DIFFERENTIAL OPERATORS ON A CUBIC CONE

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Consider in the space $\mathbf{C}^3$ with the coordinates x_1, x_2, x_3 the surface X defined by the equation $x_1^3 + x_2^3 + x_3^3 = 0$. We prove the following theorem:

THEOREM 1. *Let $D(X)$ be the ring of regular differential operators on X, and D_a the ring of germs at the point 0 of analytic operators on X. Then*

1°. *the rings $D(X)$ and D_a are not Noetherian;*

2°. *for any natural number k the rings $D(X)$ and D_a are not generated by the subspaces D_k (D_{ak}, respectively) of operators of order not exceeding k. In particular, the rings $D(X)$ and D_a are not finitely generated.*

Theorem 1 answers questions raised in Malgrange's survey article [1]. The ring $D(X)$ has an interesting structure (see Proposition 1).

We denote by $E(X)$ the ring of regular functions on $X(E(X) = \mathbf{C}[x_1, x_2, x_3]/[x_1^3 + x_2^3 + x_3^3])$ and by $D(X)$ the ring of regular differential operators on X. By D_k we denote the space of operators of order not exceeding k. Setting $a_\lambda(f)(x) = f(\lambda x)$ and $b_\lambda(\mathscr{D})(f) = a_\lambda(Da_{\lambda-1}(f))$ for $\lambda \in \mathbf{C}^*$ we define an action of the group $\mathbf{C}^*$ in the spaces $E(X)$ and $D(X)$. It is clear that $E(X) = \bigoplus_{i=0}^{\infty} E^i(X)$, where $E^i(X)$ is the finite-dimensional space of homogenous functions of degree i on X. We call an operator $\mathscr{D} \in D(X)$ homogenous of degree i ($i \in \mathbf{Z}$) if $b_\lambda(\mathscr{D}) = \lambda^i \mathscr{D}$ for all $\lambda \in \mathbf{C}^*$ (equivalent definition: $\mathscr{D}(E^n(X)) \subset E^{n+i}(X)$ for all n). We denote by D^i the space of all such operators and set $D_k^i = D^i \cap D_k$.

LEMMA 1. a) $D_k = \bigoplus_{i=-\infty}^{\infty} D_k^i$; b) $D(X) = \bigoplus_{i=-\infty}^{\infty} D^i$.

PROOF. a) Let $\mathscr{D} \in D_k$. We define the operator $\mathscr{D}^{(i)}$ in $E(X)$ as follows: if $f = E^n(X)$, then $\mathscr{D}^{(i)}f = (\mathscr{D}f)^{(n+i)}$ (that is, the component of degree of homogeneity $n + i$ of the function $\mathscr{D}f$).

It is easy to verify that $\mathscr{D}^{(i)} \in D_k$ and therefore $\mathscr{D}^{(i)} \in D_k^i$. Since any operator of order not exceeding k is defined by its values on the space $\bigoplus_{j=0}^{k} E^j(X)$, it follows that $\mathscr{D}^{(i)}$ is not equal to 0 for finite i. Clearly $\mathscr{D} = \Sigma \mathscr{D}^{(i)}$.

b) This statement follows directly from a):

We set $I = x_1 \frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial x_2} + x_3 \frac{\partial}{\partial x_3}$.

PROPOSITION 1.

1°. $D^i = 0$ *for* $i < 0$.

2°. D^0 *is generated by the elements* $1, I, I^2, \ldots$

3°. $D_k^1/(D_{k-1}^1 + ID_{k-1}^1) = \mathbf{C}^3$ $(k = 0, 1, 2, \ldots)$.

We derive Theorem 1 from Proposition 1.

It is easy to verify that $D_k \cdot D_l \subset D_{k+l}$ and $D^i \cdot D^j \subset D^{i+j}$. Moreover, if $\mathscr{D} \in D^i$, then $[I, \mathscr{D}] = I\mathscr{D} - \mathscr{D}I = i\mathscr{D}$.

