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On the support of Plancherel measure

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To my teacher Israel M. Gelfand

Abstract *Let G be a real reductive group. As follows from Plancherel formula for G, proved by Harish-Chandra, only tempered representations of G contribute to the decomposition of the regular representation in $L^2(G)$. We give a simple direct proof of this result, based on Gelfand-Kostyuchenko method. We also prove similar results for representations, which appear in the decomposition of $L^2(X)$, where X is a homogeneous G-space of polynomial growth. (See precise definition in 3.5). Important examples of such space X are semisimple symmetric spaces and quotient of G by arithmetic subgroups.*

0. INTRODUCTION

0.1. Let G be a real reductive group. Consider the decomposition of the regular representation of $G \times G$ in the space $H = L^2(G, \mu_G)$ into a direct integral of irreducible representations

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$$H = \int_{\hat{G}} H_\omega \, d\mu_{\hat{G}}(\omega), \qquad \text{where}$$

$\hat{G}$ is the set of equivalence classes of irreducible unitary representations ω, $H_\omega = \omega \otimes \omega^*$ and $\mu_{\hat{G}}$ is the Plancherel measure. This decomposition was explicitly described by Harish-Chandra. In particular, he found out that only some of the irreducible representations $\omega \in \hat{G}$, which he calls tempered, contribute to this decomposition. In other words, the Plancherel measure $\mu_{\hat{G}}$ is supported on a subset $\hat{G}_{\text{temp}} \subset \hat{G}$ of tempered representations.

This is not surprising, since, by definition, tempered representations are those, whose matrix coefficients lie "close" to $L^2(G)$. So it is natural to try to find a direct proof of this fact, without detailed study of the Plancherel measure. In this paper we give a simple proof of this result and explain the geometry behind it.

In a later paper we plan to show, that in some cases using this result one can relatively easily find explicit formulas for Plancherel measure $\mu_{\hat{G}}$.

0.2. We will consider a more general situation. Let G be a locally compact group, $\Gamma \subset G$ a closed subgroup, $X = G/\Gamma$. We assume for simplicity that X has a G-invariant measure μ_X and consider the natural representation of G in the space $H = L^2(X, \mu_X)$. By the general theorem of Gelfand-Raikov, we can decompose H into a direct integral

$$(*) \qquad H = \int_Z H_z \, d\mu_Z$$

of irreducible representations. We want to understand, which irreducible representations H_z can contribute to such a decomposition.

We will prove some results under very mild assumptions about G; but we are interested mostly in the case, where G is a real, p-adic or adelic Lie group. The following two examples give the most interesting applications.

Example 1. Let G be a real reductive group, $\sigma : G \to G$ an involutive automorphism, Γ an open subgroup in its fixed point subgroup G^σ. The space $X = G/\Gamma$ is called a semisimple symmetric space. The decomposition of $L^2(X, \mu_X)$ was described in detail by Oshima and his coauthors (see [OsMa] and subsequent papers). In the diagonal case $G = \Gamma \times \Gamma$, this reduces to Harish-Chandra's results.

Example 2. Let G be a real reductive group, $\Gamma \subset G$ an arithmetic subgroup. This case was analysed in detail by Laglands (see [Lan]).

Let $S(X)$ be the space of smooth, compactly supported functions on X (the Schwartz space of X). First of all, we prove that the decomposition $H = \int H_z \, d\mu_Z$ defines on $S(X)$ the system of projections $\alpha_z : S(X) \to H_z$, which are nonzero for almost all $z \in Z$. Hence, a unitary representation (ρ, V) can contribute to the decomposition (*) only if there exists a nonzero G-morphism $\alpha_V : S(X) \to V$.

It is convenient to move to a dual picture. Namely, each G-morphism $\alpha_V : S(X) \to V$ defines an adjoint G-morphism $\beta_V : V^\infty \to C^\infty(X)$, where V^∞ is the Garding space of V. In terms of morphism $\alpha_z : S(X) \to H_z$ and $\beta_z : H_z^\infty \to C^\infty(X)$ the decomposition (*) can be written in an explicit form

$$(**) \qquad \phi = \int_Z \phi_z \, d\mu_Z,$$

where $\phi \in S(X)$, $\phi_z = \beta_z \alpha_z(\phi) \in C^\infty(X)$.

For any representation (ρ, V) we call a morphism $\beta : V^\infty \to C^\infty(X)$ a *V-form on X*. Simple Fronenius reciprocity shows that V-forms on $X = G/\Gamma$ correspond to Γ-invariant functionals on V^∞. Thus we get the following

Algebraic necessary condition. An irreducible representation (ρ, V) can contribute to the decomposition (*) only if there exists a nonzero V-form on X, i.e., if $\mathrm{Hom}_\Gamma(V^\infty, \mathbb{C}) \neq 0$. Moreover, each contribution of V to (*) gives a V-form $\beta : V^\infty \to C^\infty(X)$ and (*) can be written in terms of such forms as (**).

It is intuitively clear, that not all V-forms $\beta : V^\infty \to C^\infty(X)$ can contribute to (**), but only "tempered" ones, for which the image $\beta(V^\infty)$ lies "close" to $L^2(X)$. In other words, we can eliminate some forms from consideration, using restrictions on the growth of functions in their image. In order to do this we need some scale function $r : X \to \mathbb{R}^+$, which would control the growth. For real, p-adic and adelic Lie groups, there is usually a natural scale function $r : G \to \mathbb{R}^+$ (for example, if $G \subset GL(n, \mathbb{R})$, we can define it by $r(g) = \log \max(\|g\|, \|g^{-1}\|)$, see details in 4.2). It gives us the scale function on X by $r(x) = \inf r(g) \mid x = gx_0$ for some fixed point $x_0 \in X$.

We say that the homogeneous space X has *polynomial growth* if it satisfies the following geometric condition:

Fix a compact neighborhood B of the identity in G. Then there exist constants $d \geqslant 0$, $C > 0$ such that for every $R > 0$ the ball $B(R) = \{x \in X \mid r(x) \leqslant R\}$ can be covered with $\leqslant C(1 + R)^d$ B-balls of the form Bx, $x \in X$.

The greatest lower bound of numbers d in this definition we will call the *rank* of X (notation $rk(X)$).

We will see that in examples 1, 2 above, the space X has polynomial growth and its rank is the rank which is usually associated with the corresponding situation.

Our main result is the following

Analytic necessary condition. Suppose X has polynomial growth. Then a form $\beta : V^\infty \to C^\infty(X)$ can contribute to the decomposition (**) above only if for each $d > rk(X)$.

(***) $\beta(\xi)(1 + r(x))^{-d/2}$ lies in $L^2(X)$ for each $\xi \in V^\infty$.

Following Harish-Chandra, we call a form $\beta : V^\infty \to C^\infty(X)$ *X-tempered* if the condition (***) holds for some $d > 0$. Then we can reformulate our result as follows:

In a decomposition (**) $\phi = \int \phi_z \, d\mu_Z$, the Plancherel measure μ_Z is supported on the subset Z_{temp} of points $z \in Z$ which correspond to X-tempered forms β_z.

Note, that it is quite possible, that a given representation V has many forms, some of them X-tempered, some of them not. This means that the notion of an X-tempered representation is not well defined. Of course, in a multiplicity free case (like the one studied by Harish-Chandra) one can talk about X-tempered representations instead of X-tempered forms.

0.3. Let us consider two typical examples.

Example 1. $G = SL(2, \mathbb{R})$, $\Gamma = SO(2)$, $X = G/\Gamma$, the hyperbolic plane. In this case, $r(x)$ is the hyperbolic distance to the unique Γ-invariant point x_0. We will see that X has polynomial growth and $rk(X) = 1$.

The algebraic necessary condition tells us that an irreducible unitary representation (ρ, V) can contribute to the decomposition of $L^2(X)$ only if it has a Γ-invariant vector ξ_V, i.e. if it is spherical. Each spherical V has one V-form $\beta : V^\infty \to C^\infty(X)$ and it is completely characterized by the function $\psi = \beta_V(\xi_V)$. It is known that unitary spherical representations V are parametrized by one parameter $s \in [0, 1] \cup i\,\mathbb{R}^+$. The corresponding spherical function $\psi_s(x)$ grows like $\exp[(Re(s) - 1/2)\, r(x)]$. Since the area of the ball $B(R)$ of radius R grows like $\exp(R)$, we see that for $Re(s) > 0$ the integral

$$\int_X |\psi_s(x)|^2 (1 + r(x))^{-d} \, d\mu_X$$

diverges for any d, i.e. the corresponding forms are not tempered. Thus, the analytic necessary condition shows that only representations with $s \in i\,\mathbb{R}$ (i.e. representations of the principle series) contribute to the spectral decomposition of $L^2(X)$.

Example 2. $G = SL(2, \mathbb{R})$, $\Gamma = SL(2, \mathbb{Z})$, $X = G/\Gamma$. The space X has polynomial growth, $rk(X) = 1$. In this case we have both a discrete and a continuous spectrum. About the discrete spectrum our results tell nothing. But in the continuous spectrum they immediately illiminate all Eisenstein series $E(s)$ for which parameter s does not lie on the unitary axis.

0.4. Our proofs are based on the Gelfand-Kostyuchenko method. The idea of the method is very simple. Suppose we have a direct integral (*) $H = \int H_z \, d\mu_Z$ representing a given Hilbert space H. This means that each vector $\eta \in H$ is represented by a function $z \mapsto \eta_z \in H_z$. However, this function is defined up to a change on a subset of measure 0, i.e. at each particular point $z \in Z$ it is not defined. In applications, the Hilbert space H usually has some additional structure. Namely one can choose some natural dense subspace $S \subset H$ of "test functions", endowed with its own topology. Gelfand and Kostyuchenko proved that under very mild assumptions on S one can choose a family of continuous morphisms $\alpha_z : S \to H_z$, such that for each $\phi \in S$ the section $z \mapsto \alpha_z(\phi)$ represents $\phi \in H$, This gives a more explicit presentation of decomposition (*).

The simplest example of this is the Fourier transform

$$f \in L^2(\mathbb{R}) \to \hat{f} \in L^2(\mathbb{R}), \quad \text{given by } \hat{f}(\xi) = \int f(x) e^{-i\xi x} \, dx.$$

This formula is well-defined for each ξ if f belongs to a subspace $S = C_c^\infty(\mathbb{R}) \subset L^2(\mathbb{R})$. But for a generic $f \in L^2(\mathbb{R})$ it does not make sense and $\hat{f}(\xi)$ is not defined for each particular ξ.

Decomposition (*) can be expressed even more explicitly, if we consider Gelfand triple $S \subset H \subset S^+$, where S^+ is the Hermitian dual of S, interpreted as a space of distributions. If we denote by $\beta_z : H_z \to S^+$ the morphism, adjoint to α_z, then for $\phi \in S$ the decomposition (*) takes the form

$$(**) \qquad \phi = \int_Z \phi_z \, d\mu_Z \quad \text{where} \quad \phi_z = \beta_z \alpha_z(\phi) \in S^+.$$

If S, H, H_z have compatible structures of modules over some algebra A (or a group G), then all α_z are morphism of A-modules.

Surprisingly, this beautifully simple idea for a long time did not find broad applications it should have found. (Regarding this fact, it is instructive to compare the remarks in [ReSi], v. III, p. 354 and [Si], p. 503).

We first apply this idea to the case $H = L^2(X, \mu_X)$, $S = S(X)$, described in 0.2. This immediately gives us the algebraic necessary condition on representations V which can appear in a decomposition of H. Note, that in slightly different terms it was earlier done in [Ma].

Now suppose we have found another subspace S between $S(X)$ and H, such that the pair $S \subset H$ satisfies Gelfand-Kostyuchenko conditions. Then for almost each $z \in Z$ the morphism $\alpha_z : S(X) \to H_z$ extends to a continuous morphism $S \to H_z$.

In other words, let us call a G-morphism $\alpha_V : S(X) \to V$ *S-tempered* if it extends to $S \supset S(X)$. Then the statement above means that only S-tempered morphisms can appear in the decomposition (**).

The natural choice for S is the Harish-Chandra Schwartz space $\mathscr{C}(X)$ of X, which is defined as follows. For each $d > 0$, consider the space $L^2(X, d) = L^2(X, (1 + r)^d \mu_X)$, where $r : X \to \mathbb{R}^+$ is the scale function, discussed in 0.2; denote its Garding space by $L^2(X, d)^\infty$ and set $\mathscr{C}(X) = \bigcap_d L^2(X, d)^\infty$.

It is easy to check that $\mathscr{C}(X)$-tempered morphism α_V correspond to X-tempered V-forms β_V (see the end of 0.2). Hence, the only thing one should check is that the pair $\mathscr{C}(X) \subset H$ satisfies the Gelfand-Kostyuchenko condition. The condition essentially is that the inclusion $\mathscr{C}(X) \to H$ is a Hilbert-Schmidt morphism or; more precisely, that it can be mapped through a Hilbert Schmidt morphism $L \to H$ for some Hilbert space L. This is the main technical result of the paper.

In the first draft of the paper I proved this result directly for each of the examples, mentioned in 0.2. Then I realized, that there is a general proof, which uses only some very general geometric property of the homogeneous space $X = G/\Gamma$ – namely, that it has polynomial growth. This in turn led me to a realization, that homogeneous spaces of locally compact groups in general have very interesting large scale geometry. I include some preliminary discussion of this geometry in section 4, but it is clear to me that this is only the beginning of the subject.

