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# Sobolev Norms of Automorphic Functionals

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## 1 Introduction

### 1.1 Motivation

It is well known that Frobenius reciprocity is one of the central tools in the representation theory. In this paper, we discuss Frobenius reciprocity in the theory of automorphic functions. This Frobenius reciprocity was discovered by Gel′fand, Fomin, and Piatetski-Shapiro in the 1960s as the basis of their interpretation of the classical theory of automorphic functions in terms of the representation theory (eventually, of adelic groups, see [7, 8, 9]). Later, Ol′shanski gave a more transparent proof of it (see [14]). However, in the subsequent rapid development of the theory of automorphic functions, Frobenius reciprocity was barely noticeable. We believe that this is due to the incompleteness of the above-mentioned results.

In this paper, we prove a general theorem (see Theorem 1.1), which we view as a quantitative version of Frobenius reciprocity. We then illustrate it by looking into the example of $SL(2, \mathbb{R})$. We think that these methods will play a more prominent role in the theory of automorphic functions.

### 1.2 Geometric functionals on representations

We consider a general problem. Suppose that we are given a representation $(\pi, G, V)$ of a locally compact group $G$ in a topological complex vector space $V$ and a morphism of representations $\nu : V \to C(X)$, where $X$ is some $G$-space and $C(X)$ the space of continuous functions on $X$. Then each point $x \in X$ defines a continuous functional $I_x$ on $V$ by $I_x(v) = \nu(v)(x)$. We would like to establish some bounds on the norm of the functional $I_x$. In

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order to do so we want to choose a norm on V which is at our disposal. Different norms could provide different information.

More precisely, suppose we are given a norm N on V (we always assume that the representation $\pi$ is continuous with respect to this norm). We would like to give an a priori estimate of the norm $\|I\|_N$ of the functional I with respect to N, where $\|I\|_N := \sup_{v \in V} |I(v)|/N(v)$.

Of course, for this, we have to know something about the morphism $\nu$.

Assume that X is a homogeneous G-space. We also assume that the image of V lies inside the space $L^2(X, \mu_X)$ of $L^2$-functions with respect to some G-invariant measure $\mu_X$ on X. Then the scalar product in $L^2(X, \mu_X)$ defines an invariant Hermitian form P on V.

We propose to bound $\|I\|_N$ in terms of the norm N and the Hermitian form P. It turns out that when the norm N is obtained from a Hermitian form Q on V, we can sometimes give a reasonable bound for $\|I\|_N$. Namely, we claim that $\|I\|_N$ can be estimated in terms of the relative trace $\mathrm{tr}(P \mid Q)$ of Hermitian forms P and Q.

More precisely, let V be a separable topological vector space, $H(V)$ the space of continuous Hermitian forms on V, and $H(V)^+ \subset H(V)$ the subset of nonnegative Hermitian forms. To any pair of forms $P, Q \in H^+(V)$, where Q is positive definite, we assign a number $\mathrm{tr}(P \mid Q)$, the relative trace of P with respect to Q, which takes values in $\mathbb{R}_+ \cup \infty$ (see Appendix A).

For example, if $P \leq cQ$ we can represent the form P by a bounded selfadjoint operator A in the Hilbert space completion H of the space V with respect to the form Q. In this case, we will have $\mathrm{tr}(P \mid Q) = \mathrm{tr}\, A$.

The number $\mathrm{tr}(P \mid Q)$ can often be effectively computed. It turns out that we can give *tight* estimates of the norm $\|I\|_N$ in terms of this number.

Namely, we will prove the following general result.

**Theorem 1.1.** The following estimate holds $\|I_x\|_N^2 \leq C \cdot \mathrm{tr}(P \mid Q)$, where $C = C(x)$ is an effectively computable constant.

If X is compact, this estimate is tight, that is, $\|I_x\|_N^2 \geq c \cdot \mathrm{tr}(P \mid Q)$, where $c > 0$ is an effectively computable constant. □

The theorem above follows from the following relation between relative traces and functionals $I_x$.

**Proposition 1.2.** The following relation holds $\mathrm{tr}(P \mid Q) = \int_X \|I_x\|_N^2 \, d\mu_X$. □

### 1.3 Automorphic functionals

Theorem 1.1 could be applied in the automorphic setting where it gives a quantitative version of Frobenius reciprocity.

Let $G$ be a Lie group and $\Gamma \subset G$ a lattice, that is, a discrete subgroup such that the quotient space $X = \Gamma \setminus G$ has finite volume with respect to an invariant measure $\mu_X$. *In order to simplify the formulas, we always normalize the measure* $\mu_X$ *so that the total volume* $\mu_X(X)$ *equals* 1.

For simplicity of exposition, we assume that the quotient space $X$ is compact (see Section 3.3 for a discussion of general lattices).

It is well known (see [8]) that for compact $X$, the space $L^2(X)$ decomposes into a direct sum of irreducible (unitary) representations of $G$. These representations are called automorphic representations.

Let $(\pi, L)$ be such a representation in a Hilbert space $L$ and $\nu : L \to L^2(X)$ the corresponding isometric embedding. Let $V \subset L$ be the space of smooth vectors of $\pi$ (i.e., $v \in L$ such that $\xi_v(g) = \pi(g)v$ is a smooth function from $G$ to $L$). It is well known that $V$ is dense in $L$ and that $\nu$ maps $V$ into the space $C^\infty(X)$ of smooth functions on $X$ (see [14]). It is easy to see that we have the following isomorphism $\mathrm{Mor}_G(L, L^2(X)) \simeq \mathrm{Mor}_G(V, C^\infty(X))$. The last space can be described using the following result (see [8, 14]).

**Frobenius reciprocity.** $\mathrm{Mor}_G(V, C^\infty(X)) \simeq \mathrm{Mor}_\Gamma(V, \mathbb{C})$. □

Namely, to every $G$-morphism $\nu : V \to C^\infty(\Gamma \setminus G)$ corresponds a $\Gamma$-invariant functional $I$ on the space $V$ given by $I(v) = \nu(v)(e)$ (here $e$ is the identity in $G$). Given $I$, we can recover $\nu$ as $\nu(v)(g) = I(\pi(g)v)$.

According to Theorem 1.1, we have estimates on automorphic functionals $I_x$, $x \in X$, in terms of relative traces. Namely, given a morphism $\nu : V \to L^2(X)$, which defines an invariant Hermitian form $P = \|\cdot\|^2$ on $V$, and given a positive definite Hermitian form $Q$ on $V$ such that the representation $\pi$ is continuous with respect to the corresponding norm $N$, we have the following estimates on the norm of corresponding functionals $I_x$:

(1) for any point $x \in X$, we have an estimate $\|I_x\|_N^2 \le C \cdot \mathrm{tr}(P \mid Q)$, where $C = C(x)$ is an effectively computable constant;

(2) if $\Gamma$ is cocompact, then $\mathrm{tr}(P \mid Q)$ is comparable to $\|I_x\|_N^2$, that is, there exist constants $C, c > 0$ such that $c \cdot \mathrm{tr}(P \mid Q) \le \|I_x\|_N^2 \le C \cdot \mathrm{tr}(P \mid Q)$.

