(\pi, i_{\alpha,\beta}(\rho)) = \mathrm{Hom}_{G_\beta}(r_{\beta,\alpha}(\pi), \rho) \neq 0$ (see 3.13b)), there exists a non-zero G_α-homomorphism $A\colon \pi \to i_{\alpha,\beta}(\rho)$. Since π is irreducible, A is an embedding.

HARISH-CHANDRA'S THEOREM

3.20. We wish to connect quasi-cuspidal representations with the finite representations studied in §2. Since the centre Z of the group $G = G_\alpha$ is non-compact, its irreducible representations cannot be finite. To remove this "defect", we restrict the representations of G to the subgroup G^0 defined as follows: if $\alpha = (n_1, \ldots, n_r)$, then $G^0 = G^0_\alpha = \Pi\, G^0_{n_i}$, where $G^0_{n_i} = \{g \in G_{n_i} | \det g \in R^*\}$ (here R^* is the group of invertible elements in R; see 3.1).

It is easy to verify the following properties of this group:

a) $G^0 \cap Z$ is compact and $\Gamma \subset G^0$.

b) G^0 is an open normal subgroup of G, G/G^0 is Abelian, and $G/G^0 \cdot Z$ is finite.

3.21. THEOREM (Harish-Chandra). *Let $(\pi, G, E) \in \mathrm{Alg}\, G$. Then the following conditions are equivalent*:

(1) *π is quasi-cuspidal.*

(2) *For any $\xi \in E$ and any congruence subgroup $N \subset G$ the set*

$K_{\xi,N} = \{g \in G \mid \pi(\varepsilon_N)\pi(g^{-1})\xi \neq 0\}$ *is compact modulo* Z.

(3) *The matrix elements of* π *are finite modulo the centre* (see 2.39).

(4) *The restriction of* π *to* G^0 *is finite.*

PROOF. (2) $\Rightarrow$ (3). If $\tilde{\xi} \in \tilde{E}^N$ and $\xi \in E$, then supp $\varphi_{\xi,\tilde{\xi}} \subset K_{\xi,N}$.

(3) $\Rightarrow$ (4). Since G^0 is open in G, the contragradient module $\tilde{E}$ does not depend on what group, G or G^0, we use to define it. Therefore, the matrix elements of $\pi|_{G^0}$ are matrix elements of π restricted to G^0 and hence have compact support.

(4) $\Rightarrow$ (2). It follows from 2.40 that $K^0_{\xi,N} = K_{\xi,N} \cap G^0$ is compact. Let $g_1, \ldots, g_k$ be coset representatives of $G/G^0 \cdot Z$. Then it is clear that

$K_{\xi,N} \subset Z \cdot (\bigcup_{i=1}^{k} g_i \cdot K^0_{\pi(g_i^{-1})\xi,N})$ is compact modulo Z.

To prove that (1) $\Longleftrightarrow$ (2) we use the following lemma:

3.22. LEMMA. *Let* $\beta < \alpha$ *and* $U = U_\beta(\alpha)$. *Let* $(\pi, G, E) \in$ Alg G *and* $\xi \in E$. *Then the following conditions are equivalent*:

(1) $\xi \in E(U)$.

(2) *For any congruence subgroup* N *in* G *there is a number* $t = t(\xi, N)$ *such that* $\pi(\varepsilon_N)\pi(\delta^{-1})\xi = 0$ *for all* $\delta \in \Delta_\alpha$ *with* $t_\beta(\delta) > t$.

PROOF. (1) $\Rightarrow$ (2). If $\xi \in E(U)$, then by 2.33 and 3.6 there is a compact subgroup $K \subset U$ such that $\pi(\varepsilon_K)\xi = 0$. If $\delta \in \Delta_\alpha$ and $t_\beta(\delta)$ is large, then $\delta^{-1}K\delta \subset N$, so that $\pi(\varepsilon_N)\pi(\delta^{-1})\xi = \pi(\varepsilon_N * \varepsilon_{\delta^{-1}K\delta})\pi(\delta^{-1})\xi =$ $= \pi(\varepsilon_N * \varepsilon_{\delta^{-1}})\pi(\varepsilon_K)\xi = 0$ (we have used 1.26).

(2) $\Rightarrow$ (1). Let N be a congruence subgroup of G such that $\xi \in E^N$. We consider the decomposition $N = N^+N^0N^-$ associated with β (see 3.11). If $\delta \in Z_\beta \cap \Delta_\alpha$, then $\delta N^0\delta^{-1} = N^0 \subset$ stab ξ. If $t_\beta(\delta)$ is large, then $\delta N^-\delta^{-1} \subset$ stab ξ (see 3.9) and $\pi(\varepsilon_N)\pi(\delta^{-1})\xi = 0$, by hypothesis. Therefore, $0 = \pi(\delta)\pi(\varepsilon_N)\pi(\delta^{-1})\xi = \pi(\delta)\pi(\varepsilon_{N^+} * \varepsilon_{N^0} * \varepsilon_{N^-})\pi(\delta^{-1})\xi = \pi(\varepsilon_{\delta N^+\delta^{-1}})\xi$, that is, $\xi \in E(U)$ (see 2.33).

3.23. We can now prove that (1) and (2) in Harish-Chandra's theorem are equivalent. If $g = \gamma_1\delta\gamma_2 (\gamma_1, \gamma_2 \in \Gamma,\ \delta \in \Delta)$, then $\pi(\varepsilon_N)\pi(g^{-1})\xi = \pi(\gamma_2^{-1})\pi(\varepsilon_N)\pi(\delta^{-1})\pi(\gamma_1^{-1})\xi$, and hence the condition "$K_{\xi,N}$ is compact modulo Z" is equivalent to the fact that $K_{\xi',N} \cap \Delta$ is compact modulo Z for all vectors ξ' in the finite set $\pi(\Gamma)\xi$. Therefore, it follows from 3.22 and 3.10 that (1) $\Longleftrightarrow$ (2).

3.24. COROLLARY. *If* (π, G, E) *is a cuspidal representation, then the contragradient representation* $\tilde{\pi}$ *is also cuspidal.*

The proof follows from 3.21, because in this case $E = \tilde{\tilde{E}}$ (see 2.13) and for any $\tilde{\xi} \in \tilde{E}$, $\xi \in E = \tilde{\tilde{E}}$ we have $\varphi_{\xi\tilde{\xi}}(g) = \varphi_{\tilde{\xi},\xi}(g^{-1})$, $g \in G$.

ADMISSIBILITY OF IRREDUCIBLE REPRESENTATIONS

3.25. THEOREM. *Let* $G = G_\alpha$ *and let* $(\pi, G, E) \in$ Alg G *be an irreducible representation. Then* π *is admissible.*

PROOF. Using 3.19 and 3.13d), we may assume that π is quasi-cuspidal. It follows from Harish-Chandra's theorem that the restriction of π to G^0 is finite. Therefore, by 2.41 and the following lemma, π is admissible.

3.26. LEMMA. *Let $(\pi, G, E) \in \mathrm{Alg}\, G$ be an irreducible representation. Then $\pi|_{G^0}$ splits into the direct sum of finitely many irreducible representations.*

PROOF. By 2.9 and 3.20b), $E = E_1 \oplus \ldots \oplus E_k$, where the E_i are irreducible $G^0 \cdot Z$-modules. It follows from Schur's lemma that the operators $\pi(z)$ for $z \in Z$ are scalar. Therefore, the E_i are irreducible G^0-modules, as required.

3.27. COROLLARY. *Let $(\pi, G_n, E) \in \mathrm{Alg}\, G_n$ be an irreducible representation. Then there exists a partition $\beta = (n_1, \ldots, n_r)$ of n and irreducible cuspidal representations (π_i, G_{n_i}, E_i) $(i = 1, \ldots, r)$ such that π is embedded in the representation $\mathrm{Ind}(G_n, P_\beta, \bigotimes_{i=1}^{r} \pi_i)$. (Here $\otimes\ \pi_i$ is the representation of $G_\beta = \Pi G_{n_i}$ constructed in 2.16, which we regard as a representation of P_β by extending it trivially to U_β.)*

This corollary follows easily from 2.16, 3.19, and 3.25.

COROLLARIES OF HARISH-CHANDRA'S THEOREM

3.28. If π is a quasi-cuspidal representation of G, then $\pi|_{G^0}$ is finite. Therefore, we can use the results of 2.40–2.44 to show that cuspidal representations can sometimes be "split off" from the rest.

PROPOSITION. *Let $G = G_\alpha$, let (ω, G, V) be an irreducible cuspidal representation, and let $(\pi, G, E) \in \mathrm{Alg}\, G$. Then E can be split into the direct sum of submodules $E = E_1 \oplus E_2$ such that E_1 is quasi-cuspidal and all its irreducible subfactor modules have the form $\psi\omega$* (see 2.25c)), *where ψ is a character of G that is trivial on G^0, and no irreducible subfactor module of E_2 has this form.*

PROOF. Let (ω_i, G^0, V_i) $(i = 1, \ldots, k)$ be the irreducible representations occurring in the restriction of ω to G^0 (see 3.26). Then they are finite and, using 2.44a) several times, we obtain $E = E_1 \oplus E_2$, where E_1 and E_2 are G^0-submodules, E_1 splits into the direct sum of modules isomorphic to $V_i (i = 1, \ldots, k)$, and no subfactor submodule of E_2 is isomorphic to any module V_i. Under the action of any $g \in G$, each ω_i is carried to the representation ω_i^g (see 2.27), which is isomorphic to one of $\omega_1, \ldots, \omega_k$. Therefore, E_1 and E_2 are G-submodules of E. It follows immediately from the next lemma that they yield the desired decomposition.

3.29. LEMMA. *Let (ω, G, V) and (ω', G, V') be irreducible algebraic representations whose restrictions to G^0 have isomorphic irreducible submodules. Then ω' is isomorphic to $\psi\omega$, where ψ is a character of G that is trivial on G^0.*

PROOF. The condition of the lemma means that the space

$W = \mathrm{Hom}_{G^0}(V', V)$ is non-zero. By 3.26 and 2.11, W is finite-dimensional. We define $(\tau, G/G^0, W)$ by $\tau(g)A = \omega(g)A\omega'(g)^{-1}$ (it is easy to verify that this is well-defined). Since G/G^0 is commutative and W is finite-dimensional, there exists an A in W that is an eigenvector for all $\tau(g)$. This means that the operator $A: V' \to V$ satisfies the condition $A\omega'(g) = \psi(g)\omega(g)A\,(g \in G)$, where ψ is a character of G that is trivial on G^0, that is, $A \in \mathrm{Hom}_G(\omega', \psi\omega)$. Since $A \neq 0$ and since ω' and $\psi\omega$ are irreducible, A defines an isomorphism of ω' and $\psi\omega$.

3.30. PROPOSITION. *Let (ω, G, V) be an irreducible cuspidal and (π, G, E) an admissible representation. We assume that V is a subfactor module of E. Then there exists a submodule (and, similarly, a factor module) of E isomorphic to V.*

PROOF. (1) Using the decomposition $E = E_1 \oplus E_2$ in 3.28, we may assume that $E = E_1$ is cuspidal and that its restriction to G^0 is completely reducible.

(2) We first take the case when $\pi(z)$ is a scalar operator for all $z \in Z$. Then E is a completely reducible $G^0 \cdot Z$-module, and since $G^0 \cdot Z$ has finite index in G, E is completely reducible as a G-module (see 2.9). The required assertion follows immediately from this.

(3) We now examine the general case. Let $\omega(z) = \theta(z) \cdot 1_V$, where θ is a character of Z (see 2.11) and N is a congruence subgroup of G such that $V^N \neq 0$. We carry out the proof by induction on $\dim E^N$.

Using (2), we may assume that there is a $z \in Z$ such that $\pi(z) \neq \theta(z) \cdot 1_E$. Then the operator $A = \pi(z) - \theta(z) \cdot 1_E$ is non-zero and commutes with the action of G. By hypothesis, V^N is a Z-subfactor module of E^N, so that the restriction of A to E^N is non-invertible. Let $E' = \mathrm{Ker}\,A$ and $E'' = \mathrm{Im}\,A \cong E/E'$. Then one of E' or E'' has a subfactor module isomorphic to V. Since $\dim E'^N < \dim E^N$ and $\dim E''^N < \dim E^N$, V can be embedded in one of E' or E'' and hence in E. Similar reasoning shows that one of $E'' \cong E/E'$ or $E''' = E/E''$, and hence E, has a factor module isomorphic to V.

REMARK. It follows from this proposition that a cuspidal submodule is a direct summand of E. This is false for $G_1 = F^*$.

§4. Some finiteness theorems

In this section we use the results of §3 to derive some finiteness theorems. The line of reasoning is basically as follows. We split off from a G-module E its cuspidal part and investigate it, restricting it to G^0, where it becomes finite. Next, with the help of the functors $r_{\beta,\alpha}$, where $\beta < \alpha$, we reduce the study of the remaining part to the study of G_β-modules.

HOWE'S THEOREM

4.1. THEOREM. *Let $G = G_\alpha$ and $(\pi, G, E) \in$ Alg G. Then the following* *conditions are equivalent*:

(1) *The G-module E has finite length* (see 2.8).

(2) π *is admissible and finitely generated.*

It follows from 3.25 that (1) $\Rightarrow$ (2). Howe (see [35]) proved that (2) $\Rightarrow$ (1). We present another proof, based on the methods of §3. To do so, we need the following theorem.

4.2. THEOREM. *Let* $(\pi, G, E) \in \operatorname{Alg} G$ *and let* N *be a congruence subgroup of* G. *We assume that the G-module* E *is generated by* E^N. *Then any G-submodule* $E' \subset E$ *is generated by* $E'^N = E' \cap E^N$.

4.3. From 4.2 we deduce that (2) $\Rightarrow$ (1) in Theorem 4.1. Let N be a congruence subgroup of G such that E is generated by E^N. By hypothesis, $\dim E^N < \infty$. If $0 = E_0 \subsetneqq E_1 \subsetneqq \ldots \subsetneqq E_k = E$ is a chain of G-modules in E, then by 4.2, $0 = E_0^N \subsetneqq E_1^N \subsetneqq \ldots \subsetneqq E_k^N = E^N$. Therefore, $k \leqslant \dim E^N$, so that $l(E) \leqslant \dim E^N$ (see 2.8).

4.4. PROOF OF THEOREM 4.2. The conclusion of the theorem is equivalent to the fact that $V^N \neq 0$ for any non-zero subfactor module (ω, G, V) of E. By 2.6, we may assume that V is irreducible. We examine two cases.

