A GENERALIZATION OF CASSELMAN'S SUBMODULE THEOREM

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1.

Let $G_{\mathbb{R}}$ be a real reductive Lie group, $\mathfrak{g}_{\mathbb{R}}$ its Lie algebra. Let M be an irreducible Harish-Chandra module. Using some fine analytic arguments, based on the study of asymptotic behavior of matrix coefficients, Casselman has proved that M can be imbedded into a principal series representation [2,3].

This statement can be formulated purely algebraically. Let $\mathfrak{g}$ be the complexification of $\mathfrak{g}_{\mathbb{R}}$ and let $\mathfrak{n}_0$ be a maximal nilpotent subalgebra of $\mathfrak{g}$, containing a maximal nilpotent subalgebra of $\mathfrak{g}_{\mathbb{R}}$. Then Casselman's theorem claims that the space $M_{\mathfrak{n}_0} = M/\mathfrak{n}_0 M$ is not equal to zero.

We want to generalize this statement and to prove it by purely algebraic methods. (Note that the first algebraic proof of Casselman's theorem is due to O. Gabber. It is based on Gabber's theorem on the integrability of the characteristic variety.) First of all, we drop the condition that M is a Harish-Chandra module. As a result we can forget about $G_{\mathbb{R}}$ and $\mathfrak{g}_{\mathbb{R}}$ and consider any $\mathfrak{g}$-module M and any maximal nilpotent subalgebra $\mathfrak{n}_0 \subset \mathfrak{g}$. We suppose M to be finitely generated, and we want to prove that $M_{\mathfrak{n}_0} \neq 0$. Of course, this is not true for any given subalgebra $\mathfrak{n}_0$ (see example in [5] where $M_{\mathfrak{n}_0} = 0$ although M is finitely generated even as an $\mathfrak{n}_0$-module). But it turns out that $M_{\mathfrak{n}_0} \neq 0$ for "almost all" $\mathfrak{n}_0 \subset \mathfrak{g}$. The set of all maximal nilpotent subalgebras of $\mathfrak{g}$ has a natural structure of an algebraic variety - its it the flag variety of $\mathfrak{g}$ and "almost all" means "contains an open dense subset in the Zariski topology."

So our aim is

THEOREM 1. *Let $\mathfrak{g}$ be a reductive Lie algebra over an algebraically closed field k of characteristic 0 and X the flag variety of $\mathfrak{g}$. Let M be a no zero finitely generated $\mathfrak{g}$-module. Then for almost all $x \in X$ (i.e. for all points x in some open dense subset $U \subset X$) the space $M_{\mathfrak{n}_x} = M/\mathfrak{n}_x M$ is not equal to 0.*

Let us check that Theorem 1 implies Casselman's result. Indeed suppose M is a Harish-Chandra module, i.e. a finitely generated $(\mathfrak{g},K)$-module, where K is the complexification of a maximal compact subgroup of $G_{\mathbb{R}}$. Consider the natural action of K on X. If points x,y belong to the same K-orbit, the spaces $M_{\mathfrak{n}_x}$ and $M_{\mathfrak{n}_y}$ are isomorphic, so $\dim M_{\mathfrak{n}_x}$ is constant along K-orbits. Since $\mathfrak{n}_0$ contains a maximal unipotent subalgebra of $\mathfrak{g}_{\mathbb{R}}$, the Iwasawa decomposition for $G_{\mathbb{R}}$ implies that the K-orbit of the corresponding point $x_0 \in X$ is open X. Hence, Theorem 1 implies that $M_{\mathfrak{n}_0} \neq 0$.

REMARK. N. Wallach explained to me that for $(\mathfrak{g},K)$-modules one can drop the condition that M is finitely generated (see [5]).

2.

For any point $x \in X$ we denote by $\mathfrak{n}_x$ and $\mathfrak{b}_x$ the corresponding nilpotent and Borel subalgebras ($\mathfrak{b}_x$ is the normalizer of $\mathfrak{n}_x$) and put $\mathfrak{h}_x = \mathfrak{b}_x/\mathfrak{n}_x$. We denote by $R_x \subset \mathfrak{h}_x^*$ the root system $\mathfrak{h}_x$ in $\mathfrak{g}$ and by R_x^+ the set of roots of $\mathfrak{h}_x$ in $\mathfrak{g}/\mathfrak{b}_x$. This ordering differs from the usual one by sign, because we study coinvariants $M_{\mathfrak{n}}$ instead of invariants $M^{\mathfrak{n}}$.

Note that all triples $(\mathfrak{h}_x, R_x, R_x^+)$ are canonically isomorphic. We will identify all these triples with an abstract Cartan triple

$(\mathfrak{h},R,R^+)$. The half-sum of positive roots we denote by ρ and the Weyl group of R by W.

Let M be a finitely generated $\mathfrak{g}$-module. We can assume that M has an infinitesimal character θ (for instance, M has an irreducible quotient M^1 and it is sufficient to prove that $M^1_{\mathfrak{n}} \neq 0$). Moreover, if $M_{\mathfrak{n}} = 0$, then for any finite-dimensional $\mathfrak{g}$-module V, $(V \otimes M)_{\mathfrak{n}} = 0$. Hence, decomposing in the usual way $V \otimes M$ with respect to infinitesimal characters, we can assume that the character θ is nondegenerate, i.e. corresponding W-orbit in $\mathfrak{h}^*$ consists of $\#W$ elements.