In order to make these estimates useful, we have to choose a Hermitian form $Q$ (it is not a simple task for general $G$ and $V$, but see [3] in this regard). We analyze, in some detail, the most simple (but interesting for automorphic functions) example of $G = SL(2, \mathbb{R})$. We also compare our method with other methods of bounding automorphic functions.

A careful reader would notice that Sobolev norms appearing in the title are not essential to the paper. This is due to the evolution of the authors' understanding during the (long) process of rewriting the paper.

## 2 Automorphic functions on $SL(2, \mathbb{R})$

Here we implement our strategy for $G = SL(2, \mathbb{R})$. We make this section self-contained in order to make the paper more accessible (hence some overlap with the previous section).

### 2.1 Setting

Let $\mathfrak{H}$ be the upper half plane with the hyperbolic metric of constant curvature $-1$. The group $G = SL(2, \mathbb{R})$ acts on $\mathfrak{H}$ by isometries.

We fix a discrete group $\Gamma \subset G$. Consider the Riemann surface $Y = \Gamma \backslash \mathfrak{H}$; we assume that $Y$ is compact. Denote by $D$ the Laplace-Beltrami operator acting in the space of functions on $Y$. We denote by $0 = \mu_0 < \mu_1 \leq \mu_2 \leq \cdots$ its eigenvalues on $Y$ and by $\phi_i$ the corresponding eigenfunctions; we normalize these eigenfunction so that $\|\phi\|_{L^2} = 1$, where the $L^2$-structure is defined by a G-invariant measure $\mu_Y$ on $\mathfrak{H}$ normalized by the condition $\mu_Y(Y) = 1$. The study of eigenfunctions $\phi$ and corresponding eigenvalues is important in many areas of representation theory, number theory, and geometry.

Here, we present a new approach to the study of eigenfunctions $\phi$ based on the version of Frobenius reciprocity formulated in Section 1.3.

### 2.2 Automorphic representations

Consider the maximal compact subgroup $K = SO(2, \mathbb{R}) \subset SL(2, \mathbb{R})$; we will identify $\mathfrak{H}$ with $G/K$.

We denote by $X$ the compact quotient $X = \Gamma \setminus G$. The group $G$ acts on $X$ and hence on the space of functions on $X$. We identify the Riemann surface $Y = \Gamma \setminus \mathfrak{H}$ with $X/K$ and consider G-invariant measure $\mu_X$ on $X$ normalized by the condition $\mu_X(X) = 1$. This induces the embedding $L^2(Y, \mu_Y) \subset L^2(X, \mu_X)$, the image consists of all K-invariant functions. For any eigenfunction $\phi$ of the Laplace operator on $Y$, we can consider a closed G-invariant subspace $L_\phi \subset L^2(X)$ generated by $\phi$ under the action of $G$. It is known that $(\pi, L) = (\pi_\phi, L_\phi)$ is an irreducible unitary representation of $G$ (see [8]).

Conversely, suppose that we fixed an irreducible unitary representation $(\pi, L)$ of the group $G$ and a K-fixed unit vector $v_0 \in L$. Then any G-morphism $\nu : L \to L^2(X)$ defines an eigenfunction $\phi = \nu(v_0)$ of the Laplace operator on $Y$; this function is normalized if $\nu$ is an isometric embedding.

Thus eigenfunctions $\phi$ correspond to the tuples $(\pi, L, v_0, \nu)$.

All irreducible unitary representations of $G$ with a K-fixed vector are classified: these are representations of principal and complementary series and the trivial representation. For simplicity, we consider only representations of principal series (these correspond to eigenfunctions as in Section 2.1 with the eigenvalue $\mu \geq 1/4$).

Such a representation $(\pi, L)$ can be realized as follows. Fix a purely imaginary number $\lambda$ and consider the natural representation $\pi_\lambda$ of the group G in the space $L_\lambda$ of (even) homogeneous functions on $\mathbb{R}^2 \setminus 0$ of degree $\lambda - 1$. Thus vectors in $L_\lambda$ are just even locally $L^2$-functions f on $\mathbb{R}^2 \setminus 0$ satisfying $f(ax, ay) = |a|^{\lambda-1} f(x, y)$ for all $a \in \mathbb{R}$. The representation $\pi$ is induced by the natural action of G on $(x, y)$.

Note that such a function is determined by its values on the unit circle $S^1$; hence we may identify the space $L_\lambda$ with the space $L^2(S^1)_{\text{even}}$ of even functions on $S^1$. The G-invariant scalar product in $L_\lambda$ is given by $P(f, g) = (1/2\pi) \int_{S^1} f\bar{g}\, d\theta$.

The K-fixed vector $\nu_0$ corresponds to the constant function 1 on $S^1$.

The eigenfunction $\phi$ of the Laplace operator, which corresponds to a representation $(\pi_\lambda, L_\lambda)$, will have the eigenvalue $\mu = (1 - \lambda^2)/4$.

Thus, we see that eigenfunctions $\phi$ on the Riemann surface Y with the given eigenvalue $\mu$ correspond to G-morphisms $\nu_\phi : L_\lambda \to L^2(X)$ (namely, $\phi = \nu_\phi(\nu_0)$). Normalization $\|\phi\| = 1$ means that $\nu$ preserves the scalar product.

Let $V_\lambda \subset L_\lambda$ be the subspace of smooth vectors (in the realization described above, $V_\lambda$ consists of smooth functions on $\mathbb{R}^2 \setminus 0$).

As we have seen, $\mathrm{Mor}_G(L_\lambda, L^2(X)) \simeq \mathrm{Mor}_G(V_\lambda, C^\infty(X))$ and we have the Frobenius reciprocity $\mathrm{Mor}_G(V_\lambda, C^\infty(X)) \simeq \mathrm{Mor}_\Gamma(V_\lambda, \mathbb{C})$ as in Section 1.3.

Thus, eigenfunctions $\phi$ of the Laplace operator on Y with eigenvalue $\mu$ correspond to $\Gamma$-invariant functionals I on the space $V_\lambda$.

## 2.3 Sobolev class of automorphic functionals

For principal series representations of $SL(2, \mathbb{R})$, we can identify $V_\lambda$ with the space $C^\infty(S^1)_{\text{even}}$ and consider I as a distribution on $S^1$. This functional is continuous, that is, it is continuous with respect to some seminorms. Then we can ask the following natural question.

