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ESTIMATES OF AUTOMORPHIC FUNCTIONS

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Abstract. We present a new method to estimate trilinear period for automorphic representations of $\mathrm{SL}_2(\mathbb{R})$. The method is based on the uniqueness principle in representation theory. We show how to separate the exponentially decaying factor in the triple period from the essential automorphic factor which behaves polynomially. We also describe a general method which gives an estimate for the average of the automorphic factor and thus prove a convexity bound for the triple period.

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1. Introduction

1.1. Maass forms. Let Y be a compact Riemann surface with a Riemannian metric of constant curvature -1 and the associated volume element dv. The corresponding Laplace–Beltrami operator is non-negative and has purely discrete spectrum on the space $L^2(Y, dv)$ of functions on Y. We will denote by $0 = \mu_0 < \mu_1 \leq \mu_2 \leq \dots$ its eigenvalues and by $\phi_i = \phi_{\mu_i}$ the corresponding eigenfunctions (normalized to have L^2 norm one). In the theory of automorphic forms the functions ϕ_{μ_i} are called automorphic functions or *Maass forms* (after H. Maass, [M]).

The study of Maass forms plays an important role in analytic number theory.

We are interested in their analytic properties and will present a new method of finding bounds for some important quantities arising from ϕ_i.

1.2. Triple products. For any three Maass forms ϕ_i, ϕ_j, ϕ_k we define the following triple product or triple period:

$$c_{ijk} = \int_Y \phi_i \phi_j \phi_k \, dv. \tag{1}$$

We would like to bound the coefficient c_{ijk} as a function of eigenvalues μ_i, μ_j, μ_k. In particular, we would like to find bounds for these coefficients when one or more of these indices tend to infinity.

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1.3. Motivation. First of all we would like to explain why this problem is interesting. The explanation goes back to pioneering works of Rankin and Selberg (see [Ra], [Se]). They discovered that in special cases triple products as above give rise to automorphic L-functions. That allowed them to obtain analytic continuation and effective bounds for these L-functions and, as an application, to obtain bounds for the Fourier coefficients of cusp forms towards the Ramanujan conjecture.

Since then the Rankin–Selberg method has had many generalizations. Recently, for Y arising from the full modular group $SL_2(\mathbb{Z})$ and for cuspidal functions ϕ, Watson (see [Wa]) proved the following beautiful formula:

$$\left|\int_Y \phi_i\phi_j\phi_k\,dv\right|^2 = G(\lambda_i,\,\lambda_j,\,\lambda_k)\frac{L(1/2,\,\phi_i\otimes\phi_j\otimes\phi_k)}{L(1,\,\phi_i,\,\mathrm{Ad})L(1,\,\phi_j,\,\mathrm{Ad})L(1,\,\phi_k,\,\mathrm{Ad})}. \tag{2}$$

Here λ_t is a natural parameter of an eigenfunction ϕ_t related to the eigenvalue by $\mu_t = \frac{1-\lambda_t^2}{4}$. The functions $L(s,\,\phi_i\otimes\phi_j\otimes\phi_k)$ and $L(s,\,\phi_t,\,\mathrm{Ad})$ are appropriate automorphic L-functions associated to ϕ_i, and the function $G(\lambda_i,\,\lambda_j,\,\lambda_k)$ is an explicit rational expression in the ordinary Γ-functions. The relation (2) can be viewed as a far reaching generalization of the original Rankin–Selberg formula. It was motivated by a work [HK] by Harris and Kudla on a conjecture of Jacquet.

1.4. Results. In this paper we will consider the following problem. We fix two Maass forms $\phi = \phi_\tau$, $\phi' = \phi_{\tau'}$ as above and consider coefficients defined by the triple period as above:

$$c_i = \int_Y \phi\phi'\phi_i\,dv \tag{3}$$

as $\{\phi_i = \phi_{\lambda_i}\}$ run over the basis of Maass forms.

Thus we see from (2) that the estimates of coefficients c_i are equivalent to the estimates of the corresponding L-functions. One would like to have a general method to estimate the coefficients c_i and similar quantities. This problem was raised by Selberg in his celebrated paper [Se].

Let us understand what kind of bounds on the left hand side of (2) one would like to have in order to estimate effectively L-functions involved in the right hand side of (2) (or at least the ratio of L-functions).

We note first that one expects that c_i have exponential decay in $|\lambda_i|$ as i goes to ∞. Namely, general experience from the analytic theory of automorphic L-functions tells us that L-functions have at most polynomial growth when $|\lambda_i| \to \infty$. Hence, analyzing the function $G(\lambda)$, one would expect from (2) and the Stirling formula for the asymptotic of Γ-function that the normalized coefficients

$$b_i = |c_i|^2 \exp\left(\frac{\pi}{2}|\lambda_i|\right) \tag{4}$$

have at most polynomial growth in $|\lambda_i|$, and hence c_i decay exponentially. However, it is difficult to see from the definition of the coefficients c_i that they have exponential decay and it is not clear what should be the rate of this decay.

The fact that an exponential decay with the exponent $\frac{\pi}{2}$ holds for a general Riemann surface was first shown by Good and Sarnak (see [G] and [Sa1]). Both proofs used ingenious analytic continuation of automorphic functions in the *variable* parameter.

In this paper we will explain how to naturally separate the exponential decay from a polynomial growth in coefficients c_i using representation theory. We also prove the following

Theorem. *There exists an effectively computable constant A such that the following bound holds for arbitrary $T > 0$:*

$$\sum_{T \leq |\lambda_i| \leq 2T} b_i \leq A. \tag{5}$$

1.5. A conjecture. The estimate in the theorem is tight but if we try to use it to get a bound for an individual term b_i we get only an inequality

$$b_i \leq A. \tag{6}$$

According to Weyl's law there are approximately cT^2 eigenvalues μ_i with λ_i between T and $2T$, so the individual bound for the coefficient b_i is definitely not tight. We would like to make the following conjecture concerning the size of coefficients b_i:

Conjecture. *For any $\varepsilon > 0$ there exists a constant $C_\varepsilon > 0$ such that*

$$b_i \leq C_\varepsilon |\lambda_i|^{-2+\varepsilon},$$

as $|\lambda_i| \to \infty$.

For Y arising from congruence subgroups this conjecture is consistent with the Lindelöf conjecture for appropriate automorphic L-functions (see [BR1], [Sa2] and [Wa] for more details). We note that the bound in the Theorem above corresponds to the so-called *convexity bound*.

1.6. The method. The first proof of the (slightly weaker) version of Theorem 1.4 appeared in [BR1]. It was based on the analytic continuation of representations from a real group to a complex group (generalizing methods of [Sa1]). The method based on the analytic continuation was extended in [KS] to the case of higher rank groups. While it gives bounds which are tight for general representations, it was not able, so far, to cover cases relevant to L-functions.

The proof we present here is based on the uniqueness of triple product in representation theory. It has an advantage that it could be generalized to higher rank groups and gives bounds which are consistent with the theory of L-functions. The present method also could be applied to p-adic groups (unlike methods of [BR1]).

It is known that the uniqueness principle plays a central role in the theory of automorphic functions (see [PS]). The impact that the uniqueness has on the analytic behavior of automorphic functions is yet another manifestation of this principle.

We describe now the general ideas behind our new proof. It is based on ideas from representation theory. Namely, we use the fact that every automorphic form ϕ generates an automorphic representation of the group $G = \mathrm{PGL}_2(\mathbb{R})$; this means that starting from ϕ we produce a smooth irreducible representation of the group G in a space V and its realization $\nu\colon V \to C^\infty(X)$ in the space of smooth functions on the automorphic space $X = \Gamma \backslash G$.