For any $x \in X$ the module $M_{\mathfrak{n}_x}$ has a natural structure of an $\mathfrak{h}_x = \mathfrak{h}$ - module. By the Harish-Chandra theorem, $M_{\mathfrak{n}_x}$ can be decomposed as

$$M_{\mathfrak{n}_x} = \bigoplus_{\chi \in \hat{\theta}} M^{\chi}_{\mathfrak{n}_x} ,$$

where $M^{\chi}_{\mathfrak{n}_x}$ consists of vectors of weight $\chi - \rho$ in $M_{\mathfrak{n}_x}$ and $\hat{\theta}$ is the W-orbit corresponding to θ. Fix a dominant weight χ_0 on the orbit, i.e. $\chi_0(h_\gamma) \neq 0, -1, -2, \ldots$, for any $\gamma \in R^+$ (here $h_\gamma \in \mathfrak{h}$ is the dual root). Then any weight $\chi \in \hat{\theta}$ can be written uniquely as $\chi = w\chi_0$ with $w \in W$. We put $\ell(\chi) = \ell(w)$; this is a distance from χ to χ_0. Note that $\ell(\chi)$ depends on the choice of a dominant weight χ_0. If χ is nonintegral, this choice is not unique.

We will prove

THEOREM 2. Let M be a finitely generated $\mathfrak{g}$-module with a nondegenerate infinitesimal character θ. Then there exist a natural ℓ and a weight $\psi \in \hat{\theta}$ with $\ell(\psi) = \ell$ such that for almost all $\mathfrak{n}$

$$M^{\chi}_{\mathfrak{n}} = 0 \ \text{for} \ \ell(\chi) < \ell \ \text{and} \ M^{\psi}_{\mathfrak{n}} \neq 0 .$$

3.

Fix a weight χ and let us study all spaces $M^{\chi}_{\mathfrak{n}_x}$ simultaneously. The key point is to understand the word "simultaneously". Studying these spaces simultaneously and separately is the same as studying the space $\prod_{x \in X} M^{\chi}_{\mathfrak{n}_x}$ - for sure this is the wrong way. Our key tool will be an algebraic object $\Delta_{\chi}(M)$ which contains all information about all spaces $M^{\chi}_{\mathfrak{n}_x}$. Roughly speaking, we consider the space M^0 of functions on X with values in M and put $\Delta(M) = M^0/\mathfrak{n}^0 M^0$, where $\mathfrak{n}^0$ is the algebra of functions $x \to \xi_x \in \mathfrak{n}_x$. Since there are very few global functions on X (we consider only regular functions), we should consider sheaves instead spaces of functions.

Now let us give precise definitions. Let $\mathcal{O}_X$ be the structure sheaf of the algebraic variety X. Quasicoherent sheaves of $\mathcal{O}_X$ - modules we shall call simply $\mathcal{O}_X$-modules. Consider $\mathcal{O}_X$-modules

$$M^0 = \mathcal{O}_X \otimes_k M , \quad \mathfrak{g}^0 = \mathcal{O}_X \otimes_k \mathfrak{g} .$$

We shall consider sections of these sheaves as functions with values in M and $\mathfrak{g}$. Put

$$\mathfrak{n}^0 = \{f \in \mathfrak{g}^0 \mid f(x) \in \mathfrak{n}_x \text{ for all } x \in X\},$$
$$\mathfrak{h}^0 = \{f \in \mathfrak{g}^0 \mid f(x) \in \mathfrak{h}_x \text{ for all } x \in X\},$$
$$\Delta(M) = M^0/\mathfrak{n}^0 M^0.$$

It is clear that $\mathfrak{h}^0/\mathfrak{n}^0 = \mathcal{O}_X \otimes \mathfrak{h}$, so we have a natural imbedding $\mathfrak{h} \to \mathfrak{h}^0/\mathfrak{n}^0$, and hence an action of $\mathfrak{h}$ on $\mathcal{O}_X$-module $\Delta(M)$. We denote by $\Delta_{\chi}(M)$ the χ-component of $\Delta(M)$, i.e. the subsheaf of sections of weight $\chi - \rho$. The Harish-Chandra theorem implies that $\Delta(M) = \oplus \Delta_{\chi}(M)$, where $\chi \in \hat{\theta}$.

LEMMA. <u>The fiber of the $\mathcal{O}_X$-module $\Delta_{\chi}(M)$ at a point $x \in X$ is naturally isomorphic to $M^{\chi}_{\mathfrak{n}_x}$.</u>

Let us recall that the fiber of $\mathcal{O}_X$-module F at x is the linear space $F_x = F/m_x F$, where m_x is the maximal ideal of $\mathcal{O}_X$ consisting of functions f such that $f(x) = 0$. The proof is straightforward.

4.

The advantage of studying $\Delta_\chi(M)$ is that this sheaf has an additional structure - the structure of a $\mathfrak{g}$-module. Indeed, let us define actions of $\mathfrak{g}$ on M^0 and $\mathfrak{n}^0$ by the Leibnitz rule (we consider the adjoint action of $\mathfrak{g}$ on $\mathfrak{g}$ and the natural action of $\mathfrak{g}$ on $\mathcal{O}_X$). Since the subsheaves $\mathfrak{n}^0, \mathfrak{b}^0 \subset \mathfrak{g}^0$ are invariant under the action of the algebraic group G, corresponding to $\mathfrak{g}$, they are $\mathfrak{g}$-invariant. Hence $\Delta(M)$ is a $\mathfrak{g}$-module. Actions of $\mathfrak{g}$ and $\mathfrak{h}$ on $\Delta(M)$ commute, hence $\Delta_\chi(M)$ is also a $\mathfrak{g}$-module.