0.5. The paper is organized as follows:

In section 1 we recall the Gelfand-Kostyuchenko theory and adapt it to representation theory.

In section 2 we prove an algebraic necessary condition and show how it can be reformulated in terms of V-forms.

In section 3 we introduce the notion of weights on X (see 3.1) and the notion of summable weights (3.2). In 3.2 we formulate the central theorem, which shows that each summable weight gives an analytic necessary condition. We prove it in 3.4 using the notion of a standard measure on X, introduced in 3.3. In 3.5 we reformulate this condition for spaces of polynomial growth. In 3.6 and 3.7 we extend these results to the case of induced representations and to the case when X has no invariant measure.

In section 4 we consider examples of homogeneous spaces of reductive groups and analyse their growth. In 4.1 and 4.2 we consider possible large scale structures on G and X. In 4.3 and 4.4 we list examples of interesting homogeneous space (in 4.3 we deal with groups over local fields, and in 4.4 with groups over adeles). In 4.5 we discuss relations between algebraic, natural and standard large scales, and in 4.6 and 4.7 we supply proofs for examples in 4.3 and 4.4.

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0.6. Notations

Throughout the paper we use the following notations:

Let f, h be positive functions on a set X. We say that f dominates h (notation $f \gg h$ or $h \ll f$) if $h \leqslant Cf$ for some $C > 0$. We say that f and h are comparable (notation $f \sim h$) if $f \ll h$ and $h \ll f$, i.e. if $C^{-1} f \leqslant h \leqslant Cf$ for some $C > 0$. Similarly, if f and h are (positive) measures.

G will denote a locally compact group (with restriction described in 2.1).

$K \subset G$ is a regular subgroup (2.1), $\mathfrak{g}$ the real Lie algebra associated with G (2.1).

$B \subset G$ is a ball (2.1).

$\Gamma \subset G$ is a closed subgroup, not necessarily discrete, $X = G/\Gamma$ (§2).

μ_X is a Haar measure on X, m_X – a standard measure on X (3.3).

$M_c(G)$ is the algebra of compactly-supported measures on G (2.2).

$S(X)$ is the Schwartz space of X (2.2).

H is a Hilbert-space; usually $H = L^2(X, \mu_X)$.

For a G-module (ρ, V), V^∞ is its Garding space (2.2); spaces $V^{K,k}$, $V^{K,\infty}$ are also described in 2.2.

For a unitary G-module (ρ, V), $\alpha_V : S(X) \to V$ is a G-morphism and $\beta_V : V^\infty \to C(X)^\infty$ the corresponding V-form (2.4).

$N \subset X$ is a net, usually a sparse net (3.2).

w is a weight on X, $L_w = L^2(X, w\mu_X)$, $S_w = L_w^\infty$ (3.1).

r is a radial function on G or on X (4.2) r_a, r_n, r_{st}-algebraic, natural and standard large scales (4.2, 4.5).

1. GELFAND-KOSTYUCHENKO METHOD

1.1. By "topological vector space" we always mean a complex topological vector space, and a morphism of such spaces is a continuous linear map. All spaces S which we condiser satisfy the following Hahn-Banach condition:

(HB) Morphisms $S \to \mathbb{C}$ separate the points of S.

In particular, all these spaces are Hausdorff.

The dual space S^* we will endow with the topology of uniform converge on bounded subsets of S (see [ReSi], v. I, ch. V, § 7).

The Hermitian dual of S (i.e. the complex conjugate of S^*) we will denoted by S^+.

Most of the topological vector spaces S which we consider are separable, i.e. have countable dense subsets.

By "topological algebra" we mean a topological vector space A, endowed with the structure of an algebra, such that the multiplication $(a, b) \to ab$ is separately continuous in a and b. An A-module S is defined as a topological vector space with the structure of an A-module, such that the multiplication $(a, \xi) \to a\xi$ is separately continuous in a and ξ.

Similarly, for a topological group G, a representation of G (or a G-module) is a topological vector space V with an action of G such that the multiplication $(g, \xi) \to g\xi$ is separately continuous in g and ξ.

1.2. Let Z be a Borel space, μ_Z a measure on Z and $z \to H_z$ a family of Hilbert spaces, parametrized by the points of Z. Suppose we are given a family F of section $\eta : z \mapsto \eta_z \in H_z$ which we call measurable. We assume that they have the following properties:

a) A section $z \mapsto \xi_z \in H_z$ lies in F iff for each section $\eta \in F$ the function $z \mapsto \langle \xi_z, \eta_z \rangle$ is measurable.

b) There exists a countable collection of section $\{\eta_\alpha\}$ in F such that for every $z \in Z$ vectors $(\eta_\alpha)_z$ span a dense subset of H_z.

In such a situation, we define a Hilbert space

$$H = \int_Z H_z \, d\mu_Z$$

(the direct integral of the family H_z) as follows:

The vector η in H is a measurable section $z \mapsto \eta_z$, for which

$$\| \eta \|^2 = \int_Z \| \eta_z \|^2 \, d\mu_Z < \infty .$$

Two such sections define the same vector in H if they differ on a subset of measure 0.

This definition is discussed in detail in [Dix], ch. II.

Suppose an algebra A acts on H and on all spaces H_z. We say that these actions are compatible if for every $\eta \in H$ and every $a \in A$ the section $a(\eta_z)$ represent the vector $a(\eta) \in H$; similarly for an action of a group. We will use the following standard facts:

LEMMA. *Let* $\eta_1, \eta_2, \ldots, \eta_k, \ldots$ *be a sequence of vectors in* H *represented by section* η_{iz}.

a) Suppose the sequence η_i *converges to a vector* $\eta \in H$, *represented by a section* η_z. *Then one can choose a subsequence* η_i' *of* η_i, *such that for almost all* $z \in Z$ η_{iz}' *converges to* η_z.

b) Suppose that $\{\eta_i\}$ *span a dense subset of* H. *Then for almost all* $z \in Z$, η_{iz} *span a dense subset of* H_z.

Proof. See [Dix], II, §1, Prop. 5 and 8. ∎

1.3. Let $H = \int H_z \, d\mu_Z$ and S be a separable topological vector space. We say that a morphism $\alpha : S \to H$ is *pointwise defined* if there exists a family of morphisms $\alpha_z : S \to H_z$ for all $z \in Z$ such that for every $\xi \in S$ the section $z \mapsto \alpha_z(\xi)$ represents the vector $\alpha(\xi) \in H$.

LEMMA. *a) The family of morphisms* $\{\alpha_z\}$ *is essentially unique, i.e., two such families* $\{\alpha_z'\}, \{\alpha_z\}$ *differ on a subset of measure 0.*

b) Suppose S, H *and* H_z *are modules over a separable algebra* A, α *is a morphism of* A*-modules and the decomposition* $H = \int H_z \, d\mu_Z$ *is compatible with the action of* A. *Then all morphisms* α_z *can be chosen to be morphisms of* A*-modules. The same also holds for an action of a separable group* G.

c) If $\alpha(S)$ *is dense in* H, *then for almost all* $z \in Z$ $\alpha_z(S)$ *is dense in* H_z

Proof. a) Fix a dense subset $\{\xi_1, \xi_2, \ldots\}$ is S. For each i set $Z_i = \{z \in Z \mid \alpha_z(\xi_i) \neq \alpha_z'(\xi_i)\}$. By definition, $\mu(Z_i) = 0$. Not set $Z_0 = \bigcup_i Z_i$. Then $\mu(Z_0) = 0$ and for every $z \in Z \setminus Z_0$ we have $\alpha_z(\xi_i) = \alpha_z'(\xi_i)$ for all i. Since both α_z and α_z' are continuous, this implies that for each $z \notin Z_0$ $\alpha_z = \alpha_z'$.

b) Fix dense subset $\{\xi_1, \ldots, \xi_k, \ldots\}$ in S and $\{a_1, \ldots, a_i, \ldots\}$ in A. In the same way as in a) we can find a subset $Z_0 \subset Z$ of measure 0 such that for $z \notin Z_0$ $\alpha_z(a_i\xi_j) = a_i\alpha_z(\xi_j)$ for all i, j. Since both sides are continuous in ξ, we have $\alpha_z(a_i\xi) = a_i\alpha_z(\xi)$ for all i and all $\xi \in S$. Since both sides are continuous in a, we have $\alpha_z(a\xi) = a \cdot \alpha_z(\xi)$ for all $\xi \in S, a \in A$. Now define the family of morphisms of A-modules, $\alpha'_z : S \to H_z$, by $\alpha'_z = \alpha_z$ for $z \notin Z_0$ and $\alpha'_z = 0$ for $z \in Z_0$.

c) Fix a dense subset $\{\xi_1, \ldots \xi_k, \ldots\}$ in S and apply lemma 1.2b to $\{\alpha(\xi_i)\}$. ■

1.4. Let $\alpha : S \to S'$ be a morphism of topological vector spaces. We say that α is *fine* if S is separable and if for each morphism $\gamma : S' \to H = \int H_z \, d\mu_Z$ the composition $\gamma \circ \alpha : S \to H$ is pointwise defined.

If α is a composition of two morphisms α_1, α_2, one of which is fine, and if the space S is separable, then clearly α is fine.

Usually we will deal with the following situation:

$$H = \int_Z H_z \, d\mu_Z$$

is a Hilbert space and $\alpha : S \to H$ is a fine ambedding with a dense image. If one interpretes S as some space of test functions, then it is natural to view its Hermitian dual S^+ as the corresponding space of distributions and consider the Gelfand triple $S \subset H \subset S^+$. Since $\alpha : S \to H$ is pointwise defined, we can choose a family of projections $\alpha_z : S \to H_z$.

If we denote by β_z the family of adjoint morphisms $\beta_z : H_z \to S^+$, then for $\phi \in S$ we have $\phi = \int \phi_z \, d\mu_Z$, where $\phi_z = \beta_z \alpha_z(\phi)$ and the equality is understood to hold in S^+.

Suppose a group G acts in a compatible way on S, H, H_z and the representations of G in H, H_z are unitary. Then we can choose all α_z to be morphisms of G-modules. If we define the action of G on S^+ as $(g^+)^{-1}$, then all morphisms β_z and the inclusion $H \subset S^+$ also will be G-equivariant.

1.5. THEOREM *(Gelfand-Kostyuchenko, see* [GeKo], [GeVi], *ch. 4 or* [Ma]).
Let L be a separable Hilbert space and $\alpha : L \to H$ a Hilbert-Schmidt morphism. Then α is fine. ■

Let us recall the definition of a Hilbert-Schmidt morphism.

DEFINITION. Let $\alpha : L \to H$ be a morphism of Hilbert spaces. We say that α is Hilbert-Schmidt if for every orthonormal basis $\{\xi_i\}$ of L the sum

$$M = \sum_i \| \alpha(\xi_i) \|^2 < \infty .$$

LEMMA. *a) The sum* M *does not depend on the choice of the basis. We will denote it by* $M(\alpha)$.

b) If α *is Hilbert-Schmidt then the adjoint morphism* $\alpha^+ : H \to L$ *is Hilbert-Schmidt and* $M(\alpha) = M(\alpha^+)$.

Proof. Choose an orthonormal basis $\{\eta_j\}$ of H. Then

$$M(\alpha) = \sum_i \| \alpha(\xi_i) \|^2 = \sum_{i,j} | (\alpha(\xi_i), \eta_j) |^2 =$$

$$= \sum_{i,j} | (\xi_i, \alpha^+ \eta_j) |^2 = \sum_j \| \alpha^+ \eta_j \|^2 = M(\alpha^+)$$

which proves both a) and b).

We will use the following

LEMMA. *Let* $\{\xi_i\}$ *be an orthonormal basis of* L *and* $\{\eta_i\}$ *a sequence of vectors in* H. *Suppose that the sum*

$$M = \sum_i \| \eta_i \|^2 < \infty .$$

Then there exists a unique Hilbert-Schmidt operator $\alpha : L \to H$ *such that* $\alpha(\xi_i) = = \eta_i$ *for all* i.