The triple product $c_i = \int_Y \phi\phi'\phi_i\, dv$ extends to a G-equivariant trilinear form on the corresponding automorphic representations $l^{\text{aut}}\colon V \otimes V' \otimes V_i \to \mathbb{C}$, where $V = V_\tau$, $V' = V_{\tau'}$, $V_i = V_{\lambda_i}$.

Then we use a general result from representation theory that such G-equivariant trilinear form is unique up to a scalar. This implies that the automorphic form l^{aut} is proportional to an explicit "model" form l^{mod} which we describe using explicit realizations of representations of the group G; it is an important fact that this last form carries no arithmetic information.

Thus we can write $l^{\text{aut}} = a_i \cdot l^{\text{mod}}$ for a constant a_i, hence $c_i = l^{\text{aut}}(e_\tau \otimes e_{\tau'} \otimes e_{\lambda_i}) = a_i \cdot l^{\text{mod}}(e_\tau \otimes e_{\tau'} \otimes e_{\lambda_i})$, where e_τ, $e_{\tau'}$, e_{λ_i} are K-invariant unit vectors in the automorphic representations V, V', V_i corresponding to the automorphic forms ϕ, ϕ' and ϕ_i.

It turns out that the proportionality coefficient a_i in the last formula carries an important "automorphic" information while the second factor carries no arithmetic information and can be computed in terms of Γ-functions using explicit realizations of representations V_τ, $V_{\tau'}$ and V_{λ_i}. This second factor is responsible for the exponential decay, while the first factor a_i has a polynomial behavior in parameter λ_i.

In order to bound the quantities a_i, we use the fact that they appear as coefficients in the spectral decomposition of the diagonal Hermitian form H_Δ on the space $E = V_\tau \otimes V_{\tau'}$ (see Sections 4.2, 4.3). This gives an inequality $\sum |a_i|^2 H_i \leq H_\Delta$ where H_i is an Hermitian form on E induced by the model trilinear form $l^{\text{mod}}\colon V \otimes V' \otimes V_i \to \mathbb{C}$ as above.

Using the geometric properties of the diagonal form and simple explicit estimates of forms H_i we establish the convexity bound for the coefficients a_i.

2. Representation Theoretic Setting

We recall the standard connection of the above setting with representation theory (see [GGPS]).

2.1. Automorphic functions and automorphic representations. Let us describe the geometric construction which allows one to pass from analysis on a Riemann surface to representation theory.

Let $\mathbb{H}$ be the upper half plane with the hyperbolic metric of constant curvature -1. The group $\mathrm{SL}_2(\mathbb{R})$ acts on $\mathbb{H}$ by fractional linear transformations. This action allows us to identify the group $\mathrm{PSL}_2(\mathbb{R})$ with the group of all orientation preserving motions of $\mathbb{H}$. For reasons explained below we would like to work with the group G of all motions of $\mathbb{H}$; this group is isomorphic to $\mathrm{PGL}_2(\mathbb{R})$. Hence throughout the paper we denote $G = \mathrm{PGL}_2(\mathbb{R})$.

Let us fix a discrete co-compact subgroup $\Gamma \subset G$ and set $Y = \Gamma\backslash\mathbb{H}$. We consider the Laplace operator on the Riemann surface Y and denote by μ_i its eigenvalues and by ϕ_i the corresponding normalized eigenfunctions.

The case when Γ acts freely on $\mathbb{H}$ corresponds precisely to the case discussed in the introduction (this follows from the uniformization theorem for the Riemann surface Y). Our results hold for general co-compact subgroup Γ (and in fact, with slight modifications, for any lattice $\Gamma \subset G$).

We will identify the upper half plane $\mathbb{H}$ with G/K, where $K = PO(2)$ is a maximal compact subgroup of G (this follows from the fact that G acts transitively on $\mathbb{H}$ and the stabilizer in G of the point $z_0 = i \in \mathbb{H}$ coincides with K).

We denote by X the compact quotient $\Gamma \setminus G$ (we call it the automorphic space). In the case when Γ acts freely on $\mathbb{H}$ one can identify the space X with the bundle of unit tangent vectors to the Riemann surface $Y = \Gamma \setminus \mathbb{H}$.

The group G acts on X (from the right) and hence on the space of functions on X. We fix the unique G-invariant measure μ_X on X of total mass one. Let $L^2(X) = L^2(X, d\mu_X)$ be the space of square integrable functions and $(\Pi_X, G, L^2(X))$ the corresponding unitary representation. We will denote by P_X the Hermitian form on $L^2(X)$ given by the scalar product. We denote by $\| \|_X$ or simply $\| \|$ the corresponding norm and by $\langle f, g\rangle_X$ the corresponding scalar product.

The identification $Y = \Gamma \setminus \mathbb{H} \simeq X/K$ induces the embedding $L^2(Y) \subset L^2(X)$. We will always identify the space $L^2(Y)$ with the subspace of K-invariant functions in $L^2(X)$.

Let ϕ be a normalized eigenfunction of the Laplace–Beltrami operator on Y. Consider the closed G-invariant subspace $L_\phi \subset L^2(X)$ generated by ϕ under the action of G. It is well-known that $(\pi, L) = (\pi_\phi, L_\phi)$ is an irreducible unitary representation of G (see [GGPS]).

Usually it is more convenient to work with the space $V = L^\infty$ of smooth vectors in L. The unitary Hermitian form P_X on V is G-invariant.

A smooth representation (π, G, V) equipped with a positive G-invariant Hermitian form P we will call a *smooth pre-unitary representation*; this simply means that V is the space of smooth vectors in the unitary representation obtained from V by completion with respect to P.

Thus starting with an automorphic function ϕ we constructed an irreducible smooth pre-unitary representation (π, V). In fact we constructed this space together with a canonical morphism $\nu\colon V \to C^\infty(X)$ since $C^\infty(X)$ is the smooth part of $L^2(X)$.

Definition. A smooth pre-unitary representation (π, G, V) equipped with a G-morphism $\nu\colon V \to C^\infty(X)$ we will call an *X-enhanced representation.*

In this note we will assume that the morphism ν is normalized, i. e., it carries the standard L^2 Hermitian form P_X on $C^\infty(X)$ into Hermitian form P on V.

Thus starting with an automorphic function ϕ we constructed

(i) an X-enhanced irreducible pre-unitary representation (π, V, ν),
(ii) a K-invariant unit vector $e_V \in V$ (this vector is just our function ϕ).

Conversely, suppose we are given an irreducible smooth pre-unitary X-enhanced representation (π, V, ν) of the group G and a K-fixed unit vector $e_V \in V$. Then the function $\phi = \nu(e_V) \in C^\infty(X)$ is K-invariant and hence can be considered as a function on Y. The fact that the representation (π, V) is irreducible implies that ϕ is an automorphic function.

Thus we have established a natural correspondence between Maass forms ϕ and tuples (π, V, ν, e_V), where (π, V, ν) is an X-enhanced irreducible smooth pre-unitary representation and $e_V \in V$ is a unit K-invariant vector.

