CATEGORY OF $\mathfrak{g}$ MODULES

I. N. Bernshtein, I. M. Gel'fand,
and S. I. Gel'fand

§1. INTRODUCTION

Let A be a finite-dimensional associative algebra with identity over a field K, and let $\mathcal{A}$ be the category of finite-dimensional A modules. An important invariant of such an algebra is its Cartan matrix, which is defined as follows. Let $L_1, \ldots, L_k$ be a complete collection of irreducible A modules. For each L_i there exists a unique (up to isomorphism) indecomposable projective A-module P_i that covers L_i; i.e., Hom $(P_i, L_i) \neq 0$. Let $c_{ij} = (P_i : L_j)$ be the number of occurrences of L_j in the Jordan–Hölder series of P_i. The integral matrix $C = \|c_{ij}\|$, $i, j = 1, \ldots, k$, is called the Cartan matrix of A (or $\mathcal{A}$).

In certain cases C is a symmetric, positive-definite matrix and, moreover, can be represented in the form $C = D^t \cdot D$, where D is some other integral matrix (not necessarily square).

This fact is ordinarily a reflection of some duality principle; to wit, the equality $C = D^t \cdot D$ means that there exists a class of modules $M_1, \ldots, M_l$, such that each P_i has a composition series with factors isomorphic to M_j, and for any i, j the number of occurrences of M_j in the series for P_i is equal to the number of occurrences of L_i in the Jordan–Hölder series for M_j. Thus, it can be said that the modules M_j occupy an intermediate position between the projective modules P_i and the simple modules L_i; they are their "mean geometric."

The elucidation of the reason why $C = D^t \cdot D$ and the intrinsic (in terms of $\mathcal{A}$) characterization of the modules M_j are highly interesting problems, approaches to which are absolutely unclear at present.

At present two classes of categories $\mathcal{A}$ are known for which the Cartan matrix C has this property.

Case 1. Let char $K = p > 0$, and let $A = KG$ be the group algebra of some finite group G, so that $\mathcal{A}$ is the category of finite-dimensional G modules over K. Let $V_1, \ldots, V_l$ be a complete collection of irreducible representations of G over the field **C** of complex numbers, and let $M_1, \ldots, M_l$ be the A modules obtained by their reduction to characteristic p (see [1]).

If $L_1, \ldots, L_k$ are a complete collection of irreducible A modules (i.e., irreducible representations of G over K) and D is the matrix with entries $d_{ij} = (M_i : L_j)$, then $C = D^t \cdot D$ (for more details see [1, §§82, 83]).

Case 2. Let $\mathfrak{g}$ be a semisimple complex Lie algebra, $\bar{\mathfrak{g}}$ the Lie algebra over a closed field of characteristic $p > 0$ obtained by the reduction of $\mathfrak{g}$, and $A = U^0(\bar{\mathfrak{g}})$ a bounded universal enveloping algebra of $\bar{\mathfrak{g}}$. Finite-dimensional A modules were studied by Humphreys [2], who constructed a collection of A modules $M_1, \ldots, M_l$ that occupy an "intermediate position" between projective and simple modules as described above and proved that the Cartan matrix of the category of A modules can be represented in the form $C = D^t \cdot D$.

The purpose of this article is to construct a category of $\mathfrak{g}$ modules having the same property for each semisimple Lie algebra $\mathfrak{g}$ over **C**. Simple objects of this category can be indexed by the elements of the Weyl group W of $\mathfrak{g}$ (to $w \in W$ corresponds a simple module L_w and a projective module P_w). In addition, to each $w \in W$ corresponds some $\mathfrak{g}$ module M_w (the so-called Verma module; see [3, 4, 5, 8]). If we now set $C = \|c_{w,w'}\|$, where $c_{w,w'} = (P_w : L_{w'})$, and $D = \|d_{w,w'}\|$, where $d_{w,w'} = (M_w : L_{w'})$, then $C = D^t \cdot D$. In addition, $\mathcal{A}$ has other good properties (D is unipotent and all objects of $\mathcal{A}$ have a finite cohomological dimension).

Note also that a detailed study of the relatively simple "finite-dimensional" category $\mathcal{A}$ makes it possible to obtain certain information about the structure of submodules of Verma modules, which is a difficult and interesting problem.

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We would like to dedicate our article to J. Dixmier, who accomplished a great deal in the study of the structure of enveloping algebras for Lie algebras. His papers and book [5] were most responsible for crystallizing this branch of mathematics into a separate and important direction.