Let us describe more thoroughly operators acting on $\Delta_\chi(M)$. Consider the sheaf of algebras U^0 generated by $\mathfrak{g}$ and $\mathcal{O}_X$ with natural relations $[A,f] = A(f)$ for $A \in \mathfrak{g}$, $f \in \mathcal{O}_X$. As $\mathcal{O}_X$-module U^0 is isomorphic to $\mathcal{O}_X \otimes_k U(\mathfrak{g})$. Since $\mathfrak{n}^0$ is $\mathfrak{g}$-invariant, the ideal $\mathfrak{n}^0 U^0$ is two-sided, and we can put $\mathcal{D}_\mathfrak{h} = U^0/\mathfrak{n}^0 U^0$. The image of $\mathfrak{h}$ under the inclusion

$$\mathfrak{h} \to \mathfrak{b}^0/\mathfrak{n}^0 \to \mathcal{D}_\mathfrak{h}$$

belongs to the center of $\mathcal{D}_\mathfrak{h}$. Let us put

$$\mathcal{D}_\chi = \mathcal{D}_\mathfrak{h}/\{H - (\chi - \rho)(H) \mid H \in \mathfrak{h}\}\mathcal{D}_\mathfrak{h} .$$

It is clear that $\Delta_\chi(M)$ is a sheaf of $\mathcal{D}_\chi$- modules. We call a sheaf of $\mathcal{D}_\chi$-modules quasicoherent (or simply a "$\mathcal{D}_\chi$-module") if it is quasicoherent as an $\mathcal{O}_X$-module. The category of $\mathcal{D}_\chi$-modules we denote by $M(\mathcal{D}_\chi)$.

Consider the category of $\mathfrak{g}$-modules with the given infinitesimal character θ. If we put $U_\theta = U(\mathfrak{g})/\{Z - \theta(Z) \mid Z \in \text{Center of } U(\mathfrak{g})\}U(\mathfrak{g})$, then this category is the category $M(U_\theta)$ of U_θ -modules. We have constructed, for any $\chi \in \hat{\theta}$, the functor

$$\Delta_\chi : M(U_\theta) \to M(\mathcal{D}_\chi) .$$

We can define the right adjoint functor $\Gamma_\chi : M(\mathcal{D}_\chi) \to M(U_\theta)$ by

$$\Gamma_\chi(F) = \Gamma(X,F) .$$

5.

Let us discuss the structure of the sheaf $\mathcal{D}_\chi$.

LEMMA (i) The sheaf of algebras $\mathcal{D}_\chi$ is locally isomorphic to the sheaf $\mathcal{D}_X$ of differential operators on X.

(ii) $\mathcal{D}_\rho = \mathcal{D}_X$. More generally, let $\lambda \in \mathfrak{h}^*$ be an integral weight and $\mathcal{O}(\lambda)$ be the corresponding invertible sheaf of $\mathcal{O}_X$ -modules. Then there is a canonical isomorphism $\mathcal{D}_{\lambda+\rho} = \mathrm{Diff}(\mathcal{O}(\lambda))$ - the sheaf of differential operators in $\mathcal{O}(\lambda)$.

In other words, although we cannot define the sheaf $\mathcal{O}(\chi)$ for non-integral χ , we can define the sheaf $\mathrm{Diff}(\mathcal{O}(\chi))$.

PROOF. Fix $x \in X$ and the nilpotent subalgebra $\overline{\mathfrak{n}}$ opposite to $\mathfrak{n}_x$. Let $\overline{N}$ be the corresponding unipotent subgroup. Then, in a neighborhood of x , the variety X is isomorphic to $\overline{N}$ and $\mathcal{D}_\chi = \mathcal{O}_X \otimes_k U(\overline{\mathfrak{n}})$. This implies (i). The proof of (ii) is analogous.

PROPOSITION. Let F be a coherent (i.e. locally finitely generated) $\mathcal{D}_\chi$-module. Then the restriciton of F on some open dense subset $U \subset X$ is free as $\mathcal{O}_U$-module.

PROOF. Restrict F to some affine open subset $V \subset X$. Then we can replace F and $\mathcal{D}_\chi$ by their global sections: $F = F(V)$ and $D = \mathcal{D}_\chi(V) = D(V)$ - the algebra of differential operators on V . Consider the filtration $D^0 \subset D^1 \subset \ldots$ of D by the degree of differential operators, and put $\Sigma = \bigoplus_{n=0}^{\infty} D^n/D^{n-1}$. Then Σ is a commutative algebra, finitely generated over k , and $\mathcal{O}_X = D^0 \subset \Sigma$.

Fix generators $f_1,\ldots,f_k$ of F and consider the filtration $\{F^i = D^i f_1 + D^i f_2 + \ldots + D^i f_k\} \subset F$. Associated graded module $F_\Sigma = \oplus F^n_\Sigma$, where $F^n_\Sigma = F^n/F^{n-1}$, is a finitely generated Σ -module. Now, we have reduced the problem to the commutative case. General results from algebraic geometry imply that, after the restriction to some open dense subset $U \subset V$, F_Σ is a free $\mathcal{O}_U$-module (see [4, lecture 8, p.2^0]). Since F^n_Σ is a direct summand of F_Σ , it is a projective $\mathcal{O}_U$-module, and hence $F^n \approx F^{n-1} \oplus F^n_\Sigma$. This implies that $F \approx F^{\Sigma}$ as $\mathcal{O}_U$-modules, i.e. F is a free $\mathcal{O}_U$-module.