For any natural number k we set $J_k = \sum_{n\geq 0} I^n D_k^1 + \sum_{i\geq 2} D^i$ and $A_k = D^0 + J_k$. It follows from Proposition 1 and the formulae above that J_k is a two-sided ideal in the ring $D(X)$, and that A_k is a subring of $D(X)$.

If $l > k$, then $J_l \supsetneq J_k$. From this it follows that the ring $D(X)$ is not Noetherian. Since $A_l \supsetneq A_k \supset D_k$ for $l > k$, the ring $D(X)$ is not generated by the subspace D_k.

Consider the ring D_a of germs at 0 of analytic differential operators on X. The group $\mathbf{C}^*$ acts in this ring. Every element can be expanded in a convergent series $\mathscr{D} = \sum_{i=-\infty}^{\infty} \mathscr{D}^{(i)}$, where $\mathscr{D}^{(i)} \in D_a$ is a homogenous operator of degree i, and the order of $\mathscr{D}^{(i)}$ does not exceed that of $\mathscr{D}$ (specifically, $\mathscr{D}^{(i)} = \frac{1}{2\pi i} \int b_\lambda(\mathscr{D})\lambda^{-i-1}\, d\lambda$, where the integral is taken over the unit circle in in the λ-plane).

If $f \in E^n(X)$, then $\mathscr{D}^{(i)} f$ is a homogenous analytic function of degree of homogeneity $n + i$; hence $\mathscr{D}^{(i)} f \in E^{n+i}(X)$. Therefore we may assume that $\mathscr{D}^{(i)} \in D^i \subset D(X)$ (it is clear that if $\mathscr{D}^{(i)} f = 0$ for all $f \in E(X)$, then $\mathscr{D}^{(i)} = 0$). It follows from Proposition 1 that every operator $\mathscr{D} \in D_a$ can be expanded in a series $\mathscr{D} = \sum_{i=0}^{\infty} \mathscr{D}^{(i)}$, where $\mathscr{D}^{(i)} \in D^i$.

Let $J_{ak} = \{\mathscr{D} \in D_a \mid \mathscr{D}^{(i)} \in J_k \text{ for all } i\}$, and let $A_{ak} = D^0 + J_{ak}$. Then J_{ak} is a two-sided ideal in the ring D_a, and A_{ak} is a subring of D_a. Since $J_{al} \supsetneq J_{ak}$ and $A_{al} \supsetneq A_{ak} \supset D_{ak}$ for $l > k$, it follows that D_a is not Noetherian and is not generated by a subspace D_{ak}, where k is any natural number. Theorem 1 is now proved.

PROOF OF PROPOSITION 1. Consider the non-singular algebraic manifold $X_0 = X \setminus 0$.

LEMMA 2. a) *The embedding of X_0 in X induces an isomorphism $E(X) \to E(X_0)$ of rings of regular functions on X and X_0.*

b) *The embedding of X_0 in X induces an isomorphism $D(X) \to D(X_0)$ of regular differential operator rings.*

PROOF. a) follows from the fact that X is a normal manifold and that codim $\{0\}$ in X is greater than 1. Since X is an affine manifold, a) implies b).

Denote by $\mathscr{D}^i_k$ the sheaf of germs of the differential operators $\mathscr{D}$ on X_0 of order not exceeding k that satisfy the condition $[I, \mathscr{D}] = i\mathscr{D}$. Then $D^i_k = \Gamma(X_0, \mathscr{D}^i_k)$.

Consider the projective manifold $\overline{X} = X_0/\mathbf{C}^*$. It is known that $\overline{X}$ is an elliptic curve. By π we denote the natural projection $\pi\colon X_0 \to \overline{X}$.

Consider the sheaves $\Delta^i_k = \pi_*(\mathscr{D}^i_k)$ on the manifold $\overline{X}$. It is clear that $\Gamma(\overline{X}, \Delta^i_k) = \Gamma(X_0, \mathscr{D}^i_k) = D^i_k$.