CASE 1. V is cuspidal. We represent E in the form $E = E_1 \oplus E_2$, as in 3.28. Here $E_1 \neq 0$, since V is a subfactor module of E. Let $p\colon E \to E_1$ be the projection. Then $E_1^N = p(E^N)$ generates E_1 as a G-module. Suppose that $0 \neq \xi \in E_1^N$, that E' is the G-submodule of E_1 generated by ξ, and that V' is an irreducible factor module of E'. It is clear that $V'^N \neq 0$. But it follows from the properties of E_1 (see 3.28) that V' and V are isomorphic as G^0-modules. Therefore, $V^N \neq 0$.

CASE 2. V is not cuspidal. We choose $\beta \lneqq \alpha$ so that $r_{\beta,\alpha}(\omega) \neq 0$, and we set $P = P_\beta(\alpha)$, $U = U_\beta(\alpha)$, and $N^0 = N \cap G_\beta$. It follows from 3.13a) that $(r_{\alpha,\beta}(\omega), G_\beta, V_U)$ is a subfactor module of $(r_{\beta,\alpha}(\pi), G_\beta, E_U)$.

By hypothesis, E is the linear span of the set $\pi(G)E^N = \pi(P)\pi(\Gamma)E^N = \pi(P)E^N$ ($G = P\Gamma$ by 3.6; $\pi(\Gamma)E^N = E^N$, since Γ normalizes N). Therefore, E_U is generated by $(E_U)^{N_0}$.

Using induction, we may assume that the conclusion of the theorem has been proved for β. Therefore, $(V_U)^{N_0} \neq 0$. Since $(V_U)^{N_0}$ is generated as a Z_β-module by the image of V^N under the projection $V \to V_U$ (see 3.16), we see that $V^N \neq 0$.

4.5. REMARK. Theorem 4.2 is no longer true if N is replaced by a maximal compact subgroup Γ.

EXAMPLE. $n = 2$, $\pi = \operatorname{ind}(G_2, P_{(1,1)}, 1) = \operatorname{ind}(G_2, P_{(1,1)}, \Delta_{G_2}/\Delta_{P(1,1)})$ (see 3.5).

ESTIMATE OF THE DIMENSION OF E^N

4.6. Let $G = G_\alpha$, N a congruence subgroup of G and $\mathscr{H}_N = \varepsilon_N * \mathscr{H}(G) * \varepsilon_N$ (see 2.10). Using 2.10, we can restate Theorem

3.25 on the admissibility of irreducible representations as follows:

Corollary of 3.25. *Every irreducible representation of* $\mathcal{H}_N$ *is finite-dimensional.*

It turns out that this assertion can be strengthened.

4.7. THEOREM. *Let* $G = G_\alpha$ *and let* N *be a congruence subgroup of* G. *Then there exists a number* $s(G, N)$ *such that the dimension of any irreducible representation of* $\mathcal{H}_N$ *does not exceed* $s(G, N)$.

This follows from Howe's results (see [36]). We present here the simpler proof, which is in [2]. In the remaining part of this section we deduce from this theorem several interesting corollaries on the structure of the representations of G.

4.8. COROLLARY. *If* $(\pi, G, E) \in \mathrm{Alg}\, G$ *is an irreducible representation, then* $\dim E^N \leqslant s(G, N)$.

4.9. PROOF OF THEOREM 4.7. We first describe the structure of $\mathcal{H}_N$. If $g \in G$, then we set $\bar{g} = \varepsilon_N * \varepsilon_g * \varepsilon_N \in \mathcal{H}_N$.

LEMMA. a) *Let* γ_i $(i = 1, 2, \ldots, p)$ *be coset representatives of* Γ/N $(= N\backslash\Gamma)$. *Then the elements* $\bar{\gamma}_i * \bar{\delta} * \bar{\gamma}_j$, *where* $i, j = 1, \ldots, p$ *and* $\delta \in \Delta$ (see 3.4), *generate* $\mathcal{H}_N$ *as a linear space.*

b) *If* $\delta_1, \delta_2 \in \Delta$, *then* $\overline{\delta_1\delta_2} = \bar{\delta}_1 * \bar{\delta}_2$.

The following description of $\mathcal{H}_N$ is a consequence of this lemma. We denote by $\mathcal{A}$ the subspace of $\mathcal{H}_N$ generated by the elements $\bar{\delta}(\delta \in \Delta)$. We choose generators $\delta_1, \ldots, \delta_l$ in Δ (it is easy to see that there are finitely many of them), and we set $a_i = \bar{\delta}_i (i = 1, \ldots, l)$. Then $\mathcal{A}$ is the commutative subalgebra of $\mathcal{H}_N$ generated by the $a_i (i = 1, \ldots, l)$ and the unit element, and $\mathcal{H}_N = \sum_{i,j=1}^{p} \bar{\gamma}_i * \mathcal{A} * \bar{\gamma}_j$. In particular, $\mathcal{H}_N$ is finitely generated.

PROOF OF THE LEMMA. a) It is easy to verify that $\mathrm{supp}\, (\bar{\gamma}_i * \bar{\delta} * \gamma_j) = N(\gamma_i\delta\gamma_j)N$. By 1.27, the restriction of any distribution $h \in \mathcal{H}_N$ to this set is proportional to $\bar{\gamma}_i * \bar{\delta} * \bar{\gamma}_j$. Since the sets $N(\gamma_i\delta\gamma_j)N$ exhaust all the cosets of $N\backslash G/N$ (this follows from the Cartan decomposition; see 3.2), $\mathcal{H}_N$ is generated by the distributions $\bar{\gamma}_i * \bar{\delta} * \bar{\gamma}_j$.

b) By 1.27, it suffices to prove that $\mathrm{supp}\, (\bar{\delta}_1 * \bar{\delta}_2) \subset N(\delta_1\delta_2)N$, that is, $N\delta_1 N\delta_2 N \subset N\delta_1\delta_2 N$ (see 1.24). We consider the decomposition $N = N^- N^0 N^+$ associated with the smallest partition $\beta = (1, 1, \ldots, 1)$ (see 3.11). It is easy to verify that $\delta_1 N^- \delta_1^{-1} \subset N^{-1}$ and $\delta_2^{-1} N^0 N^+ \delta_2 \subset N^0 N^+$, hence $N\delta_1 N\delta_2 N = N(\delta_1 N^-)(N^0 N^+ \delta_2)N \subset N\delta_1\delta_2 N$.

4.10. We used a lemma from linear algebra.

LEMMA. *Let* V *be an* m*-dimensional space over* $\mathbf{C}$ *and* $\mathcal{R} \subset \mathrm{End}\, V$ *a commutative subalgebra generated by* l *elements* $a_1, \ldots, a_l$ *(and the unit element). Then* $\dim \mathcal{R} \leqslant f_l(m)$, *where* $f_l(m) = m^{2-2^{1-l}}$.

4.11. We now deduce Theorem 4.7 from this lemma. Let $\rho\colon \mathcal{H}_N \to \mathrm{End}\, V$

be an irreducible representation of $\mathscr{H}_N$ in V. By 4.6, dim $V < \infty$. Let dim $V = m$. As follows from Burnside's theorem (see [5], Ch. VIII, §4), $\rho(\mathscr{H}_N)$ coincides with End V, that is, dim $\rho(\mathscr{H}_N) = m^2$. It follows from 4.9 and 4.10 that dim $\rho(\mathscr{A}) \leqslant f_l(m)$, and from

$\mathscr{H}_N = \sum_{i,\, j=1}^{p} \bar{\gamma}_i * \mathscr{A} * \bar{\gamma}_j$ that dim $\rho(\mathscr{H}_N) \leqslant p^2 \dim \rho(\mathscr{A})$. Thus,

$m^2 \leqslant p^2 m^{2-2^{1-l}}$, that is; m is bounded by $s(G, N) = p^{2^l}$.

4.12. PROOF OF LEMMA 4.10. (1) Reduction to the case when all the operators a_i are nilpotent. We claim that V can be split into the direct sum of $\mathscr{R}$-invariant subspaces such that a_i in each of them is the sum of a scalar and a nilpotent operator. To see so, we reduce a_1 to Jordan form and combine the blocks corresponding to the same eigenvalues. So we obtain a decomposition of V into the direct sum of $\mathscr{R}$-invariant subspaces. Decomposing each of these by means of a_2, then a_3, etc., we obtain the required decomposition into a direct sum.

Since f_l is convex, that is, $f_l(m_1 + \ldots + m_k) \geqslant f_l(m_1) + \ldots + f_l(m_k)$, it suffices to prove the lemma for each subspace. Subtracting the scalar part from a_i, we may assume that each a_i is nilpotent.

(2) Let $\varphi_l(m)$ be the largest possible dimension of $\mathscr{R}$ subject to the condition that $a_1, \ldots, a_l$ are nilpotent. We claim that

$$(*)\ \varphi_l(m) \leqslant \varphi_l([m - \varphi_l(m)/m]) + \varphi_{l-1}(m).$$

Since $f_l(m) \geqslant f_l([m - f_l(m)/m]) + f_{l-1}(m)$, the lemma follows from $(*)$ by induction on l and m.

Let $\mathfrak{J}$ be the ideal in $\mathscr{R}$ generated by $a_1, \ldots, a_l$. $\mathfrak{J}$ is a power of $\mathscr{R}$, and $V^k = \mathfrak{J}^k V$. Then $V = V^0 \supset V^1 \supset \ldots \supset V^m = 0$.

Let L be a complementary subspace to V^1 in V, and let $s = \dim L$. It is clear that $\mathfrak{J}^k L$ generates V^k modulo V^{k+1}, so that $\mathscr{R}L = V$. Hence each $a \in \mathscr{R}$ is defined by its value on L (because $a(\Sigma\, b_i \xi_i = \Sigma\, b_i(a\xi_i)$, $b_i \in \mathscr{R}$ $\xi_i \in L$). Therefore, dim $\mathscr{R} \leqslant s \cdot m$, and if we assume that dim $\mathscr{R}$ is maximal (that is, dim $\mathscr{R} = \varphi_l(m)$), then $s \geqslant \varphi_l(m)/m$. We denote by $\mathscr{R}'$ the subalgebra of $\mathscr{R}$ generated by $a_2, a_3, \ldots, a_l$ and by $\mathscr{R}''$ the ideal $a_1\, \mathscr{R} \subset \mathscr{R}$. It is clear that $\mathscr{R} = \mathscr{R}' + \mathscr{R}''$ and that dim $\mathscr{R}' \leqslant \varphi_{l-1}(m)$. Since $a_1(V) \subset V^1$, dim $\mathscr{R}''$ does not exceed the dimension of the restriction of $\mathscr{R}$ to V^1, that is, dim $\mathscr{R}'' \leqslant \varphi_l(\dim V^1) = \varphi_l(m - s) \leqslant$ $\leqslant \varphi_l([m - \varphi_l(m)/m])$. Therefore, $\varphi_l(m) = \dim \mathscr{R} \leqslant \dim \mathscr{R}' + \dim \mathscr{R}' \leqslant$ $\leqslant \varphi_l([m - \varphi_l(m)/m]) + \varphi_{l-1}(m)$, as required.

4.13. REMARK. Estimates analogous to those in 4.7 and 4.8 are also valid for G^0 and $\mathscr{H}_N(G^0)$. For suppose that $(\omega, C^0, V) \in \mathrm{Alg}\, G^0$ is irreducible. We set $\pi = \mathrm{ind}(G, G^0, \omega)$. Since G^0 is an open normal subgroup of G, it follows from the explicit construction of π (see 2.22) that $\pi|_{G^0} = \bigoplus_g \omega^g$, where g ranges over a set of coset representatives of G/G^0, and ω^g is defined in 2.27. Therefore, taking any irreducible subfactor representation (ω', G, V') in π, we find that ω occurs in $\omega'|_{G^0}$, so that $\dim V^N \leqslant \dim V'^N \leqslant s(G, N)$.

FINITENESS OF THE NUMBER OF CUSPIDAL REPRESENTATIONS

4.14. THEOREM. *Let $G = G_\alpha$ and let N be a congruence subgroup of G. Then G^0 has only finitely many non-isomorphic finite irreducible representations ω such that $\omega(\varepsilon_N) \neq 0$.*

4.15. COROLLARY. *Let N be a congruence subgroup of G and θ a character of Z. Then G has only finitely many non-isomorphic irreducible cuspidal representations (ω, G) such that $\omega(\varepsilon_N) \neq 0$ and $\omega(z) = \theta(z) \cdot 1_v$ for all $z \in Z$.*

This corollary is deduced from 4.14 with the help of 3.20b), 3.21, and 3.29.

4.16. PROOF OF THEOREM 4.14. We call an irreducible representation ω of $\mathcal{H}_N = \mathcal{H}_N(G^0)$ finite if it corresponds to a finite representation of G^0 in the sense of 2.10. We have to show that $\mathcal{H}_N$ has finitely many such representations, up to isomorphism. By virtue of 4.13, the dimensions of the irreducible representations of $\mathcal{H}_N$ are bounded. Therefore, we may treat representations of a fixed dimension s.

Let $M = M(s, \mathbf{C})$ be the algebra of matrices of order s over $\mathbf{C}$. We denote by W the set of algebra homomorphisms $w: \mathcal{H}_N \to M$. Let $b_1, \ldots, b_k$ be generators of $\mathcal{H}_N$ (it can be proved, exactly as in 4.9, that $\mathcal{H}_N$ is finitely generated). Then the mapping $w \to (w(b_1), \ldots, w(b_k))$ defines an embedding of W into the linear space $M^k = M \times \ldots \times M$.

Let ω be a finite irreducible s-dimensional representation of $\mathcal{H}_N$, $\varepsilon_N^\omega \in \mathcal{H}_N$ the distribution constructed in 2.42, and $P^\omega(x_1, \ldots, x_k)$ a non-commutative polynomial such that $P^\omega(b_1, \ldots, b_k) = \varepsilon_N^\omega$. We denote by Q^ω the polynomial function on M^k defined by $Q^\omega(x_1, \ldots, x_k) = \operatorname{tr} P^\omega(x_1, \ldots, x_k)$, $x_i \in M$. It is clear from the definition of ε_N^ω that for $w \in W$ the condition $Q^\omega(w) \neq 0$ is equivalent to the fact that the representation $w: \mathcal{H}_N \to M$ is isomorphic to ω.

In the ring of polynomial functions on M^k we consider the ideal $\mathcal{J}$ generated by the functions Q^ω for all finite s-dimensional representations ω. By Hilbert's basis theorem (see [25], Ch. VI, §2), this ideal is generated by finitely many functions $Q^{\omega_1}, \ldots, Q^{\omega_l}$. Let $w \in W$ be a finite irreducible representation. Then $Q^w(w) \neq 0$ and $Q^w \in \mathcal{J}$. Therefore, $Q^{\omega_i}(w) \neq 0$ for one of $i = 1, \ldots, l$. But this means that w is isomorphic to ω_i. Thus, each s-dimensional finite irreducible representation of $\mathcal{H}_N$ is isomorphic to one of $\omega_1, \ldots, \omega_l$, as required.