For a free $\mathcal{O}_X$-module the dimension of a fiber does not depend on a point. Hence, Theorem 2 is equivalent to the following statement about functors Δ_χ:

$$\text{supp } \Delta_\chi(M) \neq X \quad \text{for} \quad \ell(\chi) < \ell$$

$$\text{supp } \Delta_\psi(M) = X .$$

The following theorem describes the functor Δ_χ for dominant χ.

THEOREM (see [1]). <u>Suppose</u> χ_0 <u>is a dominant regular weight. Then the functors</u> Δ_{χ_0} <u>and</u> Γ_{χ_0} <u>are mutually inverse and give an equivalence of categories</u>.

$$M(U_\theta) \underset{\Gamma_{\chi_0}}{\overset{\Delta_{\chi_0}}{\rightleftarrows}} M(\mathcal{D}_{\chi_0}) .$$

In particular, if $M \neq 0$, then $\Delta_{\chi_0}(M) \neq 0$. If we denote the support of $\Delta_\chi(M)$ by S_χ this means that $S_{\chi_0} \neq \emptyset$. It is far from what we need (we need $S_\chi = X$), but at least it is something to start with.

7.

In order to prove Theorem 2 we will move from one weight χ to another in such a way that $\dim S_\chi$ will increase.

Let χ be a weight, α a simple root and $\phi = \sigma_\alpha \chi$. Suppose ϕ is α - dominant, i.e. $\phi(h_\alpha) \neq 0,-1,-2,\ldots$. We will construct the <u>intertwining functor</u>

$$I_{\chi,\phi} : M(\mathcal{D}_\phi) \to M(\mathcal{D}_\chi)$$

such that $\Delta_\chi = I_{\chi,\phi} \cdot \Delta_\phi : M(U_\theta) \to M(\mathcal{D}_\chi)$.

This functor will be described geometrically, as some operation with $\mathcal{D}$-modules. But firstly we shall describe how it changes the support of a sheaf.

Let us assign to a Borel subalgebra b_x the parabolic subalgebra $\mathfrak{p}_{x,\alpha}$ of type α by adding a root vector corresponding to α. This gives us a G-equivariant morphism $p_\alpha: X \to X_\alpha$, where X_α is an algebraic variety of parabolic subalgebras of type α. For any point $x \in X$ we denote by P_x the fiber of this morphism containing x, i.e. $P_x = p_\alpha^{-1}\, p_\alpha(x)$. As an algebraic variety P_x is isomorphic to projective line.

For any closed subset $S \subset X$ put

$$\mathrm{Env}_\alpha^+ (S) = \bigcup_{x \in S} P_x = p_\alpha^{-1} p_\alpha S \ .$$

We say that a fiber P of the morphism p_α is quasitransversal to S if it intersects S, all points of intersection are nonsingular in S and the morphism $p_\alpha|_S: S \to X_\alpha$ is an immersion at all these points.

We put $\mathrm{Env}_\alpha^-(S)$ = union of all fibers P quasitransversal to S.

STATEMENT. <u>Let</u> F <u>be a</u> $\mathcal{D}_\phi$<u>-module and</u> $S = \mathrm{Supp}\, F$. <u>Then</u>

(i) $\mathrm{Supp}\ I_{X,\phi}(F) \subset \mathrm{Env}_\alpha^+(S)$

(i) $\mathrm{supp}\ I_{X,\phi}(F) \supset \mathrm{Env}_\alpha^-(S)$.

We shall prove the statement in 11.

We shall derive Theorem 2 from Statement and the following geometric lemma.

LEMMA. <u>Let</u> S <u>be a nonempty closed subset such that</u> $\mathrm{Env}_\alpha^+ (S) = S$ <u>for any simple root</u>. <u>Then</u> $S = X$.

Indeed, let us identify X with G/B, where B is a Borel subgroup of G, and denote by $\overline{S}$ the preimage of S in G. The set $\overline{S}$ is invariant under (right) multiplication by B. The condition $\mathrm{Env}_\alpha^+(S) = S$ means that $\overline{S}$ is invariant under the multiplication by the parabolic subgroup P_α . Since the groups P_α for all simple roots α generate G , $\overline{S}$ is G-invariant, i.e. $\overline{S} = G$ and $S = X$.

8.

PROOF OF THEOREM 2. For any $\chi \in \hat{\theta}$ put $S_\chi = \mathrm{supp}\,(\Delta_\chi(M))$. If $\chi = \sigma_\alpha\phi$ and ϕ is α-dominant, we have $\Delta_\chi(M) = I_{\chi,\phi}(\Delta_\phi(M))$ and Statement 7. implies

$$\mathrm{Env}_\alpha^+\,(S_\phi) \supset S_\chi \supset \mathrm{Env}_\alpha^-\,(S_\phi)\ .$$

In particular,

(i) $\dim S_\chi \leqslant \dim S_\phi + 1$
(ii) If $\dim S_\chi = \dim S_\phi + 1$, then any irreducible component S_χ^0 of maximal dimension of S_χ is a union of fibers P_X .