We are also greatly indebted to J. Humphreys [2], who significantly instigated our paper.

§2. SOME NOTATION

$\mathfrak{g}$ is a complex semisimple Lie algebra, and $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}$.

Δ is the root system of $\mathfrak{g}$ with respect to $\mathfrak{h}$; Σ is the set of simple roots; Δ_+ is the set of positive roots; and $\rho = \frac{1}{2} \sum_{\gamma \in \Delta_+} \gamma$.

$\mathfrak{n}_+$, and $\mathfrak{n}_-$ are the subalgebras of $\mathfrak{g}$, spanned by the root vectors corresponding to the roots in Δ_+ (resp. $-\Delta_+$).

$U(\mathfrak{g})$, and $U(\mathfrak{n}_+)$ are enveloping algebras of $\mathfrak{g}$ and $\mathfrak{n}_+$, respectively, and $Z(\mathfrak{g})$ is the center of $U(\mathfrak{g})$.

$\mathfrak{h}^*$ is the space dual to $\mathfrak{h}$; $\mathfrak{h}^*_Z$ is the integral lattice in $\mathfrak{h}^*$, consisting of the weights of the finite-dimensional representations of $\mathfrak{g}$; and $C^\circ \subset \mathfrak{h}^*$ is the positive Weyl chamber.

W is the Weyl group of $\mathfrak{g}$ with respect to $\mathfrak{h}$; σ_γ is the reflection corresponding to $\gamma \in \Delta_+$; and $l(w)$ is the length of $w \in W$, i.e., the smallest number of factors in the representation $w = \sigma_{\alpha_1} \ldots \sigma_{\alpha_l}$, $\alpha_i \in \Sigma$.

$K(\chi)$ for $\chi \in \mathfrak{h}^*$ is Kostant's function, i.e., the number of representations of χ in the form $\chi = \sum_{\alpha \in \Delta_+} n_\alpha \alpha$; $n_\alpha \in \mathbf{Z}$, $n_\alpha \geqslant 0$; $\Gamma \subset \mathfrak{h}^*_Z$ is the set of elements χ such that $K(\chi) \neq 0$; for $\chi, \psi \in \mathfrak{h}^*$, $\chi < \psi$ means that $\psi - \chi \in \Gamma$.

JH(M) denotes the collection of simple modules (with multiplicities) occurring in the Jordan–Hölder series of a $\mathfrak{g}$ module M.

§3. ELEMENTARY PROPERTIES OF THE CATEGORY $\mathcal{O}$

Definition 1 (the Category $\mathcal{O}$). The objects of the category $\mathcal{O}$ are left $\mathfrak{g}$ modules M having the following properties:

1) M is a finitely generated $\mathfrak{g}$ module;

2) M is $\mathfrak{h}$ diagonalizable;

3) M is $\mathfrak{n}_+$ finite (i.e., $\dim_C U(\mathfrak{n}_+) f < \infty$ for any $f \in M$).

The morphisms of $\mathcal{O}$ are arbitrary $\mathfrak{g}$ module morphisms.

Important objects of $\mathcal{O}$ are the so-called Verma modules (the modules M_χ; see [3, 4]).

Let $\chi \in \mathfrak{h}^*$. We denote by J_χ the left ideal in $U(\mathfrak{g})$ generated by $\mathfrak{n}_+$ and $\{H - \chi(H) + \rho(H), H \in \mathfrak{h}\}$, and we set $M_\chi = U(\mathfrak{g})/J_\chi$. It is clear that $M_\chi \in \mathcal{O}$.

Let us list the basic properties of $\mathcal{O}$ and of Verma modules.

1) $\mathcal{O}$ is an Abelian category with finite direct sums. The space Hom(M, M') is finite-dimensional for any $M, M' \in \mathcal{O}$.

2) Let $M \in \mathcal{O}$, $\psi \in \mathfrak{h}^*$. We denote by $M^{(\psi)} \subset M$ the subspace of vectors of weight ψ in M. Then $\dim_C M^{(\psi)} < \infty$. Let $P(M) = \{\psi \in \mathfrak{h}^* | M^{(\psi)} \neq 0\}$. Then there exists a finite number of weights $\psi_1, \ldots, \psi_k$ such that $P(M) \subset \cup (\psi_j + (-\Gamma))$, where Γ is the semigroup generated by positive roots (for example, $P(M_\chi) = \chi - \rho - \Gamma$). In particular, if $M \neq 0$, then P(M) contains at least one maximal weight χ, i.e., such that $P(M) \cap \{\chi + \Gamma\} = \chi$.

