DIFFERENTIAL OPERATORS ON THE BASE AFFINE SPACE AND A STUDY OF $\mathfrak{g}$-MODULES

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The present work consists of two parts. In the first part we study the ring of regular differential operators on the base affine space of a complex semisimple group. By the *base affine space* of a group we mean the quotient space $A = N_+ \backslash G$ of the group G by a maximal unipotent subgroup N_+. Experience in representation theory suggests that for many problems in representation theory the solution results from a careful study of the base affine space. In particular, the structure of the ring of regular differential operators on A seems to be closely connected with the representations of the real forms of the group G. In addition to the connections with representation theory, the study of this ring yields an instructive and rather advanced example for the study of the rings of regular differential operators on algebraic varieties, an area in which not much is known so far.

We approach the study of the differential operators on A by establishing a connection between the regular functions on the group G and the regular differential operators on the base affine space A. We would also like to draw the reader's attention to Conjecture II, where the notion of the generalized Segal—Bargmann space for a representation of a compact Lie group is introduced.

The second part of the work is formally independent of the first and is devoted to the algebraic study of modules over the Lie algebra $\mathfrak{g}$ of the group G. We restrict ourselves to a category of $\mathfrak{g}$-modules, which is closely connected with the theory of highest weight. We shall call this category of $\mathfrak{g}$-modules the category O. The category O contains in a natural way every finite-dimensional representation of the Lieal gebra $\mathfrak{g}$. The fundamental result of this part lies in constructing a resolution for finite-dimensional $\mathfrak{g}$-modules. The simplest objects of the category O are the modules M_χ and it seems important that the resolution consists of modules which are direct sums of these simplest modules. The description of the modules occurring in the composition series of the modules M_χ, which is given in the Appendix, is also useful. Unfortunately, the complete structure of these composition series is not known to us yet.

We think that the methods developed in the second part of this paper may turn out to be useful in the further study of questions considered in the first part.

The fundamental content of this work is concentrated in Theorem 6.3 and Conjectures I and II in the first part and Theorems 8.12 and 10.1 in the second.

§ 1. Notations and preliminaries

$\mathfrak{g}$ is a semisimple Lie algebra of rank r over $\mathbf{C}$, $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}$. Δ denotes the root system of $\mathfrak{g}$ corresponding to $\mathfrak{h}$, with a fixed ordering, Δ_+ and Δ_- the system of the positive and negative roots, respectively, Σ the set of simple roots, and $\varrho=\frac{1}{2}\sum_{\gamma\in\Delta_+}\gamma$. $E_\gamma\in\mathfrak{g}$ is the root vector corresponding to the root $\gamma\in\Delta$. Here we have $\gamma([E_\gamma, E_{-\gamma}])=2$.

$\mathfrak{n}_+$ is the subalgebra of $\mathfrak{g}$ spanned by the vectors E_γ, $\gamma\in\Delta_+$, while $\mathfrak{n}_-$ is the subalgebra of $\mathfrak{g}$ spanned by E_γ, $\gamma\in\Delta_-$. $\mathfrak{b}=\mathfrak{h}\oplus\mathfrak{n}_+$. $U(\mathfrak{g})$, $U(\mathfrak{n}_+)$, $U(\mathfrak{n}_-)$ are the universal enveloping algebras of $\mathfrak{g}$, $\mathfrak{n}_+$, $\mathfrak{n}_-$, respectively; $Z(\mathfrak{g})$ is the centre of $U(\mathfrak{g})$.

$\mathfrak{h}^*$ is the dual space of $\mathfrak{h}$.

G is a complex semisimple Lie group with Lie algebra $\mathfrak{g}$; H, N_+, N_- and B are the subgroups of G corresponding to the subalgebras $\mathfrak{h}$, $\mathfrak{n}_+$, $\mathfrak{n}_-$ and $\mathfrak{b}$, respectively.

$A=N_+\diagdown G$ is the base affine space of the group G.

Additional notations, used in Part 2.

$\mathbf{Z}_+$ is the set of non-negative integers.

$\mathfrak{h}^*_{\mathbf{R}}$ denotes the real linear subspace of $\mathfrak{h}^*$ spanned by all roots $\gamma\in\Delta$.

$\langle , \rangle$ is the scalar product in $\mathfrak{h}^*$ constructed with the help of the Killing form of the algebra $\mathfrak{g}$; $\|\cdot\|$ is the corresponding norm in $\mathfrak{h}^*_{\mathbf{R}}$.

$\mathfrak{h}^*_{\mathbf{Z}}$ is the lattice in $\mathfrak{h}^*_{\mathbf{R}}$ consisting of those $\chi\in\mathfrak{h}^*$ for which $2\langle\chi,\gamma\rangle/\langle\gamma,\gamma\rangle\in\mathbf{Z}$ for all $\gamma\in\Delta$.

$$K=\{\chi\in\mathfrak{h}^* \mid \chi=\sum_{\alpha\in\Sigma} n_\alpha\cdot\alpha,\ n_\alpha\in\mathbf{Z}_+\};\quad K\subset\mathfrak{h}^*_{\mathbf{Z}}.$$

$\chi_1\geqq\chi_2$ means that $\chi_1-\chi_2\in K$ ($\chi_1, \chi_2\in\mathfrak{h}^*$).

W is the Weyl group of the algebra $\mathfrak{g}$, $\sigma_\gamma\in W$ is the reflexion corresponding to the root $\gamma\in\Delta$, i.e. $\sigma_\gamma\chi=\chi-2\langle\chi,\gamma\rangle\langle\gamma,\gamma\rangle^{-1}\gamma$. We note that $\sigma_\alpha\varrho=\varrho-\alpha$ for $\alpha\in\Sigma$.

$\chi_1\sim\chi_2$ for $\chi_1,\chi_2\in\mathfrak{h}^*$ means that there exists an element $w\in W$ such that $\chi_1=w\chi_2$.

$l(w)$ is the length of the element $w\in W$, i.e. the smallest possible number of factors in a decomposition $w=\sigma_{\alpha_1}\cdot\ldots\cdot\sigma_{\alpha_k}$, $\alpha_i\in\Sigma$.

$$W^{(i)}=\{w\in W \mid l(w)=i\}.$$

$\Xi_\gamma=\{\chi\in\mathfrak{h}^*_{\mathbf{R}} \mid \langle\chi,\gamma\rangle=0\}$; the connected components of $\mathfrak{h}^*_{\mathbf{R}}\diagdown(\bigcup_{\gamma\in\Delta}\Xi_\gamma)$ are called the *Weyl chambers;* $\bar{C}$ is the closure of the Weyl chamber C; C^+ is the Weyl chamber containing ϱ. The group W acts on the set of Weyl chambers simply transitively. Two Weyl chambers C_1 and C_2 are called *neighbouring* if $\dim(\bar{C}_1\cap\bar{C}_2)=\dim\mathfrak{h}^*_{\mathbf{R}}-1$. In this case there exists a unique element $\gamma\in\Delta_+$ such that $\sigma_\gamma C_1=C_2$ and the hyperplane Ξ_γ separates C_1 and C_2;

$$D=\mathfrak{h}^*_{\mathbf{Z}}\cap C^+.$$

An element $\chi \in \mathfrak{h}_{\mathrm{R}}^*$ is called *regular* if $\langle \chi, \gamma \rangle \neq 0$ for all $\gamma \in \Delta$.

Let M be a $\mathfrak{h}$-module, $\chi \in \mathfrak{h}^*$. Put

$$M^{(\chi)} = \{f \in M \mid xf = \chi(x) \cdot f \text{ for all } x \in \mathfrak{h}\};$$
$$P(M) = \{\chi \in \mathfrak{h}^* \mid M^{(\chi)} \neq 0\}.$$

Let M be a $\mathfrak{g}$-module and $0 = M_0 \subset M_1 \subset \ldots \subset M_k = M$ its Jordan—Hölder composition series, $L_i = M_i/M_{i-1}$ are simple $\mathfrak{g}$-modules. The collection of the modules L_i, with multiplicity, is called the Jordan—Hölder decomposition of M and is denoted by $JH(M)$.

PART I

DIFFERENTIAL OPERATORS

§ 2. Introduction

Let G be a connected complex semisimple Lie group of rank r, B a Borel subgroup, N_+ the unipotent radical of B, $H \subset B$ the Cartan subgroup of B. The quotient space $A = N_+ \setminus G$ plays a fundamental rôle in representation theory; it is called the *base affine space* of the group G.

The aim of this part is to study the ring $\mathscr{D}(A)$ of regular differential operators on A.

First of all we study the possible connections between the space of regular differential operators on A and the space $\mathscr{E}(G)$ of regular functions on G. More precisely, Conjecture I claims the possibility of embedding $\mathscr{E}(G)$ into $\mathscr{D}(A)$ (operation $f \to \hat{f}$); here $\mathscr{D}(A) = U(\mathfrak{h}) \otimes_{\mathbf{C}} \mathscr{E}(G)$, where $U(\mathfrak{h})$ is the universal enveloping algebra of the Lie algebra $\mathfrak{h}$ of the group H. In this part we prove a result (Theorem 6.6) weaker than Conjecture I, namely we construct an isomorphism between the L-modules $L \otimes_{\mathbf{C}} \mathscr{E}(G)$ and $L \otimes_{U(\mathfrak{h})} \mathscr{D}(A)$, where L is the quotient field of the ring $U(\mathfrak{h})$.

Further on we construct a scalar product in the ring $\mathscr{E}(A)$ of regular functions on A which is invariant under the action of the maximal compact subgroup $K \subset G$. The completion of the space $\mathscr{E}(A)$ by this scalar product consists of analytic functions on the complex manifold A. This space is a generalization of the Segal—Bargmann space. Conjecture II states that an operator adjoint to a differential operator is again a differential operator, hence the ring $\mathscr{D}(A)$ is selfadjoint with respect to the introduced scalar product. It has to be noted that the introduced involution in $\mathscr{D}(A)$ does not preserve the order of a differential operator. For instance, in the case of the group of matrices of order h, the adjoint to the simplest operator of order zero will be an operator of order $h-1$.

§ 3. Regular differential operators

In this section the rings of regular differential operators on G and A are introduced. We shall consider G and A as algebraic varieties over $\mathbf{C}$. The projection $\pi: G \to A$ is a morphism of algebraic varieties. The rings of regular functions on G and A will be denoted by $\mathscr{E}(G)$ and $\mathscr{E}(A)$, respectively. Let $\pi^*: \mathscr{E}(G) \to \mathscr{E}(A)$

be the embedding induced by the mapping π. The image $\pi^* \mathscr{E}(A)$ consists of exactly those functions which are constant on the left cosets of the subgroup N.

The variety A is non-singular and quasi-affine. More precisely, let

$$\hat{A} = \text{Spec max } \mathscr{E}(A)$$

be the affine algebraic variety corresponding to $\mathscr{E}(A)$. Then there is a natural isomorphism between A and a dense open subset of $\hat{A}$.

Let us define in the space $\mathscr{E}(G)$ the left and right representations L^G and R^G of the group G by the usual formulas

$$(L^G_{g_0} f)(g) = f(g_0^{-1} g), \quad (R^G_{g_0} f)(g) = f(gg_0), \quad g, g_0 \in G.$$

It is a common property of both of these representations that every element $f \in \mathscr{E}(G)$ is contained in some finite-dimensional invariant subspace.

The group G acts naturally on the space A (right translations). In addition one can also define the left-hand action of the Cartan subgroup on A by associating to each element $h \in H$ the transformation $x \to hx$. The element $hx \in A$ is well defined because H normalizes N.

Let us define in the space $\mathscr{E}(A)$ the representations L^A of the group H and R^A of the group G by means of the formulas

$$(L^A_h f)(x) = f(h^{-1} x), \quad (R^A_g f)(x) = f(xg); \; x \in A, \; h \in H, \; g \in G.$$

Obviously, $L^G_h \pi^* = \pi^* L^A_h$ and $R^G_h \pi^* = \pi^* R^A_h$.

By differentiating R^A we obtain for each $X \in \mathfrak{g}$ an operator $R^A_X : \mathscr{E}(A) \to \mathscr{E}(A)$. These operators determine a representation of $\mathfrak{g}$ that extends to a representation of $U(\mathfrak{g})$; here the operator corresponding to an element $X \in U(\mathfrak{g})$ will also be denoted by R^A_X. We define the similar representations R^G_X and L^G_X of the algebra $U(\mathfrak{g})$ in $\mathscr{E}(G)$ and the representation L^A_Y of the algebra $U(\mathfrak{g})$ in $\mathscr{E}(A)$.

Definition 3.1. Let X be a quasi-affine variety and $\mathscr{E}(X)$ be the ring of regular functions on X. A linear mapping $D : \mathscr{E}(X) \to \mathscr{E}(X)$ is called a *regular differential operator of order* $\leqq k$ ($k \geqq 0$) on X if it satisfies the condition

$$[f_{k+1}[f_k \dots [f_1, D] \dots]] = 0 \tag{3.1}$$

for any $f_1, f_2, \dots, f_{k+1} \in \mathscr{E}(X)$. In (3.1) f_i denotes the operator of multiplication by f_i.

The differential operators on X form a ring which will be denoted by $\mathscr{D}(X)$.

Remark. The definition given here coincides with that of a differential operator on an arbitrary algebraic variety given in [1].

It is easy to see that a differential operator of order zero is an operator of multiplication by a function $f \in \mathscr{E}(X)$.

Any vector field over X determines a differential operator of order $\leqq 1$ on X. It can be verified that on composing an operator of order $\leqq k$ with an operator of order $\leqq l$ we obtain a differential operator of order $\leqq k+l$, moreover the commutator of two such operators is of order $\leqq k+l-1$.

We shall use the following properties of differential operators (see [1]).

Lemma 3.2. 1. *If $Y \subset X$ is a dense open subset (in the Zariski topology) of a quasi-affine variety X, than every regular differential operator can be restricted to Y. More precisely, there exists a unique differential operator $D': \mathscr{E}(Y) \to \mathscr{E}(Y)$ whose restriction to $\mathscr{E}(X) \hookrightarrow \mathscr{E}(Y)$ coincides with D.*

2. *Let X be a non-singular variety and $Z_1, \ldots, Z_n$ be a system of vector fields over X which defines a basis in the tangent space at each point $x \in X$. Then every differential operator D on X has a unique representation in the form*

$$D = \sum a_{i_1 i_2 \ldots i_n}(x) Z_1^{i_1} Z_2^{i_2} \ldots Z_n^{i_n},$$

where i_j are non-negative integers, $a_{i_1 \ldots i_n}$ are regular functions only a finite number of which are different from zero.

For any element $X \in U(\mathfrak{g})$ the operators R_X^G and L_X^G are differential operators on G, moreover L_Y^A, $Y \in U(\mathfrak{h})$ and R_X^A are differential operators on A. The description of the ring of differential operators on G yields the following proposition.

Proposition 3.3. The mapping

$$\vartheta: \mathscr{E}(G) \underset{\mathbf{C}}{\otimes} U(\mathfrak{g}) \to \mathscr{D}(G)$$

given by the formula

$$\vartheta\left(\sum f_i \otimes X_i\right) = \sum f_i L_{X_i}^G$$

is an isomorphism of left $\mathscr{E}(G)$-modules.

The proof of Proposition 3.3. follows easily from Lemma 3.2.

In what follows the element $\vartheta^{-1}(D) \in \mathscr{E}(G) \otimes U(\mathfrak{g})$ will be called the *standard form* of $D \in \mathscr{D}(G)$.

Now let us turn to the study of the ring $\mathscr{D}(A)$. In this ring we can define a representation of the group G. Indeed, we put

$$D^g = R_g^A D R_{g^{-1}}^A, \quad g \in G, \quad D \in \mathscr{D}(A).$$

Similarly, we put

$${}^h D = L_h^A D L_{h^{-1}}^A, \quad h \in H, \quad D \in \mathscr{D}(A).$$

We notice that the variety A is smooth, but it is not an affine variety and there may exist differential operators on A which cannot be expressed by operators of the first order. (See e.g. Example 2.)

We shall say that a differential operator D' on G is a *lifting* of the operator D on A, if

$$D' \pi^* f = \pi^* D f, \quad f \in \mathscr{E}(A).$$

Theorem 3.4. [2]. *Every differential operator on A can be lifted to G.*

In order to prove this theorem we need the following lemma.

Lemma 3.5. *Let* $0=\pi(e)\in A$. *There exists a mapping* $\eta:\mathscr{D}(A)\to U(\mathfrak{g})/\mathfrak{n}_+U(\mathfrak{g})$ *such that for all* $D\in\mathscr{D}(A)$ *and all* $f\in\mathscr{E}(A)$

$$(R_X^A f)(0)=(Df)(0) \tag{3.2}$$

for every element $X\in U(\mathfrak{g})$ *belonging to the coset* $\eta(D)$.

Proof. It is easy to see that if $X\in\mathfrak{n}_+$ then $R_X^A f(0)=0$ for all $f\in\mathscr{E}(A)$. Therefore the validity of the equality (3.2) does not depend on the choice of the element X in a $\mathfrak{n}_+U(\mathfrak{g})$-coset.

Now let $X_1,\ldots,X_N$ be a basis in $\mathfrak{h}\oplus\mathfrak{n}_-$. Then the vector fields $R_{X_i}^A$ form a basis in the tangent space at each point of a certain affine neighbourhood V of the point $0\in A$. According to Lemma 3.2, in this neighbourhood V the operator D can be expressed in the form

$$D=\sum a_{i_1i_2\ldots i_N}(R_{X_1}^A)^{i_1}\cdot\ldots\cdot(R_{X_N}^A)^{i_N}.$$

Put

$$X=\sum a_{i_1i_2\ldots i_N}(0)X_1^{i_1}\ldots X_N^{i_N}\in U(\mathfrak{g}).$$

It is easy to see that the image $\eta(D)$ of the element X in $U(\mathfrak{g})/\mathfrak{n}_+U(\mathfrak{g})$ satisfies the condition of the lemma.

Note that the element $\eta(D)$ is uniquely determined by the equality (3.2) (see [2]).

Let $\tau:U(\mathfrak{g})\to U(\mathfrak{g})$ be an anti-automorphism such that $\tau(X)=-X$ for $X\in\mathfrak{g}$. Obviously, $\tau(\mathfrak{n}_+U(\mathfrak{g}))=U(\mathfrak{g})\mathfrak{n}_+$, so τ determines an isomorphism

$$\bar\tau:\ U(\mathfrak{g})/\mathfrak{n}_+U(\mathfrak{g})\to U(\mathfrak{g})/U(\mathfrak{g})\mathfrak{n}_+.$$

Definition 3.6. Let $D\in\mathscr{D}(A)$. Define a function $\sigma_D(g)$ on G with values in $U(\mathfrak{g})/U(\mathfrak{g})\mathfrak{n}_+$ by the formula

$$\sigma_D(g)=\bar\tau\eta(D^g).$$

One can verify [2, § 8] that $\sigma_D(g)$ is a regular function on G, i.e.

$$\sigma_D(g)\in\mathscr{E}(G)\otimes U(\mathfrak{g})/U(\mathfrak{g})\mathfrak{n}_+.$$

Now we are ready to complete the proof of the theorem. Let $D\in\mathscr{D}(A)$ and consider an arbitrary element $\sigma_D'\in\mathscr{E}(G)\otimes U(\mathfrak{g})$ which is sent into $\sigma_D(g)$ by the natural projection

$$\mathscr{E}(G)\otimes U(\mathfrak{g})\to\mathscr{E}(G)\otimes U(\mathfrak{g})/U(\mathfrak{g})\mathfrak{n}_+.$$

Put

$$D'=\vartheta(\sigma_D'(g))\in\mathscr{D}(G).$$

We shall show that D' is a lifting of D to G, i.e. that

$$(D'\pi^*f)(g)=(\pi^*Df)(g),\quad f\in\mathscr{E}(A),\ g\in G \tag{3.3}$$

(3.3) can be rewritten in the form

$$(R_g^G D' R_{g^{-1}}^G \pi^* R_g^A f)(e) = (\pi^* D^g R_g^A f)(e).$$

It is easy to see that $R_g^G D' R_{g^{-1}}^G$ is a regular differential operator on G and that $\vartheta^{-1}(R_g^G D' R_{g^{-1}}^G)$ is projected into σ_{D^g} under the natural mapping

$$\mathscr{E}(G)\otimes U(\mathfrak{g}) \to \mathscr{E}(G)\otimes U(\mathfrak{g})/U(\mathfrak{g})\mathfrak{n}_+.$$

Therefore (replacing D by D^g and f by $R_g^A f$) it suffices to prove that $(D'\pi^* f)(e) = (\pi^* Df)(e)$.