Let $\ell = \mathrm{codim}\ S_{\chi_0}$. From (i) follows that $\mathrm{codim}\,(S_\chi) \leqslant \ell - \ell(\chi)$ for any $\chi \in \hat{\theta}$ .. Let us prove by induction in i that for $i \leqslant \ell$ there exists a weight $\chi \in \hat{\theta}$ with $\ell(\chi) = i$ such that $\mathrm{codim}\,S_\chi = \ell - i$. Let ϕ be a corresponding weight with $\ell(\phi) = i-1$. Consider an irreducible component S of S_ϕ of the maximal dimension. Since $S \neq X$, the above lemma implies that there exists a simple root α such that $\mathrm{Env}_\alpha^+(S) \underset{\neq}{\supset} S$. Put $\chi = \sigma_\alpha\phi$

From (ii) we see that $\ell(\chi) = i$ and, in particular, ϕ is α-dominant. We want to prove that $\mathrm{codim}\,S_\chi = \ell - i$.

Condition $\mathrm{Env}_\alpha^+(S) \neq S$ means that $\dim p_\alpha(S) = \dim S$. Sard's lemma implies that for some dense subset $U \subset p_\alpha(S)$ the morphism p_α is an immersion on $p_\alpha^{-1}(U) \cap S$. Hence, $\mathrm{Env}_\alpha^-(S) \supset p_\alpha^{-1}(U)$.

Therefore, using the statement above, we obtain

$$S_\chi \supset \mathrm{Cl}(\mathrm{Env}_\alpha^-\,(S)) \supset \mathrm{Cl}(p_\alpha^{-1}(U)) = \mathrm{Env}_\alpha^+(S)$$

i.e.

$$\operatorname{codim} S_\chi = \operatorname{codim} S_\phi - 1 = \ell - i .$$

9.

In order to construct the functors $I_{\chi,\phi}$ we shall introduce some definitions and constructions from the theory of $\mathcal{D}$ - modules.

Let Y be a nonsingular algebraic variety, $\mathcal{O}_Y$ the structure sheaf of Y, $\mathcal{D}_Y$ the sheaf of differential operators on Y and $i : \mathcal{O}_Y \to \mathcal{D}_Y$ the standard inclusion.

DEFINITION. A twisted sheaf of differential operators (t.d.o for short) on Y is a pair $(i,\mathcal{D})$, where $\mathcal{D}$ is a sheaf of algebras on Y and $i: \mathcal{O}_Y \to \mathcal{D}$ is an inclusion of algebras; which is locally isomorphic to the standard pair $i: \mathcal{O}_Y \to \mathcal{D}_Y$.

A $\mathcal{D}$-module is a sheaf of (left) $\mathcal{D}$-modules, quasicoherent as a sheaf of $\mathcal{O}_Y$-modules. The category of $\mathcal{D}$-modules we denote by $M(\mathcal{D})$.

Examples. 1. Let L be an invertible $\mathcal{O}_Y$-module and $\mathrm{Diff}(L)$ the sheaf of differential operators in L. Then $\mathrm{Diff}(L)$ is a t.d.o.

2. Let L be an invertible $\mathcal{O}_Y$-module and $\mathcal{D}$ a t.d.o. on Y. Consider the sheaf $L \otimes_{\mathcal{O}_Y} \mathcal{D}$ and put $\mathcal{D}^L = \mathit{End}(\text{right } \mathcal{D}\text{-module } L \otimes_{\mathcal{O}_Y} \mathcal{D})$. Then $\mathcal{D}^L$ is a t.d.o.

3. Let $\mathcal{D}$ be a t.d.o and $\mathcal{D}^0$ be the opposite algebra (i.e. the same sheaf with opposite multiplication). Then $\mathcal{D}^0$ is a t.d.o.

To prove this it is sufficient to verify that $(\mathcal{D}_Y)^0$ is a t.d.o. But it is easy to check that $(\mathcal{D}_Y)^0$ is canonically isomorphic to $\mathrm{Diff}(\Omega_Y)$, where Ω_Y is the sheaf of differential forms of the highest degree. The isomorphism is given by $\xi \to -\mathrm{Lie}_\xi$, where $\xi \in \mathcal{D}_Y$ is a vector field and Lie_ξ is the Lie derivative along ξ.

In the case of the flag variety X all sheaves $\mathcal{D}_\chi$ are t.d.o., $\mathcal{D}_\chi^{\mathcal{O}(\lambda)} = \mathcal{D}_{\chi+\lambda}$ and $(\mathcal{D}_\chi)^0 = \mathcal{D}_{-\chi}$.

Constructions.

1. Shift. Let L be an invertible $\mathcal{O}_Y$-module and $\mathcal{D}$ a t.d.o on Y. Then $L \otimes_{\mathcal{O}_Y} \mathcal{D}$ is a $\mathcal{D}^L$ - $\mathcal{D}$ -bimodule. Define the functor $L: M(\mathcal{D}) \to M(\mathcal{D}^L)$ by

$$L(F) = L \otimes_{\mathcal{O}_Y} F = (L \otimes_{\mathcal{O}_Y} \mathcal{D}) \otimes_{\mathcal{D}} F .$$

2. Inverse image. Let $\pi: Y \to Z$ be a morphism of nonsingular algebraic varieties and $\mathcal{D}$ a t.d.o on Z. Consider the sheaf of $\mathcal{O}_Y$-modules $\pi^*(\mathcal{D})$ - the inverse image of $\mathcal{O}_Z$-module $\mathcal{D}$ in the category of $\mathcal{O}$-modules and denote it by $\mathcal{D}_{Y \to Z}$. Recall that, by definition, $\pi^*(F) = \mathcal{O}_Y \otimes_{\pi^\circ \mathcal{O}_Z} \pi^\circ F$ where π° is the inverse image in the category of sheaves, i.e. locally, $\pi^*(F) = \mathcal{O}_Y \otimes_{\mathcal{O}_Z} F$.