3) Let us describe all of the simple objects of $\mathcal{O}$. Let $\chi \in \mathfrak{h}^*$. For a Verma module M_χ there is a unique simple factor module (see [5]), which we denote by L_χ. The modules L_χ are pairwise nonisomorphic and are exhausted by all of the simple objects in $\mathcal{O}$.

4) Let $Z(\mathfrak{g})$ be the center of $U(\mathfrak{g})$, and let Θ be the set of characters of $Z(\mathfrak{g})$ (i.e., homomorphisms $Z(\mathfrak{g}) \to C$). For each $\theta \in \Theta$ consider the complete subcategory $\mathcal{O}_\theta$ of $\mathcal{O}$, consisting of modules M satisfying the following condition: For each $z \in Z(\mathfrak{g})$ the module M is annihilated by some power of $[z - \theta(z)]$. Then $\mathcal{O} = \bigoplus_{\theta \in \Theta}$

$\mathcal{O}_\theta$, i.e., each $M \in \mathcal{O}$ decomposes into a finite sum $M = \bigoplus_{\theta\in\Theta} M(\theta)$, $M(\theta) \in \mathcal{O}_\theta$, and for $\theta_1 \neq \theta_2$ $\mathrm{Hom}(M_1, M_2) = 0$ for any $M_1 \in \mathcal{O}_{\theta_1}$, $M_2 \in \mathcal{O}_{\theta_2}$.

5) For each $\chi \in \mathfrak{h}$ there exists a character θ_χ such that $zf = \theta_\chi(z)f$ for all $f \in M_\chi$, $z \in Z(\mathfrak{g})$. Therefore, $M_\chi \in \mathcal{O}_{\theta_\chi}$. Each $\theta \in \Theta$ has the form θ_χ for some $\chi \in \mathfrak{h}^*$; $\theta_{\chi_1} = \theta_{\chi_2} \Leftrightarrow \chi_1 = w\chi_2$ for some $w \in W$. Let $\Lambda(\theta) = \{\chi \in \mathfrak{h}^*, \theta = \theta_\chi\}$.

6) Using 1), 4), and 5), one can prove that each $M \in \mathcal{O}$ has finite length (see [5]). Therefore, each $M \in \mathcal{O}$ decomposes into the direct sum of indecomposable objects, where the summands are uniquely defined up to isomorphism and rearrangement (the Krull–Schmidt theorem [1]).

7) Let $\theta = \theta_\chi$ and $M \in \mathcal{O}_\theta$. Then $P(M) \subset \bigcup_{w\in W} \{w(\chi - \rho) - \Gamma\}$. It suffices to verify this assertion for simple objects in $\mathcal{O}_\theta$, for which it follows from 1), 3), and 5).

§4. PROJECTIVE OBJECTS OF $\mathcal{O}$

THEOREM 1. Each object in $\mathcal{O}$ is a factor object of a projective object.

We shall prove the more precise Theorem 2.

THEOREM 2. Let $\chi \in \mathfrak{h}^*$, $\theta \in \Theta$. There exist a module $Q = Q(\theta, \chi) \in \mathcal{O}_\theta$ and a vector $q \in Q^{(\chi)}$ such that for any $M \in \mathcal{O}$ the mapping $\mathrm{Hom}(Q, M) \to (M(\theta))^{(\chi)}$, defined by $\varphi \to \varphi(q)$, is an isomorphism.

It follows from Theorem 2 that the functor $M \to \mathrm{Hom}(Q, M)$ is isomorphic to the functor $M \to (M(\theta))^{(\chi)}$ and hence, is faithful; i.e., Q is projective (in $\mathcal{O}$). On the other hand, each $M \in \mathcal{O}$ is generated by a finite number of vectors $f_i \in (M(\theta_i))^{(\chi_i)}$, and by Theorem 2, M is a factor module of the projective module $\bigoplus_i Q(\theta_i, \chi_i)$. Therefore, it suffices to prove Theorem 2.