Let $X=\sigma'_D(e)\in U(\mathfrak{g})$. Clearly, then $(L_X^G f)(e)=(D'f)(e)$ for $f\in\mathscr{E}(G)$. Moreover, the image of $\tau^{-1}(X)$ in $U(\mathfrak{g})/\mathfrak{n}_+ U(\mathfrak{g})$ is equal to $\eta(D)$, hence

$$(\pi^* Df)(e) = (Df)(0) = (R_{\tau^{-1}X}^A f)(0) = (R_{\tau^{-1}X}^G \pi^* f)(e), \quad f\in\mathscr{E}(A).$$

Therefore, the required equality is implied by the following lemma.

Lemma 3.7. *If* $Y\in U(\mathfrak{g})$, $f\in\mathscr{E}(G)$ *then*

$$(L_{\tau Y}^G f)(e) = (R_Y^G f)(e).$$

Proof. If $Y\in\mathfrak{g}$ then the lemma follows from the definitions of L^G and R^G. Assume now that the lemma is valid for $Y_1, Y_2\in U(\mathfrak{g})$. Then we have

$$(R_{Y_1Y_2}^G f)(e) = (R_{Y_1}^G R_{Y_2}^G f)(e) = (L_{\tau(Y_1)}^G R_{Y_2}^G f)(e) = (R_{Y_2}^G L_{\tau(Y_1)}^G f)(e) =$$
$$= (L_{\tau(Y_2)}^G L_{\tau(Y_1)}^G f)(e) = (L_{\tau(Y_1Y_2)}^G f)(e).$$

(Here we use the fact that L_Y^G and $R_{Y'}^G$ commute for any $Y, Y'\in U(\mathfrak{g})$.) Hence we have the lemma for $Y=Y_1\cdot Y_2$, and the proof is complete.

Proposition 3.8. Let us denote by I_+ the left ideal in the ring $\mathscr{D}(G)$ spanned by the operators L_X^G, $X\in\mathfrak{n}_+$.

1) Let $D\in\mathscr{D}(G)$. Then D gives rise to a differential operator on A (i.e. $D(\mathscr{E}(A))\subset\mathscr{E}(A)$) if and only if

$$[L_X^G, D]\in I_+ \quad \text{for} \quad X\in\mathfrak{n}_+.$$

2) $D\in\mathscr{D}(G)$ gives rise to the zero operator on A if and only if $D\in I_+$.

The proof of this proposition is rather simple and is left to the reader.

We shall now describe how these conditions can be expressed in terms of the standard form of the operator D. We remark that

1) $D\in I_+$ if and only if

$$\vartheta^{-1}(D)\in\mathscr{E}(G)\otimes U(\mathfrak{g})\mathfrak{n}_+.$$

2) If $\vartheta^{-1}(D) = \sum f_i\otimes X_i\in\mathscr{E}(G)\otimes U(\mathfrak{g})$ and $X\in\mathfrak{g}$ then

$$\vartheta^{-1}([L_X^G, D]) = \sum L_X^G f_i\otimes X_i + \sum f_i\otimes[X, X_i].$$

Corollary 3.9. Let $D \in \mathscr{D}(A)$. Then it has a unique lifting D' such that

$$\vartheta^{-1}(D') \in \mathscr{E}(G) \otimes U(\mathfrak{h} \oplus \mathfrak{n}_-).$$

Definition 3.10. *1)* The element $\vartheta^{-1}(D') \in \mathscr{E}(G) \otimes U(\mathfrak{h} \oplus \mathfrak{n}_-)$ is called the *standard form* of the operator $D \in \mathscr{D}(A)$ and will be denoted by $s(D)$.

2) The *lowest term* $s_0(D)$ of an operator $D \in \mathscr{D}(A)$ is defined as the element of $\mathscr{E}(G) \otimes U(\mathfrak{h})$ which is equal to the projection of $s(D)$ under the decomposition

$$U(\mathfrak{h} \oplus \mathfrak{n}_-) = U(\mathfrak{h}) \oplus \mathfrak{n}_- U(\mathfrak{h} \oplus \mathfrak{n}_-)$$

Definition 3.11. Let us denote by *Wu* the subring of $\mathscr{D}(A)$ consisting of the operators

$$L_X^A,\ X \in U(\mathfrak{h}).$$

§ 4. Conjecture I; examples

Conjecture I. *There exists a mapping* $\mathscr{E}(G) \to \mathscr{D}(A)$, $f \to \tilde{f}$ *with the following properties:*

1) $\tilde{f} = f$ *for* $f \in \mathscr{E}(A) \subset \mathscr{E}(G)$.

2) The mapping $f \to \tilde{f}$ *commutes with the representations* R_g *and* L_h.

3) The mapping $\mathscr{E}(G) \otimes Wu \to \mathscr{D}(A)$ *given by the formula*

$$\sum f_i \otimes Z_i \to \sum \tilde{f}_i Z_i,\ f_i \in \mathscr{E}(G),\ Z_i \in Wu$$

is an isomorphism of Wu-modules.

4) Let $s_0(\tilde{f}) = \sum f_i \otimes X_i$. *Then* $f = \sum f_i \cdot X_i(-\varrho)$ (here $\varrho \in \mathfrak{h}^*$ is half-sum of the positive roots; $X_i \in U(\mathfrak{h})$ is considered as a polynomial function on $\mathfrak{h}^*$).

A weakened version of Conjecture I is

Conjecture I′. $\mathscr{D}(A)$ *is a free Wu-module.*

We remark that the mapping $f \to \tilde{f}$ (if it exists) is not uniquely determined by the properties *1)—4)*. We assume however that there exists a "natural" mapping $f \to \tilde{f}$. The following examples will perhaps illuminate to the reader what we have in mind.

We shall now present several examples illustrating the notions and facts expounded above.

Example 1. $G = SL(2, \mathbf{C})$ is the group of 2×2 matrices of determinant 1. We choose as $N_+ \subset G$ the subgroup of all matrices of the form

$$\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}.$$

In this case $A=N_+\backslash G$ can be identified with the punctured complex plane i.e. $A=\mathbf{C}^2\backslash\{(0,0)\}$, while the mapping $\pi: G\to A$ is of the form

$$g=\begin{pmatrix} u_1 & u_2 \\ z_1 & z_2 \end{pmatrix}\to(z_1, z_2).$$

Let

$$E_-=\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},\quad H=\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},\quad E_+=\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

be a basis of $\mathfrak{g}$. Then

$$L^G_{E_-}=-u_1\frac{\partial}{\partial z_1}-u_2\frac{\partial}{\partial z_2},$$

$$L^G_{E_+}=-z_1\frac{\partial}{\partial u_1}-z_2\frac{\partial}{\partial u_2},$$

$$L^G_H=z_1\frac{\partial}{\partial z_1}+z_2\frac{\partial}{\partial z_2}-u_1\frac{\partial}{\partial u_1}-u_2\frac{\partial}{\partial u_2}.$$

The algebra of regular functions $\mathscr{E}(G)$ is $\mathscr{E}(G)=\mathbf{C}(u_1, u_2, z_1, z_2)/(u_1z_2-u_2z_1-1)$ and $\mathscr{E}(A)\subset\mathscr{E}(G)$ consists of the functions $f\in\mathscr{E}(G)$ which satisfy

$$-L^G_{E_+}f=z_1\frac{\partial f}{\partial u_1}+z_2\frac{\partial f}{\partial u_2}=0.$$

The ring $Wu\subset\mathscr{D}(A)$ coincides with the ring of polynomials of the single generator

$$L^A_H=z_1\frac{\partial}{\partial z_1}+z_2\frac{\partial}{\partial z_2}.$$

Now we show how to construct the mapping $f\to\tilde{f}$. Let T be an irreducible representation of $SL(2, \mathbf{C})$. We denote by $\mathscr{E}_T$ the largest subspace of $\mathscr{E}(G)$ such that the restriction of R^G to this subspace is a multiple of T. Furthermore, for each $n\in\mathbf{Z}$ we denote by $\mathscr{E}^n_T$ the subspace of $\mathscr{E}_T$ consisting of all functions $f\in\mathscr{E}_T$ for which $L^G_H f=nf$.

Lemma 4.1. *1)* $\mathscr{E}(G)=\bigoplus_{T,n}\mathscr{E}^n_T$.

2) Let $\dim T=l+1$. *Then* $\dim\mathscr{E}^n_T=l+1$ *for* $n=-l, -l+2, \dots, l-2, l$ *and* $\mathscr{E}^n_T=0$ *for the remaining values of n. Those n for which* $\mathscr{E}^n_T\neq 0$ *are weights of T.*

3) $\mathscr{E}^n_T$ *is invariant under* R^G *and the restriction of* R^G *to* $\mathscr{E}^n_T$ *is equivalent to T.*

4) $\mathscr{E}^l_T$ *consists of vectors of highest weight with respect to* L^G *(i.e.* $L^G_X f=0$ *for all* $f\in\mathscr{E}^l_T$ *and* $X\in\mathfrak{n}_+$*).*

The proof of this lemma follows easily from simple properties of the representations of $SL(2, \mathbf{C})$.

It suffices to construct the mapping $f\to\tilde{f}$ on each space $\mathscr{E}^n_T$ separately. We may assume that the restriction of L^G to the smallest invariant subspace of $\mathscr{E}(f)$

containing f is equivalent to T. Let us put $f_i=(L^G_{E_+})^i f$. Then obviously $f_i \in \mathscr{E}_T^{n+2i}$, and in particular $f_i=0$ for $i>\frac{1}{2}(l-n)$. In accordance with property 2 of the mapping $f \to \tilde{f}$ (see Conjecture I), we shall look for the operator $\tilde{f}$ in the form

$$\tilde{f} = \sum_{i=0}^{\frac{1}{2}(l-n)} f_i (L^G_{E_-})^i \alpha_i \tag{4.1}$$

where $\alpha_i \in Wu$.

Let us assume now that the right-hand side of (4.1) defines an operator on A. Applying the relations

$$[E_+, E_-^i] = E_-^{i-1} \cdot i(H-i+1)$$

we find easily that the elements $\alpha_i \in Wu$ satisfy the equations

$$\alpha_{i-1} + i(H-i+1)\alpha_i = 0.$$

Therefore, putting $\frac{1}{2}(l-n)=p$ we obtain

$$\alpha_i = (-1)^{p-i} \frac{p!}{i!} (H-p+1) \cdot \ldots \cdot (H-i)\alpha_p.$$

Consequently, we have the operator $\tilde{f}$ if we determine α_p. It is clear that property 2) requires $\alpha_p \in \mathbf{C}$. Moreover, $s_0(\tilde{f}) = f \otimes \alpha_0$ and $H(\varrho)=1$. Therefore, to assure the validity of property 1) we have to take

$$\alpha_p = (p!)^{-2}.$$

Then all statements 1)—4) of Conjecture I will be valid.

Example 2. Let $G=SL(3, \mathbf{C})$ and N be the subgroup of upper triangular matrices with units on the diagonal.

Let T_i be the i-th fundamental representation of G, $i=1, 2$. Both representations T_i are three-dimensional and the spaces $\mathscr{E}_{T_i}$ (see Example 1) are of dimension 9. Here $\mathscr{E}_{T_1}$ consists of the linear combinations of the matrix elements g_{ij}, $1 \leqq i, j \leqq 3$ of the matrix $g \in G$, and $\mathscr{E}_{T_2}$ consists of the linear combinations of the second order minors of g. Let us construct the mapping $f \to \tilde{f}$ for $f \in \mathscr{E}_{T_1}$. Put $f_i = \alpha_1 g_{i1} + \alpha_2 g_{i2} + \alpha_3 g_{i3}$, $i=1, 2, 3$. Then for arbitrary $\alpha_1, \alpha_2, \alpha_3$ the elements f_i form a subspace of $\mathscr{E}_{T_1}$ such that the restriction of R^G to this subspace is equivalent to T_1. Here f_3 is a vector of highest weight, that is $\tilde{f}_3 = f_3$. Let us now give formulas for $\tilde{f}_1$ and $\tilde{f}_2$. Let E_{ij} $(i \neq j)$, $E_{11}-E_{22}$ and $E_{22}-E_{33}$ be the basis in $\mathfrak{g}$. We put

$$L^G_{E_{ij}} = \hat{E}_{ij}, \quad L^G_{E_{ii}-E_{jj}} = \hat{Z}_{ij}.$$

These operators act on the functions f_i as follows

$$\hat{E}_{ij} f_k = -\delta_{jk} f_i,$$

$$\hat{Z}_{ij} f_k = (\delta_{jk} - \delta_{ik}) f_k,$$

where δ_{ij} is the Kronecker symbol.

Let us consider in $\mathscr{D}(G)$ the following operators:

$$\tilde{f}_3 = f_3$$

$$\tilde{f}_2 = f_2\hat{Z}_{32} - f_3\hat{E}_{32}$$

$$\tilde{f}_1 = f_1\hat{Z}_{12}(\hat{Z}_{13}+1) + f_2\hat{E}_{21}(\hat{Z}_{13}+1) + f_3(\hat{E}_{32}\hat{E}_{21} + \hat{E}_{31}\hat{Z}_{12}).$$

It is easy to verify that the operators $\tilde{f}_1, \tilde{f}_2, \tilde{f}_3$ belong to $\mathscr{D}(A)$ (i.e. $\tilde{f}_i(\mathscr{E}(A)) \subset \mathscr{E}(A)$) and the mapping $f_i \to \tilde{f}_i$ satisfies all conditions of Conjecture I. Thus the mapping $f \to \tilde{f}$ has been constructed for $f \in \mathscr{E}_{T_1}$.

We also remark that the operator $\tilde{f}_1$ is an operator of the second order on A which cannot be expressed in term of first order operators on A.

The mapping $f \to \tilde{f}$ for $f \in \mathscr{E}_{T_2}$ can be constructed in the same way.

In Examples 1 and 2 we were dealing with representations T for which all weight subspaces were one-dimensional. It is a more difficult task to construct the operation $f \to \tilde{f}$ in the case when these subspaces are not one-dimensional. Now let us see a simple example of that kind.

Example 3. Let us put, as in Example 2, $G = SL(3, \mathbf{C})$ and T be the adjoint representation of G in its Lie algebra $\mathfrak{g}$. Let f be a vector of highest weight with respect to L^G in $\mathscr{E}_T$. We shall introduce the following notations

$$f_{13} = f, \quad f_{23} = \hat{E}_{21}f, \quad f_{12} = -\hat{E}_{32}f,$$

$$h_{12} = -\hat{E}_{21}f_{12}, \quad h_{23} = -\hat{E}_{32}f_{23},$$

$$f_{21} = -\hat{E}_{21}h_{23}, \quad f_{32} = -\hat{E}_{32}h_{12}, \quad f_{31} = \hat{E}_{32}f_{21}.$$

Then the restriction of L^G to the subspace spanned by f_{ij} and h_{ij} is equivalent to T. Here h_{12} and h_{23} generate a two-dimensional weight subspace of T. We define the operation $f \to \tilde{f}$ in the following way

$$\tilde{h}_{12} = h_{12}\left(\hat{Z}_{12}\hat{Z}_{23} + \frac{2}{3}\hat{Z}_{12} - \frac{2}{3}\hat{Z}_{23}\right) + h_{23}\left(-\hat{Z}_{12}\hat{Z}_{23} + \frac{1}{3}\hat{Z}_{12} - \frac{4}{3}\hat{Z}_{23}\right) +$$
$$+ f_{12}\hat{E}_{21}(3\hat{Z}_{23}+1) - f_{23}\hat{E}_{32}(3\hat{Z}_{12}+2) + f_{13}(3\hat{E}_{21}\hat{E}_{32} + \hat{E}_{31}).$$

$$\tilde{h}_{23} = h_{12}\left(-\hat{Z}_{12}\hat{Z}_{23} - \frac{4}{3}\hat{Z}_{12} + \frac{1}{3}\hat{Z}_{23}\right) + h_{23}\left(\hat{Z}_{12}\hat{Z}_{23} - \frac{2}{3}\hat{Z}_{12} + \frac{2}{3}\hat{Z}_{23}\right) -$$
$$- f_{12}\hat{E}_{21}(3\hat{Z}_{23}+2) + f_{23}\hat{E}_{32}(3\hat{Z}_{12}+1) - f_{13}(3\hat{E}_{21}\hat{E}_{32} + 2\hat{E}_{31}).$$

We remark that in the subspace spanned by $\tilde{h}_{12}$ and $\tilde{h}_{23}$, there is, up to multiplication, only one operator of first order namely $\tilde{h}_{12} - \tilde{h}_{23}$; other operators in this space are of order 2.

§ 5. The generalized Segal—Bargmann spaces

In this section we shall consider a generalization of Bargmann's construction [3] of representations of the group $SU(2)$.

Let K be a maximal compact subgroup of the group G, $\mathfrak{k}$ be its Lie algebra, and $\mathfrak{i}:\mathfrak{g}\to\mathfrak{g}$ be the corresponding Cartan involution. We assume that K is chosen so that $\mathfrak{i}(\mathfrak{n}_+)=\mathfrak{n}_-$. Let T_i, $1\leqq i\leqq r$ be the representations of G corresponding to the fundamental highest weights of $\mathfrak{g}$. Let f_i be a vector of highest weight in T_i. We define a function $H_i(g)$ on G by the formula

$$H_i(g) = \|T(g^{-1})f_i\|_i^2,$$

where $\|\cdot\|_i$ is a K-invariant norm on T_i such that $\|f_i\|_i=1$. Clearly, $H_i(ng)=H_i(g)$ for $n\in N_+$, hence H_i can also be considered as a function on A. Let $u(t)$ be a positive, rapidly decreasing function defined for $t>0$. Let ϱ_i $(1\leqq i\leqq r)$ be positive numbers. We define the weight function $\varrho(x)$ on A by the formula

$$\varrho(x) = u(\sum \varrho_i H_i(x)).$$

Definition 5.1. The *Segal—Bargmann space* of the group G is defined as the completion of $\mathscr{E}(A)$ with respect to the scalar product

$$\{f, g\} = \int_A f(x)\bar{g}(x)\varrho(x)\omega,$$

where ω denotes the G-invariant measure on A.