Let us define the sheaf of algebras $\mathcal{D}^\pi$ on Y as a sheaf of differential endomorphisms of $\mathcal{O}_Y$-module $\pi^*(\mathcal{D})$ commuting with the right action of $\pi^\circ(\mathcal{D})$. It is easy to verify that the sheaf $(\mathcal{D}_Z)^\pi$ is canonically isomorphic to $\mathcal{D}_Y$. Hence, for any t.d.o $\mathcal{D}$ on Z the sheaf $\mathcal{D}^\pi$ is also a t.d.o.

Sheaf $\mathcal{D}_{Y \to Z}$ is a $\mathcal{D}^\pi$ - $\pi^\circ(\mathcal{D})$ - bimodule. Using it we define the functor of inverse image $\pi^+: M(\mathcal{D}) \to M(\mathcal{D}^\pi)$ by

$$\pi^+(F) = \mathcal{D}_{Y \to Z} \underset{\pi^\circ \mathcal{D}}{\otimes} \pi^\circ F$$

(i.e. locally $\pi^+(F) = \mathcal{D}_{Y \to Z} \underset{\mathcal{D}}{\otimes} F$). As $\mathcal{O}_Y$-module $\pi^+(F)$ is canonically isomorphic to $\pi^*(F)$.

3. Direct image. We want to define the functor of direct image $\pi_*: M(\mathcal{D}^\pi) \to M(\mathcal{D})$. In order to do this, we will construct a $\pi^\circ\mathcal{D}$ - $\mathcal{D}^\pi$ - bimodule $\mathcal{D}_{Z \leftarrow Y}$ and put

$$\pi_*(H) = \pi_\circ (\mathcal{D}_{Z \leftarrow Y} \otimes_{\mathcal{D}^\pi} H),$$

where $H \in M(\mathcal{D}^\pi)$ and $\pi_\circ$ is the direct image in the category of sheaves. The functor π_* has good properties only in the case of an affine morphism π (i.e. when the preimage of open affine subset $U \subset Z$ is affine).

For the general case this functor can be correctly defined only in derived categories. We will consider here only affine morphisms, because this is enough for our purposes. In this case the functor π_* is right exact.

By definition, we put

$$\mathcal{D}_{Z \leftarrow Y} = \Omega_Y \otimes_{\mathcal{O}_Y} \pi^*(\Omega_Z^{-1} \otimes_{\mathcal{O}_Z} \mathcal{D}^0).$$

This module has a natural structure of right $\pi^\circ(\mathcal{D}^0)$-module, i.e. of left $\pi^\circ\mathcal{D}$ - module. Now, we claim that the algebra of differential endomorphisms of $\mathcal{O}_Y$-module $\mathcal{D}_{Z \leftarrow Y}$ commuting with left action of $\mathcal{D}$ is canonically isomorphic to $(\mathcal{D}^\pi)^0$, i.e. $\mathcal{D}_{Z \leftarrow Y}$ has a canonical structure of a right $\mathcal{D}^\pi$- module.

Indeed, it is sufficient to consider the case $\mathcal{D} = \mathcal{D}_Z$. Then $\Omega_Z^{-1} \otimes_{\mathcal{O}_Z} \mathcal{D}^0 = \mathcal{D}_Z \otimes_{\mathcal{O}_Z} \Omega_Z^{-1}$ and hence $\pi^*(\Omega_Z^{-1} \otimes_{\mathcal{O}_Z} \mathcal{D}^0)$ is a left $\mathcal{D}^\pi = \mathcal{D}_Y$ - module. Therefore $\mathcal{D}_{Z \leftarrow Y}$ has the structure of a left $\mathcal{D}_Y^{\Omega_Y} = \mathcal{D}_Y^0$ - module, i.e. the structure of a right $\mathcal{D}^\pi = \mathcal{D}_Y$ - module. This structure does not depend on a local isomorphism $\mathcal{D} \tilde{\to} \mathcal{D}_Z$.

REMARK. The direct image of a sheaf is often denoted by $\int F$ because the functor π_* is an algebraic version of integration along fibers (see Example 1. below).

Examples. 1. Let $Y = A \times Z$, where A is an affine line and π the projection $\pi\colon Y \to Z$. We suppose Z to be affine and identify sheaves with their global sections. Then

$$\pi^+(F) = \mathcal{O}(A) \otimes_k F = k[t] \otimes_k F$$

$$\pi_*(H) = \Omega(A) \otimes_{\mathcal{O}_Y} H / \partial_t(\Omega(A) \otimes_{\mathcal{O}_Y} H) =$$

$$= H/\partial_t H \otimes_k k(dt)$$

where t is a linear parameter on A and $\partial_t = \frac{\partial}{\partial t}$.

2. Let $\pi\colon Y \to Z$ be a closed imbedding (i.e. Y is a closed subvariety of Z). Then π^+ is the usual restriction of $\mathcal{O}$-modules. Direct image π_* in this case is an exact functor. Locally, it

can be described as follows:

Let ℓ = codim Y and let $\partial_1,\dots,\partial_\ell$ be vector fields transversal to Y . Then

$$\pi_*(F) = \{ \bigoplus_{n_1,\dots,n_\ell \in \mathbb{Z}^+} \partial_1^{n_1}\partial_2^{n_2}\dots\partial_\ell^{n_\ell} F^0 \}$$

where

$$F^0 = F \otimes_{\mathcal{O}_Y} \Omega_Y \otimes_{\mathcal{O}_Z} \Omega_Z^{-1}$$

The following technical theorem, due to Kashiwara, is often very useful.