Proof. Let $I \subset U(\mathfrak{g})$ be the left ideal generated by $H - \chi(H)$, $H \in \mathfrak{h}$, and $(\mathfrak{n}_+)^N$ for sufficiently large N (N will be chosen later). Let $\hat{Q} = U(\mathfrak{g})/I$ and let $\hat{q} \in \hat{Q}$ be the image of $1 \in U(\mathfrak{g})$ in $\hat{Q}$. It is clear that $\hat{Q} \in \mathcal{O}$.

Let us prove that if N is sufficiently large, then for any $M \in \mathcal{O}_\theta$ the mapping $(\hat{Q}, M) \to M^{(\chi)}$ $(\varphi \mapsto \varphi(\hat{q}))$ is an isomorphism. Since $\hat{q}$ is a generator of $\hat{Q}$, this mapping is an imbedding. Conversely, let $f \in M^{(\chi)}$. Consider $\alpha : U(\mathfrak{g}) \to M, \alpha(X) = Xf$. It is clear that $\alpha(H - \chi(H)) = 0$, $H \in \mathfrak{h}$. On the other hand, let $\theta = \theta_\psi$ for some $\psi \in \mathfrak{h}^*$. Then by virtue of 7), $P(M) \subset \bigcup_{w\in W} \{w(\psi - \rho) - \Gamma\}$, and hence, for sufficiently large N (not depending on M or f) $(\mathfrak{n}_+)^N f = 0$. Therefore, $\alpha((\mathfrak{n}_+)^N) = 0$; i.e., α defines a mapping $\hat{\alpha} : \hat{Q} \to M$, such that $\hat{\alpha}(\hat{q}) = f$.

It is now clear that the module $Q = \hat{Q}(\theta)$, and the projection q of $\hat{q}$ onto Q satisfy the condition of Theorem 2.

COROLLARY 1. If $P \in \mathcal{O}$ is an indecomposable projective module, then P has a unique maximal submodule P', so that the simple module L = P/P' corresponds to it. We obtain a one-to-one correspondence $P \longleftrightarrow P/P'$ between indecomposable projective objects and simple objects of $\mathcal{O}$.

We denote the projective module corresponding to the simple module L_χ by P_χ. It is called the projective covering of L_χ.

This corollary follows from Theorem 1 and Property 6) (see [1], Theorem 54.11).

Proposition 1. Let $\chi \in \mathfrak{h}^*$, $M \in \mathcal{O}$. Then $\dim \mathrm{Hom}(P_\chi, M) = (M : L_\chi)$.

Proof. Let $0 \to M_1 \to M \to M_2 \to 0$ be an exact sequence in $\mathcal{O}$. Since P_χ is a projective object, the validity of the proposition for M follows from its validity for M_1 and M_2. Therefore, we can assume that $M = L_\psi$ is a simple object. It now follows from Corollary 1 that if $\chi \neq \psi$, then both sides of the equality equal zero. But if $\chi = \psi$, then $\mathrm{Hom}(P_\chi, L_\chi) = \mathrm{Hom}(L_\chi, L_\chi) = \mathbf{C}$.

Remark. The category $\mathcal{O}$ is self-dual, i.e., equivalent to the dual category $\mathcal{O}^0$. This equivalence is defined by the following functor F. Let $i : \mathfrak{g} \to \mathfrak{g}$ be an anti-involution such that $i(H) = H$, $H \in \mathfrak{h}$. Then $i(\mathfrak{n}_+) = \mathfrak{n}_-$. Let $M \in \mathcal{O}$ and let M^* be the space of linear functionals on M. We define the action of $\mathfrak{g}$ on M^* by $(X\xi)(f) = \xi(i(X)f)$, $\xi \in M^*$, $f \in M$, $X \in \mathfrak{g}$. As F(M) we take the submodule of M^* generated by the vectors characteristic with respect to $\mathfrak{h}$. It can be proved that $F(M) \in \mathcal{O}$, and that $\mathrm{Hom}(M_1, M_2) = \mathrm{Hom}(F(M_2), F(M_1))$. In addition, F(F(M)) is naturally isomorphic to M. It is clear from the construction that $\dim(F(M))^{(\chi)} = \dim M^{(\chi)}$ for any $\chi \in \mathfrak{h}^*$. In particular, $F(L_\chi) = L_\chi$, and hence $F(\mathcal{O}_\theta) = \mathcal{O}_\theta$ for any $\theta \in \Theta$.