Proof. Uniqueness is obvious. To prove existence we define α by $\xi = \Sigma\, c_i \xi_i \mapsto \alpha(\xi) = \Sigma c_i \eta_i$. Since $\Sigma\, | c_i |^2 = \| \xi \|^2$ we have $(\Sigma\, \| c_i \eta_i \|)^2 = (\Sigma\, | c_i |\, \| \eta_i \|)^2 \leqslant \Sigma\, | c_i |^2 \cdot \Sigma\, \| \eta_i \|^2 \leqslant M \| \xi \|^2$, which shows that the sum is convergent and that α is bounded. By definition $M(\alpha) = M$. ∎

Proof of the Theorem. It is enough to check that for any decomposition $H = \int H_z \, d\mu_Z$ the morphism $\alpha : L \to H$ is pointwise defined. Choose an orthonormal basis $\{\xi_i\}$ of L, set $\eta_i = \alpha(\xi_i) \in H$ and choose some sections $z \mapsto \eta_{iz}$ representing η_i. By definition

$$M(\alpha) = \sum_i \| \eta_i \|^2 = \sum_i \int_Z \| \eta_{iz} \|^2 \, d\mu_Z .$$

Since everything is positive, we can write

$$M(\alpha) = \int_Z M_z \, d\mu_Z \quad \text{where} \quad M_z = \sum_i \| \eta_{iz} \|^2.$$

In particular, this implies that the set $Z_0 = \{z \in Z \mid M_z = \infty\}$ has measure 0. Now, let us define the family of morphisms $\alpha_z : L \to H_z$ by $\alpha_z = 0$ if $z \in Z_0$, $\alpha_z(\xi_i) = \eta_{iz}$ for all i if $z \notin Z_0$, i.e. if $M_z < \infty$. Let us show that for each vector $\xi \in L$ the section $z \mapsto \alpha_z(\xi)$ represents the vector $\alpha(\xi) \in H$. The space L' of all ξ which have this property contains the basis $\{\xi_i\}$ and hence is dense in L. Thus it is enough to check that L' is closed. Let $\phi_1 \ \phi_2, \ldots$ be a sequence of vectors in L' and $\phi_i \to \phi \in L$. Then $\alpha(\phi_i) \to \alpha(\phi)$ and hence, passing to a subsequence, we can assume that $\alpha_z(\phi_i) \to \alpha(\phi)_z$ for almost each z (see Lemma 1.2). On the other hand, for each z $\alpha_z(\phi_i) \mapsto \alpha_z(\phi)$. This shows that the section $z \mapsto \alpha_z(\phi)$ represents the vector $\alpha(\phi)$, i.e. $\phi \in L'$. Q.E.D. ■

1.6. The following lemmas are useful in proving that a morphism is Hilbert-Schmidt of fine.

LEMMA 1 *Let L be a separable Hilbert space, $H = L^2(X, \mu_X)$. Suppose that for each $x \in X$ we are given a linear functional α_x on L such that*

(i) For every $\xi \in L$ the function $x \to \alpha_x(\xi)$ is measurable.

(ii) Each α_x is bounded and $M_x = \| \alpha_x \|^2$ satisfies

$$M = \int M_x \, d\mu_X < \infty .$$

Then the morphism $\alpha : L \to H$ given by $\alpha(\xi)(x) = \alpha_x(\xi)$ is Hilbert-Schmidt and $M(\alpha) = M$.

Proof. Clearly $\| \alpha \| \leqslant M^{1/2}$. Choose an orthonormal basis $\{\xi_i\}$ in L. Then

$$M(\alpha) = \sum_i \| \alpha(\xi_i) \|^2 = \sum_i \int_X \| \alpha_x(\xi_i) \|^2 \, d\mu_X =$$

$$= \int_X \sum_i \| \alpha_x(\xi_i) \|^2 \, d\mu_X = \int_X M_x \, d\mu_X = M.$$ ■

LEMMA 2. *Let $\alpha : S \to S'$ be a morphism of topological vector spaces.*

Suppose that S is a direct limit of an increasing sequence of subspaces $S_1 \subset S_2 \subset \ldots \subset S$, *i.e.* $S = \bigcup_i S_i$ *and the topology of* S *is the weakest one in which all embeddings* $S_i \to S$ *are continuous. Suppose that for each index* i *the morphism* $\alpha_i = \alpha|_{S_i} : S_i \to S'$ *is fine. Then* α *is fine.*

Proof. We can assume $S' = H = \int H_z \, d\mu_Z$. Consider families of morphism α_{iz}. By Lemma 1.3.a) $\alpha_{iz} = \alpha_{i+1,z}$ for all z outside of a subset Z_i of measure 0. Set $Z_0 = \bigcup_i Z_i$ and define the family of morphisms α_z by $\alpha_z = \varinjlim \alpha_{zi}$ for $z \notin Z_0$, $\alpha_z = 0$ for $z \in Z_0$. ■

2. THE ALGEBRAIC NECESSARY CONDITION

Consider the following situation: G is a locally compact group, $\Gamma \subset G$ a closed sub-group, $X = G/\Gamma$. For simplicity we assume that X has a G-invariant Haar measure μ_X. We fix μ_X and consider the regular representation π of G in the space $H = L^2(X, \mu_X)$. Our goal is to find some restrictions on irreducible representations of G which can appear in the spectral decomposition of (π, H).

2.1. Local structure of G

We are interested in applications to real, p-adic and adelic groups. So we make the following assumpitons on G.

Assumption I. G has a countable base.

Assumption II. There exists a closed subgroup $K \subset G$, such that

(i) K is a profinite group (i.e. K is compact and totally disconnected).

(ii) Its normalizer $G_K = \mathrm{Norm}(K; G)$ is open in G and the quotient G_K/K is a Lie group.

A subgroup $K \subset G$ satisfying conditions (i), (ii) we call *regular.*

Example. If G is a real Lie groups we take $K = \{e\}$. For p-adic G we take K to be an open compact subgroup. For adelic G we take K to be an open compact subgroup of its nonarchimedean part.

Remark. Condition II is equivalent to the condition that G has a finite topological dimension (see [Ka]).

Let us describe the local structure of G. Fix a regular subgroup $K \subset G$, denote by G' the connected component of G_K/K, and by $\tilde{G}$ the universal covering of G'.

PROPOSITION. *There exists a unique morphism* $i : \tilde{G} \to G$ *compatible with the projection* $p : \tilde{G} \to G' = G_K/K$. *Group* $i(\tilde{G})$ *commutes with* K *and the morphism* $i : \tilde{G} \times K \to G$ *is a local homeomorphism.*

Proof: Replacing G with an open subgroup we can assume that K is normal in G and that the group G/K is connected.

Let $K_0 \subset K$ be an open normal subgroup. Consider the adjoint action of G on K. Since K_0 is compact and open in K, its stabilizer G_{K_0} is open in G. This implies that its image in $G' = G/K$ is open and, since G' is connected, it coincides with G'. Since $G_{K_0} \supset K$, this means that $G_{K_0} = G$. Thus $G/K_0 \to G/K$ is a finite covering, and hence morphism $p : \tilde{G} \to G' = G/K$ can be uniquely lifted to G/K_0. Since K is a limit $K = \varprojlim K/K_0$, where K_0 runs through open normal subgroups, we see that $G = \varprojlim G/K_0$. Hence there exists a unique morphism $i : \tilde{G} \to G$ compatible with $p : \tilde{G} \to G' = G/K$.

Since G is connected, the adjoint action of $i(g)$ on K is trivial for all $g \in \tilde{G}$, i.e. $i(\tilde{G})$ commutes with K. Locally the covering $p : \tilde{G} \to G'$ is a homeomorphism, and we denote by p^{-1} the inverse local homeomorhipsm. Then locally we can define the inverse of the morphism $i' : \tilde{G} \times K \to G$ by $g \to (\tilde{g}, g\tilde{g}^{-1})$, where $\tilde{g} = p^{-1}(g \bmod K)$. ■

Remark. It is easy to see that the group $\tilde{G}$ and the morphism $i : \tilde{G} \to G$ do not depend on the choice of K.

The Lie algebra $\underline{g} = \text{Lie}\,(\tilde{G})$ we will call the *Lie algebra of G.* We denote by $U(\underline{g})$ its universal enveloping algebra. The pair $(\underline{g}, K)$ completely determines the local structure of G.

We call a subset $B \subset G$ a *ball* if it is a compact symmetric neighborhood of identity (symmetric means that $g \in B \Rightarrow g^{-1} \in B$). If B is a ball, then for all $n \geqslant 1$ the subset $B^n = B \cdot B \cdot \ldots \cdot B$ (n factors) is also a ball. Any compact subset $\Omega \subset G$ lies in a ball (e.g. $\Omega \subset B \cup B\,\Omega\,B \cup B\Omega^{-1}B$).

Using assumption I on G, we can choose a regular subgroup K, a system of normal regular subgroups $K_1 \supset K_2, \ldots$, in K such that $\cap K_i = \{e\}$, and a sequence of K-biinvariant balls $B_0 \subset B_1 \subset \ldots$ such that $G = \cup B_i$.

2.2. Garding spaces and Schwartz spaces

Let (ρ, V) be a topological G-module. For a regular subgroup $K \subset G$ and $k \geqslant 0$ we set

$$V^{K,k} = \{\xi \in V \mid \xi \text{ is } K\text{-invariant and the function } g \to \rho(g)\xi \text{ lies in } C^k(G/K, V)\}.$$

Further we set $V^{K,\infty} = \bigcap_k V^{K,k}$ (inverse limit topology) and $V^\infty = \bigcup_K V^{K,\infty}$ (direct limit topology). The space V^∞ is called *the Garding space* of V. It is a G-module and a $U(\underline{g})$-module.

Let $X = G/\Gamma$ be a homogeneous space of G, $C(X)$ the space of continuous functions on X and π the natural action of G on $C(X)$, $\pi(g)(f)(x) = f(g^{-1}x)$. We denote by $C(X)^\infty$ the corresponding Garding space. For each compact $\Omega \subset X$ set $C_\Omega^\infty = \{f \in C(X)^\infty \mid \operatorname{supp} f \subset \Omega\}$. The space $S(X) = \cup C_\Omega^\infty$ we call the *Schwartz space* of X (another definition $S(X) = (C_c(X))^\infty$).

Similarly, we consider the G-module $(\pi, M(X))$ of locally bounded Radon measures on X, with the action $\langle \pi(g), m, f \rangle = \langle m, \pi(g^{-1}) f \rangle$. It is well known, that, for any Haar measure μ_X, one has $M(X)^\infty = C(X)^\infty \cdot \mu_X$.

The space $M_c(G)$ of compactly supported measures on G is an algebra with respect to convolution. If (ρ, V) is a complete G-module, we define the action of this algebra on V by

$$\rho(a)(\xi) = \int_G \rho(g)\, \xi \, da(g), \quad a \in M_c(G), \ \xi \in V.$$

Clearly $\rho(M_c(G)^\infty)\, V \subset V^\infty$ is dense in V.

The antiinvolution $g \mapsto g^{-1}$ on G defines an antiinvolution of the algebra $M_c(G)$, $a \mapsto a^*$. If (ρ_1, V_1), (ρ_2, V_2) have a G-invariant pairing $\langle\, , \rangle : V_1 \times V_2 \to \mathbb{C}$, then $\langle \rho_1(a) v_1, v_2 \rangle = \langle v_1, \rho_2(a^*) v_2 \rangle$.

The antilinear antiinvolution $a \mapsto a^+ = \bar{a}^*$ has an analogous property with respect to Hermitian pairings.

2.3. Algebraic necessary condition

PROPOSITION. *Let* $X = G/\Gamma$, $H = L^2(X, \mu_X)$. *Then the natural embedding* $\alpha : S(X) \to H$ *is fine (see 1.4).*

This proposition means that for any decomposition

$$H = \int_Z H_z \, d\mu_Z$$

there exists a family of morphisms of G-modules $\alpha_z : S(X) \to H_z$, *which represents* α. *We interpret it by saying that a representation* (ρ, V) *can contribute to a spectral decomposition of* H *only if there exists a nonzero G-morphism* $\alpha_V : S(X) \to V$. *This is an algebraic necessary condition of 0.2.*

Proof of Proposition. By Lemma 2 in 1.6 it is enough to check that for a fixed regular subgroup $K \subset G$ and a compact subset $\Omega \subset X$ the inclusion $C_\Omega^{K,\infty} \to L^2(X, \mu_X)$ is fine.

Choose $k \geqslant \dim \underline{g}$, consider in $U(g)$ the subspace $U(\underline{g})^k$, spanned by 1, $\underline{g}, \underline{g}^2, \ldots, \underline{g}^k$ and fix a basis $d_1, \ldots, d_r$ in $U(\underline{g})^k$. For every function $f \in C^\infty(X)$ define the function $Q(f)$ on X by

$$Q(f) = \sum_{i=1}^{r} |d_i(f)|^2,$$

and define the Q-norm of f as

$$\| f \|_Q^2 = \int_X Q(f)\, d\mu_X .$$

Let $L^2(X; Q)$ be the completion of $S(X)$ with respect to this norm and $L^2(X, Q)_\Omega^K$ the closure of the subspace $C_\Omega^{K,\infty}$ in $L^2(X, Q)$. We want to prove that the natural inclusion $\alpha : L^2(X, Q)_\Omega^K \to L^2(X, \mu_X)$ is fine, and by the Gelfand-Kostyuchenko theorem (see 1.5) it is enough to check that it is Hilbert-Schmidt.

We will prove this using Lemma 1 from 1.6: for each $x \in X$ consider the functional α'_x on $L^2(X, Q)^K$ given by $\alpha'_x(f) = f(x)$ and denote by α_x its restriction to $L^2(X, Q)_\Omega^K$. We have to show that the function $M_x = \| \alpha_x \|^2$ is integrable on X. Clearly $M_x = 0$ for $x \notin \Omega$ and $M_x \leqslant M'_x = \| \alpha'_x \|^2$. So the proposition follows from the following result which we will prove in 3.4. (*) The functional α'_x on $L^2(X, Q)^K$ is bounded and the function $M'_x = \| \alpha'_x \|^2$ is locally bounded on X. ■

2.4. Forms

Let us denote by $S(X)^+$ the Hermitian dual of $S(X)$ and consider the Gelfand triple $S(X) \subset H \subset S(X)^+$. We will interpret $S(X)^+$ as the space of distributions on X. For each G-morphism $\alpha_V : S(X) \to V$, define the adjoint morphism $\alpha_V^+ : V \to S(X)^+$ by $(\alpha_V^+(v), \phi) = (v, \alpha_V(\phi))$. We will show that $\alpha_V^+(V^\infty)$ consists of smooth distributions, i.e. there exists a morphism $\beta_V : V^\infty \to C(X)^\infty$ such that $\alpha_V^+(v) = \beta_V(v) \cdot \mu_X$ (note that β_V depends on the choice of the Haar measure μ_X). Any G-morphism $\beta : V^\infty \to C(X)^\infty$ we call a *V-form on X.*

PROPOSITION. *Fix a Haar measure* μ_X *on* X. *Then it defines an isomorphism* $\alpha_V \leftrightarrow \beta_V$ *between* $\mathrm{Hom}_G(S(X), V)$ *and* $\mathrm{Hom}_G(V^\infty, C(X)^\infty)$ *via*
$(v, \alpha_V(\phi)) = (\beta_V(v)\, \mu_X, \phi)$

Proof. (i) Clearly we can replace G by an open subgroup, so we can assume that G has arbitrarily small regular normal subgroups K (see 2.1). It is enough to check the isomorphism $\mathrm{Hom}_G(S(X)^K, V^K) = \mathrm{Hom}_G(V^{K,\infty}, C(X)^{K,\infty})$ for each of these subgroups K. Hence, replacing G by $K \backslash G$ and X by $K \backslash X$, we can assume that G is a Lie group.