Question. What is the $L^2$-Sobolev class of the functional I? In other words, for which real $s$ is the functional I continuous with respect to the $L^2$-Sobolev norm $N_s$ and how can we estimate the norm $\|I\|_{N_s}$?

We discuss the relevant definition of $L^2$-Sobolev norms $N_s$ in Appendix B.

This is a question about the regularity of the functional I in the scale of Sobolev spaces. More precisely, denote by $W_s = W_s(\pi)$ the completion of the space V with respect to the Sobolev norm $N_s$. These Sobolev spaces $W_s$ form a decreasing family of representations of G; the intersection $W^\infty = \cap W_s$ coincides with the space V of smooth vectors in the representation $\pi$, while the union $W^{-\infty} = \cup W_s$ can be interpreted as the space of distribution vectors in the representation $\pi$.

Note that the dual space of $W_s$ is naturally isomorphic to the Sobolev space $W_{-s}(\tilde{V})$ of the contragradient representation $(\tilde{\pi}, \tilde{V})$ (in the case of $SL(2, \mathbb{R})$ we can identify $\tilde{\pi}$ with $\pi$). Hence, we can consider the automorphic functional I as a distribution vector in the representation $\tilde{\pi}$, and the question above is a question to which Sobolev spaces $W_s(\tilde{\pi})$ this vector belongs.

In fact we can also ask the same question about the regularity of the functional I with respect to other classes of spaces, for example, $L^p$-Sobolev spaces, Besov spaces, Hölder spaces, and so on.

Several recent papers deal with this question for different regularity classes, see [18] for Hölder spaces and [6] for $L^p$ Sobolev spaces.

The main point of this paper is that in the case of $L^2$ Sobolev norms we can get a very simple answer, which is a special case of Theorem 1.1.

**Theorem 2.1.** Let $\Gamma \subset G$ be a cocompact discrete subgroup, $(\pi, V)$ an irreducible automorphic representation, and I the corresponding automorphic functional. Then $s$-Sobolev norm of the functional I is bounded if and only if $s > 1/2$. □

In fact, our method shows that this is true for all infinite dimensional automorphic representations of G and the corresponding functionals I.

We can restate Theorem 2.1 as a bound on automorphic functions (for a cocompact subgroup). Let $\|\phi\|_\infty = \sup_{x \in X} |\phi(x)|$ be the supremum norm on $C(X)$. We introduce the norm $N_\infty$ on the space $V_\lambda$ by $N_\infty(v) = \|\nu(v)\|_\infty$ for $v \in V_\lambda$. Then Theorem 2.1 amounts to the bound

$$N_\infty \leq CN_s, \tag{2.1}$$

on the space $V_\lambda$.

In practice, Theorem 1.1 gives more and we prove, in fact, that the functional I is bounded in an appropriate Besov norm. Namely, let $B_\mu$ be the Besov type norm on $V_\lambda$ introduced in Section 4.2 (it is equivalent to the Besov $B^{1/2}_{2,1}$ norm; see [2] for a definition of Besov norms). We have then the following proposition.

**Proposition 2.2.** Let $V_\lambda$ be an automorphic representation for a cocompact discrete subgroup $\Gamma$. There exists a constant C depending only on $\Gamma$ such that

$$N_\infty(v) \leq CB_\mu(v), \tag{2.2}$$

for all smooth vectors $v \in V_\lambda$. □

In Theorem 2.1, we assume that the space $X = \Gamma \backslash G$ is compact. In fact, we show in Section 3.2 that the upper bound holds for arbitrary X. We will also obtain some partial lower bound results in the case of nonuniform lattices Γ.

We would like to emphasize that the standard techniques from the Sobolev restriction theory imply only that $\|I\|^2_{N_s}$ is finite for $s > 1$ (this follows from the theory for elliptic operators in [19] applied to the 3-dimensional manifold X described in Section 2.2). Hence, Theorem 2.1 goes beyond the usual Sobolev type restriction theorems. This indicates that Theorem 2.1 is not a local statement but has its origin in the global geometry of X.

## 2.4 Applications

We discuss now some applications.

*2.4.1 Fourier coefficients of the functional* I. Let $(\pi_\lambda, V_\lambda)$ be a principal series representation of $SL(2, \mathbb{R})$, $\mu = (1 - \lambda^2)/4$. The automorphic functional I described in Section 2.2 is a continuous functional on $C^\infty(S^1)_{\text{even}}$. For any such functional we can define its Fourier coefficients $a_n = I(e_n)$, where $e_n = e^{in\theta}$, n is even. In terms of these coefficients Theorem 2.1 means that the sum $\sum |a_n|^2(n^2 + 1)^{-s}$ is convergent if and only if $s > 1/2$. However, from Proposition 2.2, we obtain the following stronger result.

**Corollary 2.3.** There exists an effectively computable constant A, independent of μ, such that

$$\sum_{|n| \leq N} |a_n|^2 \leq A \cdot N, \quad \text{for } N > \sqrt{\mu}. \tag{2.3}$$

□

This estimate is sharp. It is not difficult to show, using the same method, that for a cocompact subgroup, there exist effectively computable constants $\gamma > 1$ and $a > 0$, independent of μ, such that

$$\sum_{|n| \leq N} |a_n|^2 \geq a \cdot N, \quad \text{for } N > \gamma\sqrt{\mu}. \tag{2.4}$$

*2.4.2 Bounds on automorphic functions.* The coefficients $a_n$ above are easily interpreted in terms of automorphic functions. Consider the nth K-finite vector $\phi_\lambda^{(n)}$ in the automorphic representation $V_\lambda$ as a function on X. The coefficient $a_n$ is equal to the *value* of $\phi_\lambda^{(n)}$ at the point $e \in X = \Gamma \setminus G$. Hence, bounds on coefficients $a_n$ could be viewed as a part of a general question asking for bounds on automorphic functions. Recently, this question drew a lot of attention in connection with applications to analytic theory of automorphic L-functions (see [17]). Classical approaches to the problem of bounding automorphic functions (with respect to the eigenvalue or the weight) are based on Hardy-Hecke method (see Remark 2.5) or on bounds on eigenfunctions of elliptic operators (see [19]). However, these methods are not able, to the best of our knowledge, to recover Theorem 2.1.

As a corollary to Proposition 2.2, we obtain the following result on the supremum norm of the function $\phi_\lambda^{(n)}$.

**Corollary 2.4.** There exists an explicit constant $C$, depending only on $\Gamma$, such that for all $n$ and $\mu = (1-\lambda^2)/4$,

$$\left\|\phi_\lambda^{(n)}\right\|_\infty \le \begin{cases} C\cdot(1+\mu)^{1/4}, & \text{if } |n| \le |\lambda|, \\ C\cdot|n|^{1/2}, & \text{if } |n| > |\lambda|. \end{cases} \tag{2.5}$$

□

Remark 2.5. (1) From the proof of Corollaries 2.3 and 2.4, it follows that the constants in (2.3) and (2.5) are expressible in terms of the diameter of X. It is also easy to see that Corollaries 2.3 and 2.4 hold for the representations of complementary and discrete series as well.