It is clear that the modules $I_\chi = F(P_\chi)$ are indecomposable injective objects in $\mathcal{O}$, and that each indecomposable injective object in $\mathcal{O}$ is isomorphic to one of the I_χ.

§5. THE CATEGORY $\mathcal{O}_\theta$

It follows from Property 4) in §3 that many properties of $\mathcal{O}$ can be studied "locally," i.e., inside the categories $\mathcal{O}_\theta$ for various $\theta \in \Theta$. Each of the $\mathcal{O}_\theta$ is a "finite-dimensional" category in the following sense:

THEOREM 3. Let $\theta = \theta_\chi$ for some $\chi \in \mathfrak{h}^*$, and let $P = \bigoplus_{w \in W} n_w P_{w\chi}$ be a projective object in $\mathcal{O}_\theta$, such that $n_w > 0$ for all $w \in W$. Consider the finite-dimensional algebra $A = \mathrm{Hom}(P, P)$. The functor $M \to \mathrm{Hom}(P, M)$ defines an equivalence of $\mathcal{O}_\theta$ with the category of finite-dimensional right A modules.

The proof follows immediately from [6] (Theorem II.1.3) and the fact that the $P_{w\chi}$, $w \in W$, exhaust all of the indecomposable projective modules in $\mathcal{O}_\theta$.

Remark. It can be deduced from the explicit construction of $F : \mathcal{O}_\theta \to \mathcal{O}_\theta^0$ (see the remark in §4) that there exists an antiautomorphism $i : A \to A$ such that $i^2 = 1$.

Let $\theta = \theta_\chi$. If χ is a character such that $w\chi - \chi \notin \mathfrak{h}_Z^*$ for all $w \in W$ (i.e., χ is a character in general position), then any simple object in $\mathcal{O}_\theta$ is projective, and $\mathcal{O}_\theta$ is arranged very simply.

When χ is a character not in general position, the linkages appear among the various simple objects in $\mathcal{O}_\theta$. The most complicated and interesting case is when χ is a regular integral weight; i.e., $\chi \in \mathfrak{h}_Z^*$ and $w\chi \neq \chi$ for $w \in W$, $w \neq e$.

THEOREM 4. Let χ, χ' be two regular integral weights, and let $\theta = \theta_\chi$, $\theta' = \theta_{\chi'}$. Then $\mathcal{O}_\theta$ and $\mathcal{O}_\theta'$ are equivalent.

This theorem is proved by the same methods as Theorem 2 in [4] and Theorem E1 in [7]. We shall not prove it.

§6. THE CARTAN MATRIX AND THE DUALITY THEOREM

Our purpose in this section is to investigate the number of occurrences of the simple modules L_χ in the Jordan–Hölder series of an indecomposable projective module P_ψ. We shall fix $\theta \in \Theta$ and operate inside one category $\mathcal{O}_\theta$; i.e., we shall assume that $\chi, \psi \in \Lambda(\theta)$.

Definition 2. 1) The Cartan matrix $C = \| c_{\chi\psi} \|$, $\chi, \psi \in \Lambda(\theta)$, is defined by $c_{\chi\psi} = (P_\chi : L_\psi)$.

2) The decomposition matrix $D = \| d_{\chi\psi} \|$, $\chi, \psi \in \Lambda(\theta)$, is defined by $d_{\chi\psi} = (M_\chi : L_\psi)$.

THEOREM 5. 1) Let $\chi_1, \ldots, \chi_s$ be an ordering of the weights $\chi \in \Lambda(\theta)$, such that $\chi_i < \chi_j \Rightarrow i > j$. Then D is an upper triangular matrix with ones on the diagonal.

2) $C = D^t \cdot D$. In particular, C is a symmetric matrix.

The proof of 1) is obvious. To prove 2) we introduce the concept of a p filtration.

Definition 3. Let $M \in \mathcal{O}$. A filtration $0 = M_0 \subset M_1 \subset M_2 \subset \ldots \subset M_k = M$ is called a p-filtration if $M_i/M_{i-1} \simeq M_{\chi_i}$ for some $\chi_i \in \mathfrak{h}^*$. In this case we denote by $(M : M_\chi)$ the number of i such that $\chi_i = \chi$. (It follows from the first part of Theorem 5 that the numbers $(M : M_\chi)$ do not depend on the choice of p filtration.)

Proposition 2. 1) Each positive module $P \in \mathcal{O}$ admits a p filtration.

2) The duality $(P_\chi : M_\psi) = d_{\psi\chi} = (M_\psi : L_\chi)$ holds.