2.2. Decomposition of the representation $(\Pi_X, G, L^2(X))$**.** It is well known that in case when X is compact the representation $(\Pi_X, G, L^2(X))$ decomposes into a direct (infinite) sum

$$L^2(X) = \bigoplus_j (\pi_j, L_j) \tag{7}$$

of irreducible unitary representations of G (all representations appear with finite multiplicities (see [GGPS])). Let (π, L) be one of these irreducible "automorphic" representations and $V = L^\infty$ its smooth part. By definition V is given with a G-equivariant isometric morphism $\nu\colon V \to C^\infty(X)$, i. e., V is an X-enhanced representation.

If V has a K-invariant vector it corresponds to a Maass form. There are other spaces in this decomposition which correspond to discrete series representations. Since they are not related to Maass forms we will not study them in more detail.

2.3. Representations of $\mathrm{PGL}_2(\mathbb{R})$**.** All irreducible unitary representations of G are classified. For simplicity we consider those with a nonzero K-fixed vector (so called representations of class one) since only these representations arise from Maass forms. These are the representations of the principal and the complementary series and the trivial representation.

We will use the following standard explicit model for irreducible smooth representations of G.

For every complex number λ consider the space V_λ of smooth even homogeneous functions on $\mathbb{R}^2 \setminus 0$ of homogeneous degree $\lambda - 1$ (which means that $f(ax, ay) = |a|^{\lambda-1} f(x, y)$ for all $a \in \mathbb{R}\setminus 0$). The representation (π_λ, V_λ) is induced by the action of the group $\mathrm{GL}_2(\mathbb{R})$ given by $\pi_\lambda(g)f(x, y) = f(g^{-1}(x, y))|\det g|^{(\lambda-1)/2}$. This action is trivial on the center of $\mathrm{GL}_2(\mathbb{R})$ and hence defines a representation of G. The representation (π_λ, V_λ) is called *representation of the generalized principal series.*

When $\lambda = it$ is purely imaginary the representation (π_λ, V_λ) is pre-unitary; the G-invariant scalar product in V_λ is given by $\langle f, g\rangle_{\pi_\lambda} = \frac{1}{2\pi}\int_{S^1} f\bar{g}\, d\theta$. These representations are called representations of *the principal series.*

When $\lambda \in (-1, 1)$ the representation (π_λ, V_λ) is called a representation of the complementary series. These representations are also pre-unitary, but the formula for the scalar product is more complicated (see [GGV]).

All these representations have K-invariant vectors. We fix a K-invariant unit vector $e_\lambda \in V_\lambda$ to be a function which is one on the unit circle in $\mathbb{R}^2$.

Representations of the principal and the complimentary series exhaust all nontrivial irreducible pre-unitary representations of G of class one (see [GGV], [L]).

In what follows we will do necessary computations for representation of the principal series. Computations for the complementary series are a little more involved but essentially the same (compare with [BR1, Section 5.5], where similar computations are described in detail).

Suppose we are given a class one X-enhanced representation $\nu\colon V_\lambda \to C^\infty(X)$; we assume ν to be an isometric embedding. Such ν gives rise to an eigenfunction of the Laplacian on the Riemann surface $Y = X/K$ as before. Namely, if $e_\lambda \in V_\lambda$ is a unit K-fixed vector then the function $\phi = \nu(e_\lambda)$ is a normalized eigenfunction of the Laplacian on the space $Y = X/K$ with the eigenvalue $\mu = \frac{1-\lambda^2}{4}$. This explains why λ is a natural parameter to describe Maass forms.

2.4. Triple products. We now introduce our main tool.

2.4.1. *Automorphic triple products.* Suppose we are given three X-enhanced representations of G

$$\nu_j : V_j \to C^\infty(X), \quad j = 1,\, 2,\, 3.$$

We define the G-invariant trilinear form $l^{\mathrm{aut}}_{\pi_1,\pi_2,\pi_3} : V_1 \otimes V_2 \otimes V_3 \to \mathbb{C}$ by formula

$$l^{\mathrm{aut}}_{\pi_1,\pi_2,\pi_3}(v_1 \otimes v_2 \otimes v_3) = \int_X \phi_{v_1}(x)\phi_{v_2}(x)\phi_{v_3}(x)\, d\mu_X, \tag{8}$$

where $\phi_{v_j} = \nu_j(v_j) \in C^\infty(X)$ for $v_j \in V_j$.

In particular, the triple periods c_i in (3) can be expressed in terms of this form as $c_i = l^{\mathrm{aut}}_{\pi,\pi',\pi_i}(e_\tau \otimes e_{\tau'} \otimes e_{\lambda_i})$, where $e_\lambda \in V_\lambda$ is the K-fixed unit vector.

2.4.2. *Uniqueness of triple products.* The central fact about invariant trilinear functionals is the following uniqueness result:

Theorem. *Let* $(\pi_j,\, V_j)$, $j = 1,\, 2,\, 3$, *be three irreducible smooth admissible representations of* G. *Then* $\dim \mathrm{Hom}_G(V_1 \otimes V_2 \otimes V_3,\, \mathbb{C}) \leq 1$.

Remark. The uniqueness statement was proven by Oksak in [O] for the group $\mathrm{SL}(2,\,\mathbb{C})$ and the proof could be adopted for $\mathrm{PGL}_2(\mathbb{R})$ as well (see also [Mo] and [Lo]). For the p-adic GL(2) more refined results were obtained by Prasad (see [P]). He also proved the uniqueness when at least one representation is a discrete series representation of $\mathrm{GL}_2(\mathbb{R})$.

There is no uniqueness of trilinear functionals for representations of $\mathrm{SL}_2(\mathbb{R})$ (the space is two-dimensional). This is the reason why we prefer to work with $\mathrm{PGL}_2(\mathbb{R})$.

For $\mathrm{SL}_2(\mathbb{R})$ one has the following uniqueness statement instead. Let $(\pi,\, V)$ and $(\sigma,\, W)$ be two irreducible smooth pre-unitary representations of $\mathrm{SL}_2(\mathbb{R})$ of class one. Then the space of $\mathrm{SL}_2(\mathbb{R})$-invariant trilinear functionals on $V \otimes V \otimes W$ which are symmetric in the first two variables is one-dimensional. This is the correct uniqueness result needed if one wants to work with $\mathrm{SL}_2(\mathbb{R})$; this was implicitly done in [Re2], where the second author missed the absence of the uniqueness for $\mathrm{SL}_2(\mathbb{R})$. We take an opportunity to correct this gap.

We note however, that the absence of uniqueness does not pose any problem for the method we present. All that is really needed for our method is the fact that the space of invariant functionals is finite dimensional.

3. Triple Products: Exponential Decay

We now explain our method how to bound coefficients c_i. It is based on the uniqueness of trilinear functionals.

3.1. Model triple products. Let (π, V) and (π', V') be automorphic representations corresponding to Maass forms ϕ and ϕ'. Any Maass form ϕ_i gives us an automorphic representation (π_i, V_{λ_i}) and hence defines a trilinear functional

$$l^{\mathrm{aut}}_{\pi,\pi',\pi_i}\colon V \otimes V' \otimes V_{\lambda_i} \to \mathbb{C}.$$

In Section 5.1 we use an explicit model for representations π_1, π_2, π_3 to construct a model invariant trilinear functional which is given by an explicit formula. We call it the *model triple product* and denote it by $l^{\mathrm{mod}}_{\pi_1,\pi_2,\pi_3}$.