(2) We would like to compare the bound in Corollary 2.4 with other bounds on eigenfunctions.

For $n$ fixed (e.g., $n = 0$), the bound in Corollary 2.4 is the standard bound from the theory of elliptic operators (see [19]). It could be improved by $\log\mu$ for a negatively curved manifold (see [1]).

For a fixed $\lambda$, the function $\phi_\lambda^{(n)}$ on the 3-dimensional manifold $X$ is an eigenfunction of some elliptic operator (we denote it by $\Delta$). This yields the bound $\|\phi_\lambda^{(n)}(x)\|_\infty \ll |n|^{(\dim X-1)/2} = |n|^1$ as in [19]. Such a bound holds for a general Riemannian manifold. It also could be deduced from the theory of special functions via Hardy-Hecke method (see (3) below and [11, 13, 16]). However, $\phi_\lambda^{(n)}$ is also an eigenfunction of another differential operator coming from the SO(2) action and commuting with $\Delta$. This allows us to "reduce dimensions" and to obtain the better bound $\|\phi_\lambda^{(n)}\|_\infty \ll |n|^{1/2}$.

For $\lambda$ and $n$ both changing with the same rate (i.e., $\lambda = R\cdot n$) one can use symplectic reduction (in a fashion similar to [21]) to obtain the bound similar to the one in Corollary 2.4, $\|\phi_\lambda^{(n)}\|_\infty \ll |n|^{1/2}$, for a fixed $R$.

However, for a general vector in the representation $V_\lambda$, we do not see how the above methods can reproduce our bounds in Proposition 2.2 since the latter gives a bound for *all smooth* vectors in $V_\lambda$ simultaneously, with the explicit dependence on the parameter $\lambda$.

(3) *Hardy-Hecke method.* The standard method for estimating the asymptotic behavior of coefficients $a_n$ is the method which should be attributed to Hardy (but customarily called Hecke's method, see [10]). It is based on a geometric interpretation of the functional I and its Fourier coefficients $a_n$. Namely, consider another realization of the representation $\pi_\lambda$ of $SL(2,\mathbb{R})$ in the space $\mathfrak{E}_\lambda$ of all eigenfunctions of the Laplace-Beltrami operator on $\mathfrak{H}$ with the eigenvalue $\mu = (1-\lambda^2)/4$.

In this realization, the basis of K-finite vectors consists of "spherical" harmonics $F_n^\lambda(\theta, r) = e^{in\theta} f_n^\lambda(r)$, where $(\theta, r)$ are the polar coordinates in $\mathfrak{H}$ and $f_n^\lambda(r)$ are essentially equal to the hypergeometric function (see [11]).

A function $F(\theta, r) \in \mathfrak{E}_\lambda$ admits a decomposition into Fourier series, $F(\theta, r) = \sum_n c_n F_n^\lambda(\theta, r)$.

It is not difficult to see that the functional I on $V_\lambda$, when interpreted as a generalized vector in $\mathfrak{E}_\lambda$, is given by the function $\phi$ considered as a function on $\mathfrak{H}$ (see [15] for this approach). In particular, this function is bounded.

As Hardy-Hecke method shows, every bounded function $F \in \mathfrak{E}_\lambda$ has Fourier coefficients $c_n$ such that the sum $\sum |c_n|^2 (n^2+1)^{-s}$ converges for $s > 1$ (see [11, 13, 16]). The proof is based on a detailed knowledge of the behavior of hypergeometric functions; as a result, it is very difficult to see that this sum converges for $s > 1/2$. This in fact is true (for any bounded function $F \in \mathfrak{E}_\lambda$) and could be proven using a combination of a version of Hardy-Hecke method and representation theory arguments. This would give an alternative proof of the upper bound in Theorem 2.1. We also note that in the higher rank case, this method does not work due to the limited information available about spherical functions.

(4) We would like to know whether the bound in Corollary 2.4 is sharp (i.e., whether there exist infinitely many $a_n$ which do not satisfy $|a_n| \le c \cdot |n|^{1/2-\varepsilon}$, for any *fixed* $\varepsilon > 0$). We suspect that it might be sharp. This would mean that these Fourier coefficients $a_n$ are fundamentally different from the usual Fourier coefficients $u_n$ of cusp forms (these are associated to a unipotent subgroup N of G and not to K as $a_n$). For $u_n$, it is *proven* that $|u_n| \ll |n|^{1/3+\varepsilon}$ (see [5]) regardless of the arithmeticity assumption on $\Gamma$, and we can even suspect that $|u_n| \ll n^\varepsilon$ (the Ramanujan conjecture, see [16]). The reason for this discrepancy (if it exists) might be the existence of unipotent elements in $\Gamma$.

## 3 Relative traces and automorphic functionals

In this section, we prove Theorem 1.1. We use the notion of relative trace of two Hermitian forms on a topological vector space. We formulate properties of relative traces and relate them to values of Hermitian forms on automorphic functionals. We will then deduce from this our theorem.

### 3.1 Relative traces

Let $V$ be a separable topological vector space, $H(V)$ the space of continuous Hermitian forms on $V$, and $H(V)^+ \subset H(V)$ the subset of nonnegative Hermitian forms. In Appendix A, we define for any pair of forms $P, Q \in H^+(V)$, where $Q$ is positively definite, a number $\mathrm{tr}(P \mid Q)$, the relative trace of $P$ with respect to $Q$, taking values in $\mathbb{R}_+ \cup \infty$.

In Appendix A, we prove the following proposition.

**Proposition 3.1.** (1) *Linearity*. The functional $\mathrm{tr}(P \mid Q)$ is linear in $P$. It is monotonely increasing with respect to $P$ and monotonely decreasing with respect to $Q$.

(2) *Strong additivity*. Let $P_z \in H^+(V)$ be a family of forms parametrized by points of a measure space $Z$. We assume that this family is measurable, that is, for any $v \in V$ the function $z \mapsto P_z(v)$ is measurable. Fix a measure $\mu$ on $Z$ and define a Hermitian form $P$ on $V$ by $P(v) = \int P_z(v)d\mu$. We assume that all these integrals converge and define a continuous Hermitian form $P$ on $V$.

Then the relative trace $\mathrm{tr}(P_z \mid Q)$ is a measurable function with respect to the measure $\mu$ and $\int_Z \mathrm{tr}(P_z \mid Q)d\mu = \mathrm{tr}(P \mid Q)$.