It is obvious that this scalar product in $\mathscr{E}(A)$ is invariant under R_k^A, $k\in K$.

Conjecture II. *There exists a function $u(t)$ such that for any $D\in\mathscr{D}(A)$ we have $D^*\in\mathscr{D}(A)$; here D^* denotes the adjoint operator of D with respect to $\{\,,\}$.*

Let us consider the simplest case $G=SL(2, \mathbf{C})$, $K=SU(2)$. In this case A is the plane $\mathbf{C}^2$ without the point $(0, 0)$, and $\mathscr{E}(A)$ is the space of polynomials of two variables z_1, z_2. The scalar product is introduced in $\mathscr{E}(A)$ by the formula

$$\{f, g\} = \int e^{-\varrho(|z_1|^2+|z_2|^2)} f\bar{g}\, dz_1\, dz_2\, d\bar{z}_1\, d\bar{z}_2.$$

Now the ring $\mathscr{D}(A)$ is generated by the operators z_i and $\frac{\partial}{\partial z_i}$, $i=1, 2$. It is easy to verify that $\left(\frac{\partial}{\partial z_i}\right)^* = \varrho z_i$ and therefore

$$(z_i)^* = \varrho^{-1}\frac{\partial}{\partial z_i}.$$

This shows that Conjecture II is true in the present case.

The above construction for $SU(2)$ was suggested by Bargmann [3].

Using the examples given in § 4 we can show that Conjecture II is also valid for $G=SL(3, \mathbf{C})$. As $u(t)$ we take the decreasing positive solution of the equation

$$t\frac{d^2u}{dt^2}+2\frac{du}{dt}+u = 0.$$

It seems that for $G=SL(n, \mathbf{C})$ the function $u(t)$ has to satisfy the equation

$$\frac{d^{n-1}}{dt^{n-1}}(ut^{n-2})+(-1)^n u=0.$$

A more precise version of Conjecture II is the following

Conjecture II′. *Let* $g \to g^*$ *be the anti-automorphism of* G *corresponding to the anti-involution* $(-\mathfrak{i}):\mathfrak{g} \to \mathfrak{g}$. *The numbers* ϱ_i *can be chosen in such a way that for any function* $f \in \mathscr{E}(G)$ *the equality* $(\check{f})^* = (\widecheck{f^*})$ *is satisfied. Here* $f^*(g)=f(g^*)$.

§ 6. The mapping π_*

In this section we shall give a construction which yields a weaker version of Conjecture I. More precisely, for every function $f \in \mathscr{E}(G)$ a collection of regular differential operators on A will be constructed. In addition to this, we shall show that all differential operators on A can be obtained in this way. In the construction we apply an operation π_* which maps functions on G into functions on A and which, we believe, is of independent interest. This operation is an algebraic analogue of the averaging operation over a subgroup (which is unipotent in the present case). It is remarkable that π_* transfers the operation of multiplication by a function $f(g)$ into an "almost" differential operator on A. The exact formulation of these facts is given in Theorems 6.3 and 6.5.

Lemma and Definition 6.1. *There exists a unique mapping* $\pi_*:\mathscr{E}(G) \to \mathscr{E}(A)$ *such that*

1) $\pi_* R^G_g = R^A_g \pi_*$ *and* $\pi_* L^G_h = L^A_h \pi_*$

for all $g \in G$, $h \in H$,

2) $\pi_* \pi^* \varphi = \varphi$ *for all* $\varphi \in \mathscr{E}(A)$.

Proof. First we prove that $\pi_* f$ is uniquely determined by the conditions *1)* and *2)*. It is enough to consider the case when f lies in a subspace V irreducible and invariant with respect to L^G; further we can assume that f is a weight function of weight χ with respect to the restriction of L^G to H. If χ is a highest weight of the given irreducible representation, then $f \in \operatorname{Im} \pi^*$, i.e. $f=\pi^* \varphi$, hence in view of *2)*, $\pi_* f = \varphi$.

Assume now that χ is not a highest weight. Let us denote by f_0 a vector of highest weight in V and by χ_0 the corresponding highest weight. Then, under the action of R^G, f and f_0 are transformed by the same irreducible representation of G. From *1)* and the fact that every irreducible representation of G occurs in $\mathscr{E}(A)$ only once (see [2]) it follows that $\pi_* f$ and $\pi_* f_0$ belong to the same subspace, irreducible and invariant with respect to R^A. But then the weights of

$\pi_* f$ and $\pi_* f_0$ with respect to L^A coincide (see [2]). From *1)* it follows that the weight of $\pi_* f$ is equal to χ and the weight of $\pi_* f_0$ is equal to $\chi_0 \neq \chi$. Since $\pi_* f_0 \neq 0$ we have $\pi_* f=0$.

From this proof we immediately obtain a construction for π_*. Indeed, if f is a weight vector, not of highest weight, lying in a subspace, irreducible and invariant under L^G, then we put $\pi_* f=0$. On the other hand, if f is a vector of highest weight then $f=\pi^*\varphi$ and we put $\pi_* f=\varphi$. The lemma is proved.

Let us denote for each $\gamma \in \Delta$ the operator $L^G_{E_\gamma}$ by $\hat{E}_\gamma$. It follows from the construction of π_* that $\pi_*(\hat{E}_\gamma f)=0$ for all $f \in \mathscr{E}(G)$ and $\gamma \in \Delta_-$.

By means of the mapping π_* one can construct differential operators on A in the following way.

Definition 6.2. Let $f \in \mathscr{E}(G)$. We define an operator $\hat{f}$ in the space $\mathscr{E}(A)$ by the formula $\hat{f}(\varphi)=\pi_*(f\cdot\pi^*\varphi)$, $\varphi \in \mathscr{E}(A)$.

Theorem 6.3. *There exists a non-zero element $Z \in Wu$ such that $Z\hat{f}$ is a regular differential operator on A.*

Proof. A differential operator D on G is called a *chain* if it can be expressed in the form $\hat{E}_{\gamma_1}\hat{E}_{\gamma_2}\ldots\hat{E}_{\gamma_k}$, $\gamma_i \in \Delta_+$. The weight of this chain is defined to be $\gamma_1+\gamma_2+\ldots+\gamma_k \in \mathfrak{h}^*$. Let us denote by Ξ the set of all chains D such that $Df \neq 0$ and by $\Xi_0 \subset \mathfrak{h}^*$ the set of all their weights. Obviously, Ξ and Ξ_0 are finite sets. For any function $\varphi \in \mathscr{E}(A)$ and any chain D we have $D(f\pi^*\varphi)=Df\cdot\pi^*\varphi$, since $\hat{E}_\gamma\pi^*\varphi=0$ for $\gamma \in \Delta_+$. Therefore, if $D \notin \Xi$ then $D(f\pi^*\varphi)=0$. Let us denote by U the subspace of $\mathscr{E}(G)$ consisting of all functions u such that $Du=0$ for any chain $D \notin \Xi$.

Lemma 6.4. *There exist a regular differential operator T on G and an element $Z \in Wu$ such that $Tu=Z\pi^*\pi_* U$ for all $u \in U$.*

The theorem is an immediate consequence of this lemma, since for any function $\varphi \in \mathscr{E}(A)$ we have $f\cdot\pi^*\varphi \in U$, and so

$$T(f\pi^*\varphi) = Z\pi^*\pi_*(f\cdot\pi^*\varphi) = \pi^* Z\hat{f}(\varphi),$$

i.e. $T\circ f\circ\pi^*=\pi^*\circ Z\circ\hat{f}$. It follows from this equality that the differential operator $T\circ f$ preserves $\mathscr{E}(A)\subset\mathscr{E}(G)$, or in other words, that $Z\hat{f}$ is a differential operator on A.

Proof of the lemma. Let $H_1, \ldots, H_r$ denote a basis in $\mathfrak{h}$. The elements of Wu are polynomials of $H_1, \ldots, H_r$ and, as above, we may consider them as polynomial functions on $\mathfrak{h}^*$. Let Δ be the Laplace operator of the second order on G (constructed by means of the Killing form). Then there exists an element $P \in Wu$ such that for any vector $\varphi \in \mathscr{E}(A)$ the equality $\Delta\varphi=P\varphi$ is satisfied, or equivalently, $\Delta\varphi=P(\chi_0)\cdot\varphi$, where χ_0 is the weight of the vector φ.

Let B be the restriction of the Killing form of the algebra $\mathfrak{g}$ to $\mathfrak{h}$, and $Q(\chi)$ the dual quadratic form on $\mathfrak{h}^*$. It follows from results of Harish–Chandra [4] that $P(\chi)=Q(\chi+\varrho)-Q(\varrho)$, where ϱ is half-sum of the positive roots.

For an arbitrary weight β we denote by P_β and Z_β the elements of Wu corresponding to the polynomial functions

$$P_\beta(\chi) = P(\chi+\beta)$$

and

$$Z_\beta(\chi) = 2\langle\beta, \chi+\varrho\rangle-\langle\beta,\beta\rangle$$

respectively ($\langle\,,\rangle$ denotes the scalar product in $\mathfrak{h}^*$ corresponding to the quadratic form Q). Let $T=\prod\limits_\beta (P_\beta-\Delta)$, $Z=\prod\limits_\beta Z_\beta$, where β runs through $\Xi_0\diagdown\{0\}$.

We shall show that for all $u\in U$ the equality $Tu=Z\pi^*\pi_* u$ is satisfied.

Clearly, it suffices to verify this equality when u is a weight vector lying in a subspace V which is irreducible and invariant under L^G. Let u_0 be a vector of highest weight in V, χ and χ_0 be the weights of u and u_0, respectively, with respect to $\mathfrak{h}$. It follows from the uniqueness of the vector of highest weight that $u_0=cDu$ where D is an appropriate chain and $c\in\mathbf{C}$, and therefore $\chi_0-\chi\in\Xi_0$.

Case 1, $\chi\neq\chi_0$. The restriction of Δ to V is multiplication by $P(\chi_0)$. Therefore

$$(P_\beta-\Delta)u = \big(P_\beta(\chi)-P(\chi_0)\big)u = \big(P(\chi+\beta)-P(\chi_0)\big)u = 0,$$

if $\beta=\chi_0-\chi\in\Xi_0\diagdown\{0\}$. Thus $Tu=0$, and since $\pi_* u=0$ in this case, we have $Tu==Z\pi^*\pi_* u=0$.

Case 2, $\chi=\chi_0$. Then

$$(P_\beta-\Delta)u = \big(P(\chi_0+\beta)-P(\chi_0)\big)u = \big(Q(\chi_0+\beta+\varrho)-Q(\chi_0+\varrho)\big)u =$$
$$= (2\langle\chi_0+\varrho,\beta\rangle+\langle\beta,\beta\rangle)u = Z_\beta(\chi_0)u,$$

hence

$$Tu=\big(\pi(P_\beta-\Delta)\big)u = \pi Z_\beta(\chi_0)u = Zu = Z\pi^*\pi_* u.$$

The proof of the lemma and of Theorem 6.3 is complete.

It can be shown that the order of the operator $Z\hat{f}$ is equal to card Ξ_0-1, i.e. the order of Z'.

Theorem 6.5. *Every regular differential operator D on A can be written in the form $D=\sum Z_j f_j$, where $Z_j\in Wu$ and $f_j\in\mathscr{E}(G)$.*

Proof. Let D' be a lifting of D to G (Theorem 3.4). By means of simple transformations D' can be transformed to the form

$$D' = \sum Z_j f_j + \sum A_k\hat{E}_{\gamma_k} + \sum \hat{E}_{\delta_i}B_i,$$

where A_k and B_i are differential operators on G, $Z_j\in Wu$, $f_j\in\mathscr{E}(G)$, $\gamma_k\in\Delta_+$, $\delta_i\in\Delta_-$.

Since $\hat{E}_{\gamma_k}\pi^*\varphi = 0$ and $\pi_* E_{\delta_i} f = 0$, we have

$$D\varphi = \pi_*\pi^* D\varphi = \pi_* D'\pi^*\varphi = \sum Z_j \pi_*(f_j \pi^*\varphi) = \sum Z_j \hat{f}_j(\varphi),$$

which proves theorem 6.5.

Let L be the quotient field of the ring Wu. Then the operation $f \to \hat{f}$ extends to a mapping $\Omega: L \underset{\mathbf{C}}{\otimes} \mathscr{E}(G) \to L \underset{Wu}{\otimes} \mathscr{D}(A)$. Here Ω commutes with the actions of R_g and L_g.

Theorem 6.6. *Ω is an isomorphism of L-modules $L \otimes \mathscr{E}(G)$ and $L \underset{Wu}{\otimes} \mathscr{D}(A)$.*

Proof. It follows from Theorem 6.5 that Ω is a surjection.

In Theorem 6.3 we constructed for every function $f \in \mathscr{E}(G)$ an operator $Z_f \in Wu$ such that $Z_f \cdot \hat{f} \in \mathscr{D}(A)$.

Lemma 6.7.

$$s_0(Z_f \cdot \hat{f}) = f \otimes Z_f + \sum f_i \otimes Z_i,$$

where $\deg Z_i < \deg Z_f$.

Proof. The lifting of the operator $Z_f \cdot \hat{f}$ has the form $T \circ f$, where $T = \prod_\beta (P_\beta - \Delta)$ (see the proof of Theorem 6.3). It follows from [4] that

$$\Delta = P + \sum_{\gamma \in \Delta_+} c_\gamma \hat{E}_{-\gamma} \hat{E}_\gamma, \quad c_\gamma \in \mathbf{C}.$$

Consequently

$$P_\beta - \Delta = Z_\beta - \sum_{\gamma \in \Delta_+} c_\gamma \hat{E}_{-\gamma} \hat{E}_\gamma.$$

Let $K = \operatorname{card}(\Xi_0 \setminus \{0\})$. It is easy to show by induction on K that

$$T = \prod_\beta Z_\beta + \sum_\lambda c_\lambda H_\lambda X_\lambda Y_\lambda,$$

where $c_\lambda \in \mathbf{C}$, and H_λ, X_λ and Y_λ are products of suitably chosen operators $H_i \in Wu$, $\hat{E}_{-\gamma}$, $\gamma \in \Delta_+$, and $\hat{E}_\gamma$, $\gamma \in \Delta_+$, respectively. We also have $\deg H_\lambda + \deg X_\lambda \leqq K$ for all λ, and $\deg H_\lambda < K$ if $X_\lambda = 1$. Moreover, $Z_f = \prod_\beta Z_\beta$. From this it follows that the operator $T \circ f$ is of the form $f \cdot Z_f + \sum f_\mu H_\mu X_\mu Y_\mu$, where $f_\mu \in \mathscr{E}(G)$, H_μ, X_μ and Y_μ satisfy the same conditions as H_λ, X_λ, Y_λ. Therefore, $s_0(Z_f \cdot \hat{f})$ (see Definition 3.10) is of the form $s_0(Z_f \cdot \hat{f}) = f \otimes Z_f + \sum f_i \otimes Z_i$, where $\deg Z_f = K$, $\deg Z_i < K$. The lemma is proved.

Let us now consider the element $D = \sum \tilde{Z}_i \otimes f_i \in \mathscr{E}(G)$, where $\tilde{Z}_i \in L$, $\tilde{Z}_i \neq 0$, and the f_i are linearly independent. We shall show that $\Omega(D) \neq 0$. Let us multiply D by an element $Z \in Wu$ such that $ZD = \sum Z_i \otimes f_i$, where $Z_i \in W_u$ and Z_i is divisible by Z_{f_i}. Let $l = \max \deg Z_i$. Then $\Omega(ZD) \in \mathscr{D}(A)$ and by Lemma 6.7

$$s_0(\Omega(ZD)) = \sum f_i \otimes Z_i + \sum f_j' \otimes Z_j',$$

where $\deg Z_j' < l$. This implies that $s_0(\Omega(ZD)) \neq 0$, consequently $\Omega(D) \neq 0$.

We have now proved that Ω is an injection and therefore an isomorphism.

PART II

THE RESOLUTION OF A FINITE-DIMENSIONAL $\mathfrak{g}$-MODULE

§ 7. Introduction

Let $\mathfrak{g}$ be, as above, a complex semisimple Lie algebra. Let V be a finite-dimensional irreducible $\mathfrak{g}$-module. The cohomology groups $H^i(\mathfrak{n}_-, V)$ play an important rôle in the theory of representations (see [5] and [6]). They have the following properties expressed in Bott's theorem [5]. Let W be the Weyl group of the algebra $\mathfrak{g}$. Then $\dim H^i(\mathfrak{n}_-, V) = \operatorname{card} W^{(i)}$, where $W^{(i)} = \{w \in W : l(w) = i\}$ and $l(w)$ is the length of the element $w \in W$.

The fundamental result of this part is Theorem 10.1 which improves on Bott's theorem.

For any $\chi \in \mathfrak{h}^*$ we denote by M_χ the $U(\mathfrak{g})$-module generated by a vector of highest weight $\chi - \varrho$ (the exact definition is given below). In Theorem 10.1 we construct a resolution

$$0 \leftarrow V \leftarrow C_0 \leftarrow C_1 \leftarrow \ldots \leftarrow C_s \leftarrow 0$$

of the $\mathfrak{g}$-module V such that

$$C_i = \bigoplus_{w \in W^{(i)}} M_{w\chi},$$

where $\chi - \varrho$ is the highest weight of V. Bott's theorem follows from Theorem 10.1 since M_χ is a free $U(\mathfrak{n}_-)$-module with one generator.

In this part we shall make a systematic use of a certain category of $\mathfrak{g}$-modules which we call category O (see [12]). § 8 is devoted to the exposition of the properties of category O.

In § 9 several results concerning the cohomologies of Lie algebras are presented. In particular, in this section a purely algebraic proof of Bott's theorem is given which, it seems to us, is simpler than the proofs presented in [5] and [6]. This proof has several points of contact with Kostant's proof [6], but it does not make any use of the Hermitian structure. The observant reader will notice that the resolution constructed in the proof of Bott's theorem is dual to a part of the de Rham resolution well-known from the theory of formal differential forms.

In the Appendix we describe the modules occurring in the Jordan—Hölder decomposition of the modules M_χ. The study of the structure of the modules M_χ was initiated in Verma's work [7]. We remark that the works [7], [8] and [9] contain everything we know about the modules M_χ. All these facts are also contained in Theorems 8.7, 8.8, 8.12 and 10.1 of the present work.

§ 8. The category O

Definition 8.1. Category O is the full subcategory of the category of left $U(\mathfrak{g})$-modules consisting of all modules M such that

1) M is a finitely generated $U(\mathfrak{g})$-module,

2) M can be made $\mathfrak{h}$-diagonal, i.e. there exists a basis in M consisting of weight vectors,

3) M is $U(\mathfrak{n}_+)$-finite, i.e. for any $f \in M$ the space $U(\mathfrak{n}_+)f$ is finite-dimensional.

Definition 8.2. Let $\chi \in \mathfrak{h}^*$. We denote by J_χ the left ideal in $U(\mathfrak{g})$ generated by the elements E_γ, $\gamma \in \Delta_+$ and $H - \chi(H) + \varrho(H)$, $H \in \mathfrak{h}$. Let us put $M_\chi = U(\mathfrak{g})/J_\chi$. We shall denote by f_χ the image of $1 \in U(\mathfrak{g})$ in M_χ.

For the sake of convenience we formulate the elementary properties of the category O and the modules M_χ in Propositions 8.3, 8.5, 8.6. These propositions are simple consequences of Harish–Chandra's theorem about Laplace operators (see [4]).