By the uniqueness principle for representations π, π', π_i there exists a constant $a_i = a_{\pi,\pi',\pi_i}$ such that:

$$l^{\mathrm{aut}}_{\pi,\pi',\pi_i} = a_i \cdot l^{\mathrm{mod}}_{\pi,\pi',\pi_i}. \tag{9}$$

3.2. Exponential decay. This gives a formula for the triple products c_i

$$c_i = l^{\mathrm{aut}}_{\lambda_i}(e_\tau \otimes e_{\tau'} \otimes e_{\lambda_i}) = a_i \cdot l^{\mathrm{mod}}_{\lambda_i}(e_\tau \otimes e_{\tau'} \otimes e_{\lambda_i}). \tag{10}$$

Here we denoted $l^{\mathrm{aut}}_{\lambda_i} = l^{\mathrm{aut}}_{\pi,\pi',\pi_i}$, $l^{\mathrm{mod}}_{\lambda_i} = l^{\mathrm{mod}}_{\pi,\pi',\pi_i}$ and e_λ is the unit K-fixed vector in the representation V_λ.

The model triple product $l^{\mathrm{mod}}_{\lambda_i}(e_\tau \otimes e_{\tau'} \otimes e_{\lambda_i})$ constructed in Section 5.1 is given by an explicit integral. In Appendix A we evaluate this integral by a direct computation in the model. It turns out that it has an exponential decay in $|\lambda|$ which explains the exponential decay of coefficients c_i. Namely, we prove the following

Proposition. *Set* $k_\lambda := |l^{\mathrm{mod}}_\lambda(e_\tau \otimes e_{\tau'} \otimes e_\lambda)|^2$. *Then there exists a constant* $c > 0$ *such that*

$$k_\lambda = c\exp\left(-\frac{\pi}{2}|\lambda|\right) \cdot |\lambda|^{-2}(1 + O(|\lambda|^{-1}))$$

as $|\lambda| \to \infty$ *and* $\lambda \in i\mathbb{R}$.

4. Triple Products: Polynomial Bounds

We explain now how to obtain bounds on the coefficients a_i (note that these coefficients encode deep arithmetic information — values of L-functions).

Our method is based on the fact that these coefficients appear in the spectral decomposition of some geometrically defined Hermitian form on the space E which is essentially the tensor product of spaces V and V'.

More precisely, denote by L and L' the Hilbert completions of spaces V and V', consider the unitary representation $(\Pi, G \times G, L \otimes L')$ of the group $G \times G$ and denote by E its smooth part; so E is a smooth completion of $V \otimes V'$.

Denote by $\mathcal{H}(E)$ the (real) vector space of continuous Hermitian forms on E and by $\mathcal{H}^+(E)$ the cone of nonnegative Hermitian forms.

We will describe several classes of Hermitian forms on E; some of them have spectral description, others are described geometrically.

4.1. Hermitian forms corresponding to trilinear functionals. Let W be a smooth pre-unitary admissible representation of G. Any G-invariant functional $l\colon V \otimes V' \otimes W \to \mathbb{C}$ defines a G-intertwining morphism $T^l\colon V \otimes V' \to W^*$ which extends to a G-morphism

$$T^l\colon E \to \bar{W}, \tag{11}$$

where we identify the complex conjugate space $\bar{W}$ with the smooth part of the space W^*.

The standard Hermitian form (scalar product) P_W on the space W induces the Hermitian form $\bar{P}$ on $\bar{W}$. Using the operator T^l we define the Hermitian form H^l on the space E by $H^l = (T^l)^*(\bar{P})$, i.e., $H^l(u) = \bar{P}(T^l(u))$ for $u \in E$.

We note that if the representation of G in the space W is irreducible then starting with the Hermitian form H^l we can reconstruct the space W, the functional l and the morphism T^l uniquely up to an isomorphism.

Let us introduce a special notation for the particular case we are interested in. For any number $\lambda \in i\mathbb{R}$ consider the representation of the principal series $W = V_\lambda$, choose the model trilinear functional $l^{\mathrm{mod}} \colon V \otimes V' \otimes V_\lambda \to \mathbb{C}$ described in Section 5.1 and denote the corresponding Hermitian form on E by H^{mod}_λ.

4.2. Diagonal form H_Δ. Consider the space $C^\infty(X \times X)$. The diagonal $\Delta \colon X \to X \times X$ gives rise to the restriction morphism $r_\Delta \colon C^\infty(X \times X) \to C^\infty(X)$. We define a nonnegative Hermitian form H_Δ on $C^\infty(X \times X)$ by $H_\Delta = (r_\Delta)^*(P_X)$, i.e.,

$$H_\Delta(u) = P_X(r_\Delta(u)) = \int_X |r_\Delta(u)|^2 \, d\mu_X \quad \text{for } u \in C^\infty(X \times X).$$

We call H_Δ the diagonal form.

More generally, if L is a closed subspace of $L^2(X)$ and $pr_L \colon L^2(X) \to L$ the orthogonal projection onto L, we can define a Hermitian form P_L on $C^\infty(X)$ by $P_L = (pr_L)^*(P_X)$ and consider the induced Hermitian form $H_L = (r_\Delta)^*(P_L)$ on $C^\infty(X \times X)$.

Clearly the correspondence $L \mapsto H_L$ is additive (which means that $H_{L+L'} = H_L + H_{L'}$ if L and L' are orthogonal) and monotone.

4.3. First basic inequality. Let us realize the space $E = V \otimes V'$ as a $G \times G$-invariant subspace of $C^\infty(X \times X)$. We consider the restrictions of the Hermitian forms H_Δ, H_L discussed above to the space E and will denote them by the same symbols.

Claim. *Let ϕ_{λ_i} be a Maass form. Consider the G-invariant subspace $L_i \subset L^2(X)$ generated by ϕ_{λ_i} and its complex conjugate $\bar{L}_i \subset L^2(X)$.*

Then on the space E the Hermitian form $H_{\bar{L}_i}$ coincides with the form $H^{\mathrm{aut}}_{\lambda_i}$ corresponding to the automorphic trilinear form $l = l^{\mathrm{aut}}_{\pi,\pi',\pi_i} \colon V \otimes V' \otimes V_{\lambda_i} \to \mathbb{C}$.

Indeed, if we identify the space $\bar{L}_i$ with L_i^*, then the operator $pr_{\bar{L}_i} \circ r_\Delta \colon E \to \bar{L}_i$ coincides with the operator T^l corresponding to the automorphic trilinear form $l = l^{\mathrm{aut}}_{\pi,\pi',\pi_i}$.

This claim implies the first basic inequality

$$\sum_{\lambda_i} |a_i|^2 H^{\mathrm{mod}}_{\lambda_i} \le H_\Delta. \tag{12}$$

Indeed, by the uniqueness principle (9) we have:

$$H^{\mathrm{aut}}_{\lambda_i} = |a_i|^2 \cdot H^{\mathrm{mod}}_{\lambda_i}, \tag{13}$$

where $a_i = a_{\pi,\pi',\pi_i}$ are as in (9).

Since all the spaces $\bar{L}_i$ are orthogonal we have $\sum_i H^{\mathrm{aut}}_{\lambda_i} \le H_\Delta$ which proves the first basic inequality.

4.4. Second basic inequality. We would like to use the inequality (12) to bound the coefficients a_i. In order to do this we have to establish some bounds for the diagonal form H_Δ.

The group $G \times G$ naturally acts on the space of Hermitian forms on $C^\infty(X \times X)$; we denote this action by Π. We extend this action to the action of the algebra $H(G \times G) = C^\infty_c(G \times G, \mathbb{R})$ of smooth real valued functions with compact support. Note that if $h \in H(G \times G)$ is a nonnegative function then the operator $\Pi(h)$ preserves the cone of positive forms.