THE SPLITTING OFF OF THE QUASI-CUSPIDAL REPRESENTATIONS

4.17. THEOREM. *Let $G = G_\alpha$ and $(\pi, G, E) \in$ Alg G. Then E can be* *split into the direct sum $E = E_c \oplus E_c^\perp$ so that E_c is quasi-cuspidal and $E_c^\perp$ has no non-zero quasi-cuspidal subfactor modules. Obviously, this decomposition is unique.*

PROOF. We consider the restriction of π to G^0 and define $\pi(\varepsilon^c): E \to E$ by

$$\pi(\varepsilon^c)\,\xi = \sum_{\omega} \pi(\varepsilon^{\omega})\,\xi \qquad (\xi \in E),$$

where the sum is taken over all non-isomorphic finite irreducible representations of G^0 (see 2.43). Since each $\xi \in E$ is invariant under some congruence subgroup N, by Theorem 4.14 only finitely many terms in this sum are non-zero. Using 2.43 and 3.21, we can easily verify that $\pi(\varepsilon^c)$ is a projection and that the spaces $E_c = \operatorname{Im} \pi(\varepsilon^c)$ and $E_c^\perp = \operatorname{Ker} \pi(\varepsilon^c)$ satisfy the required conditions.

4.18. EXAMPLE. We identify $S(G)$ and $\mathcal{H}(G)$ by means of the Haar measure μ_G (see 1.30), and we consider the regular representation $(\gamma, G, S(G))$. In this case $S(G)_c$ coincides with the space of parabolic forms on G. (A function $f \in S(G)$ is called a parabolic form if for any non-trivial horospherical subgroup U and any $g_1, g_2 \in G$, $\int_U f(g_1 u g_2)\, d\mu_U(u) = 0$; see [33].) It is clear that $S(G)_c$ is a two-sided ideal in $S(G) = \mathcal{H}(G)$ and Theorem 4.17 states that there exists a complementary ideal $S(G)_c^\perp$ such that $S(G) = S(G)_c \oplus S(G)_c^\perp$. It is easy to show that $S(G)_c^\perp = \{h \in S(G) = \mathcal{H}(G) \mid \omega(h) = 0 \text{ for all cuspidal representations } \omega\}$.

FINITELY GENERATED G-MODULES ARE NOETHERIAN

4.19. THEOREM. *Let $G = G_\alpha$. Then any finitely generated module $(\pi, G, E) \in$ Alg G is Noetherian.*

Let us recall that a module E is called *Noetherian* if any submodule of E is finitely generated, or, equivalently, if any ascending chain of submodules $E_1 \subset \ldots \subset E_k \ldots$ of E stabilizes (see [25], Ch. VI, §1).

PROOF. We consider the decomposition $E = E_c \oplus E_c^\perp$ in 4.17. It is clear that E_c and $E_c^\perp$ are finitely generated. Therefore, it suffices to examine two cases: $E = E_c$ and $E = E_c^\perp$.

CASE 1. $E = E_c$, that is, E is quasi-cuspidal. Since $G^0 \cdot Z$ is a subgroup of finite index in G, E is finitely generated over $G^0 \cdot Z$. Let $\Omega \subset E$ be a finite set of generators of E as a $G^0 \cdot Z$-module.

Let $N \subset G$ be a congruence subgroup such that $\Omega \subset E^N$. If E' is a G-submodule of E, then by 4.2 it is generated by $E'^N = E' \cap E^N$. Therefore, it suffices to show that this space is finitely generated. We claim that it is finitely generated even over $Z' = Z \cap \Delta$ (see 3.2).

It is clear that Z' is a finitely generated discrete Abelian group; that is,

$\mathscr{H}(Z')$ is finitely generated and commutative and hence Noetherian (see [25], Ch. VI, §2). Therefore, finitely generated Z'-modules are Noetherian, and it suffices to verify that the Z'-module E^N is finitely generated.

We denote by V the G^0-submodule of E generated by Ω. Since $\pi|_{G^0}$ is finite (see 3.21), V is an admissible G^0-module (see 2.41), so that $\dim V^N < \infty$. It is easy to see that $G^0 \cdot Z = G^0 \cdot Z'$, so that E^N is generated by the vectors $\pi(\varepsilon_N)\pi(z)\pi(g)\xi = \pi(z)\pi(\varepsilon_N)\pi(g)\xi$ ($z \in Z'$, $g \in G^0$, $\xi \in \Omega$); that is, the Z'-module E^N is generated by the finite-dimensional space V^N, as required.

CASE 2. $E = E_c^{\perp}$, that is, E has no non-zero quasi-cuspidal subfactor modules. We assume that there exists a strictly ascending infinite chain of G-submodules $E_1 \subsetneqq E_2 \subsetneqq \dots \subsetneqq E_k \subsetneqq \dots$ in E. For each i the module E_{i+1}/E_i is not quasi-cuspidal, so that there is a partition $\beta = \beta_i \lneqq \alpha$ such that $(E_{i+1}/E_i)_{U_\beta(\alpha)} \neq 0$. Suppose that $\beta = \beta_i$ for infinitely many indices i and that $U = U_\beta(\alpha)$. By 3.13a), $(E_1)_U \subset (E_2)_U \subset \dots \subset (E_k)_U \dots$ is an ascending chain of submodules in E_U, where $(E_{i+1})_U/(E_i)_U = (E_{i+1}/E_i)_U \neq 0$ for infinitely many indices i; that is, the G_β-module E_U is not Noetherian. Since it is finitely generated (see 3.13e)), induction on partitions completes the proof.

4.20. COROLLARY. *For any congruence subgroup $N \subset G$ the algebra $\mathscr{H}_N$ is finitely generated and Noetherian.*

It was proved in 4.9 that $\mathscr{H}_N$ is finitely generated, and by arguments analogous to those in 2.10 it can easily be deduced from 4.19 that it is Noetherian.

UNITARY AND ALGEBRAIC REPRESENTATIONS OF THE GROUPS G_α

4.21. Let $G = G_\alpha$, N any congruence subgroup of G, and $s = s(G, N)$. Then, as follows from 4.7 and 2.12, for any non-zero $h \in \mathscr{H}_N$ there is a representation (ρ, V) of $\mathscr{H}_N$ such that $\rho(h) \neq 0$ and $\dim V \leqslant s$. The following theorem can be deduced from this fact and results of Godement [17]:

THEOREM. a) *Let π be a continuous unitary representation of $G = G_\alpha$ in a Hilbert space E. We assume that π is topologically irreducible, that is, E has no closed G-invariant subspaces. Let (π_a, G, E_a) be the algebraic part of π* (see 2.1). *Then π_a is admissible and irreducible.*

b) *If (π, G, E) and (π', G, E') are representations of the type described, then π_a and $(\pi')_a$ are isomorphic if and only if π and π' are unitarily equivalent.*

c) *Let $(\pi, G, E) \in$* Alg G *be an irreducible representation, and suppose that a G-invariant, positive-definite, Hermitian scalar product $\{,\}$ is defined on E. We denote by $\hat{\pi}$ the representation of G in $\hat{E}$ obtained from E by completion with respect to the norm $\|\xi\| = \{\xi,\xi\}^{1/2}$. Then $\hat{\pi}$ is topologically irreducible, and $\hat{E}_a = E$.*

Since unitary representations are beyond the scope of our exposition, we do not prove this theorem.

CHAPTER III

THE GEL'FAND–KAZHDAN THEORY

§5. Non-degenerate representations

THE GROUPS P_n, M_n, AND U_n

5.1. In this chapter we present another approach to the study of representations of $GL(n, F)$, which is due to Gel'fand and Kazhdan (see [13], [14]). We need the following subgroups of $G = G_n = GL(n, F)$:

$$\begin{aligned} P = P_n &= \{(p_{ij}) \in G \mid p_{nj} = 0 \text{ for } j < n,\ p_{nn} = 1\}, \\ M = M_n &= \{(m_{ij}) \in G \mid m_{ij} = \delta_{ij} \text{ for } j < n,\ m_{nn} = 1\} \subset P, \\ U = U_n &= \{(u_{ij}) \in G \mid u_{ij} = 0 \text{ for } i > j,\ u_{ii} = 1\} \subset P. \end{aligned}$$

The method of Gel'fand and Kazhdan consists in studying the restrictions of irreducible representations of G to P. It turns out that the restriction of "almost all" irreducible representations is isomorphic or "almost isomorphic" to a standard irreducible representation of P.

5.2. We denote by G' the group G_{n-1}. We assume that G' is embedded in $P \subset G$ in the standard way as the group of matrices of the form (g_{ij}), where $g_{ni} = g_{in} = 0$ for $i \neq n$ and $g_{nn} = 1$. We shall study the subgroups $P' = P_{n-1}$ and $U' = U_{n-1}$ of G'.

It is clear that M is a normal subgroup of P and that $P = G' \cdot M$ and $U = U' \cdot M$ (both products are semidirect). M is isomorphic to $F^{n-1} = F \times F \times \ldots \times F$.

5.3. We recall how the characters of F^k are constructed. We fix a non-trivial additive character ψ_0 of F. If $a = (a_1, \ldots, a_k) \in F^k$, then we define a character ψ_a of F^k by $\psi_a(x) = \psi_0(a_1x_1 + \ldots + a_kx_k)$, where $x = (x_1, \ldots, x_k) \in F^k$. We denote by $\hat{F}^k$ the group of characters of F^k. Then the mapping $a \mapsto \psi_a$ defines an isomorphism of F^k and $\hat{F}^k$ (see [10], Ch. II, §5). This isomorphism defines an l-space structure on $\hat{F}^k$.

By the Fourier transform of $T \in S_c^*(F^k)$ (see 1.10), we mean the function $\hat{T} \in C^\infty(\hat{F}^k)$ defined by $\hat{T}(\theta) = \langle T, \theta \rangle (\theta \in \hat{F}^k)$ (see 1.11). It is clear that $\widehat{(T_1 * T_2)} = \hat{T}_1 \cdot \hat{T}_2$ (pointwise multiplication).

LEMMA. a) *The mapping $h \mapsto \hat{h}$ defines an isomorphism of $\mathcal{H}(F^k)$ and $S(\hat{F}^k)$.*

b) *If N is an open compact subgroup of F^k, then the Fourier transform $\hat{\varepsilon}_N$ is equal to the characteristic function of some open compact subgroup $\hat{N} \subset \hat{F}^k$.*

c) $\bigcup_N \hat{N} = \hat{F}^k$. *In particular, F^k and $\hat{F}^k$ exhaust the compact subgroups* (see 2.33) (see [10], Ch. VII, §2).

5.4. Using 5.3, we can easily describe $\hat{M}$. We define $\Theta \in \hat{M}$ by $\Theta((m_{ij})) = \psi_0(m_{n-1,n})$.

LEMMA. *Any non-zero $\theta \in \hat{M}$ is conjugate to Θ under the action of $G' \subset P$ (see 2.27). The normalizer of Θ in P is equal to* $\mathrm{Norm}_P(M, \Theta) = P' \cdot M$ *(we recall that* $\mathrm{Norm}_P(M, \Theta) = \{p \in P \mid \Theta({}^p m) = \Theta(m)$ *for all* $m \in M\}$*). In particular,* $\hat{M} \setminus \{1_M\} \simeq P' \cdot M \setminus P \simeq P' \setminus G'$.

5.5. *We now describe the characters of $U = U_n$.*

LEMMA. *Each character θ of U has the form* $\theta((u_{ij})) = \psi_0(\sum_{i=1}^{n-1} a_i u_{i,i+1})$, *where $a_i \in F$ (notation: $\theta = \theta(a_1, \ldots, a_{n-1})$).*

PROOF. It can be verified directly that the mapping $U \to F^{n-1}$, $(u_{ij}) \to (u_{1,2}, \ldots, u_{n-1,n})$ determines an isomorphism of the commutator factor group of U with F^{n-1}. Therefore, the lemma follows from 5.3.

5.6. DEFINITION. A character $\theta = \theta(a_1, \ldots, a_{n-1})$ of U is called *non-degenerate* if all the a_i are non-zero, and *degenerate* otherwise.

We set $\Theta = \theta(1, \ldots, 1)$. It is clear that any non-degenerate character θ is conjugate to Θ under the action of $D \cap P$, where D is the subgroup of diagonal matrices in G. In addition, Θ/M is the same as the character Θ defined in 5.4.

5.7. DEFINITION. a) The representation $(\tau, G, L(G, \Theta)) = \mathrm{Ind}(G, U, \Theta)$ (see 2.21) is called the *standard* representation of $G = G_n$. We denote the restriction of τ to $S(G, \Theta) \subset L(G, \Theta)$ by τ^0 ($\tau^0 = \mathrm{ind}(G, U, \Theta)$; see 2.22).

b) An algebraic representation (π, G, E) is called *non-degenerate* if one of the following two equivalent conditions holds:

(1) $\mathrm{Hom}_G(\pi, \tau) \neq 0$;

(2) $E_{U,\Theta} \neq 0$ (see 2.30).

The definitions of the standard representation $\tau_P = \mathrm{Ind}(P, U, \Theta)$, $\tau_P^0 = \mathrm{ind}(P, U, \Theta)$, and non-degenerate representations of P are completely analogous.

Representations of G (or P) that are not non-degenerate are called *degenerate.*

5.8. PROPOSITION. *All subfactor representations of a degenerate representation are degenerate.*

This follows immediately from 2.35 and 3.6.

REPRESENTATIONS OF $M = M_n$

5.9. As follows from 2.5, Alg M consists of the modules (π, E) over $\mathscr{H}(M)$ such that $\pi(\mathscr{H}(M)) \cdot E = E$. We identify $\mathscr{H}(M)$ with $S(\hat{M})$ by the Fourier transform (see 5.3). Then E is an $S(\hat{M})$-module, where $S(\hat{M})E = E$. But, as follows from 1.14, to this module there corresponds a unique l-sheaf $\mathscr{E}$ on the l-space $\hat{M}$ such that E and $\mathscr{E}_c$ are isomorphic as $S(\hat{M})$-modules. Thus, the category Alg M is the same as that of l-sheaves on $\hat{M}$.