Proposition 8.3. *1)* The category O is closed under taking submodules, factor modules and finite direct sums.

2) Let $M \in O$. Then all the spaces $M^{(\psi)}$ $\psi \in \mathfrak{h}^*$ are finite-dimensional and $M = \bigoplus_{\psi \in \mathfrak{h}^*} M^{(\psi)}$. In addition to this $P(M)$ is contained in a finite union of sets $\chi_i - K$, $\chi_i \in \mathfrak{h}^*$.

3) Every element $M \in O$ has a finite Jordan—Hölder composition series.

4) M_χ is a free $U(\mathfrak{n}_-)$-module with f_χ as a generator.

5) $M_\chi \in O$.

6) In M_χ there exists a maximal proper submodule. The corresponding irreducible factor module will be denoted by L_χ.

7) Every irreducible module in the category O is of the form L_χ, $\chi \in \mathfrak{h}^*$.

We denote by $Z(\mathfrak{g})$ the centre of the algebra $U(\mathfrak{g})$ and by Θ the set of all homomorphisms $\vartheta : Z(\mathfrak{g}) \to \mathbf{C}$.

Definition 8.4. Let M be an arbitrary $\mathfrak{g}$-module. To each element $f \in M$ which is an eigenvector with respect to all the operators $z \in Z(\mathfrak{g})$ we can assign a homomorphism $\vartheta_f \in \Theta$ such that $zf = \vartheta_f(z) \cdot f$ for all $z \in Z(\mathfrak{g})$. The set of all such homomorphisms ϑ will be denoted by $\Theta(M)$.

Proposition 8.5. *1)* $\Theta(M_\chi)$ consists of a single element which we shall denote by ϑ_χ.

2) $\vartheta_{\chi_1} = \vartheta_{\chi_2}$ if and only if $\chi_1 \sim \chi_2$.

Proposition 8.6. Let $M \in O$. Then

1) $\Theta(M)$ is finite.

2) For any $\vartheta \in \Theta$ we put $I_\vartheta = \operatorname{Ker} \vartheta \subset Z(\mathfrak{g})$. Let $M_\vartheta^{(n)} = \{f \in M \mid I_\vartheta^n f = 0\}$. Then $M_\vartheta^{(n)}$ stabilizes for large values of n. The obtained submodule of M will be denoted by M_ϑ.

3) $\Theta(M_\vartheta) = \{\vartheta\}$.

4) $M = \bigoplus_{\vartheta \in \Theta(M)} M_\vartheta$.

5) The mapping $M \to M_\vartheta$ is an exact functor in O.

Now we shall pass on to the study of the modules M_χ. The following two theorems give a complete description of the homomorphisms between modules M_χ.

Theorem 8.7 [7]. *Let* $\chi, \psi \in \mathfrak{h}^*$. *Then either*

1) $\mathrm{Hom}_{U(\mathfrak{g})}(M_\chi, M_\psi) = 0$.

or

2) $\mathrm{Hom}_{U(\mathfrak{g})}(M_\chi, M_\psi) = \mathbf{C}$, *and every non-trivial homomorphism* $M_\chi \to M_\psi$ *is an injection.*

Theorem 8.8. *Let* $\chi, \psi \in \mathfrak{h}^*$,

then

$$\mathrm{Hom}_{U(\mathfrak{g})}(M_\chi, M_\psi) = \mathbf{C}$$

if and only if there exists a sequence of roots $\gamma_1, \ldots, \gamma_k \in \Delta_+$ *satisfying the following condition* (*A*) *for the pair* (χ, ψ).

Condition (*A*).

1) $\chi = \sigma_{\gamma_k} \cdot \sigma_{\gamma_{k-1}} \cdot \ldots \cdot \sigma_{\gamma_1} \psi$.

2) Put $\chi_0 = \psi$, $\chi_i = \sigma_{\gamma_i} \cdot \ldots \cdot \sigma_{\gamma_1} \psi$. *Then* $\chi_{i-1} - \chi_i = n\gamma_i$, *where n is a non-negative integer.*

In particular $\mathrm{Hom}_{U(\mathfrak{g})}(M_\chi, M_\psi) \neq 0$ only if $\chi \sim \psi$ and $\chi \leqq \psi$.

Theorem 8.8 was formulated in [7] as a conjecture; a proof of the sufficiency of condition (*A*) was also given there. A complete proof of Theorem 8.8 was given in [9].

The structure of the submodules of the modules M_χ is most interesting when $\chi \in D$. We shall study this case in more detail. For this purpose we introduce the following partial ordering in the Weyl group W.

Definition 8.9. If $w_1, w_2 \in W$ and $\gamma \in \Delta_+$ then $w_1 \xrightarrow{\gamma} w_2$ means that $w_1 = \sigma_\gamma w_2$ and $l(w_1) = l(w_2) + 1$. (Sometimes we shall omit the symbol γ above the arrow.) We put $w < w'$ if there exists a sequence $w_1, w_2, \ldots, w_k$ of elements of W such that

$$w \to w_1 \to w_2 \to \ldots \to w_k \to w'.$$

Theorem 8.8′. *Let* $\chi \in D$, $w_1, w_2 \in W$. *Then* $\mathrm{Hom}_{U(\mathfrak{g})}(M_{w_1\chi}, M_{w_2\chi}) = \mathbf{C}$ *if and only if* $w_1 \leqq w_2$.

In what follows all modules $M_{w\chi}$, $w \in W$, $\chi \in D$ will be considered as submodules of M_χ.

Theorem 8.8′ is an immediate consequence of Theorem 8.8 and the following lemmas.

Lemma 8.10. *Let* $\chi \in D$, $\gamma \in \Delta$, $w \in W$. *Then* $w\chi - \sigma_\gamma w\chi = n\gamma$, *where* $n \in \mathbf{Z}$, $n \neq 0$, *while* $n > 0$ *if and only if* $l(\sigma_\gamma w) > l(w)$.

Lemma 8.11. *Let* $w \in W$ *and* $\gamma \in \Delta$ *be such that* $l(\sigma_\gamma w) < l(w)$. *Then* $w < \sigma_\gamma w$.

The proofs of these lemmas will be presented in § 11.

The complete structure of the submodules of the modules M_χ is not yet known. Some information is contained in the following theorem.

Theorem 8.12. *Let* $\psi, \chi \in \mathfrak{h}^*$. *Then* $L_\chi \in JH(M_\psi)$ *if and only if there exists a sequence* $\gamma_1, \gamma_2, \ldots, \gamma_k \in \Delta_+$ *satisfying condition* (*A*) *for the pair* (χ, ψ) (see Theorem 8.8).

Since we shall need this theorem in § 10, we shall give the proof in the Appendix.

Corollary. Let $\chi \in D$. Then the Jordan—Hölder decomposition of the module $M_{w\chi}$ consists of the modules $L_{w'\chi}$, where $w' \leqq w$ (possibly counted with multiplicities). $L_{w\chi}$ occurs in this decomposition exactly once. The example in [9] shows that the modules M_ψ may contain submodules M which are not generated by submodules $M_\chi \subset M$.

In fact, even the following proposition can be proved.

Proposition 8.13. Let $\mathfrak{g} = sl(4, \mathbf{C})$. Consider the module M_ψ corresponding to a weight $\psi \in D$. Then M_ψ contains a submodule M such that

$$M \neq \sum_{M_{w\psi} \subset M} M_{w\psi}.$$

This statement is equivalent to the fact that the number of elements in $JH(M_\psi)$ is greater than the number of elements of W.

§ 9. Cohomology of Lie algebras

In this section we shall recall a number of results concerning the cohomology of Lie algebras. Moreover, Bott's theorem will also be proved here.

Let $\mathfrak{a}$ be an arbitrary complex Lie algebra and M an $\mathfrak{a}$-module. The exact sequence of $\mathfrak{a}$-modules

$$0 \longleftarrow M \longleftarrow C_0 \xleftarrow{d_1} C_1 \xleftarrow{d_2} \ldots, \qquad (9.1)$$

where each module C_i is free over $U(\mathfrak{a})$ is called a *free resolution* of M.

Let N be another $\mathfrak{a}$-module. Consider the complex

$$0 \longrightarrow \mathrm{Hom}_{\mathfrak{a}}(C_0, N) \xrightarrow{d_1'} \mathrm{Hom}_{\mathfrak{a}}(C_1, N) \xrightarrow{d_2'} \ldots$$

and put

$$\mathrm{Ext}^i(M, N) = \mathrm{Ker}\, d_{i+1}' / \mathrm{Im}\, d_i'.$$

Let $\tau: U(\mathfrak{a}) \to U(\mathfrak{a})$ be the anti-automorphism defined by the formula $\tau(X) = -X$ for $X \in \mathfrak{a}$. We denote by N^τ the right $U(\mathfrak{a})$-module whose underlying space coincides with that of N and on which the action of $U(\mathfrak{a})$ is defined by the formula $f \cdot X = \tau(X) f$ for $f \in N$, $X \in U(\mathfrak{a})$.

Now consider the complex

$$0 \longleftarrow N^\tau \underset{\mathfrak{a}}{\otimes} C_0 \xleftarrow{d_1''} N^\tau \underset{\mathfrak{a}}{\otimes} C_1 \xleftarrow{d_2''} \dots$$

and put

$$\mathrm{Tor}_i(N^\tau, M) = \mathrm{Ker}\, d_i'' / \mathrm{Im}\, d_{i+1}''.$$

The following standard facts hold (see [10]).

1) The groups $\mathrm{Tor}_i(N^\tau, M)$ and $\mathrm{Ext}^i(M, N)$ are independent of the choice of the resolution (9.1).

2) Let $\mathfrak{a}$ be a finite-dimensional Lie algebra and M, N be finite-dimensional $\mathfrak{a}$-modules. Then

a) $[\mathrm{Ext}^i(M, N)]^* = \mathrm{Tor}_i(N^*, M)$,

where $N^* = \mathrm{Hom}_{\mathbf{C}}(N, \mathbf{C})$ is a right $\mathfrak{a}$-module ([10], Chapter XI, § 3, Proposition 3.3),

b) $\mathrm{Tor}_i(N^*, M) = \mathrm{Tor}_i\big(M^\tau, (N^\tau)^*\big)$.

([10], Chapter VI, § 1).

The cohomology group of $\mathfrak{a}$ with coefficients in M is defined by the formula $H^i(\mathfrak{a}, M) = \mathrm{Ext}^i(\mathbf{C}, M)$, where $\mathbf{C}$ is the trivial one-dimensional $\mathfrak{a}$-module.

The computation of the cohomology groups is done by means of the standard resolution $V(\mathfrak{a})$ of the module $\mathbf{C}$, which is defined in the following way.

We put

$$C_k = U(\mathfrak{a}) \underset{\mathbf{C}}{\otimes} \Lambda^k(\mathfrak{a}), \quad k = 0, 1, \dots$$

Then we define a homomorphism $d_k: C_k \to C_{k-1}$ of $\mathfrak{a}$-modules by means of the formula

$$d_k(X \otimes X_1 \wedge \dots \wedge X_k) = \sum_{i=1}^{k} (-1)^{i+1} (XX_i \otimes X_1 \wedge \dots \wedge \hat{X}_i \wedge \dots \wedge X_k) +$$

$$+ \sum_{1 \le i < j \le k} (-1)^{i+j} (X \otimes [X_i, X_j] \wedge X_1 \wedge \dots \wedge \hat{X}_i \wedge \dots \wedge \hat{X}_j \wedge \dots \wedge X_k).$$

Here $X \in U(\mathfrak{a})$, $X_i \in \mathfrak{a}$ and the symbol ^ means that the corresponding element is to be omitted. Furthermore, we define $\varepsilon: C_0 \to \mathbf{C}$ by the formula $\varepsilon(X) =$ (the constant part of X), $X \in U(\mathfrak{a})$.

As was shown in [10], Chapter XIII, § 7, the sequence

$$0 \longleftarrow \mathbf{C} \xleftarrow{\varepsilon} C_0 \xleftarrow{d_1} C_0 \xleftarrow{d_2} \dots$$

is a resolution of the $\mathfrak{a}$-module $\mathbf{C}$.

Subsequently we shall need a generalization of the resolution $V(\mathfrak{a})$ for the case of relative cohomology.

Let $\mathfrak{a}$ be a complex Lie algebra and $\mathfrak{p}$ a subalgebra. The adjoint action of $\mathfrak{p}$ in $\mathfrak{a}$ yields a representation ϑ of the algebra $\mathfrak{p}$ in the linear space $\mathfrak{a}/\mathfrak{p}$. The corresponding representations of $\mathfrak{p}$ in the linear spaces $\Lambda^k(\mathfrak{a}/\mathfrak{p})$ will be denoted by the same symbol ϑ.

Let us consider for each k, $k=0, 1, 2, \dots$, the module

$$D_k = U(\mathfrak{a}) \underset{U(\mathfrak{p})}{\otimes} \Lambda^k(\mathfrak{a}/\mathfrak{p}).$$

We define the operators $d_k: D_k \to D_{k-1}$ in the following way. Let $X_1, \dots, X_k$ be elements of $\mathfrak{a}/\mathfrak{p}$. Let $Y_1, Y_2, \dots, Y_k \in \mathfrak{a}$ be arbitrary representatives for $X_1, \dots, X_k$, respectively, and put

$$d_k(X \otimes X_1 \wedge \dots \wedge X_k) = \sum_{i=1}^{k} (-1)^{i+1}(XY_i \otimes X_1 \wedge \dots \wedge \hat{X}_i \wedge \dots \wedge X_k) +$$

$$+ \sum_{1 \leqq i < j \leqq k} (-1)^{i-j}(X \otimes \overline{[Y_i, Y_j]} \wedge X_1 \wedge \dots \wedge \hat{X}_i \wedge \dots \wedge \hat{X}_j \wedge \dots \wedge X_k).$$

Here $X \in U(\mathfrak{a})$, and $\bar{Y}$ is the image of the element $Y \in \mathfrak{a}$ in $\mathfrak{a}/\mathfrak{p}$. It is easy to verify that the operator d_k is well defined, i.e. independent of the choice of the representatives Y_i.

In addition to this, we introduce the augmentation $\varepsilon: D_0 \to \mathbf{C}$ by putting $\varepsilon(X \otimes 1) =$ (the constant part of X). Thus we have constructed a sequence $V(\mathfrak{a}, \mathfrak{p})$ of $U(\mathfrak{a})$-modules

$$0 \longleftarrow \mathbf{C} \xleftarrow{\varepsilon} D_0 \xleftarrow{d_1} D_1 \longleftarrow \dots.$$

Direct computation shows that this sequence is a complex, i.e. $d_{i-1}d_i = 0$, $\varepsilon d_0 = 0$. We shall call this complex $V(\mathfrak{a}, \mathfrak{p})$ the *relative chain complex* of the algebra $\mathfrak{a}$ with respect to the subalgebra $\mathfrak{p}$. Clearly, $V(\mathfrak{a}, 0) = V(\mathfrak{a})$.

Theorem 9.1. *The complex* $V(\mathfrak{a}, \mathfrak{p})$ *is exact.*

Proof. Our proof will be similar to the proof of exactness of the standard complex $V(\mathfrak{a})$, given in [10].

We define a filtration in $V(\mathfrak{a}, \mathfrak{p})$ by writing $A \in D_k^{(l)}$ if $A \in D_k$ can be written in the form

$$A = \sum c_i (X^{(i)} \otimes X_1^{(i)} \wedge \dots \wedge X_k^{(i)}),$$

where

$$c_i \in \mathbf{C}, \; X^{(i)} \in U(\mathfrak{a}), \; X_j^{(i)} \in \mathfrak{a}/\mathfrak{p} \quad \text{and} \quad \deg X \leqq l-k.$$

It is clear that $d_k(D_k^{(l)}) \subset D_{k-1}^{(l)}$. Therefore, to prove the theorem, it is enough to show that for every l the complex

$$0 \longleftarrow M^{(l)} \longleftarrow D_0^{(l)}/D_0^{(l-1)} \xleftarrow{d_1^{(l)}} D_1^{(l)}/D_1^{(l-1)} \xleftarrow{d_2^{(l)}} \dots$$

is exact. Here $M^{(0)} = \mathbf{C}$ and $M^{(l)} = 0$ if $l > 0$. It follows from the Poincaré—Birkhoff—Witt theorem that $D_k^{(l)}/D_k^{(l-1)} = S_{l-k}(\mathfrak{a}/\mathfrak{p}) \otimes \Lambda^k(\mathfrak{a}/\mathfrak{p})$, where $S_{l-k}(\mathfrak{a}/\mathfrak{p})$ denotes the set of all homogeneous elements of degree $l-k$ in the symmetric algebra of the space $\mathfrak{a}/\mathfrak{p}$. The operator

$$d_k^{(l)} : D_k^{(l)}/D_k^{(l-1)} \to D_{k-1}^{(l)}/D_{k-1}^{(l-1)}$$

is given by the formula

$$d_k^{(l)}(X \otimes X_1 \wedge \ldots \wedge X_k) = \sum_{i=1}^{k} (-1)^{i+1} (XX_i \otimes X_1 \wedge \ldots \wedge \hat{X}_i \wedge \ldots \wedge X_k).$$

Therefore, the complex Gr $V(\mathfrak{a}, \mathfrak{p})$:

$$0 \leftarrow \mathbf{C} \leftarrow \bigoplus_l D_0^{(l)}/D_0^{(l-1)} \leftarrow \bigoplus_l D_1^{(l)}/D_1^{(l-1)} \leftarrow \ldots$$

coincides with the Koszul complex (see [10], Chapter VIII, § 4) of the space $\mathfrak{a}/\mathfrak{p}$. This means that the complex Gr $V(\mathfrak{a}/\mathfrak{p})$ and also the complex $V(\mathfrak{a}/\mathfrak{p})$ are exact. Theorem 9.1 is proved.

Proposition 9.2. Let $\mathfrak{p}$ and $\mathfrak{q}$ be subalgebras of $\mathfrak{a}$ such that $\mathfrak{a} = \mathfrak{p} \oplus \mathfrak{q}$ (as a linear space). Then $V(\mathfrak{a}, \mathfrak{p}) \approx V(\mathfrak{q})$ as complexes of $U(\mathfrak{q})$-modules.

Proof. We define a mapping of complexes $\varphi : V(\mathfrak{q}) \to V(\mathfrak{a}, \mathfrak{p})$ by the formula

$$\varphi_k(X \otimes X_1 \wedge \ldots \wedge X_k) = X \otimes \overline{X}_1 \wedge \ldots \wedge \overline{X}_k,$$

where $X \in U(\mathfrak{q})$, $X_i \in \mathfrak{q}$ and $\overline{X}_i$ is the image of X_i in $\mathfrak{a}/\mathfrak{p}$. The theorem of Poincaré—Birkhoff—Witt implies that φ is an isomorphism.

Remark 1. Proposition 9.2 owes its significance to the following fact. Assume that $\mathfrak{q}$ is a subalgebra of $\mathfrak{a}$ and that there exists in $\mathfrak{a}$ a subalgebra $\mathfrak{p}$ which is complementary to $\mathfrak{q}$. Then the action of the algebra $\mathfrak{q}$ on $V(\mathfrak{q})$ can be extended to the action of the whole algebra $\mathfrak{a}$. We remark that this extension depends essentially on the choice of $\mathfrak{p}$.