(ii) Let $\alpha : S(X) \to V$ be a G-morphism. We want to show that the adjoint morphism α^+ maps V^∞ into the subspace $C(X)\,\mu_X$, and hence into $C(X)^\infty \mu_X$.

Fix a point $x \in X$ and its relatively compact neighbourhood Ω. By definition of topology on $S(X)$, we can extend α to a morphism $\alpha : C(X)^k_\Omega \to V$ for some $k > 0$.

We will use the following standard

STATEMENT. *For every $k > 0$ we can find some $n > 0$ operators $d_i \in U(\underline{g})^n$ and measures a_i on G, supported in a small neighbourhood of identity, such that*

(i) a_i are of class C^k on G.

(ii) $\Sigma\, a_i * d_i = \delta_e$ – the δ measure at identity. (Here we identify d_i with a distribution $d_i\delta_e$ on G).

Now for each vector $v \in V^n$ and function $\phi \in C(X)^k_\Omega$ we have
$\langle \alpha^+ v, \phi\rangle = \langle \alpha^+ (\Sigma\, \rho(a_i)\, d_i v), \phi\rangle = \langle \Sigma\, \rho(a_i)\, d_i v, \alpha\phi\rangle = \Sigma\, \langle d_i v, \alpha(\pi(a_i^+)\, \phi)\, \rangle$,
where a_i^+ are measures on G of class C^k. Clearly, the right-hand side is defined when ϕ is a δ-function δ_y at some point y near x.

Thus near the point x we get a function $y \mapsto f(y) = \Sigma\, \langle d_i v, \alpha(\pi(a_i^+)\, \delta_y\,)\rangle$. It is easy to check, that f is continuous, $|f(y)| \leqslant C \cdot \|v\|_{V^n}$ (the norm of v in V^n), and that near x, $\alpha^+(v) = f\mu_X$.

(iii) Suppose we are given a G-morphism $\beta : V^\infty \to C(X)^\infty$. We want to show that it corresponds to a morphism $\alpha : S(X) \to V$. It is enough to check that for all $\phi \in S(X)$ the functional $v \mapsto \langle \beta(v)\mu_X, \phi\rangle$ is bounded in $\|\ \|_V$.

Fix a compact $\Omega \subset X$. By definition of topology on V^∞ we see that for some $k \geqslant 0$ we have a bound

$$\|\beta(v)\|_{L^2(\Omega,\, \mu_X)} \leqslant C\|v\|_{V^k}.$$

As in (ii), we can write $\delta_e = \Sigma\, a_i * d_i$. Then for $v \in V^\infty$, $\phi \in C(X)^\infty_\Omega$ we have $\langle \beta(v)\mu_X, \phi\rangle = \langle \beta(v)\mu_X, \Sigma\, \pi(a_i)\, \pi(d_i)\, \phi\rangle = \Sigma\, \langle \beta(\rho(a_i^+\, v)\, \mu_X, \pi(d_i)\, \phi\rangle$.
Using inequalities $\|\beta(\rho(a_i^+)v)\|_{L^2} \leqslant \|\rho(a_i^+)v\|_{V^k} \leqslant \|v\|_V$ and $\|d_i\,\phi\|_{L^2} \leqslant \|\phi\|_n$, where $\|\ \|_{L^2}$ and $\|\ \|_n$ are norms in $L^2(\Omega, \mu_X)$ and in $C(X)^n$, we see that $|\langle \beta(v)\mu_X, \phi\rangle| \leqslant C \cdot \|v\|_V \cdot \|\phi\|_n$. This shows that the adjoint morphism α is defined on $C(X)^\infty_\Omega$ and has a bound $\|\alpha(\phi)\|_V \leqslant C\|\phi\|_n$.
This proves the proposition. ■

2.5. Frobenius reciprocity

The space of V-forms on $X = G/\Gamma$ can be naturally identified with the space of Γ-invariant functionals on V^∞. Namely, to each G-morphism $\beta : V^\infty \to C(X)^\infty$ corresponds the functional $\gamma : V^\infty \to \mathbb{C}$ given by $\gamma(v) = \beta(v)(x_0)$, where $x_0 \in X$ is the class of Γ. Conversely, given $\gamma \in \mathrm{Hom}_\Gamma(V^\infty, \mathbb{C})$ we define β by the formula $\beta(v)(g) = \gamma(g^{-1}v)$. This identifies V-forms with $\mathrm{Hom}_\Gamma(V^\infty, \mathbb{C})$ (see details in [Ol]).

3. THE ANALYTIC NECESSARY CONDITION

3.1. Weights and tempered forms

Let (ρ, V) be an irreducible unitary G-module and $\beta : V^\infty \to C(X)^\infty$ a V-form on X. In this section we prove some analytic necessary conditions on forms β which can contribute to the spectral decomposition of the space $H = L^2(X, \mu_X)$.

Suppose we are given a G-module S in-between $S(X)$ and H, i.e. we have G-morphisms $i' : S(X) \to S$ and $\alpha' : S \to H$, such that both i', α' are embeddings, the image of i' is dense in S, and the composition $\alpha = \alpha' \circ i' : S(X) \to H$ is the standard embedding. We say that a V-form β_V is *S-tempered* if the corresponding morphism $\alpha_V : S(X) \to V$ can be extended to a morphism $\alpha'_V : S \to V$. If the morphism $\alpha' : S \to H$ is fine, then the Gelfand-Kostyuchenko method implies that only S-tempered V-forms can contribute to the spectral decomposition of H. This condition we call an analytic necessary condition.

We will choose S to be the Garding space of space $L^2(X, w\mu_X)$ for some function w on X. Since we want G to act on the space, we will impose some restrictions on w.

DEFINITION: A *weight* on X is a strictly positive function w on X which satisfies the following condition:

For every ball $B \subset G$ there exists a constant $C = C(B, w)$ such that $w(gx) \leqslant Cw(x)$ for all $g \in B$, $x \in X$.

For every continuous weight w on X we define G-module (π, L_w) by $L_w = L^2(X, w\mu_X)$ and $\pi(g)(x) = f(g^{-1}x)$, $f \in L_w$.

The Garding space L_w^∞ we denote by S_w.

If w and w' are comparable continuous weights, then the spaces L_w and $L_{w'}$ coincides as spaces of functions on X and as topological G-modules. In particular, $S_w = S_{w'}$.

Fix a continuous weight w and let us describe S_w-tempered V-forms. Let $\beta : V^\infty \to C(X)^\infty$ be a V-form, $\alpha : S(X) \to V$ the corresponding G-morphism. For each $v \in V^\infty$ the function $f = \beta(v)$ is defined by the condition that for each

$$\phi \in S(X) \quad (*) \ \langle v, \alpha(\phi) \rangle = \int_X f\bar{\phi}\, \mathrm{d}\mu_X .$$

If β is S_w-tempered, then the morphism α can be extended to $\alpha' : S_w \to V$, which would imply that the right hand side of (*) extends to $\phi \in S_w$. This implies that $f \in L^2(X, w^{-1} \cdot \mu_X) = L_{w^{-1}}$. Thus, for all S_w-tempered V-forms β we have $\beta(V^\infty) \subset L_{w^{-1}}$. Conversely, it is easy to see that this condition is equivalent to the fact that the form β is S_w-tempered.

It is convenient to define G-modules L_w and S_w for all weights. This can be done using the following

LEMMA. *For every weight w there exists a comparable continuous weight w'.*

Using this lemma, we will define $L_w = L_{w'}$. By the remark above, this definition does not depend on the choice of a continuous weight w' comparable to w.

Proof of the lemma. Fix a function $f \in C_c(G)$ such that $f(e) = 1$ and $f(g) \in [0, 1]$ for all $g \in G$, and define a function w' on X by $w'(x) = \sup\{f(g)\, w(gx) \mid g \in G\}$. Clearly, $w(x) \leqslant w'(x) \leqslant C(B, w)w(x)$, where B is a ball, containing $\operatorname{supp}(f)$. This shows that w' is comparable to w and, in particular, that w' is a weight.

Let us show that w' is multiplicatively uniformly continuous: for every $D < 1$ there exists a neighborhood of identity $U \subset G$ such that $w'(ux) \geqslant D^2 w'(x)$ for all $u \in U$, $x \in X$. Indeed since f is continuous and has compact support, for every $\epsilon > 0$ we can find a neighborhood U such that $|f(gu^{-1}) - f(g)| < \epsilon$ for all $g \in G$, $u \in U$.

By definition of $w'(x)$, we can find a $g \in G$ such that $f(g)\, w(gx)$ is close to $w'(x)$, e.g. $f(g)\, w(gx) > D\, w'(x)$. Moreover, since $g \in \operatorname{supp}(f) \subset B$, $w(gx) < Cw(x)$ with $C = C(B, w)$, i.e. we can always assume $f(g) \geqslant C^{-1}$. For all $u \in U$, $f(gu^{-1}) \geqslant f(g) - \epsilon$, i.e. for the appropriate choice of ϵ, we have $f(gu^{-1}) > D\, f(g)$. This implies that $w'(ux) \geqslant f(gu^{-1})\, w(gx) \geqslant Df(g)\, w(gx) \geqslant D^2 w'(x)$. Q.E.D. ■

3.2. Summable weights

A subset $N \subset X$ is called a *net* if there exists a ball $B \subset G$ such that $B \cdot N = X$. We say that a weight w on X is *summable* if for some countable net $N \subset X$ it satisfies

$$\sum_{n \in N} w^{-1}(n) < \infty,$$

The following theorem, which we prove in 3.4, is the central result of the paper.

THEOREM. *Let w be a summable weight on X. Then the inclusion $S_w \to H = L^2(X, \mu_X)$ is fine.*

Thus, as explained in 3.1. every summable weight w gives the following analytic necessary condition.

Cond(w): Only S_w-tempered V-form on X can contribute to the spectral decomposition of $L^2(X, \mu_X)$.

Let us discuss in more details the notion of summable weight.

Let $N \subset X$ be a net. We say that N is *sparse* if for each ball $\Omega \subset G$ the number of points in $N \cap \Omega x$ is bounded by a constant $k(N, \Omega)$ independent of x.

Criterion. a) Sparse nets exist;
b) Let N be a sparse net. Then a weight w on X is summable iff it is N-summable, i.e. iff

$$\sum_{n \in N} w(n)^{-1} < \infty,$$

Proof. a) Fix a ball B and fix a maximal (with respect to inclusion) subset $N \subset X$ such that the sets $\{Bn \mid n \in N\}$ are all disjoint. We claim that N is a sparse net. First of all, $B^2 \cdot N = X$, since otherwise one can find a point $x \notin B^2 \cdot N$ and then the ball Bx is disjoint from all the balls $\{Bn \mid n \in N\}$, which contradicts the maximality of N.

Any compact $\Omega \subset G$ can be covered with a finite number of shifted balls Bg_i, $i = 1, \ldots, k$. Then for all $x \in X$ we have

$$\#(N \cap \Omega x) \leqslant \sum_i \#(N \cap Bg_i x) \leqslant \sum_i 1 = k,$$

since for all

$$y \in X \qquad \#(N \cap By) = \# \ n \in N | \{Bn \ni y\} \leqslant 1.$$

Thus N is a sparse net.

b) We have to check that a weight w which is N'-summable for some net N' is automatically N summable. Choose a ball $B \subset G$, such that $BN' = X$ and set $C = C(B, w)$, $k = k(N, B)$. Then we have

$$\sum_{n \in N} w(n)^{-1} \leqslant \sum_{n' \in N'} \left(\sum_{n \in N \cap Bn'} w(n)^{-1} \right) \leqslant \sum_{n' \in N'} k \cdot C \cdot w(n')^{-1} < \infty ,$$

since $\# (N \cap Bn') \leqslant k$ and $w(n') \leqslant Cw(n)$ for $n \in Bn'$. ∎

3.3. Standard measure on X

LEMMA-DEFINITION. a) *There exists a strictly positive measure* m_X *on* X *satisfying*

(i) m_X *is a weight measure, i.e. for each ball* $B \subset G$ *there exists a* $C > 0$ *such that* $\pi(g) \cdot m_X \leqslant C m_X$ *for all* $g \in B$.