By $\widetilde{\mathscr{L}}$ we denote the sheaf of functions on X_0 that are homogenous of degree 1, and we set $\mathscr{L} = \pi_*(\widetilde{\mathscr{L}})$; $\mathscr{L}$ is a sheaf on $\overline{X}$.

The following facts are easy to verify:

1°. $\mathscr{L}$ and Δ^i_k are sheaves of modules over the sheaf of rings $O_{\overline{X}}$.

2°. $\mathscr{L}$ is an invertible sheaf; $\Gamma(\overline{X}, \mathscr{L}) = \mathbf{C}^3$.

3° $\Delta^i_k = \Delta^0_k \otimes \mathscr{L}^i$, this isomorphism being consistent with the natural embeddings $\Delta^i_k \to \Delta^i_l$ for $l > k$.

4°. Set $\sigma_k = \Delta^0_k/\Delta^0_{k-1}$. Then $\sigma_k = S^k(\sigma_1)$ (where S^k is the k-th symmetric power of the sheaf).

5°. By $\widetilde{N}$ we denote a subsheaf in $\mathscr{D}^0_1$, whose sections on every neighbourhood are defined as $\{f(x)I\}$, where $f(x)$ is a function of degree of homogeneity 0. Then the sheaf $N = \pi_*(\widetilde{N})$ on $\overline{X}$ is a subsheaf of Δ^0_1. We regard N as a subsheaf of $\sigma_1 = \Delta^0_1/\Delta^0_0$.

6°. The map $1 \mapsto I$ defines the isomorphism of sheaves $O_{\overline{X}} \xrightarrow{\sim} N$.

7°. Set $\mathscr{K} = \sigma_1/N$. Then $\mathscr{K}$ is an invertible sheaf naturally isomorphic to the tangent sheaf to $\overline{X}$.

The tangent sheaf on an elliptic curve $\overline{X}$ is known to be isomorphic to $O_{\overline{X}}$. We fix a certain non-zero global section k of $\mathscr{K}$.

LEMMA 3. *For every $n > 0$ there exists an exact sequence V_n of sheaves on $\overline{X}$*

$$0 \to N^n \xrightarrow{\varphi} \sigma_n \xrightarrow{\psi} \mathscr{K} \otimes \sigma_{n-1} \to 0.$$

Here the diagram

$$\begin{array}{ccccccccc} 0 \to & N^n & \to & \sigma_n & \to & \mathscr{K} \otimes \sigma_{n-1} & \to 0 \\ & \downarrow & & \downarrow & & \downarrow & \\ 0 \to & N^{n+1} & \to & \sigma_{n+1} & \to & \mathscr{K} \otimes \sigma_n & \to 0, \end{array} \tag{1}$$

where every vertical homomorphism is obtained by multiplying $I \in \Gamma(\overline{X}, N)$ by the section, commutes.

PROOF. Construct the maps φ and ψ in a neighbourhood $U \subset \overline{X}$. Let I be a global generating element of the sheaf N, and let $\tilde{k}$ be a local section over U of the sheaf σ_1 which becomes $k \in \Gamma(\overline{X}, \mathcal{K})$ under the map $\sigma_1 \to \mathcal{K}$.

Since the sequence $0 \to N \to \sigma_1 \to \mathcal{K} \to 0$ is exact, it follows that the restriction of σ_1 to U is a free sheaf over $O_{\overline{X}}$ with the generators I and $\tilde{k}$. Then $\sigma_n = S^n(\sigma_1)$ is a free sheaf on U with the generators $\tilde{k}^i I^{n-i}$ $(i = 0, 1, \ldots, n)$. We define the maps φ and ψ by the formulae: $\varphi(I^n) = I^n$, $\psi(\tilde{k}^i I^{n-i}) = ik \otimes (\tilde{k}^{i-1} I^{n-i})$.