(3) *Normalization*. Let $l$ be any continuous functional on $V$. Consider a Hermitian form $P_l$ on $V$ defined by $P_l(v) = |l(v)|^2$. Let $N$ be the norm induced by $Q$. Then $\mathrm{tr}(P_l \mid Q) = \|l\|_N^2$. □

### 3.2 Proof of Proposition 1.2

We have an isometric embedding $\nu : V \to L^2(X)$, which induces the form $P$ on $V$. For any point $x \in X$ and a vector $v \in V$, we have $I_x(v) = \nu(v)(x)$. The form $P$ on $V$ is given as an integral over $X$

$$P(v) = \|v\|^2 = \int_X \left|\nu(v)(x)\right|^2 d\mu_X = \int_X \left|I_x(v)\right|^2 d\mu_X = \int_X P_{I_x}(v)d\mu_X. \tag{3.1}$$

Hence $P = \int_X P_{I_x} d\mu_X$. Using properties (1) and (3) from Proposition 3.1, we get

$$\mathrm{tr}(P \mid Q) = \mathrm{tr}\left(\int_X P_{I_x}\, d\mu_X \mid Q\right) = \int_X \mathrm{tr}\left(P_{I_x} \mid Q\right) d\mu_X = \int_X \|I_x\|_N^2 d\mu_X. \tag{3.2}$$

This proves Proposition 1.2.

Proof of Theorem 1.1. We claim that under the assumption on the homogeneity of $X$ for any two points $x, y \in X$, $N$-norms of the corresponding functionals $I_x$ and $I_y$ are comparable.

Namely, for $g \in G$ denote by $d(g) = d_Q(g)$ the continuity constant of the operator $\pi(g)$ with respect to the form $Q$, that is, $d(g) = \|\pi(g)\|_N^2$ is the minimal constant $d$ such that $Q(\pi(g)v) \leq dQ(v)$ for all $v \in V$.

It is clear that if $x, y \in X$ and $y = g(x)$, then $\|I_x\|_N^2 \leq d(g)\|I_y\|_N^2$. Thus, if we set $d(x, y) = \min\{d(g) \mid g(x) = y\}$, we have the inequalities

$$d(x,y)^{-1}\|I_x\|_N^2 \leq \|I_y\|_N^2 \leq d(y,x)\|I_x\|_N^2. \tag{3.3}$$

Integrating these inequalities over the variable $y \in X$ with measure $\mu_X$ and using the fact that $\int \|I_y\|_N^2 d\mu_X = \mathrm{tr}(P \mid Q)$, we get the following upper and lower bounds for $\|I_x\|_N^2$.

The upper bound is $\|I_x\|_N^2 \leq C(x)\,\mathrm{tr}(P \mid Q)$, where $C(x) = (\int d(x,y)^{-1} d\mu_X)^{-1}$.

If we fix some $d \in \mathbb{R}_+$ and consider the closed "ball" $B(x, d) = \{y \in X \mid d(x, y) \leq d\}$, then the constant $C(x)$ can be estimated from above. Namely, we have $C(x) \leq d/\mu_X(B(x, d))$ (recall that we have normalized the measure so that $\mu_X(X) = 1$).

In particular, if $d$ exceeds the "radius" of $X$, that is, if $B(x, d) = X$, then $C(x) \leq d$. However, we can get sometimes a better bound taking smaller values for $d$ (e.g., for noncompact $X$).

The lower bound is $\|I_x\|_N^2 \geq c(x)\,\mathrm{tr}(P \mid Q)$, where $c(x) = (\int d(y,x) d\mu_X)^{-1}$.

For noncompact $X$ this integral is usually divergent and we get a trivial bound $c(x) = 0$.

If $X$ is compact, we get the bound $c(x) \geq d(X)^{-1}$, where $d(X)$ is the "diameter" of $X$, $d(X) = \max\{d(x, y) \mid x, y \in X\}$.

This finishes the proof of Theorem 1.1. ∎

### 3.3 Remarks

(1) In fact, the above arguments prove a slightly more general result.

Let $G$ be a locally compact group and $X$ a homogeneous space of $G$. Suppose that we are given a smooth representation $(\pi, V)$ of the group $G$ and a morphism $\nu : V \to C(X)$ of representations of the group $G$.

Fix a norm $N$ on $V$ such that $\pi$ is continuous with respect to $N$ and fix a point $x \in X$. We would like to estimate the norm $\|I\|_N$ of the functional $I = I_x$ with respect to the norm $N$.

We assume that the norm $N$ corresponds to a Hermitian form $Q$ on $V$.

Suppose that we are able to find a nonnegative Hermitian form $P$ on $V$ (not necessarily $G$-invariant), and a measure $\mu$ on $X$ with compact support of total volume 1 satisfying the inequality

$$\int_X |\nu(v)|^2 d\mu \leq P(v) \quad \forall \text{ vectors } v \in V. \tag{3.4}$$

Then there exists a constant C such that $\|I\|_N^2 \leq C \cdot \mathrm{tr}(P \mid Q)$.

Similarly, suppose that we have an estimate

$$\int_X |\nu(v)|^2 d\mu \geq P(v) \quad \forall \text{ vectors } v \in V. \tag{3.5}$$

Then there exists a constant $c > 0$ such that $\|I\|_N^2 \geq c \cdot \mathrm{tr}(P \mid Q)$.

(2) In some cases, we can use the above arguments to give some lower bounds for a noncocompact subgroup $\Gamma$. Namely, choose a ball $B \subset G$ and an (infinite) sequence of points $\{x_i\} \in G$ such that $\cup_i B \cdot x_i$ covers $X = \Gamma \setminus G$. Denote by $\mathcal{B}_i$ the image of the ball $B \cdot x_i \subset X$. If any point in X is covered by an a priori bounded number k of balls from a covering family $\{\mathcal{B}_i\}$, then we have the following lower and upper bounds:

$$c \cdot \mathrm{tr}(P \mid Q) \leq \sum_i \|I_{x_i}\|_N^2 \cdot \mu_X(\mathcal{B}_i) \leq C \cdot \mathrm{tr}(P \mid Q), \tag{3.6}$$

where $C, c > 0$ depend on B and k.

(3) Consider $G = SL(2, \mathbb{R})$ and a noncocompact lattice $\Gamma \subset G$. The upper bound from Theorem 1.1 can be translated into bounds for automorphic functions.

Assume that $\Gamma$ has a cusp at $\infty$, that is, $\Gamma$ has an infinite intersection $\Gamma_\infty$ with the upper triangular unipotent subgroup. Let $z$ be the standard complex coordinate on the upper half plane $\mathfrak{H}$. Since the function $y = \mathrm{Im}\, z$ on $\mathfrak{H}$ is invariant with respect to the group $\Gamma_\infty$, we can use it with the natural projection $G \to \mathfrak{H}$ to define a parameter $y$ near the cusp $\infty$ on the space $X = \Gamma \backslash G$.