5.10. LEMMA. *Let $\theta \in \hat{M}$. Then under the isomorphism A: $E \to \mathscr{E}_c$ in 5.9 the subspace $E(M, \theta)$ is mapped isomorphically onto the space $\mathscr{E}(\theta)$ of cross-sections in $\mathscr{E}_c$ that are equal to 0 at θ. In particular, $E_{M,\theta} \simeq \mathscr{E}_\theta$ (the stalk of $\mathscr{E}$ at θ).*

PROOF. It suffices to consider the case $\theta = 1$ (see 2.31a)). It follows from 5.3 and 2.33 that $A(E(M)) = \{\varphi \in \mathscr{E}_c \mid \chi_{\hat{N}} \cdot \varphi = 0$ for some open compact subgroup $N \subset M\}$. Since $\mathscr{E}(1) = \{\varphi \in \mathscr{E}_c \mid f(1) \neq 0$ and $f \cdot \varphi = 0$ for some $f \in S(\hat{M})\}$ (see 1.14), our assertion follows from the fact that $\bigcup_N \hat{N} = \hat{M}$ (see 5.3c)).

REPRESENTATIONS OF P

5.11. We now make a detailed study of the representations of $P = P_n$. It turns out that each such representation π can be decomposed into a subrepresentation π' connected with P' and the factor representation π/π' connected with G'. This decomposition makes it possible to study the representations of P inductively. It is accomplished with the help of the functors Φ^-, Φ^+, Ψ^-, and Ψ^+.

DEFINITION. a) Let $(\pi, P, E) \in \mathrm{Alg}\, P$. We denote by $\Phi^-(\pi)$ the representation of P' in $E_{M,\Theta} \doteq E/E(M, \Theta)$ (see 2.31b) and 5.4).

b) Let $(\tau, P', V) \in \mathrm{Alg}\, P'$. We set $\Phi^+(\tau) = \mathrm{ind}(P, P' M, \tau') \in \mathrm{Alg}\, P$, where $(\tau', P' \cdot M, V) \in \mathrm{Alg}\, (P' \cdot M)$ is defined by $\tau'(pm)\xi = \Theta(m)\tau(p)\xi$ $(m \in M,\ p \in P',\ \xi \in V)$.

c) Let $(\pi, P, E) \in \mathrm{Alg}\, P$. We denote by $\Psi^{-1}(\pi)$ the representation of G' in $E_{M,1} = E/E(M, 1)$.

d) Let $(\rho, G', V) \in \mathrm{Alg}\, G'$. We define $(\Psi^+(\rho), P, V) \in \mathrm{Alg}\, P$ by $\Psi^+(\rho)(gm) = \rho(g)$, $g \in G'$, $m \in M$.

5.12. Let us describe the properties of Φ^-, Φ^+, Ψ^-, *and* Ψ^+.

PROPOSITION. a) *The mappings $\pi \to \Phi^-(\pi)$, $\tau \to \Phi^+(\tau)$, $\pi \to \Psi^-(\pi)$, and $\rho \to \Psi^+(\rho)$ define functors Φ^-: $\mathrm{Alg}\, P \to \mathrm{Alg}\, P'$, Φ^+: $\mathrm{Alg}\, P' \to \mathrm{Alg}\, P$, Ψ^-: $\mathrm{Alg}\, P \to \mathrm{Alg}\, G'$, and Ψ^+: $\mathrm{Alg}\, G' \to \mathrm{Alg}\, P$. These functors are exact.*

b) *Φ^+ is left-conjugate to Φ^-; that is, for any $\pi \in \mathrm{Alg}\, P$ and $\tau \in \mathrm{Alg}\, P'$ there is an isomorphism*

$$(*) \qquad \mathrm{Hom}_P(\Phi^+(\tau), \pi) = \mathrm{Hom}_{P'}(\tau, \Phi^-(\pi)),$$

which depends functorially on π and τ. Similarly, Ψ^- is conjugate to Ψ^+; that is, for $\pi \in \mathrm{Alg}\, P$ and $\rho \in \mathrm{Alg}\, G'$ there is an isomorphism

$$(**) \qquad \mathrm{Hom}_P(\pi, \Psi^+(\rho)) = \mathrm{Hom}_{G'}(\Psi^-(\pi), \rho),$$

which depends functorially on π and ρ.

c) $\Phi^-\Psi^+ = 0$ *and* $\Psi^-\Phi^+ = 0$.

d) *By* b), *the morphisms*

$$i\colon\ \Phi^+\Phi^-(\pi)\to\pi,\quad i'\colon\ \tau\to\Phi^-\Phi^+(\tau),$$

$$j\colon\ \pi\to\Psi^+\Psi^-(\pi),\quad j'\colon\ \Psi^-\Psi^+(\rho)\to\rho$$

are defined (*i corresponds to the identity P'-morphism $\Phi^-(\pi)\to\Phi^-(\pi)$ in* $(*)$, *and i' corresponds to the identity P-morphism $\Phi^+(\tau)\to\Phi^+(\tau)$; similarly, j and j' correspond to the identity morphisms $\Psi^-(\pi)\to\Psi^-(\pi)$ and $\Psi^+(\rho)\to\Psi^+(\rho)$ in* $(**)$.

Then i' and j' are isomorphisms, and i and j form an exact sequence

$$0\to\Phi^+\Phi^-(\pi)\to\pi\to\Psi^+\Psi^-(\pi)\to 0.$$

e) *For $\pi\in$* Alg *P the condition $\Phi^-(\pi)=0$ is equivalent to the fact that $\pi|_M$ is the identity.*

f) *If $\tau\in$* Alg *P', then Φ^+ and Φ^- establish a bijection between τ and $\Phi^+(\tau)$. In particular, τ and $\Phi^+(\tau)$ are irreducible simultaneously. The same is true for ρ and $\Psi^+(\rho)$, $\rho\in$* Alg *G'.*

g) $\Phi^+(\tau^0_{P'})=\tau^0_P$ *and* $\Phi^-(\tau^0_P)=\tau^0_{P'}$.

PROOF. It is obvious that Φ^-, Φ^+, Ψ^-, and Ψ^+ are functorial. It follows from 2.25, 2.35, and 5.3c) that they are exact.

b) (1) Let $(\pi, P, E)\in$ Alg P. By 5.9, E can be realized as the space of finite cross-sections of some l-sheaf $\mathscr{E}=\mathscr{E}(\pi)$ on $\hat M$, where

$$(*)\qquad \pi(m)\varphi(\theta)=\theta(m)\varphi(\theta),\quad m\in M,\ \theta\in\hat M,\quad \varphi\in\mathscr{E}_c\approx E.$$

It is easy to verify that π, together with the natural action γ of P on $\hat M$ (see 2.27), defines an action of P on $\mathscr{E}$ (see 1.17). It follows from 5.9 that this realization of π is unique.

(2) By 5.4, γ has two orbits: the closed orbit Z consisting of the identity character and the open orbit $Y=\hat M\setminus Z$. It follows from 5.10 that the representation $\pi^0=\pi|_{\mathscr{E}_c(Y)}$ is isomorphic to the restriction of π to $E(M, 1)$. Moreover, the stability subgroup of $\Theta\in Y$ in P is $P'M$, and the representation of this subgroup in the stalk $\mathscr{E}_\Theta$ over this point is isomorphic to $\Phi^-(\pi)'$ (see Definition 5.11b)). Therefore, by 2.23, we have the natural isomorphism $\Phi^+\Phi^-(\pi)\approx\mathrm{ind}(P, P'M, \Phi^-(\pi)')\approx\pi^0$. In particular, to each morphism $\tau\to\Phi^-(\pi)$ there corresponds a morphism $\Phi^+(\tau)\to\Phi^+\Phi^-(\pi)\to\pi$.

(3) We now ascertain what sheaf $\mathscr{E}$ corresponds to the representation $\Phi^+(\tau)=\mathrm{ind}(P, P'M, \tau')$. This representation is realized in $S(P, \tau')$ (see 2.22), which, by 2.23, can be identified with the finite cross-sections of some l-sheaf $\mathscr{F}'$ on $P'M\setminus P\approx Y$. Here, if $f\in S(P, \tau')$ and $m\in M$, then $(\pi(m)f)(p)=f(pm)=f({}^p m\cdot p)=\Theta({}^p m)f(p)=\Theta^p(m)\cdot f(p)$. Therefore, if we look at the sheaf $\mathscr{F}$ on $\hat M$ obtained from $\mathscr{F}'$ by attaching the zero stalk over $1\in\hat M$, then P acts on this sheaf, and an action of M is defined by $(*)$ in (1). Hence $\mathscr{F}$ corresponds to $\Phi^+(\tau)$. Looking at the stalk of this sheaf at Φ, we obtain a natural isomorphism $\tau\to\Phi^-\Phi^+(\tau)$. In particular, to each morphism $\Phi^+(\tau)\to\pi$ there corresponds a morphism

$\tau \to \Phi^-\Phi^+(\tau) \to \Phi^-(\pi)$.

(4) It is easy to verify that the morphisms constructed in (2) and (3) are inverses of one another. They also specify that Φ^+ and Φ^- are conjugate. It can be established directly that Ψ^- and Ψ^+ are conjugate.

c) Since $\Psi^+(\rho)_{|M}$ is the identity, $\Phi^-\Psi^+(\rho) = 0$. As shown in the proof of b), the stalk of the sheaf $\mathcal{F}$ corresponding to $\Phi^+(\tau)$ is equal to 0 at $1 \in M$. But, by 5.10, this means that $\Psi^-\Phi^+(\tau) = 0$.

d) That i' is an isomorphism has been verified in the proof of b) in (3). It is obvious that j' is an isomorphism. As shown in the proof of b), the morphism i: $\Phi^-\Phi^+(\pi) \to \pi$ coincides with the embedding $\pi^0 \to \pi$, where $\pi^0 = \pi|_{E(M,1)}$. It is clear that $\pi/\pi^0 \approx \Psi^+\Psi^-(\pi)$, where the isomorphism is defined by j.

e) The condition that $\pi_{|M}$ is the identity is equivalent to the fact that j: $\pi \to \Psi^+\Psi^-(\pi)$ is an isomorphism. Since $\Phi^+(\tau) \neq 0$ for $\tau \neq 0$, it follows from d) that this is equivalent to the condition $\Phi^-(\pi) = 0$.

f) If $\pi = \Phi^+(\tau)$, then for any subrepresentation $\pi' \subset \pi$ we have $\Psi^-(\pi') = 0$, so that $\Phi^+\Phi^-(\pi') = \pi'$. Similarly for representations of the form $\Psi^+(\rho)$. Therefore, f) follows from d).

g) It is easy to see that $(\tau^0_{P'})' = \operatorname{ind}(P'M, U, \Theta)$. Therefore, $\Phi^+(\tau^0_{P'}) = \operatorname{ind}(P, P'M, (\tau^0_{P'})') = \operatorname{ind}(P, U, \Theta) = \tau^0_P$, because inducement is transitive (see 2.25b)). It now follows from d) that $\Phi^-(\tau^0_P) = \tau^0_{P'}$.

5.13. COROLLARY. a) *Let* $1 \leqslant k < n$ *and suppose that* $\rho \in \operatorname{Alg} G_k$ *is irreducible. Then the representation* $(\Phi^+)^{n-k-1}\Psi^+(\rho) \in \operatorname{Alg} P_n$ *is irreducible.*[1] *In particular,* $\tau^0_P = (\Phi^+)^{n-1}\Psi^+(1)$ *is an irreducible representation.*

b) *Any irreducible representation* $\pi \in \operatorname{Alg} P_n$ *is isomorphic to one of the representations constructed in* a), *and* k *and the isomorphism class of* ρ *are defined by* π.

PROOF. a) follows from 5.12f) and 5.12g). The existence of k and ρ for π in b) is proved by induction on n because, by 5.12d) and f), either $\pi = \Phi^+(\tau)$, where $\tau \in \operatorname{Alg} P'$ is irreducible, or $\pi = \Psi^+(\rho)$, where $\rho \in \operatorname{Alg} G'$ is irreducible. The uniqueness of k and ρ follows from the fact that $\Psi^-(\Phi^-)^{n-i-1}(\pi) = 0$ for $i \neq k$ and $= \rho$ for $i = k$.

5.14. COROLLARY. *Let* $(\pi, P, E) \in \operatorname{Alg} P$, $E \neq 0$. *Then there is a character* θ *of* U *such that* $E_{U,\theta} \neq 0$.

PROOF. By 5.12d), either $\Phi^-(\pi) \neq 0$ or $\Psi^-(\pi) \neq 0$. If $E' \neq 0$ is the space of the corresponding representation, then, using induction, we can find a character θ' of U' such that $E'_{U',\theta} \neq 0$. We continue θ' to a character θ on U, by setting $\theta_{|M} = \Theta$ in the first case and $\theta_{|M} = 1$ in the second. Then $E_{U,\theta} = E'_{U',\theta} \neq 0$, by 2.32.

5.15. Let $(\pi, P, E) \in \operatorname{Alg} P$. We set $\pi_i = (\Phi^+)^i(\Phi^-)(\pi)$. Using 5.13, we can easily construct embeddings $\pi_{n-1} \subset \pi_{n-2} \subset \ldots \subset \pi_1 \subset \pi_0 = \pi$. Thus,

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[1] By $(\Phi^+)^m$ we mean the functor composition $\operatorname{Alg} P_l \xrightarrow{\Phi^+} \operatorname{Alg} P_{l+1} \xrightarrow{\Phi^+} \ldots \xrightarrow{\Phi^+} \operatorname{Alg} P_{l+m}$.

with each π there is connected a filtration by P-submodules, depending functorially on π.

We are mainly interested in the submodule $\pi_{n-1} \subset \pi$, which is the "completely non-degenerate" part of π. We denote it by π^{nd}.

It is easy to deduce from 2.32 that the representation $(\Phi^-)^{n-1}(\pi)$ of $P_1 = \{e\}$ is realized in $E_{U,\Theta}$. In particular, the condition for π to be non-degenerate is equivalent to the fact that $(\Phi^-)^{n-1}(\pi) \neq 0$ or that $\pi^{nd} \neq 0$. Here π^{nd} is isomorphic to the direct sum of $\dim E_{U,\Theta}$ copies of the irreducible representation $\tau_P^0 (= (\Phi^+)^{n-1}(1))$. Furthermore, it is clear that $(\Phi^-)^{n-1}(\pi/\pi^{nd}) = (\Phi^-)^{n-1}(\pi)/(\Phi^-)^{n-1}(\pi^{nd}) = 0$; that is, π/π^{nd} is a degenerate representation.

Thus, the irreducible representation τ_P^0 of P is "semi finite": a finite irreducible representation always splits off as a direct summand (see 2.44; it can be shown that this property is equivalent to finiteness); but, in general τ_P^0 is only a submodule.

PROPOSITION. *Let* $(\pi, P, E) \in \operatorname{Alg} P$.

a) *The following conditions are equivalent*:

(1) $\pi = \pi^{nd}$.

(2) *For any non-trivial horospherical subgroup* U_β (see 3.5), $E_{U_\beta} = 0$.