It is easy to verify that the sequence (V_n) is exact. It is also clear that φ does not depend on the choice of $\tilde{k}$. Let us prove that ψ does not depend on the choice of $\tilde{k}$. In fact, let $\hat{k}$ be another section of the sheaf σ_2 on U. Then $\hat{k} = \tilde{k} + fI$, where $f \in \Gamma(U, O_X)$,

$$\psi(\hat{k}^i I^{n-i}) = \psi\Big(\Big(\sum_{j=0}^{i} C_i^j \tilde{k}^{i-j} f^j\Big) I^{n-i+j}\Big) = k \otimes \sum_j C_i^j (i-j)\, \tilde{k}^{i-j-1} I^{n-i+j} f^j =$$
$$= i\,(k \otimes (\hat{k} + fI)^{i-1} I^{n-i}).$$

So the exact sequence (V_n) is defined globally.

It follows from the construction of the homomorphisms φ and ψ that the diagram (1) is commutative. This proves Lemma 3.

We are interested in the spaces $H^0(\overline{X}, \sigma_k) = \Gamma(\widetilde{X}, \sigma_k)$ and $H^1(\overline{X}, \sigma_k)$.

LEMMA 4. $\dim H^0(\overline{X}, \sigma_1) = 1$.

PROOF. Every first-order operator $\mathcal{D}$ can be split uniquely into a sum $\mathcal{D} = f + \mathcal{D}'$, where f is the operator of multiplication by the function $f = \mathcal{D}(1)$ and $\mathcal{D}' = \mathcal{D} - f$ is differentiation in the ring of functions. From this it follows that $\Delta_1^0 = \sigma_1 \oplus \Delta_0^0$. Since $\Gamma(\overline{X}, \Delta_0^0) = \mathbf{C}$, we need only show that $\Gamma(\overline{X}, \Delta_1^0)$ is two-dimensional.

Let $\mathcal{D} \in D_1^0$. Then $\mathcal{D} = f + \mathcal{D}'$, where $f = \mathcal{D}(1)$, and where f and $\mathcal{D}$ have the degree of homogeneity 0; in particular, $f \in \mathbf{C}$.

Let us prove that $\mathcal{D}' = cI$ where $c \in \mathbf{C}$. Set $f_1' = \mathcal{D}'x_1$, $f_2' = \mathcal{D}'x_2$, $f_3' = \mathcal{D}'x_3$. Then $f_i' \in E^1(X)$ and we can extend them to linear functions f_i on $\mathbf{C}^3$. The operator $\mathcal{D}'$ coincides with $\tilde{\mathcal{D}} = f_1 \dfrac{\partial}{\partial x_1} + f_2 \dfrac{\partial}{\partial x_2} + f_3 \dfrac{\partial}{\partial x_3}$.

Therefore $\tilde{\mathcal{D}}(x_1^3 + x_2^3 + x_3^3) = 3(f_2 x_1^2 + f_2 x_2^2 + f_3 x_3^2) = c(x_1^3 + x_2^3 + x_3^3)$, where $c \in \mathbf{C}[x_1, x_2, x_3]$ (here equality is considered in the ring $\mathbf{C}[x_1, x_2, x_3]$). Since f_1, f_2, f_3 are linear functions, we have $c \in \mathbf{C}$ and $f_i = cx_i$, that is, $\mathcal{D}' = cI$.

So we have shown that every operator $\mathscr{D} \in D_1^0$ is of the form $\mathscr{D} = c_1 + cI$, where $c_1, c \in \mathbf{C}$. The lemma is now proved.

We shall use the following well-known facts on the cohomology of coherent sheaves on an elliptic curve (see [2]).

1°. If $\mathscr{F}$ is a coherent sheaf on $\bar{X}$, then $H^i(\bar{X}, \mathscr{F})$ for $i \geqslant 2$.

2°. $\dim H^0(\bar{X}, O_X) = \dim H^1(\bar{X}, O_X) = 1$.

3°. If $\mathscr{L}$ is an invertible sheaf on $\bar{X}$ and $\dim H^0(\bar{X}, \mathscr{L}) > 1$, then $H^1(\bar{X}, \mathscr{L}) = 0$ and $H^0(\bar{X}, \mathscr{L}^i) = 0$ for $i < 0$.