It is clear that the second part of Theorem 5 follows from this proposition.

Proof of Proposition 2. a) LEMMA 1. Suppose that M admits a p filtration, χ is a maximal weight in P(M), $f \in M^{(\chi)}$, and $M' = U(\mathfrak{g}) f$. Then $M' \simeq M_{\chi+\rho}$ and M/M' admits a p filtration.

Using induction on the length of a p filtration, we can assume that $M = M_k$, $f \notin M_{k-1}$. Then we obtain a nontrivial mapping of M' into $M_k/M_{k-1} \simeq M_{\chi_k}$. Since $\chi + \rho \nless \chi_k$, this mapping is an isomorphism, and $\chi_k = \chi + \rho$. Therefore, $M' = M_{\chi+\rho}$ and $M/M' \simeq M_{k-1}$ admits a p filtration.

b) If $M = M_1 \oplus M_2$ admits a p filtration, then each of M_1, M_2 admits a p filtration.

Again let χ be maximal among the weights in P(M). We can assume that $M_1^{(\chi)} \neq 0$. Choose $0 \neq f \in M_1^{(\chi)}$ and set $M' = U(\mathfrak{g}) f$. Then $M' \simeq M_{\chi+\rho}$ and $M/M' \simeq M_1/M' \oplus M_2$ admits a p filtration. Induction on the length of M completes the proof.

c) For any $\chi \in \mathfrak{h}^*$ the module $\hat{Q}$, constructed in the proof of Theorem 2, admits a p filtration.

We choose in $U(\mathfrak{n}^+)$ a collection of weight elements $x_1, \ldots, x_S$ with weights $\lambda_1, \ldots, \lambda_S$ so that the images of x_i in $U(\mathfrak{n}_+)/U(\mathfrak{n}_+)(\mathfrak{n}_+)^N$ define a basis there and $\lambda_i < \lambda_j \Rightarrow i > j$. Let $\hat{Q}_j \subset \hat{Q}$ be the submodule generated by $(x_1\hat{q}, \ldots, x_j\hat{q})$. It follows from the Poincaré–Birkhoff–Witt theorem that the $\hat{Q}_j$ form a p filtration in $\hat{Q}$. In particular, $(\hat{Q} : M_\psi) = K(\psi - \chi - \rho)$, where K is Kostant's function (see §2).

d) Since the $Q(\chi, \theta)$ (see §4) are direct summands of $\hat{Q}$, they admit a p filtration. Since any projective module $P \in \mathcal{O}_\theta$ is a direct summand of some $\bigoplus_i Q(\chi_i, \theta)$, the first part of Proposition 2 is proved.

e) By virtue of Proposition 1, to prove the second part of Proposition 2 it suffices to show that

$$(P : M_\psi) = \dim \operatorname{Hom}(P, M_\psi) \tag{1}$$

for any projective object $P \in \mathcal{O}_\theta$ and any $\psi \in \Lambda(\theta)$.

Let $Q(\chi, \theta) = \sum_{\psi \in \Lambda(\theta)} n_\psi(\chi) P_\psi$. Then $n_\psi = \dim \operatorname{Hom}(Q(\chi, \theta), L_\psi) = (L_\psi)^{(\chi)}$. In particular, $n_\psi(\chi) = 0$, if $\chi + \rho \not< \psi$, and $n_{\chi+\rho}(\chi) = 1$. Therefore, since (1) is linear in P, it suffices to verify it for $Q = Q(\chi, \theta)$.

Both sides of the equality are unchanged when Q is replaced by $\hat{Q}$. Here $(\hat{Q} : M_\psi) = K(\psi - \chi - \rho)$ and $\dim \operatorname{Hom}(\hat{Q}, M_\psi) = \dim (M_\psi)^{(\chi)} = K(\psi - \chi - \rho)$; i.e., (1) and Proposition 2 are thereby proved.

§7. COHOMOLOGICAL DIMENSION OF $\mathcal{O}$

In this section we prove that $\mathcal{O}$ has finite cohomological dimension and indicate what it equals. Let us recall the definition of cohomological dimension.

Let $\mathscr{C}$ be an Abelian category and M an object of $\mathscr{C}$. By the cohomological dimension dh(M), we mean the smallest number l, such that there exists a projective resolution of M of length l, i.e., an exact sequence

$$0 \leftarrow M \leftarrow P_0 \leftarrow P_1 \leftarrow \ldots \leftarrow P_l \leftarrow 0,$$

where the P_i are projective.