(3) *For any degenerate character* θ *of* U, $E_{U,\theta} = 0$.

b) *Any P-homomorphism* $\tau_P^0 \to \tau_P$ *is proportional to the standard embedding.*

c) *If* $\pi \subset \tau_P$ *and* $\pi \neq 0$, *then* $\pi^{nd} = \tau_P^0$. *In particular,* $\tau_P^0 \subset \pi$ *and* π/τ_P^0 *is degenerate.*

PROOF. a) (1) $\Rightarrow$ (2). We may assume that $\pi = \tau_P^0$. In addition, we may assume that β has the form $(k, n-k)$. If $k = n-1$, then $U_\beta = M$ and $E_{U_\beta} = 0$, since $\Psi^{-1}(\pi) = 0$. Let $k < n-1$ and let $\beta' = (k, n-k-1)$ be a partition of $n-1$. Let P^0 be the subgroup of P consisting of the matrices $p = (p_{ij})$ for which $p_{ij} = \delta_{ij}$ if $i \leqslant k$ or $j \leqslant k$. P^0 can be naturally identified with P_{n-k}. We consider the representation τ of P^0 in E_{U_β}. We then have $(E_{U_\beta})_{M^0,\Theta} = (E_{M_\theta})_{U_{\beta'}} = 0$ by the inductive hypothesis, because $\Phi^-(\pi) = \tau_{P'}^0$. Furthermore, $(E_{U_\beta})_{M^0} = (E_M)_{U_{\beta'}} = 0$, because $E_M = 0$. Hence $\Phi^-(\tau) = 0$ and $\Psi^-(\tau) = 0$.

By 5.12d), $\tau = 0$, that is, $E_{U_\beta} = 0$.

(2) $\Rightarrow$ (3). It is clear that for any degenerate character θ of U there is a partition β such that $U_\beta \neq \{e\}$ and $\theta|_{U_\beta} \equiv 1$. Therefore, from $E_{U_\beta} = 0$ we obtain $E_{U,\theta} = 0$.

(3) $\Rightarrow$ (1). Let $(\pi', P, E') = \pi/\pi^{nd}$. Then $F'_{U,\Theta} = 0$, and hence, by 5.6, $E'_{U,\theta} = 0$ for all non-degenerate characters θ. On the other hand, by hypothesis, $E'_{U,\theta} = 0$ for all degenerate characters θ (we use the fact that the functor $E \to E_{U,\theta}$ is exact). Therefore, it follows from 5.14 that $E' = 0$.

b) Let E be the representation space of τ_P^0. By Frobenius duality and the definition of τ_P, Hom $(\tau_P^0, \tau_P) = (E_{U,\Theta})^*$. But dim $E_{U,\Theta} = \dim (\Phi^-)^{n-1}(\tau_P^0) = 1$ by 5.12g). It follows that Hom $(\tau_P^0, \tau_P) \approx \mathbf{C}$.

c) By definition, π is non-degenerate, that is, $\pi^{nd} \neq 0$. Now π^{nd} is isomorphic to the direct sum of the τ_P^0. It follows directly from b) that $\pi^{nd} = \tau_P^0$.

WHITTAKER'S MODEL

5.16. THEOREM. *Let $G = G_n$ and let (π, G, E) be a non-degenerate, irreducible, admissible representation. Then* dim $E_{U,\theta} = 1$ *for any non-degenerate character θ of U.*

This theorem will be proved in §7.

5.17. COROLLARY. *Under the conditions of* 5.16, *there exists a unique τ-invariant subspace $W_\pi \subset L(G, \Theta)$ such that $\tau|_{W_\pi}$ is equivalent to π* (see 2.28 and 2.30). *We call this realization of π Whittaker's model* (see [23], Theorem 2.14).

KIRILLOV'S MODEL

5.18. Let (π, G, E) be a quasi-cuspidal representation. Then π is non-degenerate and $\pi|_P$ is isomorphic to the direct sum of the τ_P^0. If π is cuspidal and irreducible, then $\pi|_P = \tau_P^0$.

The proof follows from Proposition 5.15a) and Theorem 5.16.

5.19. COROLLARY. *Let (π, G, E) be an irreducible cuspidal representation. Then there exists a unique realization π in $S(P, \Theta)$ under which P acts by right translations. It is called Kirillov's model of π* (see [23], Theorem 2.13).

5.20. The concept of Kirillov's model can be generalized for non-degenerate, admissible, irreducible representations of G; that is, they can be realized in function spaces "smaller" than $L(G, \Theta)$. Let us make the question more precise.

Suppose that (π, G, E) satisfies the conditions of Theorem 5.16. It follows from 2.28 and 5.16 that there exists one and, up to a factor, only one non-zero operator $A_\pi \colon E \to L(P, \Theta)$ that commutes with the action of P.

Let us define A_π more explicitly: if $W_\pi \subset L(G, \Theta)$ is Whittaker's model of π (see 5.17), then A_π associates with each function in W_π its restriction to P.

THEOREM. *A_π is an embedding.*

This theorem can be stated differently. Let $\pi^{nd} \subset \pi|_P$ be the representation constructed in 5.15. It follows from 5.15 and 5.16 that $\pi^{nd} \approx \tau_P^0$ and that $A_\pi|_{\pi^{nd}}$ is a non-zero multiple of the standard embedding. Hence it is clear that Ker A_π can be characterized as maximal degenerate subrepresentation of $\pi|_P$, and the theorem can be restated as follows: all non-zero subrepresentations of the restriction of π to P are non-degenerate.

The theorem was stated by Gel'fand and Kazhdan as a conjecture in [13] and [14]; A proof by the authors of the present article will be published soon.

We leave to the reader as a useful exercise the verification of the theorem for $n = 2$.

G_n AND P_n ARE MODULES OF FINITE LENGTH

5.21. THEOREM (Kazhdan). *Let $G = G_n$, $(\pi, G, E) \in$ Alg G an irreducible representation, and θ an arbitrary character of U. Then* dim $E_{U,\theta} \leqslant n!$.

This theorem will be proved in §7, but we present some of its corollaries here.

5.22. COROLLARY. *Let $(\pi, G, E) \in$ Alg G. Then the following conditions are equivalent*:

(1) *π has finite length.*

(2) *$\pi|_P$ has finite length.*

(3) dim $E_{U,\theta} < \infty$ *for all characters θ of U.*

PROOF. It follows from 5.21 and 2.35 that (1) $\Rightarrow$ (3), and it is trivial that (2) $\Rightarrow$ (1).

Let us prove that (3) $\Rightarrow$ (2). It is easy to see that each character of U is conjugate to $\theta(a_1, \ldots, a_{n-1})$ under the action of P, where all the a_i are equal to 0 or 1 (see 5.5). We denote the set of these 2^{n-1} characters by Ω.

We set

$$d(\pi) = \sum_{\theta \in \Omega} \dim E_{U,\theta} < \infty.$$

If

$$0 = E_0 \subsetneqq E_1 \subsetneqq \ldots \subsetneqq E_k = E$$

is a chain of P-submodules and π_i is a representation of P in E_i/E_{i-1}, then by 2.35 $d(\pi) = \sum_{i=1}^{k} d(\pi_i)$ and by 5.14 $d(\pi_i) \geqslant 1$. Hence $k \leqslant d(\pi)$, which proves (2).

5.23. COROLLARY. *Let $(\pi, P, E) \in$ Alg P. Then the following conditions are equivalent*:

(1) *π has finite length.*

(2) dim $E_{U,\theta} < \infty$ *for all characters θ of U.*

PROOF. The fact that (2) $\Rightarrow$ (1) is established by the arguments in 5.22. Let us prove that (1) $\Rightarrow$ (2).

We may assume that π is irreducible and that $\theta \in \Omega$. Then

$E_{U,\theta} = (E_{M,\theta})_{U',\theta}$. Since $\theta|_M = 1$ or Θ, it suffices to verify the assertion for the irreducible representations $\Phi^-(\pi) \in \mathrm{Alg}\, P'$ and $\Psi^-(\pi) \in \mathrm{Alg}\, G'$. For $\Psi^-(\pi)$ it is proved in 5.22, and for $\Phi^-(\pi)$ it is proved by induction.

REPRESENTATIONS OF $G_2 = GL(2, F)$

5.24. In conclusion, we apply the results of this section to the case $n = 2$.

PROPOSITION. a) *P_2 has a unique irreducible non-degenerate representation* τ_P^0 (see 5.7). *All degenerate irreducible algebraic representations of this group are one-dimensional.*

b) *If (π, G_2, E) is an irreducible non-degenerate representation and $A_\pi: E \to L(P_2, \Theta)$ is the operator constructed in* 5.20, *then* $A_\pi(E) \supset S(P_2, \Theta)$ *and* $\dim A_\pi(E)/S(P_2, \Theta) \leqslant 2$.

PROOF. a) follows from 5.13.

b) It follows from 5.15 that $A_\pi(E) \supset S(P_2, \Theta)$ and from 5.16 and 5.21 that the number $d(\pi)$ constructed in 5.22 does not exceed 3. It is clear from the arguments in 5.22 that the length of the composition series of the restriction of π to P_2 does not exceed 3. But all the composition factors except τ_P^0 are one-dimensional, by a). Hence $\dim A_\pi(E)/S(P_2, \Theta) \leqslant 2$, as required.

§6. Theorems on invariant distributions

To prove Theorems 5.16 and 5.21 we need some assertions on invariant distributions. We state them in general form for arbitrary l-groups and l-sheaves.

GROUP ACTIONS ON TOPOLOGICAL SPACES

6.1. We study certain types of actions of l-groups on l-spaces.

Let G be an l-group, X an l-space, and $\gamma: G \times X \to X$ an action of G on X (see 1.5). We define the *graph* of γ as the subset $R_X^\gamma = \{(x, \gamma(g)x) | x \in X, g \in G\} \subset X \times X$. The subset $\gamma(G)x \subset X$ is called the *orbit* of $x \in X$. We denote by X/G the set of all G-orbits in X and by $p: X \to X/G$ the natural projection. We always assume that X/G is equipped with the quotient topology ($U \subset X/G$ is open if and only if $p^{-1}(U)$ is open in X). Clearly p is continuous in this topology. Note that X/G is not necessarily Hausdorff.

6.2. LEMMA. a) *p is an open mapping; that is, if U is open in X, then $p(U)$ is open in X/G.*

b) *If M is a G-invariant closed subset of X, then $p(M)$ is closed in X/G.*

c) *If M is a G-invariant locally closed subset of X, then $p(M)$ is locally closed in X/G.*

PROOF. a) $p^{-1}p(U) = \bigcup_{g \in G} \gamma(g)U$ is open in X.

b) Obvious.

c) Let $M = U \cap F$, where U is open and F is closed. Then $M = U \cap \bar{M}$. Since $\bar{M}$ is G-invariant, $p(M) = p(U) \cap p(\bar{M})$ is locally closed by virtue of a) and b).

6.3. DEFINITION. An action γ: $G \times X \to X$ of an l-group G in an l-space X is called *regular* if the graph R_X^γ is closed in $X \times X$ (see [31], Part II, Ch. III, §12).

6.4. LEMMA. *The following conditions are equivalent*:

1) γ *is regular.*

2) *The diagonal* $\Delta = \{(\bar{x}, \bar{x})\} \subset X/G \times X/G$ *is closed.*

3) X/G *is Hausdorff.*

Under these conditions X/G is an l-space.

PROOF. It is clear that 2) is equivalent to 3). To prove that 1) and 2) are equivalent, we consider the action γ^2 of $G \times G$ on $X \times X$, which is defined by $\gamma^2((g_1, g_2))(x_1, x_2) = (\gamma(g_1)x_1, \gamma(g_2)x_2)$. It is easy to see that $(X \times X)/(G \times G)$ is homeomorphic to $X/G \times X/G$, and under the projection p^2: $X \times X \to X/G \times X/G$ we have $(p^2)^{-1}(\Delta) = R_X^\gamma$. It now follows from 6.2b) that 1) and 2) are equivalent.

Since p is open and carries compact sets into compact sets, if X/G is Hausdorff, it is an l-space.

6.5. COROLLARY. *If G is an l-group and H is a closed subgroup of G, then $H \setminus G$, equipped with the quotient topology, is an l-space, and the projection p: $G \to H \setminus G$ is open.*

PROOF. The action γ: $H \times G \to G$ defined by $\gamma(h)g = hg$ is regular, because the graph $R_G^\gamma = \{(g_1, g_2) \in G \times G \mid g_2 g_1^{-1} \in H\}$ is closed in $G \times G$.

6.6. DEFINITION. a) A subset M of a topological space Y (here Y is not necessarily Hausdorff) is called *constructive* if it is the union of finitely many locally closed subsets.

b) An action γ: $G \times X \to X$ is called *constructive* if its graph R_X^γ is constructive in $X \times X$.

6.7. Let M be a subset of an arbitrary topological space Y. We set $U(M) = \{ y \in M \mid M$ is closed in a neighbourhood of $y \}$ and $M^1 = M \setminus U(M)$. It is clear that $U(M)$ is locally closed and that M^1 is closed in M. We now set $M^2 = (M^1)^1$, $M^3 = (M^2)^1$, $\ldots$, $M^{k+1} = (M^k)^1$.

LEMMA. *Suppose that M is constructive. Then*:

a) $M^k = \varnothing$ *starting with some k. In particular,* $M = U(M) \cup U(M^1) \cup \ldots \cup U(M^{k-1})$.

b) $U(M)$ *is dense in* M.

PROOF. a) Clearly M^k is closed in M and $(M \cap V)^k = M^k \cap V$ for any open set V. Let $M = \bigcup_{i=1}^{l} S_i$, where S_i is locally closed. We shall prove by induction on l that $M^l = \emptyset$.

We set $V_k = Y \setminus \bar{S}_k$. Then $M \cap V_k = \bigcup_{i \neq k} (S_i \cap V_k)$, and by the inductive hypothesis, $(M \cap V_k)^{l-1} = M^{l-1} \cap V_k = \emptyset$, that is, $M^{l-1} \subset \bar{S}_k$. Furthermore, S_k is open in $\bar{S}_k$, so that $S_k \cap M^{l-1}$ is open in M^{l-1}; M^{l-1} is closed in M, so that $S_k \cap M^{l-1}$ is closed in $S_k = S_k \cap M$. Therefore, $U(M^{l-1}) \subset S_k \cap M^{l-1}$, that is, $S_k \cap M^l = \emptyset$. Since this is true for all k and $M^l \subset M = \bigcup S_k$, $M^l = \emptyset$.

b) follows from a), since it is easy to see that $\overline{U(M)} \supset \overline{U(M^1)} \supset \ldots$.