LEMMA 5. *Consider the exact sequence of sheaves*

$$0 \to N^n \to \sigma_n \to \mathscr{K} \otimes \sigma_{n-1} \to 0 \qquad (V_n).$$

Then $\dim H^0(\bar{X}, \sigma_n) = 1$, *and the boundary homomorphism* $\delta_n : H^0(\bar{X}, \mathscr{K} \otimes \sigma_{n-1}) \to H^1(\bar{X}, N^n)$ *is an isomorphism.*

PROOF. We prove the lemma by induction on n.

Let $n = 1$. We write out the exact cohomology sequence

$$0 \to H^0(\bar{X}, N) \xrightarrow{\tilde{\varphi}_1} H^0(\bar{X}, \sigma_1) \xrightarrow{\tilde{\psi}_1} H^0(\bar{X}, \mathscr{K}) \xrightarrow{\delta_1}$$
$$\to H^1(\bar{X}, N) \xrightarrow{\varphi_1'} H^1(\bar{X}, \sigma_1) \xrightarrow{\psi_1'} H^1(\bar{X}, \mathscr{K}) \to 0.$$

Recall that N and $\mathscr{K}$ are isomorphic to $O_{\bar{X}}$. Hence $\tilde{\varphi}_1$ is an isomorphism (because $\dim H^0(X, \sigma_2) = \dim H^0(\bar{X}, N) = 1$). This means that $\tilde{\psi}_1 = 0$. Therefore δ_1 is an isomorphism. Hence $\varphi_1' = 0$, and ψ_1' is an isomorphism.

Suppose that the lemma has been proved for the sequence V_n; let us prove it for V_{n+1}. We write out the exact cohomology sequences that correspond to the sequences V_n and V_{n+1}, and connect them according to diagram (1) (see Lemma 3)

$$\begin{array}{ccccccccccccc}
0 \to H^0(N^n) & \longrightarrow & H^0(\sigma_n) & \to & H0(\mathscr{K} \otimes \sigma_{n-1}) & \xrightarrow{\delta_n} & H^1(N^n) & \to & H^1(\sigma_n) & \to & H^1(\mathscr{K} \otimes \sigma_{n-1}) \to 0 \\
\downarrow & & \downarrow & & \downarrow \tau & & \downarrow \eta & & \downarrow & & \downarrow \\
0 \to H^0(N^{n+1}) & \xrightarrow{\tilde{\varphi}_{n+1}} & H^0(\sigma_{n-1}) & \to & H^0(\mathscr{K} \otimes \sigma_n) & \xrightarrow{\delta_{n+1}} & H^1(N^{n+1}) & \to & H^1(\sigma_{n+1}) & \to & H^1(\mathscr{K} \otimes \sigma_n) \to 0.
\end{array}$$

It is clear that η is an isomorphism. By the inductive hypothesis δ_n is an isomorphism. Since $\eta\delta_n = \delta_{n+1}\tau \neq 0$, it follows that $\delta_{n+1} \neq 0$. Here $\mathscr{K} \otimes \sigma_n \approx \sigma_n$, and by the inductive hypothesis $\dim H^0(\bar{X}, \mathscr{K} \otimes \sigma_n) = 1$. Therefore δ_{n+1} is an isomorphism. It is now clear that $\tilde{\varphi}_{n+1}$ is an isomorphism and that $\dim H^0(\bar{X}, \sigma_{n+1}) = 1$. Lemma 5 is now proved.

Statement 2° of Proposition 1 is a direct consequence of this lemma. For it follows at once from the exact sequence $0 \to \Delta_{n-1}^0 \to \Delta_n^0 \to \sigma_n \to 0$ that $\dim D_n^0 \leqslant \dim D_{n-1}^0 + 1$ and therefore $\dim D_n^0 \leqslant n + 1$. Hence D_n^0 is generated by the elements $1, I, I^2, \ldots, I^n$.

LEMMA 6. 1°. *Let* $i < 0$. *Then* a) $H^0(\overline{X}, \sigma_n \otimes \mathcal{L}^i) = 0$; b) $H^0(X, \Delta_n^i) = 0$.