We have then the second basic inequality

Claim. *Let $h \in H(G \times G)$ be a non-negative function. Then there exists a constant C, depending on h, such that we have $\Pi(h)H_\Delta \le C \cdot P_{X\times X}$, where $P_{X\times X}$ is the standard L^2 Hermitian form on the space $C^\infty(X \times X)$.*

Proof. Let $u \in C^\infty(X \times X)$ and $f = |u|^2$. Then $P_{X\times X}(u) = \langle \mu, f\rangle$ and $\Pi(h)H_\Delta(u) = \langle \mu', f\rangle$, where $\mu = \mu_{X\times X}$ and $\mu' = \Pi(h)(\Delta_*(\mu_X))$ are two measures on $X \times X$.

Since the measure μ' is smooth it is bounded by $C\mu$. □

Note that the bound in the claim is essentially tight. Namely if the function h has large enough support, then we also have a bound in the opposite direction.

4.5. Positive functionals. We can now prove that the coefficients a_i have at most polynomial growth in $|\lambda_i|$.

We start with the inequality (12) of *non-negative forms*. We want to produce out of it an inequality for coefficients a_i. There is a standard way to do this by means of positive functionals on the space of Hermitian forms $\mathcal{H}(E)$.

Definition. A *positive functional* on $\mathcal{H}(E)$ is an additive map $\rho\colon \mathcal{H}(E)^+ \to \mathbb{R}^+ \cup \infty$.

It is easy to see that the positive functional ρ is automatically monotone and homogeneous, i.e., $\rho(H) \le \rho(H')$ if $H \le H'$ and $\rho(aH) = a\rho(H)$ for $a > 0$.

Example. Any vector $u \in E$ gives us an elementary positive functional ρ_u defined by $\rho_u(H) = H(u)$.

Fix a positive functional ρ and consider the weight function $h(\lambda) = \rho(H^{\mathrm{mod}}_\lambda)$. Then from the first basic inequality (12) we can deduce the following inequality for a weighted sum of coefficients $|a_i|^2$:

$$\sum_i h(\lambda_i)|a_i|^2 \le \rho(H_\Delta).$$

4.6. Test functional ρ_T. For any real T we construct in Section 5.2 the positive "test" functional ρ_T on $\mathcal{H}(E)$ with the properties described in the proposition below. Let us fix automorphic representations $V, V' \subset C^\infty(X)$, $E = V \otimes V' \subset C^\infty(X \times X)$ as above.

Proposition. *We can find a constant C which depends only on G and Γ and a constant T_0 which depends on V and V' such that for any $T \geq T_0$ there exists a positive functional ρ_T on $\mathcal{H}(E)$ satisfying*

$$\rho_T(H_\Delta) \leq CT^2, \tag{14}$$

$$h_T(\lambda) := \rho_T(H_\lambda^{\mathrm{mod}}) \geq 1 \quad \text{for any} \quad |\lambda| \leq 2T. \tag{15}$$

4.7. Proof of Theorem 1.4. Consider the inequality $\sum_i |a_i|^2 \rho_T(H_{\lambda_i}^{\mathrm{mod}}) \leq \rho_T(H_\Delta)$.

The right hand side $\rho_T(H_\Delta)$ is bounded by CT^2. In the left hand side we can leave only terms with $|\lambda_i| \leq 2T$. Thus we arrive at inequality

$$\sum_{|\lambda_i| \leq 2T} |a_i|^2 \leq CT^2. \tag{16}$$

This gives the desired bound for $\sum |a_i|^2$.

According to Proposition 3.2 there exists a constant B such that $b_i T^2 \leq B|a_i|^2$ for $T \leq |\lambda_i| \leq 2T$. This shows that $\sum_{T < |\lambda_i| < 2T} b_i \leq A$ for some constant A, which finishes the proof of Theorem 1.4.

4.7.1. *A conjecture.* One can show (see [Re1]) that the mean-value result in (16) is essentially sharp. One expects that for $T \leq |\lambda_i| \leq 2T$ all terms in the sum (16) are at most of order T^ε for any $\varepsilon > 0$. Hence, we have established a sharp bound on the average and a rather weak bound for each term. This is a typical situation which one often encounters in the analytic theory of L-functions, the so-called convexity bound. The major problem is thus to find a method which would allow us to obtain a better bound for a single term or for a short interval — the so-called subconvexity bounds.

We would like to make the following conjecture concerning the size of coefficients a_{π,π',π_i} which is equivalent to Conjecture 1.5:

Conjecture. *For fixed π, π' and for any $\varepsilon > 0$ there exists $C_\varepsilon > 0$ independent of λ_i such that*

$$|a_{\pi,\pi',\pi_{\lambda_i}}| \leq C_\varepsilon |\lambda_i|^\varepsilon,$$

as $|\lambda_i| \to \infty$.

5. Construction of Model Trilinear Functionals and of Test Functionals

5.1. Model trilinear functionals. For every $\lambda \in \mathbb{C}$ we denote by (π_λ, V_λ) the smooth class one representation of the generalized principal series of the group $G = \mathrm{PGL}_2(\mathbb{R})$ described in Section 2.3. We will use the realization of (π_λ, V_λ) in the space of smooth homogeneous functions on $\mathbb{R}^2 \setminus 0$ of homogeneous degree $\lambda - 1$.

For explicit computations it is often convenient to pass from plane model to a circle model. Namely, the restriction of functions in V_λ to the unit circle $S^1 \subset \mathbb{R}^2$ defines an isomorphism of the space V_λ with the space $C^\infty(S^1)^{\mathrm{even}}$ of even smooth functions on S^1 so we can think about vectors in V_λ as functions on S^1.

In this section we describe the *model* invariant trilinear functional using the geometric models. Namely, for three given complex numbers λ_j, $j = 1, 2, 3$, we construct explicitly a nontrivial trilinear functional $l^{\mathrm{mod}}\colon V_{\lambda_1}\otimes V_{\lambda_2}\otimes V_{\lambda_3}\to\mathbb{C}$ by means of its kernel.

5.1.1. *Kernel of* l^{mod}. Let $\omega(\xi,\,\eta)=\xi_1\eta_2-\xi_2\eta_1$ be the $\mathrm{SL}_2(\mathbb{R})$-invariant of a pair of vectors $\xi,\,\eta\in\mathbb{R}^2$. We set

$$K_{\lambda_1,\lambda_2,\lambda_3}(s_1,\,s_2,\,s_3)=|\omega(s_2,\,s_3)|^{(\alpha-1)/2}\;|\omega(s_1,\,s_3)|^{(\beta-1)/2}|\omega(s_1,\,s_2)|^{(\gamma-1)/2}\tag{17}$$

for $s_1,\,s_2,\,s_3\in\mathbb{R}^2\setminus 0$, where $\alpha=\lambda_1-\lambda_2-\lambda_3$, $\beta=-\lambda_1+\lambda_2-\lambda_3$, $\gamma=-\lambda_1-\lambda_2+\lambda_3$.

The kernel function $K_{\lambda_1,\lambda_2,\lambda_3}(s_1,\,s_2,\,s_3)$ satisfies two main properties:

(1) K is invariant with respect to the diagonal action of $\mathrm{SL}_2(\mathbb{R})$;
(2) K is homogeneous of degree $-1-\lambda_j$ in each variable s_j.