THEOREM 6. For any $M \in \mathcal{O}$ $\operatorname{dh}(M) \leqslant 2S$, where S is the maximum length of an element in the Weyl group W.

LEMMA 2. Let $0 \to M_1 \to M \to M_2 \to 0$ be an exact sequence. Then:

a) $\operatorname{dh}(M) \leq \max(\operatorname{dh}(M_1), \operatorname{dh}(M_2))$,

b) $\operatorname{dh}(M_2) \leq \max(\operatorname{dh}(M_1) + 1, \operatorname{dh}(M))$.

The lemma is proved by standard homological arguments (see, e.g., [6]).

We shall prove Theorem 6 in three stages.

1. Let $\chi \in C^0$. Then $\operatorname{dh}(M_{w\chi}) \leqslant l(w)$.

Proof. Consider a p filtration of $P_{w\chi}$. Its factors have the form $M_{w'\chi}$, $w' \in W$. By virtue of Theorem 5, only the $M_{w'\chi}$, such that $L_{w\chi} \in JH(M_{w'\chi})$, occur in this filtration. As follows from [7], in this case either $w' = w$, or $l(w') < l(w)$, and $w'\chi > w\chi$. Using Lemma 1, we obtain an exact sequence

$$0 \to M \to P_{w\chi} \to M_{w\chi} \to 0,$$

where M has a p filtration with factors $M_{w'\chi}$, $l(w') < l(w)$. By virtue of the induction assumption and Lemma 2a), $\operatorname{dh}(M) \leq l(w) - 1$. By Lemma 2b), $\operatorname{dh}(M_{w\chi}) \leq l(w)$.

2. $\operatorname{dh}(L_{w\chi}) \leq 2S - l(w)$.

Proof. We shall use induction up to $l(w)$. For $l(w) = S$ we have $M_{w\chi} = L_{w\chi}$, and hence $\operatorname{dh}(L_{w\chi}) \leq S$. For arbitrary w we have an exact sequence

$$0 \to M' \to M_{w\chi} \to L_{w\chi} \to 0,$$

where, by virtue of [7] (see [7], Appendix), JH(M') consists of the modules $L_{w\chi}$ with $l(w') > l(w)$. Just as before, by induction we obtain $dh(L_{w\chi}) \leq 2S - l(w)$.

3. Since each object has finite length, the theorem follows from Lemma 2a) and the inequality $dh(L) \leq 2S$, which is valid for each simple object $L \in \mathcal{O}$.

Remark. It can be shown that if χ is a regular integral highest weight, i.e., a weight such that L_χ is finite-dimensional, then $dh(L_\chi) = 2S$. More precisely, it can be shown that $\mathrm{Ext}^*_{\mathcal{O}}(L_\chi, L_\chi)$ is isomorphic to the cohomology algebra $H^*(X, C)$, where $X = G/B$ is the base projective space of a group G with Lie algebra $\mathfrak{g}$, B is the Borel subgroup of G (see [7]). In particular, $\mathrm{Ext}^{2S}_{\mathcal{O}}(L_\chi, L_\chi) = C$.

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INTEGRALS OF HIGHER-ORDER STATIONARY KdV EQUATIONS AND EIGENVALUES OF THE HILL OPERATOR

O. I. Bogoyavlenskii

The aim of this paper is to find explicit formulas connecting two sets of integrals of stationary problems for higher-order Korteweg–de Vries (KdV) equations, i.e., the Novikov integrals [1] with the Lax integrals [2] and the Gel'fand–Dikii integrals [3].

Let us recall [1, 3] that the n-th order stationary KdV equation ("Novikov's equation") is expressed by

$$\sum_{k=0}^{n+1} c_k R_k[u] = 0, \quad c_{n+1} = 1, \tag{1}$$

where the functions $R_k[u, u', u'', \ldots]$ can be obtained* by expanding the kernel of the resolvent

*We are using the notations of [3] which are connected with the notations of [1] and [4] by the following formulas:

$$\frac{\delta I_k(u)}{\delta u(x)} = \frac{\delta \chi_{2k+3}(u)}{\delta u(x)} = 2^{2k+3} R_{k+1}[u].$$

L. D. Landau Institute of Theoretical Physics, Academy of Sciences of the USSR.

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