(ii) For every ball $B \subset G$ *there exists a* $C > 0$ *such that* $C^{-1} \leqslant m_X(Bx) \leqslant C$ *for all* $x \in X$.

b) Any measure m'_X *satisfying (i) and (ii) is comparable to* m_X.

c) Fix a ball $B \subset G$ *and a left invariant Haar measure* μ_G *on* G. *Then there exist constants* $C_1, C_2 > 0$, *such that for all positive functions* Q *on* X *and for all* $x \in X$ *one has*

$$C_1 \int_B Q(g^{-1}x)\, d\mu_G(g) \leqslant \int_{Bx} Q(y)\, dm_X(y) \leqslant C_2 \int_{B^3} Q(g^{-1}x)\, d\mu_G(g)$$

The measure m_X (or rather the comparability class of this measure), we will call *a standard measure on X.* Note that it is quite different from the Haar measure μ_X (as we will see in 3.6, it exists even if μ_X does not).

We will use the inequality c) in the proof of theorem 3.2.

Proof. (i) First let us prove uniqueness. If m'_X is another measure, satisfying (*) and (**) then the ratio $w = m'_X / m_X$ is a function and, moreover, a weight. Fix a ball $B \subset G$. Then, for all $x \in X$, $m'_X(Bx) < C'$ and $m_X(Bx) \geqslant C^{-1}$, which implies that, for some $y \in Bx$, $w(y) \leqslant CC'$. Since w is a weight, it is bounded on Bx by some constant independent of x, i.e. $m'_X \ll m_X$. Similarly $m_X \ll m'_X$.

(ii) Fix a ball $B \subset G$ and define a function ν_B on X by $\nu_B(x) = \mu_X(Bx)^{-1}$. We claim that ν_B is a weight and $m_X = \nu_B \cdot \mu_X$ is a standard measure on X.

If B' is another ball, we can cover it with a finite number of shifted balls $g_i B$, $i = 1, \ldots, k$. Then $\nu_{B'}(x)^{-1} = \mu_X(B'x) \leqslant \Sigma\, \mu_X(g_i Bx) \leqslant k \cdot \mu_X(Bx) \leqslant$

$\leqslant k \cdot \nu_B(x)^{-1}$, i.e. $\nu_B \leqslant k \cdot \nu_{B'}$,
which shows that ν_B and $\nu_{B'}$ are comparable.

Let $\Omega \subset G$ be a compact. Choose a ball B' containing $B\Omega$. Then for any $g \in \Omega$ we have

$$\nu_B(gx) = \mu_X(Bgx)^{-1} \geqslant \mu_X(B'x)^{-1} = \nu_{B'}(x) \geqslant \frac{1}{k}\, \nu_B(x),$$

i.e. $\nu_B(x) \leqslant k\nu_B(gx)$. This shows that ν_B is a weight.

This fact and the definition of ν_B imply that there exists a $C > 0$ such that $C^{-1} \leqslant m_X(Bx) \leqslant C$ for all $x \in X$, where $m_X = \nu_B \cdot \mu_X$. For any other ball B' the measure $m'_X = \nu_{B'} \cdot \mu_X$ is comparable to m_X, and satisfies $C_1^{-1} \leqslant m'_X(B'x) \leqslant C_1$. This implies that $C_2^{-1} \leqslant m_X(B'x) \leqslant C_2$ for some $C_2 > 0$, which means that m_X is a standard measure.

(iii) For $i = 1, 2, 3$ consider on G measures $a_i = \chi(B^i)\, \mu_G$, where $\chi(B^i)$ is the characteristic function of the ball B^i. Set

$$Q_i = \pi(a_i)Q, \quad d = \int_{Bx} Q(y)\, dm_X(y).$$

We have to prove the inequalities $C_1 Q_1(x) \leqslant d \leqslant C_2 Q_3(x)$.

We can (and will) assume that Q is supported on Bx. Then

$$\int_X Q\, dm_X = d$$

and since m_X is a weight measure, all integrals

$$\int_X Q_i\, dm_X$$

are comparable to d.

For all $b \in B$ we have $\delta_b * a_1 \leqslant a_2$, hence $\pi(b)Q_1 \leqslant Q_2$. Thus for all $y \in Bx$ we have $Q_2(y) \geqslant Q_1(x)$, which implies that

$$\int_X Q_2(y)\, dm_X \geqslant Q_1(x)\, m_X(Bx) \geqslant C^{-1} Q_1(x).$$

Since this integral is comparable to d, we get the first inequality, $C_1 Q_1(x) \leqslant d$.

Similarly, Q_1 is supported in B^2x, and for all $y \in B^2x$, $Q_1(y) \leqslant Q_3(x)$. Thus $\int Q_1 \, dm_X \leqslant Q_3(x) \cdot m_X(B^2 \cdot x) \leqslant C\, Q_3(x)$, and since the integral is comparable to d we get the second inequality, $d \leqslant C_2 Q_3(x)$. ■

CRITERION. *A weight* w *is summable if and only if*

$$\int_X w^{-1} \, dm_X < \infty .$$

Proof. Choose a sparse net N and a ball $B \subset G$ such that $B \cdot N = X$. Then it is clear that

$$\int_X w^{-1} \, dm_X$$

is comparable to

$$\sum_{n \in N} \int_{Bn} w^{-1} \, dm_X ,$$

which in turn is comparable to

$$\sum_{n \in N} w(n)^{-1} .$$

This proves the criterion. ■

3.4. Proof of Theorem 3.2

Fix a summable weight w. We want to show that the embedding $S_w \to H$ is fine. By lemma 2 in 1.6 is enough to check that for any regular subgroup $K \subset G$ and for a sufficiently large k the embedding $\alpha : L_w^{K,k} \to H$ is fine. This follows from theorem 1.5 and the following

PROPOSITION. *Let* w *be a summable weight on* X, $K \subset G$ *a regular subgroup,* k *a natural number,* $k \geqslant \dim \underline{g}$, *where* $\underline{g}$ *is the Lie algebra associated to* G *(see 2.1). Then the natural embedding* $\alpha : L_w^{K,k} \to L^2(X, \mu_X)$ *is a Hilbert-Schmidt morphism.*

First of all let us describe the scalar product on $L_w^{K,k}$. Fix a basis $d_1, \ldots, d_r$ of $U(\underline{g})^k$ (the elements of the universal enveloping algebra of $\underline{g}$ of degree $\leqslant k$) and for each function f on X consider a new function $Q(f)$ on X given by

$$Q(f) = \sum_i |d_i f|^2.$$

Then $L_w^{K,k}$ is the completion of $S(X)^K$ with respect to the norm

$$\|f\|_{Q,w}^2 = \int_X Q(f) \cdot w \, d\mu_X,$$

KEY LEMMA. *Fix a standard measure* m_X *on* X, *a regular subgroup* $K \subset G$, *a ball* $B \subset G$ *and a* $k \geqslant \dim \underline{g}$. Then there exists a $C > 0$, such that for any function $f \in C(X)^{K,k}$ one has

$$|f(x)|^2 \leqslant C \cdot \int_{Bx} Q(f)\, dm_X.$$

This lemma implies the proposition. Indeed, let us consider for every $x \in X$ a functional $\alpha_x : S(X)^K \to \mathbb{C}$, $\alpha_x(f) = f(x)$ and set $M_x = \|\alpha_x\|_{Q,w}^2$ i.e. $M_x = \sup\{|f(x)|^2 / \|f\|_{Q,w}^2 \mid f \in S(X)^K\}$. By the key lemma, we have

$$|f(x)|^2 \leqslant C \cdot \int_{Bx} Q(f)\, dm_X \leqslant C_1 \nu(x)\, w(x)^{-1}.$$

$$\cdot \int_{Bx} Q(f)\, w d\mu_X \leqslant C_1 \nu(x) \cdot w(x)^{-1} \cdot \|f\|_{Q,w}^2,$$

where $m_X = \nu\mu_X$. Thus $M_x \leqslant C_1 \nu(x) \cdot w(x)^{-1}$, and hence

$$\int M_x \, d\mu_X \leqslant C_1 \cdot \int_X w^{-1} \nu \, d\mu_X = C_1 \int_X w^{-1} \, dm_X < \infty,$$

By lemma 1 in 1.6 this implies that $\alpha : L_w^{K,k} \to H$ is a Hilbert-Schmidt morphism.

Proof of the key lemma

Step 1. Let D be the unit ball in R^n with coordinates $x_1, \ldots, x_n$ and let $k \geqslant n$. For every function $f \in C^\infty(D)$ set

$$Q(f) = |f|^2 + \sum_{i=1}^{n} \left| \left(\frac{\partial}{\partial x_i} \right)^k f \right|^2$$

and define

$$\| f \|_Q^2 = \int_D Q(f)\, dx.$$

Then there exists a constant $C > 0$ such that for all $f \in C^\infty(D)\, |f(0)|^2 \leqslant C \| f \|_Q^2$

This is a standard fact, known as an *a priory* estimate (see e.g. [ReSi], v. I).

Step 2. For every function $f \in C(G)^\infty$ define $Q(f) = \Sigma \mid d_i f \mid^2$. Then there exists a constant $C > 0$ such that for all $f \in C(G)^{K,\infty}$ one has

$$|f(e)|^2 \leqslant C \cdot \int_B Q(f)\, d\mu_G .$$

Indeed, passing to $K \setminus G$ we can assume that G is a Lie group. Then choosing some coordinates $x_1, \ldots, x_n$ near the point $e \in G$, and expressing all operators $(\partial/\partial x)^k$ in terms of $d_1, \ldots, d_r$, we reduce the inequality to the *a priory* estimate of step 1.

Step 3. Fix $x \in X$ and consider the projection $G \to X$, $g \mapsto gx$. For each function f on X denote by f^* its lift to G, given by $f^*(g) = f(gx)$. Clearly $Q(f^*) = (Q(f))^*$.

Let $f \in C(X)^{K,\infty}$. Then

$$|f(x)|^2 = |f^*(e)|^2 \leqslant C \cdot \int_B Q(f^*)\, d\mu_G = C \cdot \int_B (Q(f))^*\, d\mu_G$$

by step 2. By lemma 3.3c), the right hand side is bounded by

$$C' \cdot \int_{Bx} Q(f)\, dm_X ,$$

where C' is independent of x. This proves the lemma and the theorem. ∎

REMARKS 1. It is instructive to prove the estimate in the key lemma directly in cases $SL(2,\mathbb{R})/SL(2,\mathbb{Z})$ and $SL(2,\mathbb{R})/SO(2)$.

2. With mild modifications, the proof above yields the following

THEOREM. Let w be a summable weight on X, $k \geqslant \dim \underline{g}$. Let w' be another weight, $m \geqslant 0$. Then the natural embedding

$$L^{K,k+m}_{w\,w'} \to L^{K,m}_{w'} \text{ is Hilbert-Schmidt and hence fine.} \qquad \blacksquare$$

3.5. Scales. Spaces of polynomial growth

Let w and w; be weights on X, such that $w \leqslant w'$. If w is summable, then w' is also summable and $S_{w'} \subset S_w$. This means that the analytic necessary condition Cond(w) is stronger than the condition Cond(w'). Thus it is natural to try to find a summable weight w which is minimal or almost minimal.

Since it depends on too many parameters, it is not clear how to look for a minimal w. But in applications the homogeneous space X is usually given with a large scale structure. We will discuss this notion in more detail in section 4. For now, we will use only one piece of this structure – the radial function $r: X \to \mathbb{R}^+$, which roughly measures the distance to a basic point $x_0 \in X$. This function has the following properties

(*) r is positive, locally bounded and proper, i.e. for any $R \in \mathbb{R}^+$ the "ball" $B(R) = \{x \in X \mid r(x) \leqslant R\}$ is relatively compact in X.

(**) For every ball $B \subset G$ there exists a constant $C > 0$ such that $|r(gx) - r(x)| < C$ for all $g \in B$, $x \in X$.

The radial function is defined up to the following equivalence:

(***) $r \sim r'$ if $(1+r)$ and $(1+r')$ are comparable.

This means that the value of r is relevant only for large distances and only up to a fixed factor.

Let us fix a radial function r on X, and use it to construct a "small" summable weight. Namely, we will consider only weights, which are functions of r, i.e. weights of the form $w(x) = u(r(x))$ for some function u on $\mathbb{R}^+$.

Fix a sparse net N and consider a counting function

$$\pi(t) = \pi_{N,r}(t) = \#\,(B(t) \cap N) = \#\{n \mid r(n) \leqslant t\}.$$

We call π a *growth function* of X. For large t it is comparable to the function $\pi_r(t) = m_X(B(t))$. In particular, it does not depend on the choice of N. The change of r by an equivalent function leads essentially to a linear rescaling of an argument in π.

Let w be a weight of the form $w = u(r)$. We claim that if for large t $u(t) \gg \pi(t)^{1+\epsilon}$ or even $u(t) \gg \pi(t)(\log \pi(t))^{1+\epsilon}$ for some $\epsilon > 0$, then the weight w is summable. Indeed, let us order the points of N in such a way that the sequence $r_i = r(n_i)$ is increasing. Then for large i, $\pi(r_i) \geqslant i$ and $\pi(r_i) \equiv i$ if all r_i are distinct. Hence

$$\sum_{i=1}^{\infty} w(n_i)^{-1} = C + \sum_{i>k} w(n_i)^{-1} \leqslant C + \sum_{i>k} (i(\log i)^{1+\epsilon})^{-1} < \infty.$$

The same calculation shows that if $u(t) \ll \pi(t)$, then usually the weight w is not summable.