Hence if f_j are homogeneous functions of degree $-1+\lambda_j$, then the function

$$F(s_1,\,s_2,\,s_3)=f_1(s_1)f_2(s_2)f_3(s_3)K_{\lambda_1,\lambda_2,\lambda_3}(s_1,\,s_2,\,s_3),$$

is homogeneous of degree -2 in each variable $s_j\in\mathbb{R}^2\setminus 0$.

5.1.2. *Functional* l^{mod}. To define the model trilinear functional l^{mod} we notice that on the space $\mathcal{V}$ of functions of homogeneous degree -2 on $\mathbb{R}^2\setminus 0$ there exists a natural $\mathrm{SL}_2(\mathbb{R})$-invariant functional $\mathfrak{L}\colon\mathcal{V}\to\mathbb{C}$. It is given by the formula $\mathfrak{L}(f)=\int_\Sigma f\,d\sigma$ where the integral is taken over any closed curve $\Sigma\subset\mathbb{R}^2\setminus 0$ which goes around 0, and the measure $d\sigma$ on Σ is given by the area element inside of Σ divided by π; this last normalization factor is chosen so that $\mathfrak{L}(Q^{-1})=1$ for the standard quadratic form Q on $\mathbb{R}^2$.

Applying $\mathfrak{L}$ separately to each variable $s_i\in\mathbb{R}^2\setminus 0$ of the function $F(s_1,\,s_2,\,s_3)$ above we obtain the G-invariant functional

$$l^{\mathrm{mod}}_{\pi_1,\pi_2,\pi_3}(f_1\otimes f_2\otimes f_3):=\langle\mathfrak{L}\otimes\mathfrak{L}\otimes\mathfrak{L},\,F\rangle.\tag{18}$$

In the circle model this functional is expressed by the following integral:

$$l^{\mathrm{mod}}_{\pi_1,\pi_2,\pi_3}(f_1\otimes f_2\otimes f_3)=(2\pi)^{-3}\iiint f_1(x)f_2(y)f_3(z)K_{\lambda_1,\lambda_2,\lambda_3}(x,\,y,\,z)\,dx\,dy\,dz,\tag{19}$$

where $x,\,y,\,z$ are the standard angular parameters on the circle.

Remark. The integral defining the trilinear functional is often divergent and the functional should be defined using a regularization of this integral. There are standard procedures how to make such a regularization (see, e. g., [GS]).

Fortunately in the case of unitary representations, all integrals converge absolutely so we need not discuss the regularization procedure.

5.2. Construction of test functionals. In this section we will present a construction of a family of test functionals ρ_T on the space $\mathcal{H}(E)$.

Fix smooth irreducible pre-unitary representations of class one, $V=V_\tau$, $V=V_{\tau'}$ and denote by E the smooth completion of $V\otimes V'$ as in Section 4. We will do the computations only for representations of the principal series; complementary series are treated similarly.

For computations we will identify the spaces V and V' with $C^\infty(S^1)^{\mathrm{even}}$.

Our aim is to prove the following

Proposition. *There exist constants* T_0*,* C*,* $c > 0$ *such that for any* $T \geq T_0$ *there exists a positive functional* ρ *on* $\mathcal{H}(E)$ *satisfying*

$$\rho(H_\Delta) \leq CT^2, \tag{20}$$

$$h_T(\lambda) := \rho(H_\lambda^{\mathrm{mod}}) \geq c \quad \text{for any} \quad |\lambda| \leq 2T. \tag{21}$$

The functional $\rho_T = c^{-1}\rho$ is the required test functional in Section 4.6.

5.3. Proof of Proposition 5.2. We will construct a functional ρ as an integral of elementary functionals. Namely, we find a positive function $h \in H(G \times G) \subset C^\infty(G \times G)$ and a vector $u \in E$ and define $\rho(H) = \rho_u(\Pi(h)(H))$, where ρ_u is the elementary functional on the space $\mathcal{H}(E)$ corresponding to the vector u.

5.3.1. *Construction of function* h*.* We construct the function h independent of parameter T. Let $D_1 \subset \mathrm{SL}_2(\mathbb{R}) \subset G$ be the subset of matrices g with $\|g\| \leq 2$. We consider the subset $D = D_1 \times D_1 \subset G \times G$ and choose a positive function $h \in H(G \times G) = C_c^\infty(G \times G)$ which is ≥ 1 on the subset D and is supported in some neighborhood of D. We also assume that the function h is invariant under left and right translations by elements of the maximal compact subgroup $K \times K$.

5.3.2. *Construction of vector* u*.* Let us identify the space $E = V \otimes V'$ with the space of smooth functions $C^\infty(S^1 \times S^1)^{\mathrm{even}}$. Let S be a disc in $S^1 \times S^1$ of radius $(100T)^{-1}$. We construct u as a smooth non-negative real valued function on $S^1 \times S^1$ supported in S such that

(i) $\int u\,dx\,dy = 1$,

(ii) $\|u\|_{L^2}^2 \leq 10^5 T^2$.

We would like to show that the functional ρ constructed in Section 5.3 satisfies conditions formulated in Proposition 5.2.

5.3.3. *Geometric bound.* We have

$$\rho(H_\Delta) = \rho_u(\Pi(h)(H_\Delta)) \leq C'\rho_u(P_E) = C'P_E(u) \leq CT^2,$$

see Claim 4.4 and Section 5.3.2.

5.3.4. *Spectral bound.* First we would like to give another description of the Hermitian form H_λ^{mod}. Consider the model trilinear functional $l = l_{\pi,\pi',\pi_i}^{\mathrm{mod}}$ described in Section 5.1 and the corresponding operator $T^l \colon E \to \bar{V}_\lambda$.

We will identify the space $\bar{V}_\lambda$ with the space $C^\infty(S^1)^{\mathrm{even}}$. Fix a point $z \in S^1$. Consider the functional δ_z on $\bar{V}_\lambda$ given by evaluation at the point z and the corresponding rank-one Hermitian form $P_z(v) = |\delta_z(v)|^2$. Define a functional f_z on E by $f_z(u) = \delta_z(T^l(u))$ and consider the corresponding rank-one Hermitian form $P_f(u) = |f_z(u)|^2$.

We claim that $H_\lambda^{\mathrm{mod}} = \int_K (\Pi(k, k)(P_f))\,dk$. This immediately follows from the fact that the standard invariant Hermitian form $P_{\bar{V}_\lambda}$ on $\bar{V}_\lambda \simeq C^\infty(S^1)^{\mathrm{even}}$ is equal to the integral $P_{\bar{V}_\lambda} = \int_K \pi(k)(P_z)\,dk$.

Since we assumed the function $h \in H(G \times G)$ to be $K \times K$-invariant we see that

$$\rho(H_\lambda^{\mathrm{mod}}) = \rho_u(\Pi(h)(P_f)).$$

Thus we see that in order to prove a lower bound for $\rho(H_\lambda^{\mathrm{mod}})$ it is enough to establish a lower bound for $\rho_u(\Pi(g)P_f) := |\langle \Pi(g)f,\, u\rangle|^2$, for a subset of $g \in D$ of a measure bounded from below by a constant.