THEOREM. Let π: Y → Z be a closed imbedding. Then π_* defines an equivalence of the category $M(\mathcal{D}^\pi)$ and the subcategory $M_Y(\mathcal{D}) \subset M(\mathcal{D})$ consisting of sheaves supported on Y .

10.

Now we can define intertwining functor $I_{\chi,\phi}$.

We have fixed simple root α and weights ϕ and $\chi = \sigma_\alpha\phi$ such that ϕ is α-dominant. Consider the projection $p_\alpha : X \to X_\alpha$ and put

$$N = \{(x,x^1) \in X \times X \mid p_\alpha(x) = p_\alpha(x^1),\ x \neq x^1\}$$

Denote by pr_1 , pr_2 : N → X projections of N on the first and second factor respectively. They both are G - equivariant fibrations with fibers isomorphic to the affine line A . Denote by L the invertible sheaf of $\mathcal{O}_N$ - modules corresponding to the tangent bundle to fibers of the projection pr_1 .

LEMMA. $\mathcal{D}_\chi^{pr_1}$ is canonically isomorphic to $(\mathcal{D}_\phi^{pr_2})^L$.

We will not prove the lemma, but explain it in the case when χ is integral. We want to check that

$$pr_2^*(\mathcal{O}(\phi - \rho)) = L \otimes pr_1^*(\mathcal{O}(\chi - \rho)) .$$

Since N is a homogeneous G - space, it is sufficient to prove the equality at one point $n = (x,y) \in N$. The stationary subgroup G_n is equal to $B_x \cap B_y$. Let us choose a Cartan subgroup $H \subset B_x \cap B_y$ and compare weights of the fibers of both sheaves at the point n. We identify H with a standard Cartan group using the subgroup B_y. If we use the subgroup B_x, we obtain weights changed by the automorphism σ_α.

$pr_2^*(\mathcal{O}(\phi - \rho))_n$ has a weight $\phi - \rho$,

$pr_1^*(\mathcal{O}(\chi - \rho))_n$ has a weight $\sigma_\alpha(\chi - \rho)$

L_n has a weight $\sigma_\alpha(-\alpha) = \alpha$ (recall that $\alpha \notin \{$roots of $\mathfrak{h}_x$ in $\mathfrak{n}_x\}$).

The equality $(\phi - \rho) + \alpha = \sigma_\alpha(\chi - \rho)$ implies the lemma.

11.

DEFINITION. <u>Define the intertwining functor</u> $I_{\chi,\phi}: M(\mathcal{D}_\phi) \to M(\mathcal{D}_\chi)$ <u>by</u>

$$I_{\chi,\phi}(F) = (pr_1)_* (L \otimes pr_1^+ F) .$$

Informally

$$(I_{\chi,\phi}F)_x = \int_{P_x \setminus x} F|P_x ,$$

i.e. it really is intertwining.
The functor $I_{\chi,\phi}$ is right exact.

PROOF of Statement 7. Let F be a $\mathcal{D}_\phi$ -module, $S = \operatorname{supp} F$. Put $F' = L \otimes pr_2^+(F)$, $S' = pr_2^{-1}(S)$. It is clear that $S' = \operatorname{supp} F'$. We replace X by a small open subset Z and N by

$Y = Z \times A$, such that the natural morphism $\pi: Y \to Z$ is the projection pr_1 . We want to prove that

(i) supp $(\pi_*(F'))$ $\subset (S')$

(ii) If $\pi|S': S' \to Z$ is an immersion, then supp $\pi_*(F') = \pi(S')$.

The assertion (i) is trivial and (ii) follows from Kashiwara's theorem, because F' is the direct image of a $\mathcal{D}_{S'}$-module F'' and hence $\pi_*(F') = (\pi|S')_*(F'')$ has support $\pi(S')$.

12.

THEOREM. $\Delta_\chi = I_{\chi,\phi} \cdot \Delta_\phi$.

We will sketch the proof of the theorem. (i) Fix a U_θ -module M and let us first prove that, for any point $x \in X$, the fibers $(\Delta_\chi(M))_x$ and $(I_{\chi,\phi}(\Delta_\phi(M)))_x$ are canonically isomorphic. It means that

$$(\Delta_\chi(M))_x = \int_{P_x \setminus x} (L \otimes \Delta_\phi(M)|P_x \setminus x) .$$

Consider the parabolic subalgebra $\mathfrak{p}_\alpha = \mathfrak{p}_{x,\alpha}$ (see 7) and its nilpotent radical $\mathfrak{n}_\alpha$ and put $\mathfrak{g}_\alpha = \mathfrak{p}_\alpha/\mathfrak{n}_\alpha$. The semisimple part of $\mathfrak{g}_a$ is isomorphic to $\mathfrak{sl}(2)$. The flag variety for $\mathfrak{g}_\alpha$ is naturally isomorphic to P_x .

Put $Q = M/_{\mathfrak{n}_\alpha M}$. We will consider Q as an $\mathfrak{g}_\alpha$-module. It is easy to see that the restriction of the sheaf $\Delta_\chi(M)$ to P_x is naturally isomorphic to $\Delta_{\chi'}(Q)$, where $\chi' = \chi + (\rho - \rho(\mathfrak{g}_\alpha))$, and the same for ϕ . Using this fact we can (and will) reduce the problem to the case $\mathfrak{g} = \mathfrak{sl}(2)$, $X = \mathbb{P}^1$.