In this case, Theorem 1.1 gives the following estimate for the function $\phi$ on X corresponding to a vector $v \in V$ in terms of Sobolev norms $N_s$:

$$|\phi(x)| \leq C(s) y^{1/2} N_s(v), \quad \text{for any } s > \frac{1}{2}. \tag{3.7}$$

These bounds hold for an arbitrary representation $V \subset L^2(X)$. For a cuspidal representation $\pi$, we prove a much better bound,

$$|\phi(x)| \leq C(s,t) y^{1/2-t} N_{s+t}(v), \quad \text{for any } s > \frac{1}{2},\ t \geq 0. \tag{3.8}$$

We hope to discuss it in a future paper (see also [5] for a related discussion via non-Hermitian norms).

## 4 Norms on representations of $SL(2,\mathbb{R})$

### 4.1 Continuous Hermitian norms

In order to effectively use Theorem 1.1, we have to understand the structure of continuous Hermitian norms on unitary representations of a real reductive group. We do not know how to think about this for general groups. But in the case of the group $SL(2,\mathbb{R})$, we are able to exhibit a large family of continuous Hermitian norms by elementary means.

Let $(\pi, V)$ be a smooth representation of the group $G = SL(2,\mathbb{R})$ equipped with an invariant positive definite Hermitian form $P$; we denote by $\|\cdot\|$ the corresponding norm on $V$.

Fix the maximal compact subgroup $K = SO(2) \subset G$. Then for any continuous Hermitian form $H$ on $V$, we can construct an equivalent $K$-invariant Hermitian form $Q$, namely, $Q$ is an average of $H$ over $K$, $Q = \int_K k \cdot H\, dk$. Thus up to equivalence, we can (and will) always assume our form to be $K$-invariant.

A $K$-invariant Hermitian form $Q$ has a nice presentation in terms of its coefficients. Namely, since an irreducible representation of $SL(2,\mathbb{R})$ has $K$ multiplicities $0$ or $1$, we can choose an orthonormal basis $e_n$ of $V$ such that $e_n$ changes according to the $n$th character of $K$. Then the form $Q$ is completely characterized by the coefficients $Q(n) = Q(e_n)$, since for a vector $v = \sum a_n e_n$ we have $Q(v) = \sum Q(n)|a_n|^2$. As follows from Appendix A.3 in this case, we have $\mathrm{tr}(P \mid Q) = \sum Q(n)^{-1}$.

For the construction below, we need one *continuous* norm of a finite trace. We describe its construction.

#### *4.1.1 The form* $Q_{\mathfrak{g},r}$.

Fix the standard $K$-invariant scalar product on $\mathfrak{g} = sl(2,\mathbb{R})$ $(X, Y \mapsto \mathrm{tr}\, XY^t)$. Let $\{X_1, X_2, X_3\}$ be an orthonormal basis of $\mathfrak{g} = sl(2,\mathbb{R})$. For $r > 0$, we define (the first Sobolev) $K$-invariant Hermitian form $Q_{\mathfrak{g},r}$ on $V$ by the formula

$$Q_{\mathfrak{g},r}(v) = \sum_i P\big(X_i v\big) + r^2 P(v). \tag{4.1}$$

We denote by $N_{\mathfrak{g},r}$ the corresponding norm. The form $Q_{\mathfrak{g},r}$ is $K$-invariant.

We will see later that the representation $(\pi, V)$ is continuous with respect to $N_{\mathfrak{g},r}$. More precisely, for every $g \in G$, we have $\|\pi(g)\|_{N_{\mathfrak{g},r}} \leq \rho(g)$, where $\rho(g) = \|g\|_{\mathrm{Ad}}$ is the norm of the adjoint action of $g$ with respect to the chosen scalar product on $\mathfrak{g}$ (see Appendix B). In particular, continuity constants $\rho(g)$ are *independent* of the representation $\pi$.

Example 4.1. Consider a representation of the principal series $(\pi_\lambda, V_\lambda)$. Choose an orthonormal basis $\{e_n\}$, $n$ is even, consisting of eigenvectors of $K$. In the realization above $V_\lambda \simeq C^\infty_{\text{even}}(S^1)$, we have $e_n = e^{in\theta}$, $n$ is even.

The coefficients $Q(n)$ of the form $Q_{\mathfrak{g},r}$ are given by the formula $Q(n) = 2\mu + r^2 + n^2$.

*4.1.2 A family of forms.* Let $(\pi_\lambda, V_\lambda)$ be a representation of the principal series. Example 4.1 suggests that it is useful to consider a family of $K$-invariant Hermitian forms $Q_t$ on $V = V_\lambda$ parametrized by real numbers $t$, where the form $Q_t$ has coefficients $Q_t(n) = n^2 + t^2$.

For $t > t_0 = \sqrt{2\mu}$ the form $Q_t$ is of the form $Q_{\mathfrak{g},r}$ for an appropriate $r$ and hence all such forms $Q_t$ are continuous with the same continuity constants $\rho(g)$ independent of $\pi$ and $t$. We consider only such forms $Q_t$ (with $t > t_0$) and the corresponding norms $N_t$.

We can normalize the forms $Q_t$ to have roughly the same trace. Namely, consider the normalized form $q_t = t^{-1}Q_t$; it has coefficients $q_t(n) = t + n^2/t$ and an easy computation shows that $\operatorname{tr}(P \mid q_t)$ is of order 1 (more precisely, for $t \geq 1$, we have $1/2 \leq \operatorname{tr}(P \mid q_t) \leq 2$).

We denote by $n_t$ the norm corresponding to the form $q_t$. Theorem 1.1 now reads: there exists an explicit constant $C(\Gamma)$, depending only on $\Gamma$ such that for any point $x \in X$ and any vector $v \in V$,

$$\left|I_x(v)\right| \leq \sqrt{2}C(\Gamma) \cdot n_t(v). \tag{4.2}$$

## 4.2 The infimum norm

We now apply general considerations about infimum of a family of (semi-)norms (see [5, Appendix A]). Namely, we have the following general lemma applicable to any family of norms.

**Lemma 4.2.** A nonempty family of seminorms $N_u$ on $V$ has an exact lower bound $N = \inf_u N_u$. □

Applying this to the family of norms $\{n_t\}$, $t > t_0$, we obtain a seminorm $\mathfrak{n}$. This seminorm is a norm since all norms $n_t$ are bounded from below by the invariant Hermitian norm $\|\cdot\|$.

Hence, we have

$$\left|I_x(v)\right| \leq \sqrt{2}C(\Gamma) \cdot \mathfrak{n}(v) \tag{4.3}$$

for any point $x \in X$ and all vectors $v \in V$.

We claim that the norm $\mathfrak{n}$ is equivalent (with effective constants) to an explicitly described norm $B_\mu$. We construct now the norm $B_\mu$.