6.8. PROPOSITION. *Suppose that an action γ: $G \times X \to X$ is defined.*

a) *The condition that γ is constructive is equivalent to the fact that the diagonal $\Delta \subset X/G \times X/G$ is constructive.*

b) *If γ is constructive and X is not empty, then there is a G-invariant non-empty open subset $U \subset X$ on which γ is regular.*

c) *If γ is constructive, then all its orbits are locally closed.*

PROOF. a) If M is a G-invariant constructive subset of X, then all subsets $U(M^i)$ in the decomposition $M = U(M) \cup U(M^1) \cup \ldots \cup U(M^k)$ are locally closed and G-invariant. Therefore, by 6.2c), $p(M)$ is constructive in X/G. The subsequent reasoning is the same as in 6.4.

b) Let $(\bar{x}, \bar{x})$ be a point of $\Delta \subset X/G \times X/G$ in a neighbourhood of which Δ is closed. Its existence follows from a) and 6.7b). If U is an open neighbourhood of $\bar{x}$ in X/G such that Δ is closed in $U \times U$, then, as follows from 6.4, the action of G on $p^{-1}(U)$ is regular.

c) If S is an orbit, then it is constructive, that is, $U(S) \neq \emptyset$. By virtue of G-invariance, $S = U(S)$ is locally closed.

THEOREMS ON INVARIANT DISTRIBUTIONS

6.9. THEOREM. *Suppose that an action γ of an l-group G on an l-sheaf $(X, \mathcal{F})$ is defined. We assume that:*

a) *the action of G on X is constructive.*

b) *there are no non-zero G-invariant $\mathcal{F}$-distributions on any G-orbit in X (this makes sense, since by* 6.8c) *all orbits are locally closed).*

Then there are no non-zero G-invariant $\mathcal{F}$-distributions on X.

PROOF. (1) If the action of G on X is regular, then our assertion follows from 2.37 and 6.4. We reduce the case of a constructive action to the regular case.

(2) Let T be a non-zero G-invariant $\mathcal{F}$-distribution on X. Replacing X by supp T, we may assume that $X = \operatorname{supp} T$. By 6.8b), there exists a non-empty open G-invariant subset $Y \subset X$ on which γ is regular. By virtue of (1), $i_Y^*(T) = 0$ (see 1.9 and 1.15). But this contradicts the fact that $X = \operatorname{supp} T$. The theorem is now proved.

6.10. THEOREM (Gel'fand and Kazhdan). *Let γ be an action of an l-group G on an l-sheaf $(X, \mathcal{F})$ and σ: $(X, \mathcal{F}) \to (X, \mathcal{F})$ an isomorphism.*

We assume that:

a) *the action of G on X is constructive.*

b) *for each* $g \in G$ *a* $g^\sigma \in G$ *can be found such that* $\gamma(g) \circ \sigma = \sigma \circ \gamma(g^\sigma)$. *In particular,* σ *rearranges G-orbits in X.*

c) *there exist a natural number n and a* $g_0 \in G$ *such that* $\sigma^n = \gamma(g_0)$.

d) *if there exists a non-zero G-invariant* $\mathscr{F}$*-distribution T on a G-orbit S, then* $\sigma S = S$ *and* $\sigma T = T$.

Then each G-invariant $\mathscr{F}$*-distribution on X is* σ*-invariant.*

PROOF. We prove the theorem by contradiction. Let T be a G-invariant $\mathscr{F}$-distribution on X such that $\sigma T \neq T$. For each n-th root ζ of 1 we set $T_\zeta = \sum_{i=0}^{n-1} \zeta^{-i} \cdot \sigma^i(T)$. Then $\sigma T_\zeta = \zeta \cdot T_\zeta$, since $\sigma^n(T) = \gamma(g_0)T = T$. It is easy to see that $\sum_\zeta T_\zeta = nT$ and that $\sum_\zeta \zeta T_\zeta = n \cdot \sigma T$. It follows that $\sum_\zeta (\zeta - 1)T_\zeta = n(\sigma T - T) \neq 0$; that is, there exists a root $\zeta \neq 1$ such that $T_\zeta \neq 0$. By b), T_ζ is G-invariant.

We consider the isomorphism σ_ζ of $(X, \mathscr{F})$ that acts on X exactly as σ and on the cross-sections of $\mathscr{F}$ by the formula $\sigma_\zeta(\varphi) = \zeta \cdot \sigma(\varphi)$. It is clear that T_ζ is invariant under σ_ζ.

Let G' be the group of isomorphisms of $(X, \mathscr{F})$ generated by $\gamma(G)$ and σ_ζ (we assume that it is discrete), and let γ' be the natural action of G' on $(X, \mathscr{F})$. Clearly $R_X^{\gamma'} = \bigcup_{i=0}^{n-1} \sigma_1^i R_X^\gamma$, where σ_i: $X \times X \to X \times X$ is the homeomorphism defined by $\sigma_1(x_1, x_2) = (\sigma x_1, x_2)$. Therefore, the action γ': $G' \times X \to X$ is constructive.

Let S' be a G'-orbit in X. We claim that there are no non-zero G'-invariant $\mathscr{F}$-distributions on S'. For if there are no non-zero G-invariant $\mathscr{F}$-distributions on any G-orbit $S \subset S'$, then by 6.9 there are also none on S'. But if such a distribution exists on some G-orbit S, then by virtue of condition d) of the theorem, $\sigma S = S$, so that $S = S'$. But it then follows from d) that there are no G'-invariant $\mathscr{F}$-distributions on S'.

Since T_ζ is a non-zero G'-invariant $\mathscr{F}$-distribution on X, we obtain a contradiction to 6.9, and the theorem is proved.

VERIFICATION OF THE CONDITIONS OF THEOREM 6.10

6.11. In the remainder of §6 we are concerned with the problem of verifying that the conditions of Theorem 6.10 hold.

In verifying d), it is useful to use Frobenius duality. Let G be an l-group that is countable at infinity, and let γ be an action of G on the l-sheaf $(X, \mathscr{F})$ such that $(\gamma, G, \mathscr{F}_c)$ is algebraic. Let S be a G-orbit in X, $s \in S$, H the stationary subgroup of s, and $(\rho, H, \mathscr{F}_s)$ the representation of H in the stalk of $\mathscr{F}$ over s. By 1.6, $S = H \backslash G$, and it follows from

2.23 and 2.29 that the space of G-invariant $\mathscr{F}$-distributions on S is isomorphic to $\mathrm{Hom}_H(\Delta_H/\Delta_G \cdot \rho, 1_H)$.

6.12. EXAMPLE. Suppose that $\mathscr{F}_s$ is one-dimensional. Then under condition d) we have to verify that if $\sigma S \neq S$, then $\rho \neq \Delta_G/\Delta_H$. This is true, for example, if $\rho(h)$ is not positive for some $h \in H$. If $\sigma S = S$ and $\rho = \Delta_G/\Delta_H$, that is, if non-zero G-invariant $\mathscr{F}$-distributions exist on S, then all of them are proportional to T_0, which is defined by

$\langle T_0, \varphi\rangle = \int\limits_{H\setminus G} (\gamma(g)\varphi)(s)\, d\nu_{H\setminus G}(g)$ (see 1.21 and 2.29), and it suffices to verify that $\sigma T_0 = T_0$. It follows from 6.10b) that σT_0 is G-invariant, that is, $\sigma T_0 = c \cdot T_0$, and it suffices to verify that $c = 1$. This equality holds, for example, when $\sigma s = s$ and σ acts in $\mathscr{F}_s$ as multiplication by a positive number. In fact, since T_0 is positive, the coefficient c in this case is positive, and from 6.10c) we obtain $c^n = 1$ and hence $c = 1$.

Taking the l-sheaf $C^\infty(X)$ (see 1.14) for $\mathscr{F}$ we obtain the following theorem:

6.13. THEOREM. *Suppose that an action γ of an l-group G, countable at infinity, on an l-space X and a homeomorphism σ: $X \to X$ are defined. We assume that:*

a) *γ is constructive.*

b) *for each $g \in G$ there is a $g^\sigma \in G$ such that $\gamma(g)\sigma = \sigma\gamma(g^\sigma)$.*

c) *for some natural number n and $g_0 \in G$, $\sigma^n = \gamma(g_0)$.*

d) *σ carries each G-orbit into itself.*

Then each G-invariant distribution on X is invariant under σ.

6.14. Let us now show how to verify that an action γ: $G \times X \to X$ of an l-group G on an l-space X is constructive. We state a convenient reduction lemma.

LEMMA. *Let X and Y be l-spaces, γ: $G \times X \to X$ and γ': $G \times Y \to Y$ actions, and q: $Y \to X$ a continuous open surjection, where $q(\gamma'(g)y) = \gamma(g)q(y)$ for all $y \in Y$ and $g \in G$. We assume that for any $x \in X$ the set $q^{-1}(x)$ lies on a G-orbit in Y. Then if γ' is constructive, so is γ.*

PROOF. It is clear that q induces a homeomorphism q': $Y/G \to X/G$. Therefore, the lemma follows from 6.8a).

6.15. Next we quote a general criterion for constructiveness, which together with Lemma 6.14 comprises all the cases of interest to us.

Let F be a local field (see 3.1), X an algebraic variety defined over F, and $X(F)$ the sets of its F-points. The locally compact topology of F induces a topology on $X(F)$ in which $X(F)$ is an l-space. If G is a linear algebraic F-group, then $G(F)$ is an l-group. If α: $G \times X \to X$ is an F-rational action, then the corresponding mapping α_F: $G(F) \times X(F) \to X(F)$ defines a continuous action of $G(F)$ on $X(F)$. In the Appendix we shall prove the following theorem:

THEOREM A. *The action α_F of the l-group $G(F)$ on the l-space $X(F)$ is constructive.*

6.16. EXAMPLE. Let G be the subgroup of $GL(n, F)$ distinguished by polynomial equations with coefficients in F. Let α be a linear representation of G in F^k whose matrix elements are rational functions on G with coefficients in F (the denominators of these functions do not vanish on G). Then the action α of G on F^k (and hence on any of its G-invariant locally closed subsets) is constructive.

§7. Proof of Theorems 5.16 and 5.21

REALIZATION OF THE CONTRAGRADIENT REPRESENTATION

7.1. In this section we denote by g' the transpose of a matrix $g \in G = G_n$. We define a matrix $s_n \in G$ and an automorphism $g \mapsto {}^s g$ of G by $(s_n)_{ij} = (-1)^i \cdot \delta_{i,n+1-j}$ and ${}^s g = s_n g'^{-1} s_n^{-1}$.

LEMMA. a) ${}^s U = U$ *and* $\Theta^s = \Theta$ (see 2.27 and 5.6).

b) $s_n' = s_n^{-1} = \alpha \cdot s_n$, *where* $\alpha = (-1)^{n+1} \in Z$ (see 3.1).

7.2. LEMMA. *If (π, G, E) is an irreducible admissible representation, then π^s* (see 2.27) *is also irreducible and admissible, where $\widetilde{\pi}^s = (\widetilde{\pi})^s$. The space $E(U, \Theta)$ does not depend on which of π or π^s we treat* (see 2.30).

7.3. THEOREM (Gel'fand and Kazhdan). *Under the conditions of* 7.2, *π^s and $\widetilde{\pi}$ are equivalent.*

PROOF. By virtue of 2.15, 2.20, and 7.2 it suffices to prove that $\operatorname{tr} \widetilde{\pi} = \operatorname{tr} \pi^s$.

For each $f \in S(G)$ we define $f^-, f^s \in S(G)$ by $f^s(g) = f({}^s g)$ and $f^-(g) = f(g^{-1})$. Then $\operatorname{tr} \pi^s(f) = \operatorname{tr} \int_G \pi^s(g) f(g) d\mu_G(g) = \operatorname{tr} \int_G \pi(g_1) f^s(g_1) d\mu_G(g_1)$ (we have substituted $g_1 = {}^s g$), that is, $\operatorname{tr} \pi^s(f) = \operatorname{tr} \pi(f^s)$.

On the other hand, it follows from 2.14b), 1.19c), and the fact that G is unimodular that $\widetilde{\pi}(f\mu_G) = (\pi(f^-\mu_G))^*|_{\widetilde{E}}$. Since $A = \pi(f^-\mu_G)$ has finite rank and $\operatorname{Im} A^* \subset \widetilde{E}$ (see 2.14a) and 2.17), we see that $\operatorname{tr} \widetilde{\pi}(f) = \operatorname{tr} \widetilde{\pi}(f\mu_G) = \operatorname{tr} A^*|_{\widetilde{E}} = \operatorname{tr} A^* = \operatorname{tr} A = \operatorname{tr} \pi(f^-)$.

Thus, we have to prove that $\operatorname{tr} \pi(f^-) = \operatorname{tr} \pi(f^s)$. Since $f^s(g) = f^-(s_n g' s_n^{-1})$, our assertion follows immediately from 2.18, the fact that G is unimodular, and the following fact: a distribution on G invariant under conjugation (that is, the action of G on itself by inner automorphisms) is invariant under transposition.

To prove this, we apply Theorem 6.13 in the following situation: $G = G_n$, $X = G$, $\gamma(g)x = gxg^{-1}$, and $\sigma(x) = x'$. Let us verify that the conditions of 6.13 hold.

a) That γ is constructive follows from 6.15 and 6.16.

b) We set $g^\sigma = g'^{-1}$. It is clear that $\gamma(g)\sigma = \sigma\gamma(g^\sigma)$.

c) $\sigma^2 = id$.

d) Transposed matrices are conjugate in G.

The theorem is now proved.

7.4. REMARK. It can be verified directly that the action of G on itself by inner automorphisms is constructive. We leave this to the reader.

7.5. COROLLARY. *Under the conditions of* 7.2, *if* π *is non-degenerate, then so is* $\widetilde{\pi}$.■

PROOF OF THEOREM 5.16

7.6. Let (π, G, E) be a non-degenerate irreducible admissible representation. By 5.6, it suffices to prove that $\dim E_{U,\Theta^{-1}} = 1$. Now
$(E_{U,\Theta^{-1}})^* = \mathrm{Hom}_G(\pi, \mathrm{Ind}(G, U, \Theta^{-1})) = \mathrm{Hom}_G(\widetilde{\pi, \mathrm{ind}(G, U, \Theta)}) =$
$= \mathrm{Hom}_G(\mathrm{ind}(G, U, \Theta), \widetilde{\pi})$ (see 2.30, 2.28, and 2.25c)).

Therefore, we need to prove that the space of operators $S(G, \Theta) \to \widetilde{E}$ that commute with the action of G is one-dimensional.

Let A be such an operator, $A \neq 0$. By Schur's lemma (see 2.11), A is defined uniquely up to a factor by its kernel $\mathrm{Ker}\, A \subset S(G, \Theta)$. We choose a non-zero operator $A' \in \mathrm{Hom}_G(\mathrm{ind}(G, U, \Theta), \pi)$ (this can be done by virtue of 7.5).