Remark 2. Let A be a complex Lie group, P a Lie subgroup of A, $\mathfrak{a}$ and $\mathfrak{p}$ the Lie algebras of A and P respectively. Let us consider the de Rham complex $\Omega = \{\Omega^k\}$ of formal analytic differential forms at the point e on the space A/P. More precisely, let $z_1, \ldots, z_n$ be a system of coordinates on the complex manifold A/P in a neighbourhood of the point e. Then Ω^k consists of the forms

$$\omega = \sum a_{i_1 i_2 \ldots i_k}(z)\, dz_{i_1} \wedge \ldots \wedge dz_{i_k},$$

where $a_{i_1 i_2 \ldots i_k}(z)$ are formal power series of the variables $z_1, \ldots, z_n$.

The group A acts on the space A/P. Of course, it cannot act on the complex Ω. However, the Lie algebra $\mathfrak{a}$ acts on Ω. It is easy to verify that the complex Ω is dual to the complex $V(\mathfrak{a}, \mathfrak{p})$ constructed above. Therefore the exactness of $V(\mathfrak{a}, \mathfrak{p})$ is also a consequence of the exactness of de Rham's complex Ω.

In what follows we shall be interested in the case when $\mathfrak{a} = \mathfrak{g}$ is a complex semisimple Lie algebra, $\mathfrak{p} = \mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}_+$ is a Borel subalgebra of $\mathfrak{g}$. We shall study the structure of the members D_k of the complex $V(\mathfrak{g}, \mathfrak{b})$.

Lemma 9.3. *Let V be a $\mathfrak{b}$-module and let $V^{\mathfrak{g}}=U(\mathfrak{g}) \underset{U(\mathfrak{b})}{\otimes} V$. The mapping $V \to V^{\mathfrak{g}}$ generates an exact functor from the category of $\mathfrak{b}$-modules to the category of $\mathfrak{g}$-modules. If in addition V is a one-dimensional module, $Hv=\chi(H)v$, $E_\gamma v=0$ for $H\in\mathfrak{h}$, $\gamma\in\Delta_+$, $v\in V$ then $V^{\mathfrak{g}}=M_{\chi+\varrho}$.*

The proof follows easily from the fact that $U(\mathfrak{g})$ is a free $U(\mathfrak{b})$-module. The second statement follows from the definition of $M_{\chi+\varrho}$.

Lemma 9.3 enables us to present an easy description of the modules D_k occuring in the complex $V(\mathfrak{g}, \mathfrak{b})$.

Definition 9.4. Let Ψ be a finite collection of weights (we allow that some of them coincide). We shall say that the module M is of *type* Ψ if there exists a filtration $0=M^{(0)}\subset M^{(1)}\subset\ldots\subset M^{(l)}=M$ such that $M^{(i)}/M^{(i-1)}=M_{\psi_i}$ and the collection of weights $\{\psi_i\}$ coincides with Ψ.

Lemma 9.5. *Let N be a finite-dimensional $\mathfrak{h}$-diagonalizable $\mathfrak{b}$-module, $\Psi(N)=\{\varphi+\varrho\}$, where φ runs through all weights of N (with multiplicities). Then the module $U(\mathfrak{g}) \underset{U(\mathfrak{b})}{\otimes} N$ is of type $\Psi(N)$.*

The proof is an immediate consequence of Lemma 9.3.

Corollary. The module D_k in the complex $V(\mathfrak{g}, \mathfrak{b})$ is of type $\Psi(\Lambda^k(\mathfrak{g}/\mathfrak{b}))$.

Since the Lie algebra $\mathfrak{g}$ acts on the complex $V(\mathfrak{g}, \mathfrak{b})$, we can distinguish in it a subcomplex corresponding to the "zero" eigenvalues of the elements of $Z(\mathfrak{g})$. More precisely, let $\vartheta=\vartheta_\varrho\in\Theta$. We consider the subcomplex $V_\vartheta(\mathfrak{g}, \mathfrak{b})$ of $V(\mathfrak{g}, \mathfrak{b})$ consisting of the submodules $(D_k)_\vartheta\subset D_k$ and the module $\mathbf{C}_\vartheta=\mathbf{C}$. It follows from Proposition 8.6, §5 that the complex $V_\vartheta(\mathfrak{g}, \mathfrak{b})$ is exact.

Proposition 9.6. Let $\Psi_k=\{w\varrho | w\in W^{(k)}\}$. Then $(D_k)_\vartheta$ is of type Ψ_k.

First we prove the following lemma.

Lemma 9.7. *Let M be a module of type Ψ and $\vartheta\in\Theta$. Then the module M_ϑ is of type Ψ_ϑ, where Ψ_ϑ is the collection of all weights $\psi\in\Psi$ such that $\vartheta_\psi=\vartheta$.*

Proof. Let $0=M^{(0)}\subset M^{(1)}\subset\ldots\subset M^{(k)}=M$ be a filtration of M for which $M^{(i)}/M^{(i-1)}=M_{\psi_i}$, $\psi_i\in\Psi$. From the exactness of the functor $M\to M_\vartheta$ it follows that the modules $M_\vartheta^{(i)}$ form a filtration of M_ϑ and $M_\vartheta^{(i)}/M_\vartheta^{(i-1)}=(M_{\psi_i})_\vartheta$. It follows from Proposition 8.5 that $(M_{\psi_i})_\vartheta=M_{\psi_i}$ if $\vartheta_{\psi_i}=\vartheta$ and $(M_{\psi_i})_\vartheta=0$ if $\vartheta\neq\vartheta_{\psi_i}$. This implies Lemma 9.7.

It follows from Lemma 9.7 that $(D_k)_\vartheta$ is of type $[\Psi(\Lambda^k(\mathfrak{g}/\mathfrak{b}))]_\vartheta$. Now we shall study this set.

Let Φ be a subset of Δ. Put $|\Phi|=\sum_{\gamma\in\Phi}\gamma$. Since the set of weights of $\mathfrak{g}/\mathfrak{b}$ coincides with Δ_-, the collection of weights of $\Lambda^k(\mathfrak{g}, \mathfrak{b})$ (with multiplicities) coincides with the collection of weights of the form $-|\Phi|$ for all $\Phi\subset\Delta_+$ such that card $\Phi=k$. Therefore,

$$[\Psi(\Lambda^k(\mathfrak{g}/\mathfrak{b}))]_\vartheta=\{\varrho-|\Phi|\,|\,\Phi\subset\Delta_+,\ \text{card}\,\Phi=k,\ (\varrho-|\Phi|)\sim\varrho\}.$$

For any element $w \in W$ we put $\Phi_w = \{\gamma \in \Delta_+ | w^{-1}\gamma \subset \Delta_-\}$. Then card $\Phi_w = k$, for $w \in W^{(k)}$ (Lemma 11.1). Thus Proposition 9.6 is a consequence of the following lemma.

Lemma 9.8. *Let* $w \in W$, $\Phi \subset \Delta_+$. *Then* $\varrho - w\varrho = |\Phi|$ *if and only if* $\Phi = \Phi_w$.

A proof of this lemma is presented in § 11, see also [6].

Now we can formulate the main theorem of this section.

Theorem 9.9. *Let* V *be an irreducible finite-dimensional* $\mathfrak{g}$-*module with highest weight* λ. *Then there exists an exact sequence of* $U(\mathfrak{g})$-*modules*

$$0 \leftarrow V \leftarrow B_0^V \leftarrow B_1^V \leftarrow \ldots \leftarrow B_s^V \leftarrow 0$$

where $s = \dim \mathfrak{n}_-$ *and* B_k *is a module of type* $\Psi_k(\lambda) = \{w(\lambda+\varrho) | w \in W^{(k)}\}$.

Proof. In Proposition 9.6 the required exact sequence of the $\mathfrak{g}$-modules $B_k^{\mathbf{C}}$ was constructed for the case $V = \mathbf{C}$ (i.e. $\lambda = 0$). In the general case we consider the exact sequence $B_*^{\mathbf{C}} \underset{\mathbf{C}}{\otimes} V$ and put

$$B_k^V = (B_k^{\mathbf{C}} \otimes V)_{\theta_{\lambda+\varrho}}.$$

Now we prove that the sequence

$$0 \leftarrow V \leftarrow B_0^V \leftarrow B_1^V \leftarrow \ldots$$

satisfies the conditions of the theorem. Its exactness follows from the fact that $M \to (M \otimes V)_\theta$ is an exact functor. Now we show that B_k^V is of type $\Psi_k(\lambda)$.

Lemma 9.10. *Let* $\chi \in \mathfrak{h}^*$, V *be a finite-dimensional* $\mathfrak{g}$-*module. Let us denote by* Ψ *the set* $\{\lambda + \chi\}$ *where* λ *runs through all weights of* V *with the corresponding multiplicities. Then* $M_\chi \otimes V$ *is of type* Ψ.

Proof. Let $e_1, \ldots, e_l$ be a basis in V consisting of weight vectors and $\lambda_1, \lambda_2, \ldots, \lambda_l$ be the corresponding weights. We choose an enumeration of the vectors e_i such that $\lambda_i < \lambda_j$ implies $i > j$. Let $a_i = f_\chi \otimes e_i \in M_\chi \otimes V$ and $M^{(k)} = U(\mathfrak{g})(a_1, \ldots, a_k)$. Then $0 = M^{(0)} \subset M^{(1)} \subset \ldots \subset M^{(l)}$.

To prove Lemma 9.10 it will suffice to show that

$$M^{(i)}/M^{(i-1)} = M_{\lambda_i+\chi} \quad \text{and} \quad M^{(l)} = M_\chi \otimes V.$$

Let $\bar{a}_k$ denote the image of a_k in $M^{(k)}/M^{(k-1)}$. It is obvious that $\bar{a}_k$ is a generator of $M^{(k)}/M^{(k-1)}$ of weight $\chi + \lambda_k - \varrho$ and $E_\gamma \bar{a}_k = 0$ for $\gamma \in \Delta_+$. Therefore $M^{(k)} = U(\mathfrak{n}_-)(a_1, \ldots, a_k)$. We shall show that $M^{(k)}$ is a free $U(\mathfrak{n}_-)$-module with generators $a_1, \ldots, a_k$. Let $X_i \in U(\mathfrak{n}_-)$, $1 \leq i \leq k$ and let p be the largest among the degrees of X_i (with respect to the natural filtration in $U(\mathfrak{n}_-)$). Then

$$\sum_{i=1}^{k} X_i a_i = \sum_{i=1}^{k} X_i f_\chi \otimes e_i + \sum_{j=1}^{l} Y_j f_\chi \otimes e_j \neq 0,$$

since the degrees of the elements Y_j are less than p and M_χ is a free $U(\mathfrak{n}_-)$-module.

This means that $M^{(k)}/M^{(k-1)}$ is a free $U(\mathfrak{n}_-)$-module with the single generator $\bar{a}_k$, i.e. $M^{(k)}/M^{(k-1)} \approx M_{\chi+\lambda_k}$.

Similar considerations will show that $M^{(l)} = M_\chi \otimes V$. This completes the proof of Lemma 9.10.

It follows from this lemma that $B_k^{\mathbf{C}} \otimes V$ is of type Ψ, where $\Psi = \{\lambda_i + w\varrho | \lambda_i$ are the weights of V (with multiplicities), $w \in W^{(k)}\}$. Therefore, Theorem 9.9 follows from Lemma 9.7 and the following lemma.

Lemma 9.11. *Let V be a finite-dimensional irreducible* $\mathfrak{g}$*-module with highest weight λ. Then for every $w \in W$ there exists exactly one weight $\mu \in P(V)$ such that $\mu + w\varrho \sim \lambda + \varrho$. Moreover the weight μ has multiplicity one in V.*

Proof. Let $\mu \in P(V)$, $w_1, w_2 \in W$ be such that $w_1(\mu + w\varrho) = \lambda + \varrho$; since $w_1\mu$ is a weight of V, we have $w_1\mu \leqq \lambda$.

In addition, $w_1 w\varrho \leqq \varrho$. Therefore $w_1\mu = \lambda$ and $w_1 w\varrho = \varrho$. This implies $w_1 = w^{-1}$ and thus $\mu = w\lambda$. The last statement of the lemma follows from the fact that the multiplicity of the weight μ in V is equal to the multiplicity of the highest weight λ.

Corollary (Bott's theorem [5], [6]). *Let V be a finite-dimensional irreducible* $\mathfrak{g}$*-module. Then*

$$\dim H^i(\mathfrak{n}_-, V) = \operatorname{card} W^{(i)}.$$

Proof. We know that

$$H^i(\mathfrak{n}_-, V) = \operatorname{Ext}^i_{\mathfrak{n}_-}(\mathbf{C}, V) = \operatorname{Tor}_i^{\mathfrak{n}_-}(V^*, \mathbf{C})^* = \operatorname{Tor}_i^{\mathfrak{n}_-}(\mathbf{C}, (V^\tau)^*)^*.$$

Let us construct the resolution $\{B_i^{V_1}\}$ for the module $V_1 = (V^\tau)^*$. Then $\operatorname{Tor}_i^{\mathfrak{n}_-}(\mathbf{C}, V_1)$ will be the homologies of the complex

$$0 \longleftarrow \bar{B}_0^{V_1} \overset{\bar{d}_1}{\longleftarrow} \dots \longleftarrow \bar{B}_s^{V_1} \longleftarrow 0.$$

The algebra $\mathfrak{h}$ acts on this complex in a natural way. In view of Theorem 9.9 we have here that $\bar{B}_i^{V_1} = B_i^{V_1}/\mathfrak{n}_- B_i^{V_1}$ is a finite-dimensional space whose weights with respect to $\mathfrak{h}$ are equal to $w(\lambda+\varrho)$, $w \in W^{(i)}$, and each of these weights occurs with multiplicity one. Therefore $\dim \bar{B}_i^{V_1} = \operatorname{card} W^{(i)}$, and every $\bar{d}_i$ is the null-mapping. The corollary is proved.

§ 10. Construction of the resolution of a finite-dimensional $\mathfrak{g}$-module

The present section is devoted to the proof of Theorem 10.1, which yields a sharpening of Theorem 9.9.

Theorem 10.1. *Let V be an irreducible finite-dimensional* $\mathfrak{g}$*-module with highest weight λ. Then there exists an exact sequence of* $\mathfrak{g}$*-modules*

$$0 \longleftarrow V \overset{\varepsilon}{\longleftarrow} C_0^V \overset{d_1}{\longleftarrow} C_1 \longleftarrow \dots \longleftarrow C_s^V \longleftarrow 0$$

where

$$s = \dim \mathfrak{n}_-, \quad C_k = \bigoplus_{w \in W^{(k)}} M_{w(\mu+\varrho)}.$$

First we shall present an explicit construction of the mappings d_i and ε. Let us put $\chi=\lambda+\varrho$. Then $\chi\in D$ and by Theorem 8.8′ every submodule of the module $M_{w\chi}$, $w\in W$ can be considered as a submodule of M_χ. By Theorem 8.7, any mapping $M_{w_1\chi}\to M_{w_2\chi}$ is a multiple of the canonical imbedding for $w_1<w_2$, hence it can be determined by a complex number $c_{w_1w_2}$. Therefore, any mapping $C_i^V\to C_{i-1}^V$ can be represented by a complex matrix $(c_{w_1w_2})$, $w_1\in W^{(i)}$ and $w_2\in W^{(i-1)}$. Thus in order to construct the mappings d_i it will suffice to define the corresponding matrices $(d^{(i)}_{w_1w_2})$.

Definition 10.2. Let us call a quadruple (w_1, w_2, w_3, w_4) of elements of W a *square* if

$$w_1\to w_2\to w_4 \quad\text{and}\quad w_1\to w_3\to w_4$$

(see Definition 8.9).

It will be convenient to consider the finite directed graph corresponding to W, whose vertices are the members of W and in which an arc leads from w_1 to w_2 if $w_1\to w_2$.

Lemma 10.3.* *Let $w_1, w_2\in W$ and $l(w_1)-2=l(w_2)$. Then the number of elements $w'\in W$ such that $w_1\to w'\to w_2$ is equal to either zero or two.*

Lemma 10.4. *To each arrow $w_1\to w_2$ we can assign a number $s(w_1, w_2)=\pm 1$ in such a way that for every square (w_1, w_2, w_3, w_4) the product of the numbers assigned to the four arrows occurring in it is equal to -1.*

The proofs of Lemmas 10.3 and 10.4 will be given in § 11.

Now we can improve on Theorem 10.1 in the following way.

Theorem 10.1′. *With the notation of Theorem* 10.1. *we define the mapping $d_i\colon C_i\to C_{i-1}$ by means of the matrix $(d^{(i)}_{w_1w_2})$, $w_1\in W^{(i)}$, $w_2\in W^{(i-1)}$, where $d^{(i)}_{w_1w_2}=s(w_1, w_2)$ if $w_1\to w_2$ and $d^{(i)}_{w_1w_2}=0$ otherwise. Let us denote by $\varepsilon\colon C_0\to V$ the natural surjection. Then the sequence*

$$0\longleftarrow V\xleftarrow{\varepsilon} C_0^V\xleftarrow{d_1} C_1^V\longleftarrow\dots\xleftarrow{d_s} C_s^V\longleftarrow 0 \qquad (10.1)$$

is exact.

Proof. It follows immediately from Lemmas 10.3 and 10.4 that $d_i\circ d_{i+1}=0$ for $i=1, \dots, s-1$.

We remark that $W^{(1)}=\{\sigma_\alpha, \alpha\in\Sigma\}$. Therefore, Harish-Chandra's theorem on ideals [4] implies the exactness of the sequence (10.1) at its members V and C_0.

Assume now that we have already shown the exactness of the sequence at the members $C_0, \dots, C_{i-1}$. We shall prove that it is also exact at the member C_i, i.e. that $\operatorname{Ker} d_i=\operatorname{Im} d_{i+1}$. Let us put $K=\operatorname{Ker} d_i$. The desired equality $d_{i+1}(C_{i+1})=K$ is obviously a consequence of the following three lemmas.

* Lemma 10.3 follows easily from certain unpublished results of D.-N. Verma concerning the Möbius function on the Weyl group.

(We recall that $C_{i+1} = \bigoplus_{w \in W^{(i+1)}} M_{w\chi}$ is a free $U(\mathfrak{n}_-)$-module.)

Lemma 10.5. *Let C be a free $U(\mathfrak{n}_-)$-module with generators $f_1, \ldots, f_n$ and $\zeta: C \to K$ be a homomorphism of $U(\mathfrak{n}_-)$-modules such that $\zeta(f_i)$ is a weight vector in K (with respect to $\mathfrak{h}$). Then ζ is a surjection if and only if the induced mapping $\bar{\zeta}: C/\mathfrak{n}_- C \to K/\mathfrak{n}_- K$ is surjective.*

Lemma 10.6. *The mapping*

$$\bar{d}_{i+1}: C_{i+1}/\mathfrak{n}_- C_{i+1} \to K/\mathfrak{n}_- K$$

is an injection.

Lemma 10.7.

$$\dim_{\mathbf{C}} C_{i+1}/\mathfrak{n}_- C_{i+1} = \dim_{\mathbf{C}} K/\mathfrak{n}_- K < \infty.$$

Proof of Lemma 10.5. Clearly if ζ is surjective then so is $\bar{\zeta}$. Conversely, assume that $\bar{\zeta}$ is surjective but ζ is not. Consider the weight vector f in K of weight ψ having the following properties:

a) $f \notin \operatorname{Im} \zeta$,

b) any vector f' of weight $\psi' > \psi$ belongs to $\operatorname{Im} \zeta$.