Let V be a unitary representation of G and $V = \oplus_{l\in\mathbb{Z}} V_l$, the K-type decomposition. Let $\{I_k\}$, $k = 0, 1, 2, \ldots$ be dyadic segments in $\mathbb{Z}$: $I_k = [-2^k, 2^k]$. Suppose that the Casimir operator $\Delta$ in V has the eigenvalue $\mu = (1-\lambda^2)/2$. We denote by $k_0$ the minimal integer k such that $|\lambda|$ lies inside the segment $I_k$.

We consider the following dyadic decomposition $\{J_\alpha\}$, $\alpha \in \mathbb{N}$, $\alpha \geq k_0$ of $\mathbb{Z}$, $\mathbb{Z} = \cup_\alpha J_\alpha : J_{k_0} = I_{k_0}$ and $J_\alpha = I_\alpha \setminus I_{\alpha-1}$ for $\alpha \geq k_0 + 1$. We denote by $h(\alpha)$ the maximal element in $J_\alpha$.

For every index $\alpha \geq k_0$, consider the space $V_\alpha = \oplus_{l\in J_\alpha} V_l$; for a vector $v \in V$, we denote by $pr_\alpha(v)$ the projection of $v$ to the space $V_\alpha$.

Definition 4.3. Let $(\pi, V)$ be a unitary representation of G where Casimir acts by $\mu$; $k_0$ and $pr_\alpha$ are as above. Denote by $B_\mu$ the norm on V given by

$$B_\mu(v) = \sum_{\alpha\geq k_0} h(\alpha)^{1/2} \cdot \left\| pr_\alpha(v) \right\|, \quad \text{for any } v \in V. \tag{4.4}$$

Remark 4.4. Note that this is not a Hermitian norm. It is easy to see that if we change $k_0$ to any (nonzero) multiple of it, we obtain an equivalent norm. We can see from the definition that for a representation $V_\lambda$ of the principal series, the norm $B_\mu$ viewed on $V_\lambda \simeq C^\infty(S^1)$ is an appropriately modified Besov norm $B^{1/2}_{2,1}$; see [2] for the definition of Besov norms.

By an easy computation (see Section 4.2.1), we have the following proposition.

**Proposition 4.5.** The infimum norm $\mathfrak{n} = \inf \mathfrak{n}_t$ on the representation V satisfies the inequality

$$\mathfrak{n} \leq \sqrt{2} \cdot B_\mu. \tag{4.5}$$

□

In fact, it is not difficult to show that $B_\mu \leq 2\mathfrak{n}$, that is, the norms $\mathfrak{n}$ and $B_\mu$ are equivalent.

**Corollary 4.6.** There exists a universal effectively computable constant $C(\Gamma)$ depending only on $\Gamma$ such that

$$\left|I_x(v)\right| \leq 2C(\Gamma) \cdot B_\mu(v), \tag{4.6}$$

for any point $x \in X$ and vector $v \in V$. □

This immediately implies Corollaries 2.3 and 2.4.

*4.2.1 Proof of Proposition 4.5.* Consider the family of K-invariant forms $\{q_t\}$, $t \geq t_0$, having coefficients $q_t(n) = t + n^2/t$. We will prove that $\mathfrak{n} = \inf \mathfrak{n}_t \leq \sqrt{2}B_\mu$.

Consider the decomposition $\{J_\alpha\}$ of $\mathbb{Z}$ and the corresponding decomposition $V = \oplus_\alpha V_\alpha$ of the space V as above. For a vector $v \in V$, we have $v = \sum_\alpha v_\alpha$, $v_\alpha \in V_\alpha$.

By the definition of the norm $B_\mu$, we have $B_\mu(v) = \sum_\alpha B_\mu(v_\alpha)$. On the other hand, for any vector $v_\alpha \in V_\alpha$, we have $\mathfrak{n}(v_\alpha)^2 = \inf_t q_t(v_\alpha) \leq q_{h(\alpha)}(v_\alpha) \leq 2(B_\mu(v_\alpha))^2$. Hence, $\mathfrak{n} \leq \sqrt{2} \cdot B_\mu$.

## Appendices

## A Relative traces

### A.1 Construction of relative traces

Let V be a separable topological complex vector space, H(V) the space of continuous Hermitian forms on V, and $H^+(V) \subset H(V)$ the subset of nonnegative Hermitian forms. Let $P, Q \in H^+(V)$ and Q be positive definite. In this situation, we define a number, the relative trace $\mathrm{tr}(P \mid Q)$, taking value in $\mathbb{R}_+ \cup \infty$.

First of all, define relative traces for finite dimensional spaces. Let $V = W$, $\dim W < \infty$. Then Q, P define homomorphisms $Q, P : W \to W^+$ (to the Hermitian dual of the space $W$), moreover, $Q^{-1}$ exists (since Q is positive definite). We define the relative trace by $\mathrm{tr}(P \mid Q) = \mathrm{tr}_W(Q^{-1}P)$.

It is clear from this definition that the relative trace is a continuous function of Q and P.

From this definition, we can easily deduce a formula for the relative trace. Namely, if we chose a basis $\{e_i\}$ of the space $W$ orthogonal with respect to the form Q, then $\mathrm{tr}(P \mid Q) = \sum_i P(e_i)/Q(e_i)$.

This formula implies that the relative trace is monotone with respect to a subspace, that is, if we restrict forms P and Q to a subspace $L \subset W$, then we have $\mathrm{tr}(P_L \mid Q_L) \leq \mathrm{tr}(P \mid Q)$.

For an infinite dimensional space V, we define $\mathrm{tr}(P \mid Q)$ as a supremum of relative traces $\mathrm{tr}(P_W \mid Q_W)$ of restrictions of Q and P to all finite dimensional subspaces $W \subset V$: $\mathrm{tr}(P \mid Q) = \sup_{W \subset V} \mathrm{tr}(P_W \mid Q_W)$ (note that the supremum could be infinite).

### A.2 Properties of relative traces

Here we prove the properties of the relative traces we listed in Proposition 3.1. Linearity and values on functionals are immediate since these are obviously true for finite dimensional spaces.

We assume that $V$ is separable, that is, there exists a sequence $\{W_i\}$ of finite dimensional subspaces, $W_1 \subset W_2 \subset \cdots$ such that the closure of $\bigcup W_i$ is equal to $V$. We have then the following lemma.