Theorem 5.16 follows immediately from the equality

$$(*) \qquad \mathrm{Ker}\, A = \{f \in S(G, \Theta) \mid f^s \in \mathrm{Ker}\, A'\},$$

which shows that the kernel of A, in fact, does not depend on A (here $f^s(g) = f({}^s g)$; see 7.1).

7.7. To prove $(*)$ we define a G-invariant bilinear form B on $S(G, \Theta)$ by $B(f, f') = \langle Af, A'f' \rangle$. Clearly $\mathrm{Ker}\, A$ coincides with the left kernel of B, that is, with the set of functions $f \in S(G, \Theta)$ such that $B(f, f') = 0$ for all $f' \in S(G, \Theta)$. Similarly, $\mathrm{Ker}\, A'$ coincides with the right kernel of B. Therefore, $(*)$ is an immediate consequence of the following lemma:

7.8. LEMMA. *Let B be an arbitrary G-invariant bilinear form on $S(G, \Theta)$. Then $B(f_1, f_2) = B(f_2^{\alpha s}, f_1^s)$ for any $f_1, f_2 \in S(G, \Theta)$, where $f_1^s(g) = f_1({}^s g)$ and $f_2^{\alpha s}(g) = f_2(\alpha \cdot {}^s g)$* (see 7.1).

PROOF. We restate the assertion of the lemma in order to apply Theorem 6.10.

It is easy to see that $S(G, \Theta) \otimes S(G, \Theta) = S(G \times G, \theta_0)$, where θ_0 is the character of $U \times U$ defined by $\theta_0((u_1, u_2)) = \Theta(u_1 \cdot u_2)$. By 2.23, $S(G \times G, \theta_0)$ is the space of finite cross-sections of the l-sheaf $(U \times U \backslash G \times G, \mathscr{F}^{\theta_0})$, and the bilinear form on $S(G, \Theta)$ defines a distribution on this sheaf. Thus, it remains to apply Theorem 6.10 in the following situation: $G = G_n$, $X = U \times U \backslash G \times G$, $\mathscr{F} = \mathscr{F}^{\theta_0}$, and the action of G and the homeomorphism σ on X are induced by the action of G and σ on $G \times G$ defined by $\gamma(g)\,(g_1, g_2) = (g_1 g^{-1}, g_2 g^{-1})$ and

$\sigma(g_1, g_2) = (\alpha^s g_2, {}^s g_1)$. The action of G and σ on $\mathscr{F}$ is defined by

$$(\gamma(g)f)(g_1, g_2) = f(\gamma^{-1}(g)(g_1, g_2)) = f(g_1 g, g_2 g),$$
$$(\sigma f)(g_1, g_2) = f(\sigma^{-1}(g_1, g_2)) = f({}^s g_2, \alpha.{}^s g_1),$$
$$(f \in S(G \times G, \theta_0) = \mathscr{F}_c).$$

Let us verify that the conditions of Theorem 6.10 hold.

a) We set $Y = G \times G$ and define an action γ' of $G' = U \times U \times G$ on Y by $\gamma'(u_1, u_2, g)(g_1, g_2) = (u_1 g_1 g^{-1}, u_2 g_2 g^{-1})$. This action is constructive, by 6.15. The action of G on X is constructive by the reduction lemma 6.14 applied to the natural projection of Y onto X.

b) We set $g^\sigma = {}^s g$. Then $\gamma(g)\sigma = \sigma\gamma(g^\sigma)$.

c) $\sigma^4 = id$.

7.9. We now verify 6.10d).

(1) It follows from the Bruhat decomposition (see 3.2) that the G-orbits in X can be indexed by the elements of $D \times W$; namely on each G-orbit in X there is exactly one point of the form $x_{(d,w)} = p((dw, e))$, where $p\colon G \times G \to X$ is the natural projection, $d \in D$, $w \in W$, and e is the unit element of G.

(2) Let us see for what d and w on the appropriate orbit there exists a non-zero G-invariant distribution.

It is easy to see that the stability subgroup of $x_{(d,w)}$ is $H = U \cap w^{-1} Uw$, that the stalk of $\mathscr{F}_x$ over this point is one-dimensional, and that $(\rho, H, \mathscr{F}_x)$ in 6.11 is defined by $\rho(h) = [\Theta^{dw} \cdot \Theta](h)$. Since $\Delta_G / \Delta_H = 1$ (see 1.19 and 3.6), the arguments in 6.12 show that there exists a non-zero G-invariant distribution on an orbit if and only if $\rho(h) = 1$.

Let $w_{ij} = \delta_{i,\omega(j)}$ and $d_{ij} = d_i \cdot \delta_{ij}$ where ω is a permutation. It is easy to see that $H = U \cap w^{-1} Uw = \{ h \in G \,|\, h_{ii} = 1,\ h_{ij} = 0$ for $i > j$ or $\omega(i) > \omega(j)\}$ and that $\rho(h) = \psi_0(\sum_{i=1}^{n-1} h_{i,i+1} + \sum_{i=1}^{n-1} d_i \cdot d_{i+1}^{-1} h_{\omega^{-1}(i),\omega^{-1}(i+1)})$.

For $\rho(h) \equiv 1$ to hold, the following condition is necessary: if $\omega(i) < \omega(i+1)$, that is, there exist elements $h \in H$ such that $h_{i,i+1} \neq 0$, then the term $h_{i,i+1}$ in the last sum must be cancelled by one of the terms of the form $d_j \cdot d_{j+1}^{-1} \cdot h_{\omega^{-1}(j),\, \omega^{-1}(j+1)}$; that is, the following equalities must hold:

$$(**) \qquad \omega(i+1) = \omega(i) + 1; \quad d_{\omega(i+1)} = -d_{\omega(i)}.$$

Thus, if $\rho(h) \equiv 1$, then ω satisfies the following condition: if $\omega(i) < \omega(i+1)$, then $\omega(i+1) = \omega(i) + 1$. It is easy to describe all such permutations: they are defined by collections of integers $0 = t_0 < t_1 < \ldots < t_s = n$, where the permutation ω corresponding to

$(t_0, \ldots, t_s)$ acts on each segment $I_k = [t_k + 1, t_{k+1}]$ simply by a translation, reversing the order of succession of these segments. The second equality in (**) shows that if i_1 and i_2 lie in the same segment I_k, then $(-1)^{\omega(i_1)}d_{\omega(i_1)} = (-1)^{\omega(i_2)}d_{\omega(i_2)}$.

(3) Let us now show that all the orbits constructed in (2) are σ-invariant. We have $\sigma x_{(d,w)} = \sigma(p(dw, e)) = p(\alpha, d^s w^s) = \gamma((d^s w^s)^{-1})$ $p(\alpha(d^s w^s)^{-1}, e)$. It is easy to see that $\alpha \cdot (d^s w^s)^{-1}$ belongs to $D \times W$. Therefore, for the orbit of $x_{(d,w)}$ to be σ-invariant, it is necessary and sufficient that $\alpha(d^s w^s)^{-1} = dw$. A direct verification shows that this holds for the elements d and w described in (2).

(4) It remains to show that all G-invariant distributions on the orbits in (2) are invariant under σ. But it follows from (3) that $x_{(d,w)}$ is invariant under the automorphism $\gamma(d^s w^s)\sigma$, and it is easy to see that this automorphism acts trivially in the stalk of $\mathscr{F}$ over this point. By 6.12, G-invariant distributions on the orbit of $x_{(d,w)}$ are invariant under $\gamma(d^s w^s)\sigma$ and are consequently σ-invariant. This completes the verification of 6.10d).

Theorem 5.16 is now proved in full.

7.10. REMARK. Using the Bruhat decomposition, we can verify directly that the action γ from 7.8 is constructive, without resorting to the "non-elementary" Theorem 6.15. We leave this to the reader.

PROOF OF THEOREM 5.21

7.11. Let $(\pi, G, E) \in \mathrm{Alg}\, G$ be an irreducible representation. By 3.27 there exist a partition $\beta = (n_1, \ldots, n_r)$ of n and an irreducible cuspidal representation (ρ, G_β, V) such that π is a subrepresentation of $(\tau, G, L) = i_{(n),\beta}(\rho)$. It follows from 2.35 that $\dim E_{U,\theta} \leqslant \dim L_{U,\theta}$ for all characters θ of $U = U_n$. Therefore, Theorem 5.21 follows from the following more precise theorem:

7.12. THEOREM. *Let $\beta = (n_1, \ldots, n_r)$ be a partition of n, (ρ, G_β, V) an irreducible cuspidal representation, and $(\pi, G, E) = i_{(n),\beta}(\rho) \in \mathrm{Alg}\, G$* (see 3.12). *Then* $\dim E_{U,\theta} \leqslant n!$ *for all characters θ of U. But if θ is non-degenerate* (see 5.6), *then* $\dim E_{U,\theta} = 1$.

PROOF. (1) By 2.23, π is realized in the space $\mathscr{F}_c$ of finite cross-sections of the l-sheaf $\mathscr{F} = \mathscr{F}^\rho$ on $X = P_\beta \backslash G$. It follows from the Bruhat decomposition (see 3.2) that the number of U-orbits in X does not exceed $n!$. Applying 1.5 several times, we can index the U-orbits in X so that each orbit X_i is open in $\bigcup_{j \geqslant i} X_j$.

Applying 2.35, 1.9, and 1.16 several times, we obtain $\dim E_{U,\theta} = \sum_i \dim \mathscr{F}_c(X_i)_{U,\theta}$. Therefore, it suffices to prove that $\dim \mathscr{F}_c(X_i)_{U,\theta} \leqslant 1$; if θ is non-degenerate, then $\dim \mathscr{F}_c(X_i)_{U,\theta} = 1$ for exactly one X_i. We need an explicit description of the orbits X_i.

(2) We denote by W_β the set of elements $w \in W$ such that the corresponding permutation w leaves all segments I_k of β fixed, and by $W^{(\beta)}$ the set of elements $w \in W$ such that ω^{-1} preserves the order of succession of the numbers in each I_k. It is clear that $W_\beta = W \cap P_\beta$ and that $W^{(\beta)}$ is a family of coset representatives of $W_\beta \backslash W$. It is clear from the Bruhat decomposition that there is a point of the form $P_\beta w$, $w \in W^{(\beta)}$, on each U-orbit in $P_\beta \backslash G$. For it can readily be shown that such a point is unique; that is, the U-orbits in $P_\beta \backslash G$ are indexed by the elements of $W^{(\beta)}$. But we do not even need this.

(3) We denote by X_w the U-orbit of $P_\beta \cdot w$, $w \in W^{(\beta)}$, and by π_w the representation of U in $\mathscr{F}_c(X_w)$. The stability subgroup of $P_\beta w \in X_w$ is $H_w = U \cap w^{-1} P_\beta w$, and the stalk over this point is identified with V, where (ρ_w, H_w, V) acts according to the formula $\rho_w(h) = \rho(whw^{-1})$. By 2.23, we have $\pi_w \cong \operatorname{ind}(U, H_w, \rho_w)$, hence $(\mathscr{F}_c(X_w)_{U,\theta})^* \cong \operatorname{Hom}_U(\pi_w, \theta) \cong \operatorname{Hom}_{H_w}(\rho_w, \theta) \cong \operatorname{Hom}_{wH_w}(\rho, \theta^{w^{-1}})$ (see 2.30, 2.29, and 2.27).

(4) Let us prove that ${}^wH_w \supset U \cap G_\beta$ (see 3.5) for all $w \in W^{(\beta)}$. It suffices to verify that $U \cap G_\beta \subset {}^wU$, but this follows immediately from the definition of $W^{(\beta)}$.

(5) By (4), $\operatorname{Hom}_{wH_w}(\rho, \theta^{w^{-1}}) \subset \operatorname{Hom}_{U \cap G_\beta}(\rho, \theta^{w^{-1}}) = (V_{U \cap G_\beta, \theta^{w^{-1}}})^*$. Now $U \cap G_\beta = \Pi\, U_{n_i}$ and $\theta^{w^{-1}}|_{U \cap G_\beta} = \Pi\, \theta_i$, where θ_i is a character of U_{n_i}. Recalling that ρ is irreducible and cuspidal, we see, exactly as in 5.18, that if at least one of the θ_i is degenerate, then $V_{\Pi U_{n_i}, \Pi \theta_i} = 0$. But if all the θ_i are non-degenerate, then $\dim V_{\Pi U_{n_i}, \Pi\theta_i} = 1$.

It follows from (3) that $\dim \mathscr{F}_c\ (X_w)_{U,\theta} \leqslant 1$; hence by (1) we obtain the first conclusion of the theorem.

(6) Now suppose that θ is non-degenerate. Since the restriction of ρ to U_β is trivial, the following assertion is valid: if there exists a $u \in H_w \cap {}^{w^{-1}}U_\beta$ such that $\theta(u) \neq 1$, then $\operatorname{Hom}_{H_w}(\rho_w, \theta) = 0$. It is easy to see that we can choose such a u if there exists an i such that $\omega(i) < \omega(i+1)$ and $\omega(i)$ and $\omega(i+1)$ lie in different segments of β. Therefore, if $\operatorname{Hom}_{H_w}(\rho_w, \theta) \neq 0$, then the following condition must hold: if $\omega(i) < \omega(i+1)$, then $\omega(i)$ and $\omega(i+1)$ lie in the same segment of β.

On the other hand, it can be readily verified that all the characters θ_i in (5) are non-degenerate if and only if ω satisfies the following condition: whenever i and $i+1$ lie in one segment of β, then $\omega^{-1}(i+1) = \omega^{-1}(i) + 1$. This means that ω^{-1} acts on each segment I_k of β simply by a translation.

Combining this condition with the preceding one, we find that there exists exactly one permutation ω_0 satisfying both of them; namely, ω_0^{-1} acts by translation on each I_k, reversing the order of their succession.

We have seen that $\mathscr{F}_c\ (X_w)_{U,\theta} = 0$ for $\omega \neq \omega_0$. On the other hand, it is easy to verify that ${}^{w_0}H_{w_0} = U \cap G_\beta$; that is, for $\omega = \omega_0$ the inclusion $\mathrm{Hom}_{wH_w}(\rho, \theta^{w-1}) \subset (V_{\Pi U_{n_i}, \Pi\theta_i})^*$ in (5) becomes an equality. Therefore, $\dim \mathscr{F}_c\,(X_{w_0})_{U,\theta} = 1$.

The remaining part of the theorem now follows from (1).

7.13. COROLLARY. *Under the conditions of* 7.12, *a P_n-module E has finite length not exceeding $2^{n-1}\cdot n!$. If $0 \subsetneqq E_1 \subsetneqq \ldots \subsetneqq E_l = E$ is a maximal chain of P_n-submodules, then exactly one of the modules E_i/E_{i+1} is non-degenerate* (see 2.8).