DEFINITION. Let X be a homogeneous space with a radial function r. We say that X has polynomial growth if for some $d \geqslant 0$, $\pi_r(t) \ll (1 + t^d)$. The greatest lower bound of such numbers d we call the rank of X and denote it by $rk(X)$.

Let X be a homogeneous space of polynomial growth. Then for every $d > rk(X)$ the weight $w(x) = (1 + r(x))^d$ is summable and hence only S_w-tempered forms can contribute to the decomposition of $L^2(X, \mu_X)$.

Usually it is more convenient to consider a weaker condition. Namely, following Harish-Chandra, we define a Harish-Chandra Schwartz space $\mathscr{C} = \mathscr{C}(X)$ by

$$\mathscr{C} = \bigcap_d S_{(1+r)^d} = \bigcup_{K} \bigcap_{m,d} L^2(X, (1+r)^d \mu_X)^{K,m},$$

where $m, d > 0$, $K \subset G$ is a regular subgroup.

We call a V-form $\beta : V^\infty \to C(X)^\infty$ *X-tempered* if it is $\mathscr{C}$-tempered. In other words, β is X-tempered if for some $d > 0$ $(1+r)^{-d} \beta(v) \in L^2(X, \mu_X)$ for all $v \in V^\infty$ (See 3.1). As we have shown, only such forms contribute to the spectral decomposition of $L^2(X, \mu_X)$.

Note that the Harish-Chandra Schwartz space $\mathscr{C}$ is nuclear, since for $d > rk(X)$, $k \geqslant \dim \underline{g}$ the embedding $L^{K,k+m}_{(1+r)^d+\ell} \to L^{K,m}_{(1+r)^\ell}$ is Hilbert-Schmidt for all m, ℓ and K (See remark at the end of 3.4).

3.6. Generalization. The case of an induced representation

Let (σ, E) be a finite-dimensional Γ-module. We want to study the induced G-module $\operatorname{Ind}_\Gamma^G (E)$. Let us denote by C_X the sheaf of germs of continuous functions on X, and let $\zeta = \operatorname{Ind}(E)$ be the sheaf of C_X-modules, whose sections are given by functions f on G with values in E, satisfying $f(g\gamma) = \sigma(\gamma)^{-1} f(g)$, $\gamma \in \Gamma$.

Clearly ζ is a G-equivariant locally free and finitely generated sheaf of C_X-

modules. Conversely, each such sheaf arises from a finite-dimensional Γ-module.

We denote by $C(X, \zeta)$ the space of continuous sections of ζ, by $C_c(X, \zeta)$ the subspace of sections with compact support, and by $S(X, \zeta)$ its Garding space.

Now suppose (σ, E) is a unitary Γ-module. We will introduce a G-invariant scalar product on $S(X, \zeta)$ by

$$\langle \phi, \psi \rangle = \int_X \langle \phi_x, \psi_x \rangle \, d\mu_X$$

and denote by $H = L^2(X, \zeta)$ the completion of $S(X, \zeta)$ with respect to this scalar product.

All the results and proofs of §2, 3 remain valid for the unitary G-module (π, H) with the following modifications.

Contributions of (ρ, V) to the spectral decomposition of H are given by morphisms $\alpha_V : S(X, \zeta) \to V$, or equivalently, by V-forms $\beta_V : V^\infty \to C(X, \zeta)^\infty$, or, equivalently, by Γ-morphisms $V^\infty \to E$.

For each weight w we denote by $L_w(\zeta)$ the completion of $S(X, \zeta)$ with respect to the scalar product $\langle \phi, \psi \rangle_w = \int \langle \phi_x, \psi_x \rangle \, w \, d\mu_X$, and set $S_w(\zeta) = L_w(\zeta)^\infty$. A form $\beta_V : V^\infty \to C(X, \zeta)^\infty$ is $S_w(\zeta)$-tempered iff $\beta_V(V^\infty) \subset L_{w^{-1}}(\zeta)$. If w is a summable weigth, then only S_w-tempered forms can contribute to the spectral decomposition of H.

REMARK 1. Sometimes it happens, that for a nontrivial Γ-module (σ, E) the bound in key lemma 3.4 can be strengthened, namely

$$|f(x)|^2 \leqslant C \cdot u_E(x) \cdot \int_{Bx} Q(f) \, dm_X,$$

where $u_E(x) \leqslant 1$ is some weight, depending on E (see examples 4.3.4 and 4.3.5 below).

We say that a weight w is E-summable if the weight $u_E^{-1} \cdot w$ is summable (but w itself is not necessarily summable). Then, repeating the proof in 3.4, one checks that theorem 3.2 and psoposition 3.4 remain valid for such a weight, and hence one gets a stronger analytic necessary condition, Cond(w).

REMARK 2. It would be interesting to analyse the case of an infinite-dimensional Γ-module E, but I do not known how to do it. The natural approach would be to assume that E is given together with a Gelfand pair, i.e. with a fine morphism of Γ-modules $\alpha_E : E' \to E$. After this one has to consider $S(X, \zeta')$ and complete it with respect to some scalar product in order to define $L_w(\zeta')$. However, since the action of Γ on E' is not unitary, it is not clear how to define $\langle\,,\rangle_w$ on $S(X, \zeta')$.

It can be done in some simple cases, but the general pattern is unclear. On the other hand, I do not know examples of interesting applications in this case, so may be this is just the wrong question.

3.7. Generalization. The case when X has no invariant measure

Let Δ_X be the sheaf on continuous measures on X ($m \in \Delta_X$ if locally it has a form $m = f(\pi(a)\delta_x), f \in C(X), a \in M_c(G)^\infty, x \in x$). This is a G-equivariant invertible sheaf of C_X-modules, which is isomorphic to $\mathrm{Ind}(\Delta, \mathbb{C})$, where Δ is the character of Γ equal to $\Delta_G/\Delta_\Gamma, \Delta_G, \Delta_\Gamma$ – moduli of G and Γ.

We denote by δ the sheaf of half-measures on X, i.e. an invertible sheaf of C_X-modules with a positivity structure and an isomorphism $\delta \otimes \delta = \Delta_X$. This sheaf can be constructed as $\mathrm{Ind}(\Delta^{1/2}, \mathbb{C})$. Let (σ, E) be a finite-dimensional unitary Γ-module, $\zeta = \mathrm{Ind}(E)$ the corresponding sheaf and $\tilde{\zeta} = \zeta \otimes \delta$. The scalar product on E defines a natural pairing $\langle\,,\rangle: \tilde{\zeta} \times \tilde{\zeta} \to \Delta_X$. Using this pairing we define the scalar product on $S(X, \tilde{\zeta})$ by

$$\langle \phi, \psi \rangle = \int_X \langle \phi(x), \psi(x) \rangle.$$

The completion of $S(X, \tilde{\zeta})$ is a G-module (π, H), unitary induced from (σ, E).

All the results of Sections 2 and 3 remain valid with the following modifications:

Contributions of V to H correspond to G-morphisms $\alpha_V : S(X, \tilde{\zeta}) \to V$ or, equivalently, to V-forms $\beta_V : V^\infty \to C(X, \tilde{\zeta})^\infty$ or, equivalently, to Γ-morphisms $(\rho, V^\infty) \to (\Delta^{1/2}\, \sigma, E)$.

For every weight w we define $L_w(\tilde{\zeta})$ and $S_w(\tilde{\zeta})$ using scalar product

$$\langle \phi, \psi \rangle_w = \int_X \langle \phi(x), \psi(x) \rangle\, w(x).$$

A form β_V is S_w-tempered if $\beta_V(V^\infty) \subset L_{w-1}(\tilde{\zeta})$.

The definition of a standard measure remains the same as in 3.3, but the proof of its existence has to be modified, since in the proof in 3.3 we used the Haar measure μ. In fact, the only thing we used about μ was that it is a weight measure. Hence in order to modify the proof in 3.3 we need the following

LEMMA. There exists a weight measure μ on X. ∎

Proof. Fix a sparse net $N \subset X$ and a ball $B \subset G$ such that $B \cdot N = X$. Choose smooth positive measures $a_1, a_2 \in M_c(G)^\infty$ such that $a_1 \neq 0$ and for all $g \in B$ $\pi(g)a_1 \leqslant a_2$. Set

$$\mu_1 = \sum_{n \in N} \delta_n$$

the sum of δ-measures at $n \in N$ and $\mu = \pi(a_2^*)\mu$ (see 2.2). We claim that μ is a weight measure.

Fix a ball $\Omega \subset G$. We want to show that $\pi(g)\mu \leqslant C \cdot \mu$ for all $g \in \Omega$, i.e. for every positive function Q on X $\langle \pi(g)\mu, Q\rangle \leqslant C \cdot \langle \mu, Q\rangle$. Without loss of generality, we can assume that Q is supported in a ball Bx for some point $x \in X$.

The function $Q_1 = \pi(a_1)Q$ is continuous, positive, and has compact support. Let us denote by A its maximum. We want to show that $\langle \mu, Q\rangle \geqslant A$, and $\langle \pi(g)\mu, Q\rangle \leqslant C \cdot A$.

By definition, $\langle \mu, Q\rangle = \langle \mu_1, Q_2\rangle$, where $Q_2 = \pi(a_2)Q$.

By the choice of $a_1, a_2, Q_2 \geqslant \pi(g)Q_1$ for all $g \in B$. In particular, if $Q_1(x_0) = A$ then $Q_2 \mid_{Bx_0} \geqslant A$, and hence $\langle \mu, Q\rangle \geqslant A$.

We can choose a positive measure $a_3 \in M_c(G)^\infty$ such that, for all $g^{-1} \in \Omega$, $a_2^* \delta_g \leqq a_3 * a_1$. Then $\langle \pi(g^{-1})\mu, Q\rangle = \langle \mu_1, \pi(a_2 * \delta_g)Q\rangle \leqslant \langle \mu_1, Q'\rangle$, where $Q' = \pi(a_3)\pi(a_1)Q = \pi(a_3)Q_1$.

Clearly, Q' is bounded by $C'A$ and is supported in $B'x$ where $B' = \operatorname{supp}(a_3) \cdot \operatorname{supp}(a_1) \cdot B$. Hence $\langle \pi(g^{-1})\mu, Q\rangle \leqslant \langle \mu_1, Q'\rangle \leqslant k \cdot C' \cdot A$, where k is a bound on $\#(B'x \cap N)$ (see 3.2).

The rest of 3.3 goes as before.

In the proof of Theorem 3.2, as given in 3.4, the proof of the key lemma remains the same, but now we need a modified version of the lemma. Namely let us consider the standard measure m_X as a section of the sheaf Δ_X. Then the following version of the key lemma remain true.

(*) Let $B \subset G$ be a ball, $K \subset G$ a regular subgroup, and $k \geqslant \dim g$. Then there exists a $C > 0$ such that for all $\phi \in C(X, \zeta)^{K,\infty}$

$$|\phi(x)|^2 \leqslant C \cdot m_X(x) \cdot \int_{Bx} Q(\phi).$$

Proof. Choose a smooth positive measure $a \in M_c(G)^\infty$, and replace m_X by a new measure $\pi(a)m_X$. This again is a standard measure, but it is already smooth in the sense that it is fixed by some regular subgroup K and for all $d \in U(g)$ $| \pi(d)m_X | \leqslant C(d) \cdot m_X$.

Using the section $m_X^{1/2}$ of the sheaf δ we will identify δ with C_X and $\tilde{\zeta}$ with ζ ($\phi \in \tilde{\zeta} \to f_\phi = m_X^{-1/2}\, \phi \in \zeta$) Since m_X is smooth, we have $Q(\phi) \ll m_X \cdot Q(f_\phi)$ Hence, using the inequality of the key lemma:

$$| f(x) |^2 \ll \int_{Bx} Q(f)\, dm_X .$$

we can deduce the inequality (*)

The remainder of the proof of proposition 3.4 remains essentially the same. Namely, using the isomorphism of $\tilde{\zeta}$ with ζ we will identify H with $L^2(\zeta, dm_X)$. For every point $x \in X$ let us denote by α_x the morphism $\alpha_X : S(X, \tilde{\zeta})^K \to E$, given by $\alpha_x(\phi) = f_\phi(x)$. Using inequality (*) we see that $M_x = \| \alpha_x \|^2_{w,Q} \leqslant C \cdot w(x)^{-1}$. This implies, that the Hilbert-Schmidt norm M of the embedding $\alpha : L_w(X, \tilde{\zeta})^{K,k} \to H$ is bounded by $C \cdot \dim E \cdot \int w^{-1}(x)\, dm_X < \infty$, which proves proposition 3.4 and theorem 3.2. ■

4. EXAMPLES

4.1. Large scale spaces

Let M be a metric space with a distance function $d(x, y)$. This function, in fact, defines two structures on M. One is a small scale structure, which takes only small distances into account – for instance, it will not change if we replace d by $d_1 = \min(d, 1)$. Only this structure is responsible for the topology of M. Another is a large scale structure, which takes into account only large distances – for instance, it will not change if we replace d by $d_2 = \max(d(x, y), 1)$ for $x \neq y$. This structure was used by many mathematicians (see e.g. [Gro1], [Mos]), mostly to analyze the global effects of hyperbolicity. Let us describe some basic features of this structure, which we use as an intuitive background for the discussion below.

We define a semimetric space as a set M with a distance function $d(x, y)$ such that

(i) $d(x, y) = d(y, x) \geqq 0$, $d(x, x) = 0$ for $x, y \in M$

(ii) $d(x, z) \leqq d(x, y) + d(y, z)$, $x, y, z \in M$.