The desired lower bound follows from the following

Lemma. *Let T_0 be large enough. Then there exists an open non-empty subset $D_0 \subset D$ such that for $T \geq T_0$ and for $g \in D_0$ we have $|\langle \Pi(g)f,\, u\rangle| \geq 1/2$.*

Proof. As before we identify the space $E = V \otimes V'$ with the space $C^\infty(S^1 \times S^1)^{\mathrm{even}}$. Denote by $(x,\, y)$ coordinates on $S^1 \times S^1$. It follows from definition (19) that the functional $f = f_z$ corresponding to the point z is given by the function $f = f(x,\, y)$ on $S^1 \times S^1$ described by

$$f(x,\, y) = |\sin(y - z)|^{(\alpha-1)/2}\, |\sin(x - z)|^{(\beta-1)/2}\, |\sin(x - y)|^{(\gamma-1)/2},$$

where $\alpha,\, \beta,\, \gamma \in i\mathbb{R}$.

Let $D_0 \subset D$ be the subset of elements $g \in D$ such that the restriction of $\Pi(g)(f)$ to the subset $S \subset S^1 \times S^1$ has the absolute value ≤ 10. (Note that the absolute value $|\Pi(g)(f)|$ is bounded from *below* for any $g \in D$ by a constant depending only on D.) It is easy to see that, for large T, the set D_0 is a non empty subset of $D \subset G \times G$ of a measure bounded from below by a constant which is independent of T.

On the other hand, for $g \in D_0$ we see that the gradient of the function $\Pi(g)(f)$ on the subset S is bounded by $3T$. We note now that the diameter of S is bounded by $(100T)^{-1}$ and hence the lower bound on $|\langle \Pi(g)f,\, u\rangle|$ for $g \in D_0$ is a direct consequence of the following easy

Claim. *Let S be a set with a measure ν and u, h be two measurable functions on S. Let us assume that*

(i) *u is real valued positive function and $\int u\, d\nu = 1$.*

(ii) *$\sup |h(s)| \geq 1$ and the variation $\mathrm{Var}(h) := \sup |h(s) - h(s')|$ is bounded by $1/2$.*

Then $\left|\int hu\, d\nu\right| \geq 1/2$.

The lemma is proved. □

5.4. Construction of test functionals via Sobolev norms. In this section we outline another, slightly more conceptual, construction of test functionals. This construction uses the notion of Sobolev norms on representation spaces (see [BR2]).

5.4.1. *Sobolev norms.* Let G be a Lie group and $(\pi,\, G,\, V)$ a smooth pre-unitary representation. Then we can construct a family of positive definite Hermitian forms on the space V as follows.

Fix a basis $\{X_j \colon j = 1,\, \ldots,\, r\}$ of the Lie algebra $\mathfrak{g}$ of the group G. Then for any natural number l and any $T > 0$ we define a Hermitian form $Q_{l,T}$ on V by

$$Q_{l,T}(v) = \sum_\nu T^{2(l-|\nu|)} P(X^\nu(v)).$$

Here the sum is over all multi indexes $\nu = (n_1,\, \ldots,\, n_r)$ with the norm $|\nu| := \sum n_j$ bounded by l, and $P = P_V$ is the Hermitian form defining the unitary structure on V.

5.4.2. *Positive functionals defined by forms.* Every positive definite Hermitian form Q on V defines a positive functional ρ_Q on $\mathcal{H}(V)$ by $\rho_Q(H) = \mathrm{tr}(H|Q)$. Here $\mathrm{tr}(H|Q)$ denotes the relative trace of forms H and Q; by definition it is equal to the square of the Hilbert–Schmidt norm of the identity operator on V considered as a morphism of pre-Hilbert spaces $(V, Q) \to (V, H)$. This notion is discussed in detail in [BR2].

5.4.3. *Construction of Sobolev test functionals.* Let us apply these constructions to the representation $(\Pi, G \times G, E)$ discussed in Section 4.

Fix l and T, consider the Sobolev Hermitian form $Q = Q_{l,T}$ on the space E and define the positive functional ρ on $\mathcal{H}(E)$ to be $\rho = \rho_Q$.

Proposition. *Suppose $l \geq 2$. Then*
(i) $\rho(H_\Delta) \leq CT^{2-2l}$,
(ii) *There exists $c > 0$ such that $\rho(H^{\mathrm{mod}}_\lambda) \geq cT^{-2l}$ for $|\lambda| \leq 2T$.*

This gives another proof of Proposition 4.6.

5.4.4. *Sketch of the proof of Proposition 5.4.3.* (i) Since the representation Π is continuous with respect to the form $Q_{l,T}$ the second basic inequality (Claim 4.4) implies that $\rho(H_\Delta) \leq C'\rho(P_E)$. The proof of the inequality $\rho(P_E) \leq C''T^{2-2l}$ is the same as in [BR2, Section 4].

In order to prove (ii) it is enough to find a vector $u \in E$ such that $Q_{l,T}(u) \leq T^{2l}$ and $|\langle f, u\rangle| \geq c$, where $f = f_z$ is the function described in Section 5.3.4. We can take a function $u \in C^\infty(S^1 \times S^1)$ of the form $u = \phi f$ where ϕ is a smooth cut-off function which equals 0 around singularities of the function f.

We leave details to the reader.

Appendix A.

In this appendix we prove the Proposition 3.2 which describes the asymptotic behavior of the function k_λ.

A.1. Computation of l^{mod} for K-fixed vectors. One can prove this proposition applying the stationary phase method directly to the integral (19). To do this we need to consider the complexification of the functions $e_\lambda(s_i)$ and the function $K_{\lambda_1,\lambda_2,\lambda_3}(s_1, s_2, s_3)$ in the variables s_i and move contour of integration towards the singularities of the complexified integral. This could be done either in a classical language or using analytic continuation of representations in the spirit of [BR1].

A.2. Computation of the integral. We prefer to prove this proposition in a different way. Namely we explicitly compute the value of the model functional on the unit vectors in terms of Γ- functions and then prove the proposition by applying the Stirling formulas for the asymptotic behavior of Γ-functions.

Let π_{λ_i}, $i = 1, 2, 3$, be three representations of the generalized principal series, and e_{λ_i} be the corresponding K-fixed unit vectors (they correspond to function 1 in the circle model). Set $A(\lambda_1, \lambda_2, \lambda_3) := l^{\mathrm{mod}}_{\pi_{\lambda_1},\pi_{\lambda_2},\pi_{\lambda_3}}(e_{\lambda_1} \otimes e_{\lambda_2} \otimes e_{\lambda_3})$.

In Sections A.4, A.5 we explicitly compute the function $A(\lambda_1, \lambda_2, \lambda_3)$ (see the final expression in Section A.5).

A.3. Gaussian. We would like to compute our integral by comparing it with Gaussian integrals which are much easier to manipulate.

Suppose we are given a finite-dimensional Euclidean vector space L. Then we introduce the Gaussian probability measure G on L by $dG = \pi^{-\dim L/2}\exp(-Q)\,dl$, where Q is the quadratic form which defines the Euclidean structure on L and dl is the standard Euclidean measure on L.

We are interested in the quantities $\langle f, G\rangle = \langle f, G\rangle_L := \int f\,dG$ for various (usually homogeneous) functions f on L. The main properties of the Gaussian which we use are the following:

(i) *Normalization.* $\langle 1, G\rangle = 1$.

(ii) *Product formula.* Suppose that the Euclidean space L is a product of Euclidean spaces L_1 and L_2. Then the Gaussian measure G on L is the product of Gaussian measures G_1 and G_2 on L_1 and L_2. In particular, if a function f decomposes as a product of functions f_1 and f_2 on L_1 and L_2 we have $\langle f, G\rangle = \langle f_1, G_1\rangle\langle f_2, G_2\rangle$.