There always exists such a vector f since $K \in O$, and therefore for any weight $\psi \in \mathfrak{h}^*$ there are finitely many weights $\psi' > \psi$ such that $K^{(\psi')} \neq \{0\}$.

Let $\bar{f}$ be the image of f in $K/\mathfrak{n}_- K$. Then $\bar{f} = \sum c_i \bar{\zeta}(f_i)$. Since $\mathfrak{n}_- K$ is invariant under $\mathfrak{h}$, $\mathfrak{h}$ acts on $K/\mathfrak{n}_- K$. Therefore we can assume that $c_i \neq 0$ only for those indices i for which the weight of $\zeta(f_i)$ is equal to ψ. Moreover, $g = f - \sum c_i \zeta(f_i)$ is a weight vector lying in $\mathfrak{n}_- K$, and thus $g = \sum_{\gamma \in \Delta_+} E_{-\gamma} g_\gamma$, where the weight of g_γ is $\psi + \gamma > \psi$. According to the construction of f, $g_\gamma \in \operatorname{Im} \zeta$, and therefore $f \in \operatorname{Im} \zeta$. Lemma 10.5 is proved.

Proof of Lemma 10.6. The quotient space $C_{i+1}/\mathfrak{n}_- C_{i+1}$ is a linear space over $\mathbf{C}$ for which $\{\bar{f}_{w\chi} | w \in W^{(i+1)}\}$ forms a basis.* Since the homomorphism $\bar{d}_{i+1}$ commutes with the action of $\mathfrak{h}$ and all of the vectors $\bar{f}_{w\chi}$ have different weights it will suffice to prove that $\bar{d}_{i+1}(\bar{f}_{w\chi}) \neq 0$ for any $w \in W^{(i+1)}$.

The proof of this proposition is divided in a natural way into two steps.

Lemma 10.6a. *The irreducible modules occurring in the Jordan—Hölder decomposition of the module K are of the form $L_{w\chi}$, $l(w) > i$.*

Proof. For the proof of Lemma 10.6a we shall make use of the exact sequence

$$0 \leftarrow V \leftarrow B_0^V \leftarrow B_1^V \leftarrow \ldots \leftarrow B_s^V \leftarrow 0 \tag{10.2}$$

constructed in § 9. It is clear that for all j

$$JH(B_j^V) = \bigcup_{w \in W^{(j)}} JH(M_{w\chi}) = JH(C_j^V).$$

* Let us recall, that $f_{w\chi}$ is the generator of $M_{w\chi}$ (see Def. 8.2).

Since both of the sequences (10.1) and (10.2) are exact at their members with numbers less than i, we have $JH(K)=JH(K_B)$, where K_B denotes the kernel of the mapping $B_i^V \to B_{i-1}^V$. As the sequence (10.2) is exact at its member B_i^V, K_B is equal to the image of B_{i+1}^V. Therefore

$$JH(K_B) \subset JH(B_{i+1}^V) = \bigcup_{w \in W^{(i+1)}} JH(M_{w\chi}),$$

and thus Lemma 10.6a is implied by the Corollary of Theorem 8.12.

Lemma 10.6b. *Let* $w_0 \in W$ *and the module* $M \in O$ *be given. We assume that* $l(w) \geqq l(w_0)$ *for any* $L_{w\chi}$ *occurring in* $JH(M)$. *Let* $\tau: M_{w_0\chi} \to M$ *be a homomorphism such that* $\tau(f_{w_0\chi}) \neq 0$. *Then the image of* $\tau(f_{w_0\chi})$ *in* $M/\mathfrak{n}_- M$ *is not* 0.

Proof. We shall use induction on the number of elements in $JH(M)$. Let $f \in M$ be an element of maximal weight ψ and $N \subset M$ be the submodule generated by f. Then the module N is isomorphic to a factor module of the module M_ψ. We shall distinguish two cases.

Case 1. $\tau(f_{w_0\chi}) \in N$. In this case

$$L_{w_0\chi} \subset JH(N) \subset JH(M_\psi).$$

Therefore (as it follows from Theorem 8.12) $\psi = w_1\chi$, where $w_1 \geqq w_0$. On the other hand

$$L_\psi \subset JH(N) \subset JH(M).$$

Thus, according to the condition of the lemma $w_1 = w_0$, i.e. $\psi = w_0\chi$. Since ψ is the maximal weight of M,

$$\tau(f_{w_0\chi}) \notin \mathfrak{n}_- M.$$

Case 2. $\tau(f_{w_0\chi}) \notin N$. In this case the statement of the lemma for M can be reduced to the similar statement for the module M/N. Since $JH(M/N) \subsetneqq JH(M)$, we can apply the induction hypothesis. Lemma 10.6b is proved.

To complete the proof of Lemma 10.6 it suffices to apply Lemma 10.6b to the module $M=K$.

Proof of Lemma 10.7. The module K has only a finite number of generators (as a $U(\mathfrak{n}_-)$-module). Therefore $\bar{K} = K/\mathfrak{n}_- K$ is a finite-dimensional space over **C**. Let us choose weight vectors $f_1, \ldots, f_n \in K$ whose images in $\bar{K}$ form a basis in $\bar{K}$. Let us consider the free $U(\mathfrak{n}_-)$-module C with n generators $g_1, \ldots, g_n$ and define a homomorphism of $U(\mathfrak{n}_-)$-modules $\vartheta: C \to K$ by the formula $\vartheta(g_i) = f_i$. By virtue of Lemma 10.5 ϑ is surjective.

Let us consider the exact sequence

$$0 \longleftarrow V \longleftarrow C_0^V \xleftarrow{d_1} \ldots \longleftarrow C_i^V \xleftarrow{\vartheta} C$$

of $U(\mathfrak{n}_-)$-modules.

Since C_j^V and C are free $U(\mathfrak{n}_-)$-modules, this sequence can be augmented to a free resolution

$$0 \longleftarrow V \longleftarrow C_0^V \longleftarrow \ldots \longleftarrow C_i^V \xleftarrow{\vartheta} C \xleftarrow{\eta} D \longleftarrow \ldots$$

of the $U(\mathfrak{n}_-)$-module V.

Now, consider the sequence

$$\bar{D} \xrightarrow{\bar{\eta}} \bar{C} \xrightarrow{\bar{\vartheta}} \bar{C}_i^V \xrightarrow{\bar{d}_i} \bar{C}_{i-1}^V$$

where, for any $U(\mathfrak{n}_-)$-module M, $\bar{M}$ denotes $1 \underset{U(\mathfrak{n}_-)}{\otimes} M = M/\mathfrak{n}_- M$. By definition,

$$\mathrm{Tor}_i^{\mathfrak{n}_-}(\mathbf{C}, V) = \mathrm{Ker}\, \bar{\vartheta}/\mathrm{Im}\, \bar{\eta}.$$

We shall show that $\bar{\vartheta}$ and $\bar{\eta}$ are equal to 0.

From the exact sequence

$$D \xrightarrow{\eta} C \xrightarrow{\tau} K \longrightarrow 0$$

we obtain the exact sequence

$$\bar{D} \xrightarrow{\bar{\eta}} \bar{C} \xrightarrow{\bar{\tau}} \bar{K} \longrightarrow 0.$$

But $\bar{\tau}$ is an isomorphism, hence $\bar{\eta}=0$.

Furthermore, we have the exact sequence

$$C \xrightarrow{\vartheta} C_i^V \xrightarrow{d_i} K_{i-1} \longrightarrow 0$$

and thus the sequence

$$\bar{C} \xrightarrow{\bar{\vartheta}} \bar{C}_i^V \xrightarrow{\bar{d}_i} \bar{K}_{i-1} \longrightarrow 0,$$

where $K_{i-1} = \mathrm{Ker}\, d_{i-1} = \mathrm{Im}\, d_i$. Applying Lemma 10.6 to the mapping $d_i: C_i \to K_{i-1}$ we see that $\bar{d}_i$ is an isomorphism, hence $\bar{\vartheta}=0$.

Thus we have

$$\dim \mathrm{Tor}_i^{\mathfrak{n}_-}(\mathbf{C}, V) = \dim \bar{C} = \dim (K/\mathfrak{n}_- K).$$

On the other hand, the Corollary to Theorem 9.9 shows that

$$\dim \mathrm{Tor}_i^{\mathfrak{n}_-}(\mathbf{C}, V) = \mathrm{card}\, W^{(i)} = \dim (C_i/\mathfrak{n}_- C_i).$$

Lemma 10.7 is proved.

This also completes the proof of Theorem 10.1′ and Theorem 10.1.

§ 11. Proof of the lemmas

First we shall present several results clarifying the properties of the function $l(w)$ in more detail (see [11]).

Let us put $\Phi_w = \Delta_+ \cap w(\Delta_-)$.

Lemma 11.1.

1) $l(w) = \operatorname{card} \Phi_w$.

2) If $\gamma_1, \gamma_2 \in \Phi_w$ *and* $\gamma_1 + \gamma_2 = n\gamma$ *for* $n \in \mathbf{Z}_+$, *then* $\gamma \in \Phi_w$.

Proof. Statement *1)* is proved in [11]. To verify statement *2)* it is enough to prove it separately for Δ_+ and $w\Delta_-$, and it is obvious in both cases.

Proof of Lemma 8.10. Plainly,

$$w\chi - \sigma_\gamma w\chi = \frac{2\langle w\chi, \gamma\rangle}{\langle \gamma, \gamma\rangle}\gamma.$$

Since $\chi \in \mathfrak{h}^*_{\mathbf{Z}}$ and the element χ is regular,

$$n = \frac{2\langle w\chi, \gamma\rangle}{\langle \gamma, \gamma\rangle} \in \mathbf{Z} \setminus 0.$$

We assume that $n > 0$ and prove then that $l(\sigma_\gamma w) > l(w)$. By *1)* of Lemma 11.1, it suffices to prove that $\operatorname{card} \Phi_w < \operatorname{card} \Phi_{\sigma_\gamma w}$.

a) $\gamma \notin \Phi_w$. Indeed, $\langle \chi, w^{-1}\gamma\rangle = \langle w\chi, \gamma\rangle > 0$, and therefore $w^{-1}\gamma \notin \Delta_-$.

b) $\gamma \in \Phi_{\sigma_\gamma w}$. Indeed,

$$(\sigma_\gamma w)^{-1}\gamma = w^{-1}(-\gamma) = -w^{-1}\gamma \in \Delta_-.$$

c) Let $\delta \in \Phi_w$ and $\sigma_\gamma\delta \in \Delta_+$. Then $\sigma_\gamma\delta \in \Phi_{\sigma_\gamma w}$. In fact,

$$(\sigma_\gamma w)^{-1}(\sigma_\gamma\delta) = w^{-1}\sigma_\gamma^{-1}\sigma_\gamma\delta = w^{-1}\delta \in \Delta_-.$$

d) Let $\delta \in \Phi_w$, $\sigma_\gamma\delta \in \Delta_-$. We shall show that $\delta \in \Phi_{\sigma_\gamma w}$. Assume, on the contrary that $w^{-1}\sigma_\gamma\delta \in \Delta_+$. Then $w^{-1}(-\sigma_\gamma\delta) \in \Delta_-$, and thus $-\sigma_\gamma\delta \in \Phi_w$. But $\delta + (-\sigma_\gamma\delta) = \frac{2\langle \gamma, \delta\rangle}{\langle \delta, \delta\rangle}\gamma$, while $\frac{2\langle \gamma, \delta\rangle}{\langle \delta, \delta\rangle} > 0$. By *2)* of Lemma 11.1 this implies $\gamma \in \Phi_w$, and that contradicts a).

Propositions a)—d) imply that if $n > 0$ then $l(w) < l(\sigma_\gamma w)$.

By interchanging w and $\sigma_\gamma w$ we obtain that if $n < 0$ then $l(w) > l(\sigma_\gamma w)$. Lemma 8.10 is proved.

Lemma 11.2.

1) Let $w \in W$, $\alpha \in \Sigma$. *Then* $\alpha \in \Phi_w$ *implies* $w \xrightarrow{\alpha} \sigma_\alpha w$, *and* $\alpha \notin \Phi_w$ *implies* $\sigma_\alpha w \xrightarrow{\alpha} w$.

2) If $\sigma_\alpha w \xrightarrow{\alpha} w$ *for all* $\alpha \in \Sigma$, *then* $w = 1$.

3) There exists a unique element $s \in W$ *such that* $s \to \sigma_\alpha s$ *for all* $\alpha \in \Sigma$.

This lemma follows immediately from the theorems proved in [11].

Lemma 11.3. *Let* $w_1, w_2 \in W$, $\gamma \in \Delta_+$ *and* $\alpha \in \Sigma$, $\alpha \neq \gamma$.

We put

$$\sigma_\alpha w_1 \xrightarrow{\alpha} w_1 \quad \textit{and} \quad \sigma_\alpha w_1 \xrightarrow{\gamma} w_2. \tag{11.1}$$

Then

$$w_2 \xrightarrow{\alpha} \sigma_\alpha w_2 \quad \textit{and} \quad w_1 \xrightarrow{\sigma_\alpha \gamma} \sigma_\alpha w_2. \tag{11.2}$$

Conversely, (11.2) *implies* (11.1)

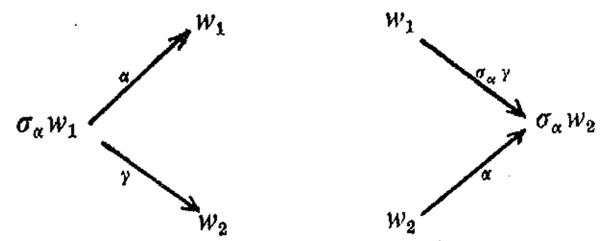


Diagram 1 Diagram 2

Proof. We have to prove that Diagram 1 is equivalent to Diagram 2. Let $\gamma' = \sigma_\alpha \gamma$. Then $\gamma' \in \Delta_+$ and $\sigma_\alpha w_2 = \sigma_\gamma w_1$. Since $l(w_1) = l(w_2)$, formula (11.2) is equivalent to $l(\sigma_\alpha w_2) = l(w_2) - 1$. Let $\chi \in D$. Then, by Lemma 8.10,

$$w_2\chi - \sigma_\alpha w_1 \chi = n\gamma, \quad n > 0.$$

Applying σ_α to this equality we obtain

$$\sigma_\alpha w_2 \chi - w_1 \chi = n\gamma'.$$

Using Lemma 8.10 again we obtain that $l(\sigma_\alpha w_2) < l(w_1)$. Therefore, (11.1) implies (11.2).

We can prove similarly that (11.2) implies (11.1).

Proof of Lemma 8.11. The proof will be performed by induction on $l(w)$. If $l(\sigma_\gamma w) = l(w) - 1$, then by definition $\sigma_\gamma w > w$. Since $l(w)$ and $l(\sigma_\gamma w)$ are of different parities, it remains to consider the case in which $l(\sigma_\gamma w) \leqq l(w) - 3$.

Let $\alpha \in \Sigma$ be a root such that $w \to \sigma_\alpha w$. Obviously, $\alpha \neq \gamma$ and $\sigma_\alpha \sigma_\gamma w = \sigma_{\gamma'} \sigma_\alpha w$, where $\gamma' = \sigma_\alpha \gamma \in \Delta_+$.

Here we have

$$l(\sigma_{\gamma'} \sigma_\alpha w) = l(\sigma_\alpha \sigma_\gamma w) \leqq l(\sigma_\gamma w) + 1 \leqq l(w) - 2 < l(\sigma_\alpha w).$$

By the induction hypothesis there exists a chain

$$\sigma_\alpha w \to w_1 \to w_2 \to \ldots \to w_2 \to \sigma_\alpha \sigma_\gamma w$$

and thus also a chain

$$w_{-1} \xrightarrow{\gamma_0} w_0 \xrightarrow{\gamma_1} w_1 \longrightarrow \ldots \longrightarrow w_k \xrightarrow{\gamma_k + 1} \sigma_\alpha \sigma_\gamma w \tag{11 3}$$

(where $w_{-1} = w$, $w_0 = \sigma_\alpha w$ and $\gamma_0 = \alpha$). There are two possibilities.

Case 1. $\sigma_\alpha \sigma_\gamma w \to \sigma_\gamma w$. Then there exists a chain

$$w = w_{-1} \to w_0 \to w_1 \to \ldots \to w_k \to \sigma_\alpha \sigma_\gamma w \to \sigma_\gamma w,$$

hence $w < \sigma_\gamma w$.

Case 2. $\sigma_\gamma w \to \sigma_\alpha \sigma_\gamma w$. Let i be the largest number such that in the chain (11.3) $\gamma_i = \alpha$. Applying Lemma 11.3 several times we obtain that

$$\sigma_\alpha w_k \xrightarrow{\sigma_\alpha \gamma_{k+1}} \sigma_\gamma w,$$

$$\sigma_\alpha w_{k-1} \xrightarrow{\sigma_\alpha \gamma_k} \sigma_\alpha w_k, \ldots, \sigma_\alpha w_i \xrightarrow{\sigma_\alpha \gamma_{i+1}} \sigma_\alpha w_{i+1}.$$

But $\sigma_\alpha w_i = w_{i-1}$. Hence we obtain a chain

$$w = w_1 \xrightarrow{\gamma_0} w_0 \xrightarrow{\gamma_1} \ldots \longrightarrow w_{i-1} \xrightarrow{\sigma_\alpha \gamma_{i+1}} \sigma_\alpha w_{i+1} \longrightarrow \ldots \xrightarrow{\sigma_\alpha \gamma_{k+1}} \sigma_\gamma w.$$

Therefore, $w < \sigma_\gamma w$, and Lemma 8.11 is proved.

Proof of Lemma 9.8. We want to prove that if $w \in W$, $\Phi \subset \Delta_+$ and $\varrho - w\varrho = |\Phi|$, then $\Phi = \Phi_w = \Delta_+ \cap w\Delta_-$. If $w = e$, this is obvious. We shall perform the proof by induction on $l(w)$. Let $l(w) = i > 0$. We choose an $\alpha \in \Sigma$ such that $l(\sigma_\alpha w) = i - 1$, i.e. $\alpha \in \Phi_w$. Then

$$|\sigma_\alpha \Phi| = \sigma_\alpha \varrho - \sigma_\alpha w \varrho = \varrho - \sigma_\alpha w \varrho - \alpha.$$

$\Phi \subset \Delta_+$ implies $\alpha \notin \sigma_\alpha \Phi$. Therefore, putting $\sigma_\alpha w = w'$, we have

$$\varrho - w'\varrho = \varrho - \sigma_\alpha w \varrho = |\sigma_\alpha \Phi \cup \{\alpha\}|.$$

We shall show that $\alpha \in \Phi$. Indeed, assume that this is not true. Then $\sigma_\alpha \Phi \cup \{\alpha\} \subset \Delta_+$ and by the induction hypothesis $\Phi_{w'} = \sigma_\alpha \Phi \cup \{\alpha\}$, i.e. $\alpha \in \Phi_{w'}$. According to Lemma 11.2. 1 $\sigma_\alpha w \to w$, which contradicts the choice of α.