We say that two distance functions d_1, d_2 on the same set M are equivalent (notation $d_1 \sim d_2$) if there exists a constant $C > 0$ such that $C^{-1}(d_1 + 1) \leqslant (d_2 + 1) < C(d_1 + 1)$.

A set M with a class of equivalent distance functions we call a *large scale space.*

Let M, N be two large scale spaces. *A large space map* $f: M \to N$ is a map, such that for some constant $C > 0$

$$d(f(x), f(y)) \leqq C(d(x, y) + 1), \quad x, y \in M.$$

Two large scale maps $f_1, f_2 : M \to N$ are called *equivalent* (notation $f_1 \sim f_2$) if the distances $d(f_1(x), f_2(x))$ are bounded by some constant $C > 0$ for all $x \in M$.

Clearly these notions are well defined, the composition of large scale maps is a large scale map and $f_1 \sim f_2$, $h_1 \sim h_2$ implies $f_1 \circ h_1 \sim f_2 \circ h_2$.

A large scale map $f: M \to N$ is called a *large scale equivalence* if there exists a large scale map $h: N \to M$ such that $f \circ h \sim Id_N$, $h \circ f \sim Id_M$.

Example: The embedding $\mathbb{Z}^n \to \mathbb{R}^n$ is a large scale equivalence.

Let M be a large scale space with a distance function d. Let $R \in \mathbb{R}^+$. For each point $x \in M$ consider the ball $B(x, R)$ of radius R around x, i.e. $B(x, R) = \{y \in M \mid d(x, y) \leqq R\}$. For any subset $N \subset M$ we define its R-neighborhood $B(N, R)$ by

$$B(N, R) = \bigcup_{n \in N} B(n, R).$$

We say that two subsets $N, N' \subset M$ are equivalent if for some $R > 0$ $B(N, R) \supset N'$ and $B(N', R) \supset N$. Clearly, in this case large scale spaces (N, d) and (N', d) are canonically equivalent.

We say that a subset $N \subset M$ is a *net* if it is equivalent to M, i.e. $M = B(N, R)$ for some $R > 0$.

A net $N \subset M$ is called sparse if for any $R > 0$ the number of points in $N \cap B(x, R)$ for $x \in M$ is uniformly bounded by a constant $k = k(N, R)$.

For a fixed point $x \in M$ and a sparse net $N \subset M$ we consider a counting function $\tau_{N,x,d}(t) = \#\{n \in N \mid d(x, n) \leqq t\}$.

The following statement is straightforward:

(*) Let (M, d), (M', d') be equivalent large scale spaces, $x \in M$ $x' \in M'$, $N \subset M$, $N' \subset M'$ sparse nets. Then for some constant $C > 0$

$$1 + \pi_{N',x',d'}(t) \leqq C(1 + \pi_{N,x,d}(Ct))$$

This shows that the function π, which we call a *growth* function of the space M, is well defined up to comparability and linear rescaling of the argument.

Esamples 1: $\mathbb{R}^n$ has polynomial growth. Namely, $\pi_{\mathbb{R}^n}(t) \sim t^n$, i.e. $rk(\mathbb{R}^n) = n$

2. The hyperbolic space H^n has exponential growth. Namely, for large t $\exp(C_1 t) \ll \pi_{H^n}(t) \ll \exp(C_2 t)$

Remark 1. It would be useful to have some geometric picture for a large scale space defined up to a large scale equivalence. One of the approaches can be to use Gromov's limit procedure, described in [Gro2] Namely, let (M, d) be a large scale space. Fix a point $x_0 \in M$, and consider the family $\{M_\lambda, x_0\}$ of semimetric spaces, where $\lambda \in \mathbb{R}^{+*}$ and M_λ is the set M with the distance function λd. Suppose that for $\lambda \to 0$ the family $\{M_\lambda, x_0\}$ has a limit (Y, d_Y) in a sense of [Gro2].

Then this limit is a metric space (Y, x_0) defined up to a Lipschitz isomorphism, and it contains some information about the original space M.

Remark 2. In fact, the objects we have discussed should be called connected large scale spaces. The general notion of a large scale space should be based on a distance function d which takes on values in $\mathbb{R}^+ \cup \infty$.

4.2. Large scale structures on G and X

DEFINITION: Let G be a locally compact group. A *radial function* on G is a locally bounded function $r : G \to \mathbb{R}^+$ such that

$$(*) \qquad r(g) = r(g^{-1}) \geqslant 0, \ r(g_1 \cdot g_2) \leqslant r(g_1) + r(g_2), \ g, g_1, g_2 \in G.$$

Two radial functions r and r' are called equivalent if $(r' + 1)$ is comparable to $(r + 1)$, i.e. for some $C > 0$ $C^{-1}(r + 1) \leqslant r' + 1 \leqslant C(r + 1)$.

Given a radial function r on G, we will define a distance function on every homogeneous G-space X by $d(x, y) = \inf\{r(g) \mid gx = y)\}$ for $x \neq y$. The equivalence class of functions r defines a large scale structure on X.

Remark: If X is a nonhomogeneous G-space, this definition is still applicable if we allow $d(x, y)$ to take on infinite values. The resulting large scale space will be disconnected.

Usually we will fix a point $x_0 \in X$ and consider a function $r_X(x) = d(x, x_0)$, which we also call a radial function on X. We say that the function r_X on X is *proper* if for any $R \in \mathbb{R}^+$ the ball $B(R) = \{x \mid r_x(x) \leqslant R\}$ is relatively compact.

Note, that we can always replace r by an equivalent continuous radial function (see the trick in 3.1). Then every ball $B(R)$ will be closed.

A weight w on X is called r-admissible if there exists a $C > 0$ such that

$$w(gx) \leqslant e^{Cr(g)} w(x) \text{ on } G \times X.$$

In other words, w is r-admissible if the map $w: X \to \mathbb{R}^{+*}$ is a large scale map, where the distance on $\mathbb{R}^{+*}$ is defined via the isomorphism log: $\mathbb{R}^{+*} \to \mathbb{R}$. Comparable weights correspond to equivalent large scale maps.

Given a proper radial function r_X on X, we define the growth function $\pi_X(t)$ as in 4.1. In other words, we fix a sparse net $N \subset X$ and set $\pi_N(t) = \#\{n \in N \mid r_X(n) \leqq t\} = \#(N \cap B(t))$. As we saw in 3.5, it also can be defined as $\pi_X(t) = m_X(B(t))$, where m_X is the standard measure on X.

There are several ways to introduce a radial function on G. Let us discuss some of them.

1. Natural large scale

Suppose that G is compactly generated. Fix a ball $B \subset G$, generating G, and consider the radial function $r_n(g) = \min\{k \mid g \in B^k\}$.

It is easy to see that r_n is a proper radial function. Up to equivalence, this function does not depend on the choice of B, and it defines a large scale on G which we call natural.

Note that r_n dominates all the other radial functions on G. Indeed, if r is another radial function, then on B it is bounded by some constant C. Hence $r|_{B^k} \leqq kC$, i.e. $r \leqq C \cdot r_n$.

2. Algebraic large scale structures

Let F be a local field, $\|$ – the standard norm on F. For every n we introduce the norm $\| \ \|$ on the vector space F^n by $\| v = (v_1, \ldots, v_n) \| = \max |v_i|$, and consider the operator norm on the group $GL(n, F)$, $\| g \| = \sup\{\| gv \| / \| v \|, \ v \in F^n \setminus 0\}$. It is easy to see that if $g = (g_{ij})$, then $\| g \| = \max_{ij} |g_{ij}|$ in the non-Archimedean case and $\| g \| \sim \max_{ij} |g_{ij}|$ in any case.

We define a proper radial function r on $GL(n,F)$ by $r(g) = \max(\log \| g \|, \log \| g^{-1} \|)$. Let $\underline{G}$ be an algebraic group over F, $G = \underline{G}(F)$ the locally compact group of its F-points. Choose a faithful representation $\rho: \underline{G} \to GL(n)$ for some n, and define a radial function r_ρ on G by $r_\rho(g) = r(\rho(g))$.

LEMMA. *(See 4.5)a) r_ρ is a proper radial function on G. Its equivalence class does not depend on the choice of ρ. We call it an algebraic large sclae on G (notation r_a).*

b) Let G be a reductive F-group. Then G is compactly generated, and the algebraic large scale r_a is equivalent to the natural large scale r_n. ∎

The case of adelic groups we will discuss in 4.4.

4.3. Examples. Reductive groups over local fields

Let F be a local field, G a reductive algebraic F-group, $r = r_a$ the algebraic large scale on G (it is also equivalent to the natural large scale r_n).

We consider several examples of subgroup $\Gamma \subset G$, and describe the growth of homogeneous spaces $X = G/\Gamma$. The proofs are given in 4.6.

Example 1: Let G_0 be a reductive F-group, $G = G_0 \times G_0$, $\Gamma = \Delta G_0$ (the diagonal subgroup), $X = G/\Gamma \approx G_0$. Then X has polynomial growth, $rk(X)$ equals the split rank of G_0.

Example 2: $\Gamma = K$ – the maximal compact subgroup of G (or Γ is an open subgroup of K). Then X has polynomial growth, $rk(X)$ equals the split rank of G.

Example 3: $\Gamma = \{e\}$ and G is not compact. Then $X = G$ has exponential growth.

Example 4: G is a reductive group over R, $\Gamma \subset G$ an arithmetic subgroup. Then $X = G/\Gamma$ has polynomial growth. If Γ arises from an algebraic group $\underline{G}$ over $\mathbb{Q}$ of split rank d, then $rk(X) = d$.

Example 4': In example 4, consider an induced G-module $H = \mathrm{Ind}_\Gamma^G (E)$, where E is a unitary Γ-module. Then some weights w which are not summable can be E-summable (see Remark 1 in 3.6).

For example, consider the case $G = SL(2, \mathbb{R})$, $\Gamma = SL(2, \mathbb{Z})$. Then the weight $w = (1 + r)^d$ is summable iff $d > 1$. Suppose E is a Γ-module, which does not have vectors invariant with respect to the subgroup

$$\Gamma_\infty = \left\{ \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} \right\} \subset \Gamma$$

Then it is easy to check that in this case all the weights are E-summable. In other words, in this case the passage from the trivial Γ-module to Γ-module E effectively reduces the rank of the problem from 1 to 0.

Example 5: $\Gamma = U$ – a maximal unipotent subgroup of G. Then X has polynomial growth, $rk(X)$ equals the split rank of G.

This space appears in the study of principle series representations and in the study of Whittaker models. Note, that in the last case when we study $H = \mathrm{Ind}_U^G (\psi)$, where $\psi : U \to \mathbb{C}^*$ is a nondegenerate character, there exist weights w which are ψ-summable but not summable (see Remark 1 in 3.6).

For example, consider the case $G = SL(2, \mathbb{R})$. Using Iwasawa decomposition $G = KAU$, where $K = SO(2)$, $A = \{a(y) = \mathrm{diag}(y, y^{-1}) \mid y > 0\}$, we see that as a large scale space $X = G/U$ is equivelent to A and

$$r_X(x) \sim |\log y| \quad \text{for } x = ka(y).$$

Let $w(y)$ be any weight on A (and hence on X). It is easy to check that w is summable if

$$\int_0^\infty w^{-1}(y)\,dy < \infty ,$$

and w is ψ-summable if

$$\int_0^1 w^{-1}(y)\,dy < \infty .$$

Example 6: Let $\sigma : G \to G$ be an involutive automorphism, Γ an open subgroup of finite index in the group G^0 of its fixed points, $X = G/\Gamma$ a semisimple symmetric space. Then, if $F = \mathbb{R}$, X has polynomial growth, $rk(X)$ equals the split semisimple rank of the symmetric pair (G, Γ). Probably the same is true for any local field F provided $\operatorname{char}(F) \neq 2$ (see 4.6).

4.4. Examples, Reductive groups over adeles

Let F be a global field, $\{p\}$-places of F,

$$\mathbb{A} = \prod_p{}' F_p$$

the adeles of F.

For each n we introduce a radial function r on $GL(n, \mathbb{A})$ by $r = \Sigma\, r_p$, where for $g = (g_p)$ $r_p(g) = r(g_p)$.

Let $\underline{G}$ be an algebraic F-group, $G = \underline{G}(\mathbb{A})$ – the locally compact group of its adelic points. Let us choose a faithful representation $p : \underline{G} \to GL(n)$ over F, and define a radial function r_ρ on G by $r_\rho(g) = r(\rho(g))$.

LEMMA: (see 4.5). r_ρ is a proper radial function on G, whose equivalence class does not depend on ρ. We call it an *algebraic scale on* G (notation r_a). ■

In the examples below, we condiser a reductive group G with large scale r_a.

Example 1: Let $\underline{G}$ be a reductive F-group, $G = \underline{G}(\mathbb{A}), \Gamma = \underline{G}(F)$. Then $X = G/\Gamma$ has polynomial growth, $rk(X)$ equals the split rank of $\underline{G}$.

Example 2: Let $\underline{P} \subset \underline{G}$ be a parabolic subgroup, $\underline{U}$ its unipotent radical. Set $G = \underline{G}(A)$, $\Gamma = \underline{P}(F) \cdot \underline{U}(\mathbb{A})$. Then $X = G/\Gamma$ has polynomial growth, $rk(X)$ equals the split rank of $\underline{G}$.