**Lemma A.1.** For $V$ separable and $\{W_i\}$ as above, the following relation holds:

$$\operatorname{tr}(P \mid Q) = \lim_i \operatorname{tr}\left(P_{W_i} \mid Q_{W_i}\right). \tag{A.1}$$

□

Proof of Proposition 3.1(2). Lemma A.1 and the assumption that all functions $\operatorname{tr}(P_{z,W_i} \mid Q_{W_i})$ are measurable imply that the limit of a monotone sequence of positive measurable functions, $\operatorname{tr}(P_z \mid Q) = \lim \operatorname{tr}(P_{z,W_i} \mid Q_{W_i})$, is a measurable function. We also assumed that $P = \int_Z P_z \, d\mu$ is a continuous Hermitian form, which implies that $\operatorname{tr}(P_W \mid Q_W) = \int_Z \operatorname{tr}(P_{z,W} \mid Q_{z,W})$. This implies that

$$\begin{aligned} \operatorname{tr}(P \mid Q) &= \lim_i \operatorname{tr}\left(P_{W_i} \mid Q_{W_i}\right) = \lim_i \int_Z \operatorname{tr}\left(P_{z,W_i} \mid Q_{W_i}\right) d\mu \\ &= \int_Z \left\{\lim_i \operatorname{tr}\left(P_{z,W_i} \mid Q_{W_i}\right)\right\} d\mu = \int_Z \operatorname{tr}\left(P_z \mid Q_z\right), \end{aligned} \tag{A.2}$$

since the limit and the integral are interchangeable for a monotone sequence of positive measurable functions. This finishes the proof of Proposition 3.1. ■

Proof of Lemma A.1. Obviously $\operatorname{tr}(P \mid Q) \geq \lim_i \operatorname{tr}(P_{W_i} \mid Q_{W_i})$. To prove the converse, we have to show that for any finite dimensional subspace $W \subset V$ and any $\varepsilon > 0$, there exists $i$ such that $\operatorname{tr}(P_W \mid Q_W) - \varepsilon \leq \operatorname{tr}(P_{W_i} \mid Q_{W_i})$. Let $\{\psi_\alpha\}$, $\alpha \in [0,1]$ be a continuous family of maps $\{\psi_\alpha : W \to V\}$. Then $\operatorname{tr}(P_W \mid Q_W) = \operatorname{tr}_W(\psi_\alpha^* P \mid \psi_\alpha^* Q)$ and it is continuous in $\alpha$. By the assumption on separability, the union $\bigcup W_i$ is dense in $V$. Hence we can take a perturbation $\psi' = \psi_\alpha$ of $\psi = \psi_0$ as small as we wish such that $\psi'(W) \subset W_i$ for some $i$. Continuity of tr in $P$, $Q$, $\psi_\alpha$ and monotonicity with respect to subspaces imply the lemma. ■

## A.3 Computation of relative traces

**Proposition A.2.** Let $\{e_i\}$ be a topological basis of the space $V$ which is orthogonal with respect to the Hermitian form $Q$. Then $\operatorname{tr}(P \mid Q) = \sum_i P(e_i)/Q(e_i)$. □

The proof immediately follows from Lemma A.1 and from the formula for relative trace on finite dimensional spaces.

## B Sobolev norms

The purpose of this appendix is to define Sobolev norms in a form appropriate for the representation theory.

Derived norms. We recall the definition of Sobolev or derived norms. Let $G = SL(2, \mathbb{R})$ and $(\pi, G, V)$ be a smooth representation of G. Fix a norm N on the space V such that the representation $\pi$ is continuous with respect to N. Using derivations, we can produce a derived norm $N' = DN$ on V as follows.

Fix a basis $\{X_i\}$ of the Lie algebra $\mathfrak{g} = \mathrm{Lie}(G)$ and a positive number r. Then we define the norm $N'$ on V by $N'(v)^2 = \sum N(X_i v)^2 + N(rv)^2$.

It is easy to check that the representation $\pi$ is continuous with respect to the norm $N'$ with explicit continuity constants, $\|\pi(g)\|_{N'} \leq C(g)\|\pi(g)\|_N$, where $C(g)$ depends on the choice of the basis $X_i$ but does not depend on $\pi$ or N.

Different choices of the basis $X_i$ and of the number r lead to equivalent norms.

We are mostly interested in the case when the norm N comes from a positive definite Hermitian form H on V. In this case, it is easy to check that the norm $N'$ also corresponds to a Hermitian form $H'$. The form $H'$ depends only on the quadratic form q on $\mathfrak{g}$ defined by the basis $\{X_i\}$ and on the constant r.

Moreover, in this case, we have $C(g) \leq \max\{1, \|\mathrm{Ad}(g)\|_q\}$. This follows from the following argument. Consider the vector space $E = (\mathfrak{g} \otimes V^*) \oplus V^*$ and the morphism $\nu : E \to V^*$ given by $\nu(x \otimes v, y) = x \cdot v + ry$. The dual morphism $\nu^*$ embeds V into the space $E^* = (\mathfrak{g}^* \otimes V) \oplus V$. Let T be the Hermitian form on $E^*$ coming from the form $q^*$ on $\mathfrak{g}^*$ and the form H on V. It is easy to see that the derived norm on V corresponds to a Hermitian form, which is induced by $\nu^*$ from the Hermitian form T. The form T obviously has the desired continuity constants and hence the induced norm on V also has these continuity constants.

Now, let $(\pi, G, V)$ be a smooth representation equipped with an invariant positive definite Hermitian form P. We define Sobolev norms $N_0, N_1, N_2, \ldots$ on V by the inductive formula $N_{i+1} = DN_i$, where $N_0$ is the norm corresponding to P.

We can use interpolation of Hermitian norms to extend this family of norms to a family of norms $N_s$ defined for all real numbers $s \geq 0$.

In a more detail, it is shown in [12, Chapter 4, Theorem 1.13] that for any pair of positive definite Hermitian forms P and Q on V, there is an interpolating family of forms $Q^s$, $0 \leq s \leq 1$, such that $Q^0 = P$ and $Q^1 = Q$. Moreover, we can also see that if an operator $A : V \to V$ preserves the form P and has a continuity constant C with respect to Q, then it has a continuity constant $C^s$ with respect to the form $Q^s$ [12, Theorem 1.11].

Applying this to $P$ and the Hermitian form $Q = Q_i$, which corresponds to $N_i$, we obtain a family of Hermitian norms $N_s$, $0 \leq s \leq i$, with continuity constants which are *independent* of representation $\pi$; namely, $\|\pi(g)\|_{N_s} \leq C(g)^{|s|}$ with $C(g)$ described above.

Note that the families of norms $N_s$ obtained in such a way using different norms $N_i$ and $N_j$, $j < i$ do not coincide for $0 \leq s \leq j$. For the group $SL(2, \mathbb{R})$, they are equivalent; for general groups, this might not be true.

In general, although the procedure described above works for any Lie group $G$, it is not clear that the norms it produces are useful for estimates. The reason is that it gives only a 1-parameter family of norms, while we expect that the Sobolev norms are parametrized by $l$ parameters, where $l$ is the split rank of $G$ (see [3]).


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