The following integrals are classical

Proposition. *Let $L = \mathbb{R}^n$ be the standard Euclidean space.*

(i) *Let r denote the radius function on L. Then $\langle r^s, G\rangle = \Gamma((s+n)/2)/\Gamma(n/2)$.*

(ii) *Let h be a linear functional on L. Then $\langle |h|^s, G\rangle = \|h\|^s\Gamma((s+1)/2)/\Gamma(1/2)$.*

(iii) *Let L be the space $M_{2,2}$ of 2×2 matrices with the standard Euclidean structure. Then $\langle |\det|^s, G\rangle = \Gamma((s+1)/2)\Gamma(s/2+1)/\Gamma(1/2)$.*

Proof. In (i) passing to spherical coordinates we get the integral

$$2c\int r^{s+n-1}\exp(-r^2)\,dr = c\int u^{(s+n)/2}\exp(-u)\,du/u = c\Gamma((s+n)/2).$$

The normalization at $s = 0$ defines the constant $c = 1/\Gamma(n/2)$.

The proof of (ii) is reduced to the one variable case using the product formula and then it follows from (i).

In (iii) we can write L as a product of two column spaces L_1 and L_2. Then we have

$$\langle |\det|^s, G\rangle = \int |\omega(x, y)|^s\,dG_1(x)\,dG_2(y) = \int\left(\int |\omega(x, y)|^s\,dG_1(x)\right)dG_2(y)$$
$$= \frac{\Gamma((s+1)/2)}{\Gamma(1/2)}\cdot\int |y|^s\,dG_2(y) = \frac{\Gamma((s+1)/2)\,\Gamma(s/2+1)}{\Gamma(1/2)}$$

since $\Gamma(1) = 1$. □

A.4. Reduction 1. Proposition A.3 allows us to write the integrals which we would like to compute as some Gaussian integrals.

Corollary. *For any function $h \in V_{-\lambda}$ we have*

$$\langle h, G\rangle = \Gamma((1-\lambda)/2)\cdot\mathfrak{L}(h\cdot e_\lambda),$$

where the functional $\mathfrak{L}$ is defined in Section 5.1.2.

Indeed, after averaging h with respect to the action of $SO(2)$ we can assume that it is proportional to the function $e_{-\lambda}$. Then the formula follows from Proposition A.3(i).

Using this corollary we can rewrite the integral for the function $A(\lambda_1, \lambda_2, \lambda_3)$.

Proposition. *Consider the Euclidean space* $L=\mathbb{R}^2\times\mathbb{R}^2\times\mathbb{R}^2$ *and define the function* $B(\lambda_1, \lambda_2, \lambda_3)$ *by the Gaussian integral* $B(\lambda_1, \lambda_2, \lambda_3) := \langle K_{\lambda_1,\lambda_2,\lambda_3}(s_1, s_2, s_3), G\rangle$. *Then*

$$B(\lambda_1, \lambda_2, \lambda_3) = A(\lambda_1, \lambda_2, \lambda_3) \cdot \Gamma((1-\lambda_1)/2)\,\Gamma((1-\lambda_2)/2)\,\Gamma((1-\lambda_3)/2).$$

A.5. Reduction 2. Let us rewrite the integral defining the function B. First, we identify the Euclidean space L in Section A.4 with the space $M_{2,3}$ of 2×3 matrices. We consider the Euclidean space $W \approx \mathbb{R}^3$ and define the map $\nu\colon M_{2,3} \to W$ using 2×2 minors. Let us define the function f on W by the formula

$$f(w_1, w_2, w_3) = |w_1|^{(\alpha-1)/2}|w_2|^{(\beta-1)/2}|w_3|^{(\gamma-1)/2}.$$

We can write $K_{\lambda_1,\lambda_2,\lambda_3} = \nu^*(f)$ and hence $B(\lambda_1, \lambda_2, \lambda_3) = \langle \nu^*(f), G\rangle$ (here $\alpha = \lambda_1-\lambda_2-\lambda_3$, $\beta = -\lambda_1+\lambda_2-\lambda_3$, $\gamma = -\lambda_1-\lambda_2+\lambda_3$ as in Section 5.1.1).

Now we will use the following general lemma which we prove in Section A.6.

Lemma. *Let h be a homogeneous function on the space W of homogeneous degree s. Then*

$$\langle \nu^*(h), G\rangle_L = \langle h, G\rangle_W \cdot \Gamma(s/2+1).$$

From this lemma we see that the computation of the function $B(\lambda_1, \lambda_2, \lambda_3)$ is reduced to the computation of the function $C(\alpha, \beta, \gamma) := \langle f, G\rangle$.

Since the Gaussian G on W is a direct product of three one-dimensional Gaussians and the function f is a product of functions depending only on one coordinate, we deduce that the integral $\langle f, G\rangle$ is a product of three one-dimensional integrals which can be computed using Proposition A.3.

Thus we obtain $C(\alpha, \beta, \gamma) = \Gamma((\alpha+1)/4)\Gamma((\beta+1)/4)\Gamma((\gamma+1)/4)/\Gamma(1/2)^3$.

The final expression for the function $A(\lambda_1, \lambda_2, \lambda_3)$ is

$$A(\lambda_1, \lambda_2, \lambda_3) = \frac{\Gamma((\alpha+1)/4)\,\Gamma((\beta+1)/4)\,\Gamma((\gamma+1)/4)\,\Gamma((\delta+1)/4)}{\Gamma(1/2)^3\,\Gamma((1-\lambda_1)/2)\,\Gamma((1-\lambda_2)/2)\,\Gamma((1-\lambda_3)/2)},$$

where $\alpha = \lambda_1-\lambda_2-\lambda_3$, $\beta = -\lambda_1+\lambda_2-\lambda_3$, $\gamma = -\lambda_1-\lambda_2+\lambda_3$, $\delta = -\lambda_1-\lambda_2-\lambda_3$.

A.6. Proof of Lemma A.5. Consider the natural actions of the group $SO(3)$ on the Euclidean spaces $M_{2,3} \approx W\times W$ and W; these actions preserve Gaussian measures.

The map $\nu\colon M_{2,3} \to W$ is $SO(3)$-equivariant; it is nothing else than the exterior product map $W\times W \to \bigwedge^2(W) = W^* = W$. Hence we can replace the function h by its average with respect to the action of the group $SO(3)$, i.e., up to some constant by a function $h = r^s$. This shows that $\langle \nu^*(h), G\rangle = a(s)\langle h, G\rangle$, where $a(s)$ depends on s but not on h.

In order to compute the function $a(s)$ we can consider the identity above for the function $h(w) = |w_3|^s$. According to Proposition A.3(ii) we have $\langle h, G\rangle = \Gamma((s+1)/2)/\Gamma(1/2)$.

On the other hand it is clear that the function $\nu^*(h)$ depends only on four variables and hence the integral $\langle \nu^*(h), G\rangle$ coincides with the integral $\langle h', G\rangle$ over the space $M_{2,2}$ of 2×2 matrices, where $h'(m) = |\det(m)|^s$.

From Proposition A.3(iii) we deduce that $a(s) = \Gamma(s/2 + 1)$.

A.7. Proof of Proposition 3.2. According to the Stirling formulas, for any fixed σ and large t, we have $\Gamma(\sigma + it) = \sqrt{2\pi}\exp(-\frac{\pi}{2}|t|)|t|^{\sigma-1/2}(1 + O(|t|^{-1}))$.

This and the explicit formula for the function $A(\lambda_1, \lambda_2, \lambda_3)$ implies the proposition.

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