Thus $\alpha \in \Phi$. Let us put $\Phi = \Phi - \{\alpha\}$. Then $\varrho - \sigma_\alpha w \varrho = |\sigma_\alpha \Phi'|$ and $\sigma_\alpha \Phi' \subset \Delta_+$. By the induction hypothesis $\Phi_{w'} = \sigma_\alpha \Phi'$, i.e. $\Phi = \sigma_\alpha \Phi_{w'} \cup \{\alpha\}$. It is easy to verify that $\sigma_\alpha \Phi_{w'} \cup \{\alpha\} = \Phi_w$. Lemma 9.8 is proved.

Proof of Lemma 10.3. We shall prove this by induction on $l(w_1)$. Since $l(w_1) \geqq 2$, we can choose an $\alpha \in \Sigma$ such that $w_1 \to \sigma_\alpha w_1$. There are two possibilities.

Case 1. $w_2 \to \sigma_\alpha w_2$. Let us assign to every chain

$$w_1 \xrightarrow{\gamma_1} w \xrightarrow{\gamma_2} w_2 \tag{11.4}$$

a chain

$$\sigma_\alpha w_1 \xrightarrow{\delta_1} w' \xrightarrow{\delta_2} \sigma_\alpha w_2 \tag{11.5}$$

in the following way.

a) If $\gamma_1 \neq \alpha$, then put $\delta_1 = \sigma_\alpha \gamma_1$, $\delta_2 = \sigma_\alpha \gamma_2$, $w' = \sigma_\alpha w$. It follows from Lemma 11.3 that

$$\sigma_\alpha w_1 \xrightarrow{\delta_1} w' \xrightarrow{\delta_2} \sigma_\alpha w_2.$$

b) If $\gamma_1 = \alpha$, then put $\delta_1 = \gamma_2$, $\delta_2 = \alpha$, $w' = w_2$. Using Lemma 11.3 again, it is easy to see that we have constructed a one-to-one correspondence between the chains of form (11.4) and the chains of form (11.5). As $l(\sigma_\alpha w_1) < l(w_1)$, we can apply the induction hypothesis.

Case 2. $\sigma_\alpha w_2 \to w_2$. If $w_1 \xrightarrow{\gamma_1} w \xrightarrow{\gamma_2} w_2$, then by Lemma 11.3, we have either $\gamma_1 = \alpha$ or $\gamma_2 = \alpha$. The same lemma shows us that $w_1 \to \sigma_\alpha w_2$ is equivalent to $\sigma_\alpha w_1 \to w_2$. Therefore, either there is no chain of form (11.5) or there exist exactly two chains $w_1 \to \sigma_\alpha w_1 \to w_2$ and $w_1 \to \sigma_\alpha w_2 \to w_2$. Lemma 10.3 is proved.

Proof of Lemma 10.4. For any element $w \in W$ we put $I(w) = \{w' \in W \mid w' \geqq w\}$. We shall prove by induction on $l(w)$ that $s(w_1, w_2)$ can be defined for $w_1, w_2 \in I(w)$ in such a way that the condition of the lemma be satisfied for any square (w_1, w_2, w_3, w_4) with $w_1 \in I(w)$. This will imply the statement of Lemma 10.4 since $I(s) = W$ (the element $s \in W$ is defined in Lemma 11.2. 3).

Thus assume $w \in W$. We choose an $\alpha \in \Sigma$ such that $w \to \sigma_\alpha w$. By the induction hypothesis, the function $s(w_1, w_2)$ has been defined for $w_1, w_2 \in I(\sigma_\alpha w)$.

1) Let

$$w = w_0 \xrightarrow{\gamma_1} w_1 \longrightarrow \ldots \xrightarrow{\gamma_k} w_k,$$

where $\gamma_i \neq \alpha$ for all i. Then by Lemma 11.3 $\sigma_\alpha w_0 \to \sigma_\alpha w_1 \to \ldots \to \sigma_\alpha w_k$ and $w_1 \to \sigma_\alpha w_i$ for all i. In particular, $\sigma_\alpha w_i \in I(\sigma_\alpha w)$ for all i.

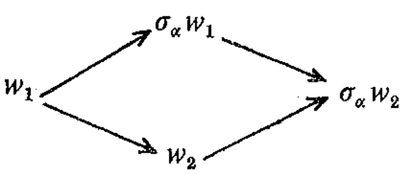
2) Let $w' \in I(w) \setminus I(\sigma_\alpha w)$ and let

$$w = w_0 \xrightarrow{\gamma_1} w_1 \longrightarrow \ldots \xrightarrow{\gamma_k} w_k = w'$$

be an arbitrary chain leading from w to w'. Then $\gamma_i \neq \alpha$ for all i, because otherwise $w' \in I(\sigma_\alpha w)$ would hold in view of *1)*. In particular, we have $w' \to \sigma_\alpha w'$ and $\sigma_\alpha w' \in I(\sigma_\alpha w)$.

3) Let us now define the function $s(w_1, w_2)$ for all arrows $w_1 \xrightarrow{\gamma} w_2$, $w_1 \in I(w)$. Here we assume that $s(w_1, w_2)$ has already been defined if $w_1 \in I(\sigma_\alpha w)$.

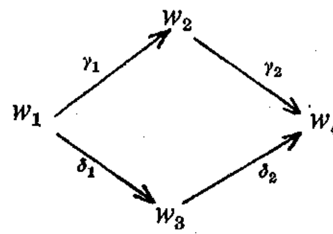
Let $w_1 \in I(w) \setminus I(\sigma_\alpha w)$. If $w_2 = \sigma_\alpha w_1$, then we put $s(w_1, w_2) = 1$. Let $w_2 \neq \sigma_\alpha w_1$. Then Lemma 11.3.2 implies



where $\sigma_\alpha w_1, \sigma_\alpha w_2 \in I(\sigma_\alpha w)$. In this case we put

$$s(w_1, w_2) = -s(w_1, \sigma_\alpha w_2) s(w_2, \sigma_\alpha w_2) s(\sigma_\alpha w_1, \sigma_\alpha w_2). \tag{11.6}$$

4) Now we prove that the function s defined in this manner satisfies the conditions of Lemma 10.4. For every square A



we put

$$p(A)=s(w_1, w_3)s(w_1, w_4)s(w_2, w_4)s(w_3, w_4).$$

We have to prove that $p(A)=-1$. There are the following possibilities.

a) $w_1 \in I(\sigma_\alpha w)$. In this case $p(A)=-1$ by the induction hypothesis.

b) $w_1 \notin I(\sigma_\alpha w)$ and $\gamma_1=\delta=\alpha$. Then $p(A)=-1$ in view of formula (11.6).

c) $w_1 \notin I(\sigma_\alpha w)$ and $\gamma_1 \neq \alpha$, $\delta_1 \neq \alpha$. Since $w_2 \neq w_3$, we have $\gamma_2 \neq \delta_2$, and therefore one of the elements γ_2, δ_2 must be different from α. Then by Lemma 11.3 $w_4 \to \sigma_\alpha w_4$, and thus $\gamma_2 \neq \alpha$ and $\delta_2 \neq \alpha$. According to Lemma 11.3 we have five squares

$$A_1=(w_1, w_2, \sigma_\alpha w_1, \sigma_\alpha w_2),$$

$$A_2=(w_1, w_3, \sigma_\alpha w_1, \sigma_\alpha w_3),$$

$$A_3=(w_2, w_4, \sigma_\alpha w_2, \sigma_\alpha w_4),$$

$$A_4=(w_3, w_4, \sigma_\alpha w_3, \sigma_\alpha w_4),$$

$$A_5=(\sigma_\alpha w_1, \sigma_\alpha w_2, \sigma_\alpha w_3, \sigma_\alpha w_4).$$

Here $p(A_i)=-1$, $i=1, \ldots, 5$, in view of a) and b). It is easy to verify that $p(A)\cdot\prod_{i=1}^{5} p(A_i)=1$, hence $p(A)=-1$, too.

d) $\gamma_1=\alpha$, $\delta_2 \neq \alpha$.

By Lemma 11.3, $w_3 \to \sigma_\alpha w_3$, $w_4 \to \sigma_\alpha w_4$ and we have three squares

$$A_1=(w_1, w_2, w_3, \sigma_\alpha w_3),$$

$$A_2=(w_3, w_4, \sigma_\alpha w_3, \sigma_\alpha w_4),$$

$$A_3=(w_2, w_4, \sigma_\alpha w_3, \sigma_\alpha w_4).$$

In view of a) and b) we have $p(A_i)=-1$, $i=1, 2, 3$. Since $p(A)p(A_2)=p(A_1)p(A_3)$, we also have $p(A)=-1$.

Thus we have considered all the possibilities for the roots $\gamma_1, \gamma_2, \delta_1, \delta_2$ (up to interchanging γ_i with δ_i and w_2 with w_3). The proof of Lemma 10.4 is complete.

APPENDIX

Let $\chi, \psi \in \mathfrak{h}^*$, $\gamma_1, \ldots, \gamma_k \in \Delta_+$. We shall say that the sequence $\gamma_2, \ldots, \gamma_k$ satisfies condition (A) for the pair (χ, ψ) if

1) $\chi=\sigma_{\gamma_k} \cdots \sigma_{\gamma_1} \psi$,

2) Let $\chi_0=\psi$ and $\chi_i=\sigma_{\gamma_i} \cdots \sigma_{\gamma_1} \psi$. Then for all i we have $\chi_{i-1}-\chi_i=n_i\gamma_i$, where

$$n_i=\frac{2\langle\chi_{i-1}, \gamma_i\rangle}{\langle\gamma_i, \gamma_i\rangle} \in \mathbf{Z}_+.$$

In the Appendix we are going to prove the following theorem.

Theorem A1. *Let* $L_\chi \in JH(M_\psi)$. *Then there exists a sequence* $\gamma_1, \dots, \gamma_k$ *which satisfies condition* (A) *for the pair* (χ, ψ).

In the proof of Theorem A1 we shall use the following lemma.

Lemma A2. *Let* $\chi \in \mathfrak{h}^*$. *Then*

$$JH(M_\chi) = \bigcup_{\substack{\varphi \sim \chi \\ \varphi \leqq \chi}} c_\varphi L_\varphi, \quad c_\varphi \in \mathbf{Z}_+,$$

where $c_\chi = 1$.

This lemma follows easily from Propositions 8.5 and 8.6.

Proof of Theorem A1. Let $\lambda \in \mathfrak{h}^*_{\mathbf{Z}}$. Let us denote by F_λ an irreducible finite-dimensional $\mathfrak{g}$-module for which λ is an extremal weight. We recall that if $\mu \in P(F_\lambda)$, then $\|\mu\| \leqq \|\lambda\|$ and $\|\mu\| = \|\lambda\|$ implies $\mu \sim \lambda$.

Definition A3. Let $\chi \in \mathfrak{h}^*$, $\lambda \in \mathfrak{h}^*_{\mathbf{Z}}$. The pair (χ, λ) is called *admissible* if there are no $w \in W$ and $\mu \in P(F_\lambda)$ such that

a) $w\chi < \chi$,

b) $w(\chi+\mu) \geqq \chi+\lambda$,

c) $w(\chi+\mu) \sim \chi+\lambda$.

The meaning of Definition A3 is illuminated by the following proposition, whose proof we postpone until later.

Proposition A4. *Suppose* $\chi, \psi \in \mathfrak{h}^*$, $\lambda \in \mathfrak{h}^*_{\mathbf{Z}}$ *and the pair* (χ, λ) *is admissible. Furthermore, let* $L_\chi \in JH(M_\psi)$. *Then there exists a weight* $\mu \in P(F_\lambda)$ *such that* $L_{\chi+\lambda} \in JH(M_{\psi+\mu})$.

Proposition A4 includes all the information about the modules M_χ and L_χ that we need to prove Theorem A1. The subsequent arguments will only concern the geometric structure of the space $\mathfrak{h}^*$.

Lemma A5. *1) There exists a constant* $c_1 > 0$ *with the following property: if* $\varphi > \psi$, $\varphi + \vartheta_1 < \psi + \vartheta_2$ $(\varphi, \psi, \vartheta_1, \vartheta_2 \in \mathfrak{h}^*)$, *then* $c_1(\|\vartheta_1\| + \|\vartheta_2\|) > \|\varphi - \psi\|$.

2) There exists a constant c_2 *with the following property: let* $\varphi \in \mathfrak{h}^*_{\mathbf{Z}}$; *then one can find a sequence of weights* $0 = \varphi_0, \varphi_1, \dots, \varphi_k = \varphi$, $\varphi_i \in \mathfrak{h}^*_{\mathbf{Z}}$, *such that*

$$\|\varphi_i - \varphi_{i-1}\| < c_2 \quad \text{and} \quad d(\varphi_i, [0, \varphi]) < c_2,$$

where $[0, \varphi]$ *is the segment in* $\mathfrak{h}^*$ *connecting* 0 *with* φ, *and* $d(\ ,\ )$ *denotes the distance in* $\mathfrak{h}^*_{\mathbf{R}}$.

3) Let $\varphi, \psi \in \bar{C}^+$, $w \in W$. *Then* $\langle w\varphi, \psi \rangle \leqq \langle \varphi, \psi \rangle$. *If in addition* $\varphi, \psi \in C^+$, *then the equality is satisfied for* $w = e$ *only.*

4) If C *is an arbitrary Weyl chamber,* $\varphi, \psi \in C$ *and* $\varphi \sim \psi$, *then* $\varphi = \psi$.

5) Let $\sum n_i \gamma_i = c\gamma$, *where* $\gamma_i, \gamma \in \Delta$, $n_i \in \mathbf{Z}$. *Then* $c \in \mathbf{Z}$ *as well.*

Proof. 1) It is well-known that $\langle \varrho, \alpha\rangle > 0$ for all $\alpha \in \Sigma$. Therefore there exists a $c>0$ such that $\langle \varrho, \alpha\rangle > c\|\alpha\|$ for all $\alpha \in \Sigma$. This implies that for any $\varkappa \in K$ we have $\langle \varrho, \varkappa\rangle \geqq c\|\varkappa\|$. Moreover, $\varphi-\psi>0$ and $\vartheta_1-\vartheta_2>\varphi-\psi$. Consequently,

$$\|\varrho\|(\|\vartheta_1\|+\|\vartheta_2\|) \geqq \langle \varrho, \vartheta_1-\vartheta_2\rangle > \langle \varrho, \varphi-\psi\rangle \geqq c\|\varphi-\psi\|.$$

Therefore we can put $c_1=\|\varrho\|c^{-1}$.

2) It suffices to put $c_2=2d$, where d is the diameter of an arbitrary fundamental domain of the group $\mathfrak{h}_{\mathbf{Z}}^*$ in $\mathfrak{h}_{\mathbf{R}}^*$.

3) Let us write w in the form $w=\sigma_{\alpha_k}\dots\sigma_{\alpha_1}$ with $k=l(w)$ and put $\varphi_0=\varphi$, $\varphi_i=w_i\varphi$, where $w_i=\sigma_{\alpha_i}\dots\sigma_{\alpha_1}$. Then $\varphi_i-\varphi_{i+1}=c_i\alpha_i$ where

$$c_i=\frac{2\langle\varphi_i, \alpha_i\rangle}{\langle\alpha_i, \alpha_i\rangle}=\frac{2\langle\varphi, w_i^{-1}\alpha_i\rangle}{\langle\alpha_i, \alpha_i\rangle}.$$

It follows from Lemma 11.2.1 that $c_i\geqq 0$. It is also clear that $c_i\neq 0$ if $\varphi\in C$. Therefore $\langle\varphi, \psi\rangle-\langle w\varphi, \psi\rangle=\sum c_i\langle\alpha_i, \psi\rangle$. This completes the proof.

4) It can be shown that $C=C^*$. Now let $\psi=w\varphi$. According to 3, $\langle\varphi, \psi\rangle\geqq \geqq\langle\psi, \psi\rangle$. But $\|\varphi\|=\|\psi\|$, hence $\varphi=\psi$.

5) We can assume that $\gamma\in\Sigma$. In this case the lemma follows from the fact that the elements of Σ form a basis in which every root has integer coordinates.

The proof of Lemma A5 is complete.

Let $\chi\in\mathfrak{h}^*$. We shall denote by $Y(\chi)$ the following proposition:

$Y(\chi)$. *For every* $\psi\in\mathfrak{h}^*$ *such that* $L_\chi\in JH(M_\psi)$ *there exists a sequence* $\gamma_1, \dots, \gamma_k\in\Delta_+$ *which satisfies condition* (A) *for the pair* (χ, ψ).

Theorem A1 says that $Y(\chi)$ is valid for all $\chi\in\mathfrak{h}^*$. We shall prove it in several steps. For any $\gamma\in\Delta_+$ we shall denote Ξ_γ the hyperplane in $\mathfrak{h}_{\mathbf{R}}^*$ orthogonal to γ.

Definition A6. Let $c_3=3c_1c_2$. The element $\varphi\in\mathfrak{h}^*$ is called *strongly regular* if $d(\operatorname{Re}\varphi, \Xi_\gamma)>c_3$ for all $\gamma\in\Delta_+$.

Step 1. Let $\chi, \varphi\in\mathfrak{h}^*$ be strongly regular while $\chi-\varphi\in\mathfrak{h}_{\mathbf{Z}}^*$ and $\operatorname{Re}\chi$, $\operatorname{Re}\varphi$ belong to the same Weyl chamber. Then propositions $Y(\chi)$ and $Y(\varphi)$ are equivalent.

Step 2. Let $\varphi, \chi\in\mathfrak{h}^*$ be strongly regular elements with $\chi-\varphi\in\mathfrak{h}_{\mathbf{Z}}^*$. In addition to this, let $\operatorname{Re}\chi\in C$, $\operatorname{Re}\varphi\in\sigma_\gamma C$, where C and $\sigma_\gamma C$ $(\gamma\in\Delta_+)$ are neighbouring Weyl chambers and $\langle\operatorname{Re}\chi, \gamma\rangle<0$. Then $Y(\varphi)$ implies $Y(\chi)$.

It is easy to see that Steps 1 and 2 enable us to reduce the proof of $Y(\chi)$ for a strongly regular $\chi\in\mathfrak{h}^*$ to the case $\operatorname{Re}\chi\in C^+$. Now we are going to prove $Y(\chi)$ in this case. Let $\psi\in\mathfrak{h}^*$ and $L_\chi\in JH(M_\psi)$. It follows from Lemma A2 that $\chi\sim\psi$ and $\chi\leqq\psi$. This means that $\psi=\chi+\varkappa$, where $\varkappa\in K$. Then $\langle\varphi, \psi\rangle=\langle\chi, \chi\rangle+\langle\varkappa, \varkappa\rangle+2\langle\chi, \varkappa\rangle$. Now

$$\langle\psi, \psi\rangle=\langle\chi, \chi\rangle, \quad \operatorname{Re}\langle\chi, \varkappa\rangle=\langle\operatorname{Re}\chi, \varkappa\rangle\geqq 0,$$

and $\langle\varkappa, \varkappa\rangle\geqq 0$. Consequently $\langle\varkappa, \varkappa\rangle=0$, hence $\chi=\psi$.

Therefore Steps 1 and 2 prove $Y(\chi)$ for all the strongly regular weights $\chi\in\mathfrak{h}^*$. The general case is reduced to this one by means of the following Step 3.

Step 3. Let $\chi\in\mathfrak{h}^*$. Then there exists a strongly regular element $\varphi\in\mathfrak{h}^*$ such that $Y(\varphi)$ implies $Y(\chi)$.

Thus the proof of Theorem A1 reduces to the proofs of Steps 1, 2, 3 and Proposition A4.

