MODELS OF REPRESENTATIONS OF COMPACT LIE GROUPS

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1. Let U be a compact Lie group. A representation σ of the group U will be called a model if every irreducible representation π of the group U enters into σ exactly one time. The theory of highest weight provides one method to construct such models. Namely, σ can be realized in the space of analytic functions on the fundamental affine space of the group U, i.e., on the factor space G/N of the complex cover G of the group U by the maximal unipotent subgroup $N \subset G$. The defect of this construction is that we must require that our functions be analytic. On the other hand, for the simplest group SO(3) there is another classical realization of a model in the space of all square-integrable functions on the two-dimensional sphere. In this note we introduce an analogous construction of a model for an arbitrary compact Lie group. Our model will be realized in a space of vector functions on the compact symmetric space of maximum rank corresponding to the group U.

2. Let U be a connected compact Lie group, let T be the maximal torus in U, and let Λ be the lattice of characters of T. We shall fix a Cartan involution, i.e., an anti-automorphism θ: $U \to U$ such that $\theta^2 = 1$ and $\theta(t) = t$ for all $t \in T$. Let us set $K = \{u \in U \mid \theta(u) = u^{-1}\}$. We shall call K the involutive subgroup in U; this subgroup is determined by the group U uniquely up to an inner automorphism of U. An important role will be played by the group $S = T \cap K$. It is easy to check that S consists of all elements of order two in T, so that S is a finite commutative group of order 2^r, where r is the rank of U.

Example. If $U = U(n)$ and θ is a transposition, then $K = O(n)$ and S is the group of diagonal matrices with the numbers ± 1 on the main diagonal.

3. Let τ be a finite-dimensional representation of the group K. Our goal is to study how the representation $\mathrm{Ind}_K^U(\tau)$ of the group U, induced by the representation τ of the subgroup K, breaks up into irreducible components. Let $C \subset \Lambda$ be the set of all highest weight irreducible representations of U (with respect to some ordering).

PROPOSITION 1. Let π be an irreducible representation of U with highest weight λ: $T \to C^*$. Then†

$$(\mathrm{Ind}_K^U(\tau), \pi)_U \leqslant (\tau|_S, \lambda|_S)_S. \qquad (*)$$

COROLLARY. If $\tau|_S$ has spectrum of multiplicity one (i.e., decomposes into a direct sum of pairwise inequivalent irreducible representations), then $\mathrm{Ind}_K^U(\tau)$ also has spectrum of multiplicity one.

4. In what follows we shall study the conditions under which equality holds in formula (*).

PROPOSITION 2. a) For every representation τ of the group K we can find $\mu \in C$ such that for every irreducible representation π of the group U with highest weight $\lambda \in \mu + C$, equality holds in formula (*).

b) If $\tau = 1$ is the identity representation of K, then equality holds in formula (*). In other words, an irreducible representation π of the group U enters into $\mathrm{Ind}_K^U(1)$ if and only if its highest weight λ is even, i.e., $\lambda \in 2\Lambda$.

†$(\rho_1, \rho_2)_G$ denotes the number of times the irreducible representation ρ_2 of the group G occurs in the representation ρ_1.

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5. We shall describe those representations τ for which equality always holds in formula (*).

Let G be the complexification of U, let $H \subset G$ be the complexification of T, and let α be a root of G with respect to H. We choose a homomorphism φ_α: SL(2, C) $\to$ G so that the image under φ_α lies in the standard three-dimensional subgroup of G corresponding to the root α, and moreover, φ_α (diag $(a, a^{-1})) \subset$ H and $\varphi_\alpha(SO(2)) \subset K$. Let ψ_α: SO(2) $\to$ K be the restriction of φ_α.

We note that the group SO(2) is isomorphic to the circle, and so its irreducible representations are one-dimensional and are defined by a single integer, the degree of the representation.

Definition. A representation τ of the group K will be called fine if, for every root α, the representation $\tau \circ \psi_\alpha$ of the group SO(2) decomposes into a direct sum of one-dimensional representations of degrees 0, 1, and -1.

We observe that it suffices to check this condition on one root from each orbit of the Weyl group.

THEOREM 1. If τ is a fine representation of the group K, then, for every irreducible representation π of the group U, equality holds in formula (*). This property holds only for fine representations.

THEOREM 2. For every connected compact Lie group U there exists a fine representation τ of the subgroup $K \subset U$ such that $\tau|_S$ is a regular representation of S.

If τ is the representation mentioned in Theorem 2, then it follows from Theorem 1 that the representation $\mathrm{Ind}_K^U(\tau)$ is a model for the group U. This representation is realized in the space of sections of a vector bundle over a compact symmetric space U/K of maximal rank. The fiber of this bundle has dimension 2^r.

We note that, in general, the representation τ in Theorem 2 is not uniquely determined. It can be shown, however, that if the factor-group Z/Z^0 of the center Z of the group U by the connected component of the identity Z^0 does not contain elements of order two (for example, if the center Z is connected), then the representation τ in Theorem 2 is uniquely defined.

6. In this section we shall indicate for every classical compact Lie group U a fine representation τ of the subgroup $K \subset U$ for which $\mathrm{Ind}_K^U(\tau)$ is a model. We shall denote by ρ_n the natural representation of the group U(n) in the space $\wedge^*(C^n) = \bigoplus_{i=0}^{n} \wedge^i(C^n)$.

a) U = U(n), K = O(n); $\tau = \rho_n|_K$ is the natural representation of K in the space $\wedge^*(C^n)$.

a') U = SU(n), K = SO(n); the representation τ of the group K is the restriction of the representation ρ_n on some 2^{n-1}-dimensional subspace $L \subset \wedge^*(C^n)$. To construct the space L we consider the operator B: $\wedge^*(C^n) \to \wedge^*(C^n)$ such that $B^2 = 1$ and for every $o \in O(n)$ we have $B \cdot \rho_n(o) = \det o \cdot \rho_n(o) \cdot B$, and we set $L = \{x \in \wedge^*(C^n) \mid Bx = x\}$ (the operator B is easily constructed from the ordinary operator* (see [3], p. 33)). For odd n we can take $L = \bigoplus_{i<n/2} \wedge^i(C^n)$.

b) U = SO (2n+ε), where $\varepsilon = 0, 1$, $K = (O(n) \times O(n+\varepsilon)) \cap SO(2n+\varepsilon)$; the representation τ in the space $\wedge^*(C^n)$ is defined by the formula $\tau(o \times o') = \rho_n(o)$, $o \in O(n)$, $o' \in O(n+\varepsilon)$.

c) U = U(n, H) (the unitary quaternion group), K = U(n); $\tau = \rho_n$ is the natural representation of K in the space $\wedge^*(C^n)$.

7. Detailed proofs of the results stated above, and in particular, a complete construction of the models for the spinor groups and the basic groups of type G_2, F_4, E_6, E_8 are contained in [1].

Obviously, the construction of the models can be generalized to arbitrary semisimple Lie groups. A different construction of models, connected with the choice of other subgroups, is examined in [2] for the case of the finite Chevalley groups.

Added in Proof. D. P. Zhelobenko has brought to our attention that similar results have been obtained by Yu. B. Dzyadyk; part of these have been published in Dokl. Akad. Nauk SSSR, 220, No. 5, 1019-1020; No. 6, 1259-1262 (1975).

LITERATURE CITED

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