Let $x \in X = \mathbb{P}^1$, $A = X \setminus x$ is the affine line. We shall prove that $\Delta_\chi(M)_x = (M/\mathfrak{n}_x M)$ coincides with $\int_A (L \otimes \Delta_\phi(M))$.

Put $F = \Delta_\phi(M) \in M(\mathcal{D}_\phi)$, $H = F|_A$, F_* - the direct image of H on X . We have an exact sequence

$$0 \to K \to F \to F_* \to C \to 0$$

where $\mathcal{D}_\phi$-modules K and C are supported in x. Since ϕ is a dominant weight with respect to $\mathfrak{g}_\alpha$ we have the exact sequence

$$0 \to \Gamma(K) \to \Gamma(F) \to \Gamma(F_*) \to \Gamma(C) \to 0$$

and $\Gamma(F) = M$ (see 6). Using Kashiwara's theorem, one can verifv that, for any $\mathcal{D}_\phi$ - module E supported in x, we have $H_0(\mathfrak{n}_x, \Gamma(E))^\chi = \Gamma(E)^\chi_{\mathfrak{n}_x} = 0$. Hence $M^\chi_{\mathfrak{n}_x} = \Gamma(F)^\chi_{\mathfrak{n}_x} = \Gamma(F_*)^\chi_{\mathfrak{n}_x}$.

Since $\Gamma(F_*) = \Gamma(A,H)$, we should prove

$$\int_A L \otimes H = \Gamma(A,H)^\chi_{\mathfrak{n}_x} .$$

Let e,h be generators of the Lie algebras $\mathfrak{n}_x$ and $\mathfrak{h} \subset \mathfrak{b}_x$ such that $[h,e] = -e$. Then there exists a unique linear coordinate t on $A = X \setminus x$ such that e and h act on A as ∂_t and $t\partial_t$. Then

$$\int_A L \otimes H = (\Omega(A) \otimes L \otimes H)/\partial_t(\Omega_A \otimes L \otimes H) = H/\partial_t H = \Gamma(A,H)_{\mathfrak{n}_x} .$$

Let us check that h acts on this space as multiplication by $(\chi - \rho)(h) = \chi(h) + \frac{1}{2}$. Indeed, h acts on H as $-t\partial_t - (\phi(h) + \frac{1}{2}) = -\partial_t t + 1 - \phi(h) - \frac{1}{2} = -\partial_t t + (\chi(h) + \frac{1}{2})$. Hence on the quotient space $H/\partial_t H$ the element h acts as $\chi(h) + \frac{1}{2}$.

ii) We have proved that the fibers of the sheaves $\Delta_\chi(M)$ and $I_{\chi,\phi}(\Delta_\phi(M))$ at any point x are canonically isomorphic. From this it follows that these sheaves are isomorphic in the case when M is a G-equivariant $\mathfrak{n}$-module and hence $\Delta_\chi(M)$ and $I_{\chi,\phi}(\Delta_\phi(M))$ are G-equivariant. In particular, these sheaves are canonically isomorphic for any free U_θ-module. Since both functors Δ_χ and $I_{\chi,\phi} \cdot \Delta_\phi$ are right exact, they are isomorphic for any U_θ - module M.

This finishes the proof of Theorem 12 and hence of Theorem 2.

13. SEVERAL REMARKS ON $\mathfrak{n}$-HOMOLOGY.

THEOREM. The intertwining functor $I_{\chi,\phi}$ has homological dimension ≤ 1. The corresponding derived functor $L(I_{\chi,\phi})\colon D(\mathcal{M}(\mathcal{D}_\phi)) \to D(\mathcal{M}(\mathcal{D}_\chi))$ is an equivalence of derived categories.

COROLLARY. Functors Δ_χ and Γ_χ have homological dimensions $\leq \ell(\chi)$. The corresponding derived functors

$$L\Delta_\chi : D(\mathcal{M}(U_\theta)) \to D(\mathcal{M}(\mathcal{D}_\chi))$$

and

$$R\Gamma_\chi : D(\mathcal{M}(\mathcal{D}_\chi)) \to D(\mathcal{M}(U_\theta))$$

are mutually inverse and give an equivalence of derived categories.

COROLLARY. Let M be a finitely generated U_θ-module and $\ell = \operatorname{codim} \operatorname{supp} \Delta_{\chi_0}(M)$. Then for almost all $x \in X$

$$H_i(\mathfrak{n}_x, M)^\chi = 0 \quad \text{for} \quad \ell(\chi) < \ell \quad \text{and any} \quad i,$$

$$H_i(\mathfrak{n}_x, M)^\chi = 0 \quad \text{for} \quad \ell(\chi) = \ell \quad \text{and} \quad i > 0$$

(and, as we have seen, there exists ψ with $\ell(\psi) = \ell$ such that $H_0(\mathfrak{n}_x, M)^\psi \neq 0$).

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REFERENCES

[1] A. Beilinson, J.N. Bernstein, Localisation de $\mathfrak{g}$-modules, C.R. Acad. Sci. Paris, 292, (1981), 15-18.

[2] W. Casselman, Differential equations satisfied by matrix coefficients, preprint.

[3] D. Miličić, Notes on asymptotics of admissible representations of semi-simple Lie groups, Institute for Advanced Study, 1976.

[4] D. Mumford, Lectures on curves on an algebraic surface, Ann. of Math. Studies, Vol. 59, Princeton University Press, 1966.

[5] J.T. Stafford, N.R. Wallach, The restriction of admissible modules to parabolic subalgebras, Trans. Amer. Math. Soc. 272(1982), 333-350.