In the proof of Steps 1, 2, 3 it will be convenient for us to make use of the following lemma.

Lemma A7. *Let C_1 and C_2 be two Weyl chambers, $\gamma, \psi\in\mathfrak{h}^*$ such that* $\operatorname{Re}\gamma\in\bar{C}_1$, $\operatorname{Re}\psi\in\bar{C}_2$, $\chi\sim\psi$ *and* $\lambda\in\mathfrak{h}^*$, $\mu\in\mathfrak{h}^*_Z$ *be chosen so that* $\operatorname{Re}\chi+\lambda\in C_1$, $\operatorname{Re}\psi+\mu\in C_2$. *Suppose that the sequence of roots* $\gamma_1, \ldots, \gamma_k\in\Delta_+$ *satisfies condition* (A) *for the pair* $(\chi+\lambda, \psi+\mu)$. *Then it satisfies condition* (A) *for the pair* (χ, ψ) *as well.*

Proof. Item *2)* of condition (A) follows immediately from the fact that the sign of $\langle\chi, \gamma\rangle$ is constant in a Weyl chamber. In order to verify item *1)* we put $\chi_k=\sigma_{\gamma_k}\ldots\sigma_{\gamma_1}\psi$. Then $\chi\sim\psi\sim\chi_k$ and $\operatorname{Re}\chi, \operatorname{Re}\chi_k\in C_1$. By Lemma A5. 4, $\operatorname{Re}\chi=\operatorname{Re}\chi_k$. Moreover, $\chi-\chi_k\in\mathfrak{h}^*_Z$ and consequently $\operatorname{Im}\chi=\operatorname{Im}\chi_k$, i.e. $\chi=\chi_k$.

Proof of Step 1. Lemma A5. 2 enables us to reduce the proof of Step 1 to the proof of the following proposition. Let $\chi\in\mathfrak{h}^*$, $\lambda\in\mathfrak{h}^*_Z$ be such that $d(\operatorname{Re}\chi, \Xi)>2c_1c_2$, where $\Xi=\bigcup_{\gamma\in\Delta_+}\Xi_\gamma$ and $\|\lambda\|<c_2$. Then $Y(\chi+\lambda)$ implies $Y(\chi)$.

First we show that such a pair (χ, λ) is admissible. Indeed, assume that there exist $w\in W$ and $\mu\in P(F_\gamma)$ for which $w\chi<\chi$ and $w(\chi+\mu)>\chi+\lambda$. Then by Lemma A5. 1,

$$2c_2>2\|\lambda\|\geqq\|\lambda\|+\|\mu\|>c_1^{-1}\|\chi-w\chi\|.$$

On the other hand

$$\|\chi-w\chi\|\geqq d(\operatorname{Re}\chi, \Xi)>2c_1c_2.$$

Therefore, the pair (χ, λ) is indeed admissible.

Now let $L_\chi\in JH(M_\psi)$. Then according to Proposition A4, $L_{\chi+\lambda}\in JH(M_{\psi+\mu})$ for a certain $\mu\in P(F_\lambda)$. It follows from $Y(\chi+\lambda)$ that there exists a sequence of roots $\gamma_1, \ldots, \gamma_k\in\Delta_+$ which satisfies condition (A) for the pair $(\chi+\lambda, \psi+\mu)$. Since $\|\mu\|<c_2$, ψ and $\psi+\mu$ lie in the same Weyl chamber. Lemma A7 implies then that the same sequence $\gamma_1, \ldots, \gamma_k$ satisfies condition (A) for the pair (χ, ψ) as well.

Proof of Step 2. Applying Step 1, we can reduce the proof to the case when

$$3c_1\|\varphi-\chi\|<d(\operatorname{Re}\varphi, \Xi_\delta) \tag{A.1}$$

for all

$$\delta\in\Delta_+\setminus\gamma.$$

We put $\lambda=\varphi-\chi$ and prove that the pair (χ, λ) is admissible. Indeed, suppose we have $w\in W$ and $\mu\in P(F_\lambda)$ such that

$$w\chi<\chi,\quad w(\chi+\mu)>\chi+\lambda.$$

$w\chi < \chi$ implies $w \neq \sigma_\gamma$. Therefore,

$$\|w\chi - \chi\| > d(\operatorname{Re}\varphi, \bigcup_{\delta\in\gamma} \Xi_\delta) > 2c_1\|\lambda\|.$$

But then, by Lemma A5. 1,

$$2\|\lambda\| \geqq \|\lambda\| + \|\mu\| \geqq c_1^{-1}\|w\chi - \chi\| > 2\|\lambda\|.$$

Thus the pair (χ, λ) is indeed admissible.

Now assume $L_\chi \in JH(M_\psi)$, $\psi \in \mathfrak{h}^*$. Then $L_{\chi+\lambda} \in JH(M_{\psi+\mu})$ for some $\mu \in P(F_\lambda)$. $Y(\chi+\lambda)$ implies the existence of a sequence $\gamma_1, \ldots, \gamma_k \in \Delta_+$ satisfying condition (A) for the pair $(\chi+\lambda, \psi+\mu)$. To complete the proof of Step 2 we have to construct a sequence of roots which satisfies condition (A) for the pair (χ, ψ).

Let $w \in W$ be such that $\psi = w\chi$. Then $w^{-1}(\psi+\mu) = \chi + w^{-1}\mu$. Since $\|w^{-1}\mu\| \leqq \|\lambda\|$, we have either $\operatorname{Re} w^{-1}(\psi+\mu) \in C$ or $\operatorname{Re} w^{-1}(\psi+\mu) \in \sigma_\gamma C$. Let us consider these two cases separately.

1) $\operatorname{Re} w^{-1}(\psi+\mu) \in C$. We have

$$w^{-1}(\psi+\mu) \sim \chi+\lambda \sim \sigma_\gamma(\chi+\lambda)$$

and

$$\operatorname{Re}\sigma_\gamma(\chi+\lambda) \in C.$$

By Lemma A5. 4,

$$\sigma_\gamma(\chi+\lambda) = w^{-1}(\psi+\mu) = \chi + w^{-1}\mu.$$

Consequently

$$\sigma_\gamma\chi - \chi = w^{-1}\mu - \sigma_\gamma\lambda.$$

Since $w^{-1}\mu$ and $\sigma_\gamma\lambda$ are weights of F_λ, we have

$$\sigma_\gamma\chi - \chi = c\gamma = \sum_{\gamma_i\in\Delta} n_i\gamma_i,$$

where $n_i \in \mathbf{Z}$. By Lemma A5. 5 $c \in \mathbf{Z}$, and in addition

$$c = -\frac{2\langle\chi, \gamma\rangle}{\langle\gamma, \gamma\rangle} > 0.$$

By Lemma A7, the sequence $\gamma_1, \ldots, \gamma_k$ satisfies condition (A) for the pair $(\sigma_\gamma\chi, \psi)$, hence the sequence $\gamma_1, \ldots, \gamma_k, \gamma$ satisfies condition (A) for the pair (χ, ψ).

2) $\operatorname{Re} w^{-1}(\psi+\mu) \in \sigma_\gamma C$. Let us put

$$\chi_i = \sigma_{\gamma_i} \ldots \sigma_{\gamma_1}\psi,$$

$$\tilde{\chi}_i = \sigma_{\gamma_i} \ldots \sigma_{\gamma_1}(\psi+\mu).$$

We remark that all χ_i and $\tilde{\chi}_i$ are congruent modulo $\mathfrak{h}^*_{\mathbf{Z}}$. Thus for each i we have either $\chi_i < \chi_{i-1}$ or $\chi_i > \chi_{i-1}$. If $\chi_i < \chi_{i-1}$ for all i, then the sequence $\gamma_1, \ldots, \gamma_k$ satisfies condition (A) for the pair (χ, ψ). In the opposite case we denote by i_0 the smallest index such that $\chi_{i_0} \geqq \chi_{i_0-1}$. We shall show that the sequence of roots

$$\gamma_1, \ldots, \gamma_{i_0-1}, \gamma_{i_0+1}, \ldots, \gamma_k, \gamma$$

satisfies condition (A) for the pair (χ, ψ).

First of all Re $\bar{\chi}_{i_0}$ and Re χ_{i_0-1} lie in the same Weyl chamber. Indeed,

$$\|\chi_{i_0-1}-\bar{\chi}_{i_0-1}\| < \|\lambda\|,$$

$$\|\chi_{i_0}-\bar{\chi}_{i_0}\| \leqq \|\lambda\|,$$

and by Lemma A2. 1,

$$\|\bar{\chi}_{i_0-1}-\bar{\chi}_{i_0}\| < 2c_1\|\lambda\|.$$

In addition to this, the real parts of the weights

$$\bar{\chi}_{i_0-1} \quad \text{and} \quad \bar{\chi}_{i_0} = \sigma_{\gamma_{i_0}}\bar{\chi}_{i_0-1}$$

as well as of the weights

$$\chi_{i_0-1} = \sigma_{\gamma_{i_0-1}}\dots\sigma_{\gamma_1}\psi \quad \text{and} \quad \bar{\chi}_{i_0-1} = \sigma_{\gamma_{i_0-1}}\dots\sigma_{\gamma_1}(\psi+\mu)$$

belong to different Weyl chambers. However, on account of condition (A1) the ball of radius $2c_1\|\lambda\|$ about the point Re $\bar{\chi}_{i_0-1}$ intersects exactly two Wey chambers

$$\sigma_{\gamma_{i_0}}\dots\sigma_{\gamma_k}C \quad \text{and} \quad \sigma_{\gamma_{i_0}}\dots\sigma_{\gamma_k}\sigma_\gamma C.$$

Therefore, Re $\bar{\chi}_{i_0}$ and Re χ_{i_0-1} lie in the same Weyl chamber.

By Lemma A7 the sequence $\gamma_1, \dots, \gamma_{i_0-1}$ satisfies condition (A) for the pair (χ_{i_0-1}, ψ), and the sequence $\gamma_{i_0+1}, \dots, \gamma_k$ satisfies condition (A) for the pair $(\sigma_\gamma\chi, \chi_{i_0-1})$. In exactly the same way as in case 1) we can show that $\chi < \sigma_\gamma\chi$. Consequently the sequence of roots

$$\gamma_1, \gamma_2, \dots, \gamma_{i_0-1}, \gamma_{i_0+1}, \dots, \gamma_k, \gamma$$

satisfies condition (A) for the pair (χ, ψ).

Proof of Step 3. Let us first consider the case Re $\chi=0$. Then the assumption $L_\chi \in JH(M_\psi)$ and Lemma A2 imply $\psi=\chi$.

Assume now that Re $\chi\neq 0$. We can choose a weight $\lambda\in\mathfrak{h}^*_{\mathbf{Z}}$ in such a way that the following conditions are satisfied.

a) Re χ and λ belong to the closure $\bar{C}$ of the same Weyl chamber C.

b) The weight $\chi+\lambda$ is strongly regular.

c) Let $c_4=\min\{\|\varkappa\| \mid \varkappa\in K\setminus\{0\}\}$. Then

$$\|\lambda - n\,\mathrm{Re}\,\chi\| \leqq nc_4/2c_1 \tag{A.2}$$

for some $n\in\mathbf{Z}_+$.

Lemma A8. *Let* $\nu\in P(F_\lambda)$ *such that* $\chi+\nu\sim\chi+\lambda$. *Then and* ν Re χ *lie in the same Weyl chamber and*

$$\|\nu - n\,\mathrm{Re}\,\chi\| \leqq nc_4/2c_1. \tag{A.3}$$

Proof. Let $w_1, w_2\in W$ be chosen in such a way that $w_1C=C^+$ and $w_1(\chi+\lambda)=w_2(\chi+\nu)$. Then $w_1\lambda$ is the highest weight of the representation F_λ and therefore $w_1\lambda-w_2\nu\in K$. Thus

$$\langle\varrho, w_1\lambda\rangle \geqq \langle\varrho, w_2\nu\rangle,$$

where the equality holds only if $w_2\nu = w_1\lambda$. On the other hand, by Lemma A2. 3,

$$\mathrm{Re}\langle \varrho, w_2\chi\rangle \leqq \mathrm{Re}\langle \varrho, w_1\chi\rangle.$$

Comparing these inequalities with the equality $w_1(\chi+\lambda) = w_2(\chi+\nu)$, we obtain $w_1\lambda = w_2\nu$ and $w_1\chi = w_2\chi$. Now applying the element $w_2^{-1}w_1$ to the inequality (A2) we obtain the statement of the lemma.

Now we prove that the pair (χ, λ) is admissible. Indeed, let $w \in W$ and $\mu \in P(F_\lambda)$ such that

$$w\chi < \chi, \quad w(\chi+\mu) \sim \chi+\lambda, \quad w(\chi+\mu) > \chi+\lambda.$$

Since $w\chi < \chi$, we have $\|w\chi - \chi\| \geqq c_4$. This implies

$$\|w(n+1)\chi - (n+1)\chi\| \geqq (n+1)c_4$$

and

$$w((n+1)\chi) < (n+1)\chi.$$

Applying Lemma A5. 1 we obtain that

$$c_1(\|\lambda - n\chi\| + \|\mu - n\chi\|) \geqq (n+1)c_4,$$

with contradicts (A2) and (A3). Consequently, the pair (χ, λ) is admissible.

Now let $L_\chi \in JH(M_\psi)$. Then in view of Proposition A4, $L_{\chi+\lambda} \in JH(M_{\psi+\mu})$ for some $\mu \in P(F_\lambda)$. Applying Lemma A8 again we find that ψ and $\psi+\mu$ lie in the same Weyl chamber. Let $\gamma_1, \ldots, \gamma_k \in \Delta_+$ be a sequence of roots which satisfies condition (A) for the pair $(\chi+\lambda, \psi+\mu)$. Then by Lemma A7 it also satisfies condition (A) for the pair (χ, ψ).

Proof of Proposition A4. Let $\lambda = \mu_1, \mu_2, \ldots, \mu_l$ be the weights of the module F_λ (with the corresponding multiplicities). Then, as follows from Lemma 9.10,

$$JH(M_\chi \otimes F_\lambda) = \bigcup_{i=1}^{l} JH(M_{\chi+\mu_i}).$$

Lemma A9. *Let $M_1, M_2 \in O$, while $JH(M_1) \subset JH(M_2)$ and F be a finite-dimensional* g*-module. Then*

$$JH(M_1 \otimes F) \subset JH(M_2 \otimes F).$$

The proof follows easily from the fact that tensor product by F is an exact functor.

Now let the pair (χ, λ) be admissible and $L_\chi \subset JH(M_\psi)$. Then

$$JH(L_\chi \otimes F_\lambda) \subset JH(M_\psi \otimes F_\lambda) = \bigcup_{i=1}^{l} JH(M_{\psi+\mu_i}).$$

Therefore, in order to show $L_{\chi+\lambda} \subset JH(M_{\psi+\mu})$, for some weight $\mu \in P(F_\lambda)$ it suffices to prove that

$$L_{\chi+\lambda} \subset JH(L_\chi \otimes F_\lambda).$$

Let M be a maximal proper submodule in M_χ such that $L_\chi = M_\chi/M$. Then

$$L_\chi \otimes F_\lambda = M_\chi \otimes F_\lambda / M \otimes F_\lambda.$$

Since $L_{\chi+\lambda} \in JH(M_\chi \otimes F_\lambda)$, it suffices to prove that $L_{\chi+\lambda} \notin JH(M \otimes F_\lambda)$.

By Lemma A2,
$$JH(M) \subset \bigcup_{\substack{\varphi \sim \chi \\ \varphi < \chi}} JH(M_\varphi),$$
consequently
$$JH(M \otimes F_\lambda) \subset \bigcup_{\substack{\varphi \sim \chi \\ \varphi < \chi}} c_\varphi JH(M_\varphi \otimes F_\lambda).$$

Thus it suffices to show that for any $\varphi \in \mathfrak{h}^*$ with $\varphi \sim \chi$, $\varphi < \chi$ and any $\mu \in P(F_\lambda)$, $L_{\chi+\lambda} \notin JH(M_{\varphi+\mu})$.

Assume, on the contrary, that $L_{\chi+\lambda} \in JH(M_{\varphi+\mu})$. Let $\varphi = w\chi$, $w \in W$. Then by lemma A2,
$$\chi + \lambda \sim \varphi + \mu = w(\varphi + w^{-1}\mu)$$
and
$$\chi + \lambda < w(\varphi + w^{-1}\mu).$$
But $w^{-1}\mu \in P(F_\lambda)$ which contradicts our assumption on admissibility.

This completes the proof of Proposition A4 and thus of Theorem A1.

Theorem A10. *Let* $\chi, \psi \in \mathfrak{h}^*$ *and* $\gamma_1, \ldots, \gamma_k \in \Delta_+$ *be a sequence of roots satisfying condition* (A) *for the pair* (χ, ψ). *Then* $\mathrm{Hom}\,(M_\chi, M_\varphi) = \mathbf{C}$.

A proof of Theorem A10 is contained in [8] and [9].

Theorems 8.8 and 8.10 follow immediately from Theorems A1 and A10.

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QUASI-ADMISSIBLE REPRESENTATION OF $\mathfrak{p}$-ADIC GROUPS

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Introduction

The notion of a quasi-admissible representation of a $\mathfrak{p}$-adic group is due to Harish–Chandra. It is closely related to the notion of a quasi-simple representation of a real connected semisimple Lie group. Following some fruitful ideas of Harish–Chandra, we recently made some progress in the theory of "asymptotic expansions" as described in [2], Part IV, without assuming Conjecture III ([1(c)], § 1). Quasi-admissible representations occur very naturally there. The ideas are worked out in this note. The main applications of the theory, which we have in mind, are concerned with the asymptotic expansions of the matrix coefficients of the representations of the discrete series of G. They should belong to the $\mathfrak{p}$-adic analogue of the (Schwartz-) space $\mathscr{C}(G)$, introduced by Harish–Chandra for real semisimple Lie groups with finitely many connected components (cf. [1 (c)], § 9). We do not discuss the applications here. In § 1 we discuss admissible representations of $\mathfrak{p}$-adic groups and derive a few properties, most of which are well known. § 2 is concerned with quasi-admissible representations. The results should justify the relationship to quasi-simple representations of real groups, mentioned above. It concludes with a theorem which was inspired by ideas of Jacquet. The result turns out to be basic for the theory of the "asymptotic expansions", which is described in § 3.

§ 0. Some notations and conventions

Throughout this paper, Ω will be a $\mathfrak{p}$-adic field, i.e. a locally compact field with a non-trivial discrete valuation. Let us fix an additive Haar measure $d\omega$ on Ω. The valuation (or absolute value) on Ω is assumed to be normalized by requiring $d(\alpha\omega)=|\alpha|\, d\omega$ $(\alpha \in \Omega^*)$. Let $\mathfrak{O}$ denote the ring of integers of Ω.

By G we mean a reductive $\mathfrak{p}$-adic group, i.e. the group of Ω-rational points of a connected, reductive, linear algebraic group $\mathbf{G}$ defined over Ω. Then $G \subset GL_n(\Omega)$ for a suitable $n \geqq 0$. $GL_n(\Omega)$, being an open subset of a vector space over Ω of dimension n^2 is a locally compact group. Since G is closed in $GL_n(\Omega)$, it is also locally compact. Moreover G is unimodular. We keep mainly to the notations and terminology of [2].

For a subset S of a topological space V, we shall denote by $Cl(S)$ the closure of S in V.