Bearing in mind that $H^0(\overline{X}, \mathcal{L}^i) = 0$, it is easy to prove the lemma by induction over n.

Lemma 6 implies Statement 1° of Proposition 1.

LEMMA 7. *For any* n ($n = 0, 1, \ldots$) we have

1°. $H^1(\overline{X}, \sigma_n \otimes \mathcal{L}) = 0$.

2°. *We consider the natural map* θ: $\sigma_{n-1} \otimes \mathcal{L} \to \sigma_n \otimes \mathcal{L}$ (*multiplication by I*) *and denote by* θ' *the corresponding cohomology map* θ': $H^0(\overline{X}, \sigma_{n-1} \otimes \mathcal{L}) \to H^0(\overline{X}, \sigma_n \otimes \mathcal{L})$. *Then* θ' *is an embedding, and* $H^0(\overline{X}, \sigma_n \otimes \mathcal{L})/\mathrm{Im}\,\theta'$ *is three-dimensional.*

3°. $H^1(\overline{X}, \Delta_n^1) = 0$.

4°. $\dim (D_n^1/(D_{n-1}^1 + ID_{n-1}^1)) = 3$.

PROOF. From the exact sequence of sheaves

$$0 \to \sigma_{n-1} \otimes \mathcal{L} \xrightarrow{\theta} \sigma_n \otimes \mathcal{L} \to \mathcal{K}^n \otimes \mathcal{L} \to 0$$

we obtain the exact cohomology sequence

$$0 \to H^0(\overline{X}, \sigma_{n-1} \otimes \mathcal{L}) \xrightarrow{\theta'} H^0(\overline{X}, \sigma_n \otimes \mathcal{L}) \to H^0(\overline{X}, \mathcal{K}^n \otimes \mathcal{L}) \to$$
$$\to H^1(\overline{X}, \sigma_{n-1} \otimes \mathcal{L}) \to H^1(\overline{X}, \sigma_n \otimes \mathcal{L}) \to H^1(\overline{X}, \mathcal{K}^n \otimes \mathcal{L}) \to 0.$$

Since $H^1(\overline{X}, \mathcal{K}^n \otimes \mathcal{L}) = H^1(\overline{X}, \mathcal{L}) = 0$, we find by induction on n that $H^1(\overline{X}, \sigma_n \otimes \mathcal{L}) = 0$ ($n = 0, 1, \ldots$). Here θ' is an embedding and $H^0(\overline{X}, \sigma_n \otimes \mathcal{L})/\mathrm{Im}\,\theta' = H^0(\overline{X}, \mathcal{K}^n \otimes \mathcal{L}) = H^0(\overline{X}, \mathcal{L}) = \mathbf{C}^3$.

From the exact cohomology sequence corresponding to the exact sequence of sheaves

$$0 \to \Delta_{n-1}^1 \to \Delta_n^1 \to \sigma_n \otimes \mathcal{L} \to 0$$

we find by induction on n that $H^1(\overline{X}, \Delta_n') = 0$ and $D_n^1/D_{n-1}^1 = H^0(\overline{X}, \sigma_n \otimes \mathcal{L})$ ($n = 0, 1, \ldots$). Therefore $D_n^1/(D_{n-1}^1 + ID_{n-1}^1) = H^0(\overline{X}, \sigma_n \otimes \mathcal{L})/\mathrm{Im}\,\theta' = \mathbf{C}^3$. The lemma is now proved.

This lemma contains Statement 3° of Proposition 1.

NOTE. By the same method as in Lemma 7 we can show that $D_k^i = x_1 D_k^{i-1} + x_2 D_k^{i-1} + x_3 D_k^{i-1}$ for $i > 1$ and for any k.

References

[1] B. Malgrange, Analytic spaces.
= Uspekhi Mat. Nauk **27** : 1 (1972), 147–184.

[2] J. Serre, Groupes algébriques et corps de classes, Hermann & Cie, Paris 1959.