The proof is in 5.22.

APPENDIX

AN ALGEBRAIC THEOREM

In this Appendix we prove Theorem A of 6.15. We assume that the reader is familiar with the basic concepts of algebraic geometry to the extent of Borel's [3] chapter on algebraic geometry. In what follows, references to this chapter will be denoted by (AG, . . .).

A.1. Let F be a local field with a non-Archimedean valuation, and let $K = \bar{F}$ be its algebraic closure.

If X is an F-variety, then, following (AG), we identify X as a set with the set $X(K)$ of its K-points. We denote by $X(F)$ the set of F-points of X. If $\alpha\colon X \to Y$ is an F-morphism of F-varieties, then we denote the induced morphism $X(F) \to Y(F)$ by α_F.

We introduce a locally compact topology on $X(F)$ so that:

1) if U is an F-open subset of X, then $U(F)$ is open in $X(F)$;

2) if U is an affine F-variety, then the induced topology on $U(F)$ is the weakest in which the functions $f_F\colon U(F) \to F$ are continuous for all $f \in F[U]$ (on F we take the usual locally compact topology).

If X is a closed F-subvariety in an affine space A^n, then $X(F)$ is closed in $A^n(F) = F^n$, and the topology in $X(F)$ is induced by the usual topology of F^n.

It is clear that if $\alpha\colon X \to Y$ is an F-morphism, then $\alpha_F\colon X(F) \to Y(F)$ is a continuous mapping; moreover, $(X \times Y)(F)$ is naturally homeomorphic to $X(F) \times Y(F)$.

A.2. Theorem A from 6.15 is an immediate consequence of the following theorem:

THEOREM. *Let $\alpha\colon X \to Y$ be an F-morphism of F-varieties. Then $\alpha(X(F))$ is a constructive subset of $Y(F)$.*

A.3. The main tool for studying the topological structure of algebraic mappings is the following lemma:

LEMMA. *Let $\alpha\colon X \to Y$ be an F-morphism of smooth affine F-varieties. We assume that α is a coregular morphism (that is, the tangent mapping $(d\alpha)_x$ is surjective at all $x \in X$). Then $\alpha_F\colon X(F) \to Y(F)$ is an open mapping.*

PROOF. We embed X in an affine space A^n, Let $x \in X(F)$. Since x is a simple point of X, there exist polynomials $f_1, \ldots, f_k \in F[A^n]$ whose zeros define X in a neighbourhood of x, and the differentials $(df_i)_x$ are linearly independent. Similarly for Y. It follows that $X(F)$ and $Y(F)$ are analytic varieties over F and that α_F is a coregular morphism (for definitions and proofs see [31], PartII, Ch.III, §§10 and 11). By [31], Part II, Ch. III, §10, Theorem 2, α_F is an open mapping.

A.4. We need a criterion to determine when an open subset of an F-variety X is F-open. Let $F_s \subset K$ be the separable closure of F, and let Γ be the Galois group of F_s over F. The action of Γ on F_s can be uniquely extended to an action of Γ on $K = F_s^{p^{-\infty}}$ (p = char F). Now, exactly as in (AG, 14.3), we can define an action of Γ on K – the points of X.

LEMMA. a) *An open subset $U \subset X$ is F-open if and only if it is Γ-invariant (similarly for closed sets).*

b) *If $M \subset X(F)$, then the closed subvariety $\bar{M}$ (the closure of M in the K-topology) is defined over F.*

The lemma follows from (AG, 12.1, 14.3, 14.4).

A.5. LEMMA. *Let α: $X \to Y$ be an F-morphism of F-varieties. Then there is a non-empty F-open subvariety U in X such that $\alpha_F(U(F))$ is constructive in $Y(F)$.*

Let us show how Theorem A.2 follows from this lemma. We set $X_1 = X(F) \setminus U(F)$. This is an F-variety, by Lemma A.4b). Using Noetherian induction, we may assume that $\alpha_F(X_1(F))$ is constructive in $Y(F)$. But then $\alpha_F(X(F))$ is also constructive, as required.

A.6. In the proof of Lemma A.5 we may assume that $Y = \overline{\alpha(X)}$, and we can replace X and Y by arbitrary open F-subsets of them. Using this, we can reduce the proof to the case (*): X and Y are affine irreducible smooth F-varieties, $Y = \overline{\alpha(X)}$, and all stalks of α have the same dimension. This follows from (AG, 10.1, 17.2) and Lemma A.4a).

A.7. If under the conditions (*) α is separable, then by (AG, 17.3), X can be replaced by an F-open subset such that α becomes a coregular morphism. In this case Lemma A.5 follows from A.3.

If char $F = 0$, this completes the proof.

A.8. To study non-separable morphisms, we give a convenient local criterion for separability.

DEFINITION. Let α: $X \to Y$ be a morphism of varieties, and let $x \in X$ and $y = \alpha(x)$. We set $O_{x,\alpha} = O_x/\alpha_0(m_y) \cdot O_x$, where O_x and O_y are the local rings of x and y, m_y is the maximal ideal of O_y, and α_0 is the co-morphism corresponding to α (see (AG, 5.1)). $O_{x,\alpha}$ is called the *local ring* of α at x. We say that α is separable at x if $O_{x,\alpha}$ is a regular ring (see (AG, 3.9)).

It is clear that $O_{x,\alpha}$ is the local ring of $\alpha^{-1}(y)$ at x (we regard this stalk as a scheme). In particular, if we denote by dim X/Y the maximum of the dimensions of the stalks of α, then dim $X/Y = \max_{x \in X} \dim O_{x,\alpha}$ (on the right there stands the Krull dimension (see (AG, 3.4)).

A.9. LEMMA. *Let* α: $X \to Y$ *be a morphism of smooth varieties, and let* $x \in X$, $y = \alpha(x) \in Y$. *Then the following conditions are equivalent*:

a) *The tangent mapping* $(d\alpha)_x$ *is surjective.*

b) α *is separable at* x, *and* $\dim O_{x,\alpha} = \dim O_x - \dim O_y$.

PROOF. Let $m_{x,\alpha}$ be the maximal ideal of $O_{x,\alpha}$. Using the fact that O_x and O_y are regular, we can easily show that both a) and b) are equivalent to the equality $\dim m_{x,\alpha}/m_{x,\alpha}^2 = \dim O_x - \dim O_y$.

A.10. COROLLARY. *Under the conditions* $(*)$ *in* A.6 *the following conditions are equivalent*: a) α *is separable*; b) α *is separable at some* $x \in X$; c) *there exists a non-empty open subset* $U \subset X$ *such that* α *is separable at all points of it.*

The corollary follows from A.9 and (AG, 17.3 and 10.1).

A.11. We prove Theorem A.2 and Lemma A.5 by induction on $\dim X/Y$. For the remainder of the Appendix we fix $\dim X/Y = k$ and assume that Theorem A.2 is valid for all morphisms α': $X' \to Y'$ with $\dim X'/Y' < k$.

We say that a morphism α is *k-separable* if α is separable at all $x \in X$ such that $\dim O_{x,\alpha} = k$.

PROPOSITION. *Theorem* A.2 *is valid for k-separable morphisms.*

PROOF. Let α: $X \to Y$ be a k-separable morphism. It is easy to see that if X' is an F-subvariety of X, then the restriction of α to X' is also k-separable. Therefore, it suffices to show that the conclusion of Lemma A.5 is valid for α. Here we may assume that condition $(*)$ in A.6 holds. If $\dim X/Y < k$, then the lemma follows from the inductive hypothesis, and if $\dim X/Y = k$, then α is separable by A.10, and the lemma was proved in A.7.

A.12. Since we are interested not in varieties themselves, but only in the sets of their F-points, the plan for proving A.5 is as follows: we replace X and Y by other varieties so that α becomes a k-separable morphism, but the set of F-points do not reflect this.

We recall some facts about sheaf products.

DEFINITION. Let α: $X_1 \to Y$ and α_2: $X_2 \to Y$ be morphisms of affine K-schemes. Then by the *sheaf product* of X_1 and X_2 over Y (denoted by $X_1 \times {}_Y X_2$) we mean the affine scheme X corresponding to the K-algebra $K[X] = K[X_1] \otimes_{K[Y]} K[X_2]$, together with the mappings p_1: $X \to X_1$ and p_2: $X \to X_2$ corresponding to the natural embeddings $K[X_1] \to K[X]$ and $K[X_2] \to K[X]$ (see (AG, 5.2)).

A scheme X can be naturally identified with a closed subscheme in $X_1 \times X_2$ (see (AG, 6.1)). Here $X = \{(x_1, x_2) \in X_1 \times X_2 \mid \alpha_1(x_1) = \alpha_2(x_2)\}$. We note that even if X_1, X_2, and Y are varieties, X is not necessarily a variety: the ring $K[X]$ can have nilpotent elements.

It is easy to see that: a) if X_1. X_2, Y, α_1, and α_2 are defined over F, then $X_1 \times {}_Y X_2$, p_1, and p_2 are also defined over F; b) if $x_1 \in X_1$, $x_2 \in X_2$, $\alpha_1(x_1) = \alpha_2(x_2)$, and $x = (x_1, x_2) \in X_1 \times {}_Y X_2$, then the local rings O_{x,p_2} and O_{x_1,α_1} are naturally isomorphic.

A.13. LEMMA. *Let X be an irreducible affine F-variety, A^n an affine space, and α: $X \to A^n$ an F-morphism, where* $\dim X/A^n = k$. *Then there exist a non-empty F-open subset $X' \subset X$, an affine F-variety V, and an F-morphism β: $V \to A^n$ such that*:
a) β_F: $V(F) \to A^n(F) = F^n$ *is a homeomorphism*; b) *the morphism p_2*: Red $(X' \to_{A^n} V) \to V$ *is separable at all points* (*here* Red $(X' \times_{A^n} V)$ *is the reduced subscheme corresponding to the embedding of $X' \times_{A^n} V$ in* $X' \times V$ (see (AG, 6.3)); *this variety is no longer necessarily defined over F*).

Let us deduce Lemma A.5 from this lemma. Let α: $X \to Y$ be an F-morphism of F-varieties. We may assume that condition $(*)$ in A.6 holds. We embed Y in A^n and apply Lemma A.13 to the morphism α: $X \to A^n$. We wish to prove that $\alpha_F(X'(F))$ is a constructive subset of $A^n(F)$.

It is easy to see that the set $M = (X' \times_{A^n} V)(F)$ is equal to $\{(x, v) \in X'(F) \times V(F) | \alpha_F(x) = \beta_F(v)\}$. Therefore, by a), M is homeomorphic to $X'(F)$, and it suffices to prove that $p_2(M)$ is constructive in $V(F)$. Let $Z = \bar{M}$ be the closure of M in $X' \times V$; this is an F-variety by Lemma A.4b) and is contained in Red $(X' \times_{A^n} V)$. It follows from b) that the restriction of p_2 to Z is k-separable (see A.11), and by A.11, $p_{2F}(Z(F)) = p_{2F}(M)$ is constructive in $V(F)$, as required.

A.14. PROOF OF LEMMA A.13. 1. We first construct a morphism β': $V' \to A^n$ and a subset $X' \subset X$ so that b) holds.

LEMMA. *Let L be a field of characteristic p and M a finitely generated extension of L. Then there is a $q = p^N$ such that the factor ring $\bar{C}$ of $C = M \otimes_L L^{1/q}$ by the nilpotent elements is a separable extension of $L^{1/q}$* ($L^{1/q}$ *is a field by* (AG, 2.1)).

The lemma will be proved in A.15.

We apply this lemma to the case when $M = K(X)$ and L is the quotient field of $\alpha_0(K[A^n]) \subset K[X]$, and we look for an appropriate q. We set $V' = A^n$ and define an F-morphism β': $V' \to A^n$ by $\beta'(t_1, \ldots, t_n) = (t_1^q, \ldots, t_n^q)$. It is clear that β' is a homeomorphism in the F-topology, and hence p_1': $X \times_{A^n} V' \to X$ is also a homeomorphism. We set $S =$ Red $(X \times_{A^n} V')$ and $T = \overline{p_2'(S)}$. It then follows from the lemma that the morphism p_2': $S \to T$ is separable. Reasoning exactly as in A.6 and using A.10, we can find an open subset $S' \subset S$ such that p_2' is separable at each point of S'. We set $X' = p_1'(S')$. Since S is defined over $F^{p^{-\infty}}$ and since we may assume that S' is defined over $F^{p^{-\infty}}$, X' is $F^{p^{-\infty}}$-open in X and hence F-open in X (see (AG, 12.1)). Then the morphism p_2': Red $(X' \times_{A^n} V') \to V'$ is separable at all points; that is, b) holds.

2. Our morphism β' does not satisfy a). We rectify this as follows. Let $e_1, \ldots, e_q$ be a basis of F over F^q. We consider the affine space V of dimension nq with coordinates (a_{ij}, $i = 1, \ldots, n$; $j = 1, \ldots, q$) and define

an F-morphism β: $V \to A^n$ by $\beta((a_{ij})) = (t_1, \ldots, t_n)$, where $t_i = \sum_{j=1}^{q} a_{ij}^q e_j$. It is easy to see that β_F: $V(F) \to A^n(F)$ is a homeomorphism. To verify condition b) in A.13, we note that $\beta = \beta' \circ \gamma$, where γ: $V \to V'$ is the morphism defined by $\gamma((a_{ij})) = (t_1, \ldots, t_n)$, and $t_i = \sum_{j=1}^{q} a_{ij} e_j^{1/q}$ (this morphism is not defined over F). If $(x, v) \in X' \times_{A^n} V$, then it is easy to verify that the local ring at (x, v) of $p_2|_{\mathrm{Red}(X' \times_{A^n} V)}$ is isomorphic to the local ring at $(x, \gamma(v))$ of $p_2'|_{\mathrm{Red}(X' \times_{A^n} V')}$; that is, b) holds. Lemma A.13 is now proved.

A.15. PROOF OF LEMMA A.14. Let Ω be an algebraically closed extension of L containing M. Then $\bar{C}$ can be identified with the compositum $L^{1/q}(M)$ of M and $L^{1/q}$ in Ω. Let S be a transcendence basis of M over L; that is, the extension $L(S)$: L is purely transcendental, and M: $L(S)$ is finite (see [25], Ch. X, §1). It is easy to see that the extension $L^{1/q}(S^{1/q})$: $L^{1/q}$ is purely transcendental, so that it suffices to verify that $L^{1/q}(M)$ is separable over $L^{1/q}(S^{1/q})$; that is, everything reduces to the case when M: L is finite. In this case the conclusion of the lemma is obvious, because each element of Ω algebraic over L is separable over $L^{p^{-N}}$ for large N.

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