Example 3: $G = \underline{G}_0(\mathbb{A}) \times \underline{G}_0(\mathbb{A})$, $\Gamma = \underline{G}_0(\mathbb{A})$ the diagonal subgroup. Then $X = G/\Gamma$ has exponential growth.

Example 4: $G = \underline{G}(\mathbb{A})$, $\Gamma = \underline{U}(\mathbb{A})$, where $\underline{U}$ is a maximal unipotent subgroup of G. Then $X = G/\Gamma$ has exponential growth.

4.5. Algebraic, natural and standard large scales

In this section we discuss the relation between algebraic and natural large scales for groups over local fields and adeles. We also introduce the standard radial function, which is usually equivalent to the algebraic and natural ones, but has the advantage of a more rigid definition.

First of all, let us prove lemmas 4.2 and 4.4. Let $\underline{G}$ be an algebraic group over a local field F, $\rho : G \to GL(n)$ a faithful (algebraic) representation, r_ρ the corresponding radial function on G, $r_\rho(g) = r(\rho(g))$.

Since r is a proper radial function on $GL(n, F)$, r_ρ is proper on G. In order to prove that functions r for different ρ are equivalent, it is enough to check that for any representation $\sigma : G \to GL(m)$ one has $r_\sigma \ll (1 + r_\rho)$.

Let us denote by L the class of all the representations σ which satisfy this condition. It is easy to see that if $\sigma \in L$ then $\sigma^* \in L$, and that any representation σ' isomorphic to a subquotient of σ belongs to L. Also, if $\sigma, \tau \in L$, then $\sigma \oplus \tau \in L$ and $\sigma \otimes \tau \in L$.

Any representation σ is a submodule of a direct sum of several copies of the regular representation $(R, F[G])$ in the space of regular functions on G. Consider a submodule $(\tau, M) \subset (R, F[G])$, spanned by the matrix coefficients of ρ and ρ^*. Since τ is a quotient of a direct sum of n copies of $\rho \oplus \rho^*$, it belongs to the class L. Since $\underline{G} \subset GL(n)$, M generates $F[G]$ as an algebra. The properties of the class L imply that any G-submodule $\sigma \subset F[G]$ belongs to L, and hence any G-module belongs to L.

Now consider the adelic case (lemma 4.4). The function $r = \Sigma\, r_p$ on $GL(n, \mathbb{A})$ is proper, since for any $C > 0$ there exist only a finite number of places p, such that r_p can take on values between 0 and C. Hence if $\underline{G}$ is an F-group and $\rho : \underline{G} \to GL(n)$ a faithful algebraic representation, then the function $r_\rho(g) = r(\rho(g))$ on $G = \underline{G}(\mathbb{A})$ is proper.

The proof that up to equivalence r_ρ does not depend on ρ is the same as in the local case, but one has to be a little bit more careful with tensor products. Namely, let σ, τ be two representations of $\underline{G}$. Then it is clear that $r_{\sigma\ \tau, p} \leqslant r_{\sigma,p} + r_{\tau,p} + C_p$ where C_p is a constant, which does not depend on $g \in G$ and $C_p = 0$ if p is a non-Archimedean place. Since there is only a finite number of Archimedean places, $r_{\sigma\otimes\tau} \leqslant r_\sigma + r_\tau + C$, and the rest of the proof is the same as above.

Now let us prove lemma 4.2b. Let G be a reductive group over a local field F. Choose a maximal split F-torus $A \subset G$ and find a lattice (i.e. a finitely generated discrete free abelian subgroup) $L \subset A$ such that A/L is compact. By

Cartan decomposition, there exists a ball $B \subset G$ such that $G = B \cdot L \cdot B$. In particular, G is compactly generated and it is enough to check the equivalence $r_l \sim r_n$ on the lattice L.

Suppose r_a is defined using a representation $\rho : G \to GL(n, F)$. Since A is split, we can diagonalize it in some basis, i.e. we can assume that $\rho(A)$ consists of diagonal matrices. Then it is clear that on L $r_n \ll (1 + r_a)$. Since r_n dominates any radial function, we see that $r_a \sim r_n$, Q.E.D.

It would be nice to have an analogous lemma for adelic groups. At first glance this seems impossible, since adelic groups are not compactly generated. But it turns out that in the most interesting examples, 4.4.1 and 4.4.2, it is possible to do.

DEFINITION: Let G be a locally compact group, $X = G/\Gamma$ its homogeneous space. We say that X is compactly generated if

(*) There exists a ball $B \subset G$, such that the subgroup $G_B = U_k B^k$, generated by B, acts transitively on X. We denote by d_B the corresponding distance function on X, $d_B(x, y) = \min\{k \mid y \in B^k x\}$.

(**) For large enough balls B the corresponding distance functions d_B are equivalent.

Given a compactly generated X, we define the natural large scale on X by using the distance function d_B for sufficiently large B. In turn, we define a radial function r_n on G by $r_n(g) = \sup\{d_B(gx, x) \mid x \in X\}$. Note that the radial function $r_X(x) = d_B(x_0, x)$ is proper, while the function r_n on G is not necessarily proper.

LEMMA. *In examples 4.4.1 and 4.4.2 the space X is compactly generated and the natural large scale on X is equivalent to the algebraic large scale. Moreover, the natural radial function is equivalent to the algebraic one on the group $G/\underline{Z}(F)$, where $\underline{Z}$ is the center of $\underline{G}$.* ■

Let G be a locally compact group, $\Gamma \subset G$, $X = G/\Gamma$. Let us assume that X has a G-invariant measure μ_X. As shown in 3.3, one can also construct the standard measure m_X on X, which is canonically defined up to comparability.

The ratio $\nu = m_X/\mu_X$ is a weight, which we call the *standard weight* on X. Let us define the standard radial function r_{st} on X by $r_{st}(x) = |\log \nu(x)|$. This function is defined canonically up to the addition of a bounded function, which is much more rigid than our constructions of functions r_a and r_n. In some interesting cases, notably in examples 4.3.1, 4.3.2, 4.3.4, and 4.4.1, for semisimple groups G, this function is equivalent to r_n and r_a. This shows that in these cases there is a natural choice of a radial function on X, which is defined up to the addition of a bounded function. The corresponding radial function on G can be chosen as $r(g) = \sup_x | \log \nu(gx) - \log \nu(x) |$.

4.6. Proofs for 4.3

In this section we analyze examples of homogeneous spaces X listed in 4.3. In each case we will try to construct a model of X as a large scale space. Given such a model, it is easy to describe the growth function of X.

We are mostly interested in the case where X has polynomial growth. In these cases we will model X on the following "elementary" space.

Let $\underline{a}$ be a finite dimensional Euclidean space, $W \subset \mathrm{Aut}(\underline{a})$ a finite reflection group $\underline{a}^+ \subset \underline{a}$ Weyl chamber for W. We consider $\underline{a}$ as a large scale space with the standard distance and $\underline{a}^+$ as its subspace. Note that $\underline{a}^+ \sim \underline{a}/W$.

Let G be a locally compact group with a radial function r, $X = G/\Gamma$ its homogeneous space with the corresponding distance function d_X. We will call an $\underline{a}^+$-model of X a large scale map $m : X \to \underline{a}$ which satisfies the condition (M) below. This condition essentially requires that X could be covered by a finite number of subsets $S_1, \ldots, S_k$ (which we call SIegel domains), such that for any i $m : S_i \to \underline{a}$ would be a large scale equivalence of S_i with $\underline{a}^+ \subset \underline{a}$.
Remark 1: We will not check it, but in the examples below one can choose Siegel domains $S_1, \ldots, S_k$ in such a way that for $i \neq j$ $m(S_i \cap S_j)$ lies in a neighborhood of a wall of $\underline{a}^+$. Moreover, it seems that X is glued from k copies of $\underline{a}^+$ in such a way that all the glueings are along the walls. This vague statement probably can be made precise if we replace X and $\underline{a}^+$ by their Gromov's limits.
Remark 2: The model map $X \to \underline{a}$ is useful in detailed harmonic analysis on X. The reason is that any weight function w on $\underline{a}^+$ gives a weight on X. Thus we have more freedom than just considering weights which are functions of r_X (see 3.5). In particular, by choosing appropriate weights, we can analyze functions which have different growth along different faces of $\underline{a}^+$.

To describe a Siegel domain S in X is the same as to describe a large scale section $\kappa : \underline{a}^+ \to X$. We want to have some explicit description of such a section. One of the difficulties here is that in the case of p-adic fields the space X is totally disconnected, so it is difficult to represent κ by a map. So we adopt the following:

DEFINITION: Let $X = G/\Gamma$ be a homogeneous space, $p : G \to X$ the natural projection. Let $m : X \to \underline{a}$ be a large scale map. A *Siegel section* of m is any pair (L, κ) consisting of a compact lattice $L \subset \underline{a}$ and a group homomorphism

$\kappa : L \to G$, such that on the semigroup $L^+ = L \cap \underline{a}^+$ the composition $m \cdot p \cdot \kappa : L^+ \to \underline{a}$ is equivalent to the standard embedding $L^+ \to \underline{a}$.

Given a Siegel section $\varkappa : L \to G$ and a ball $B \subset G$ (preferably large) we define a *Siegel domain* $S = S(\varkappa) \subset X$ as $S = B \cdot p \cdot \kappa(L^+)$. Clearly $p\kappa : L^+ \to S$ and $m : S \to \underline{a}$ give mutually inverse large scale equivalences of $L^+ \sim \underline{a}^+$ and S.

We say that $m : X \to \underline{a}$ is an $\underline{a}^+$-model of X if it satisfies the following condition:

(M) There exist Siegel sections κ_i of M, $i = 1, 2, \ldots, k$, $k \geqslant 1$, and a ball $B \subset G$, such that the corresponding Siegel domains S_i cover X.

Let $m : X \to \underline{a}$ be an $\underline{a}^+$-model. Then the following facts are obvious.

(i) X has polynomial growth, $\pi_X(t) \sim t^d$, where $d = \dim \underline{a}$.

(ii) Replacing M by an equivalent map, we can assume that $m(X) \subset \underline{a}^+$.

(iii) The collection of Siegel sections $\kappa_1, \ldots, \kappa_k$ completely determines m up to a large scale equivalence.

Thus in order to construct a model of X we have to find an appropriate triple $(\underline{a}, W, \underline{a}^+)$; define a large scale map $m : X \to \underline{a}$ describe sections $\kappa_i : L \to G$ and prove that they are Siegel sections; and prove that for some ball B Siegel domains S_i would cover X (usually this is the most difficult part). Let us describe step by step, how we are going to do this.

We fix a connected reductive group $\underline{G}$ over a local field F, set $G = \underline{G}(F)$ and consider some homogeneous space $X = G/\Gamma$.

Step 1. The elementary space $\underline{a}$ is in fact a model of a split torus. Suppose we are given a split torus $\underline{A}$ over F. Consider the lattice $L = \mathrm{Hom}(G_M, \underline{A})$ of cocharacters (or one-parameter subgroups) of $\underline{A}$ and the dual lattice $L^* = \mathrm{Hom}(\underline{A}, G_m) = \mathrm{Hom}(L, \mathbb{Z})$ of characters of $\underline{A}$.

We want to construct a large space equivalence between L and A. In order to do this, fix an element $c \in F^*$ such that $|c| > 1$ and define an embedding $i : L \to A$ by $i(\ell) = \ell(c)$; sometimes we will identify L with its image in A. Since A/L is compact, i is a large scale equivalence. In order to describe an inverse map, let us consider the linear space $\underline{a} = L \otimes \mathbb{R}$, which is equivalent to L as a large scale space.

It is easy to check that there exists a unique homomorphism $j : A \to \underline{a}$ which satisfies the following condition:

(*) For any $\lambda \in L^*$ $\langle \lambda, j(a) \rangle = \log |\lambda(a)| / \log |c|$. It is also clear that $j \cdot i : L \to L$ is the identify homomorphism, so i and j give mutually inverse large scale equivalences between $L \sim \underline{a}$ and A.

Now suppose se are given a root system $\Sigma \subset L^*$ and a positive root system $\Sigma^+ \subset \Sigma$. We denote by W the Weyl group of Σ, acting on $\underline{a}$, and by $\underline{a}^+$ its Weyl chamber, corresponding to Σ^+, i.e. $\underline{a}^+ = \{t \in \underline{a} \mid \langle \gamma, t \rangle \geq 0$ for all $\gamma \in \Sigma^+\}$. We also set $L^+ = L \cap \underline{a}^+$, $A^+ = j^{-1}(\underline{a}^+)$.

Step 2. Suppose we are given $(\underline{A}, \Sigma, \Sigma^+)$ and a reductive group $\underline{G}$. Let us fix a family of embedding $\kappa : \underline{A} \to \underline{G}$, which are all conjugate under the adjoint action of $\underline{G}(F)$ and have the following property:

$\kappa : \underline{A} \to \kappa(\underline{A})$ gives an isomorphism of $\Sigma \subset L^*$ with the root system $\Sigma(\kappa(\underline{A}), G) \subset L^*(\kappa(\underline{A}))$.

The embedding κ from our family we will call sections. Each section κ defines (and is determined by) a homomorphism $\kappa : L \to G$, $\ell \to \kappa(i(\ell))$.

Let (ρ, V) be an algebraic $\underline{G}$-module. For any κ the representation $\rho \cdot \kappa : \underline{A} \to GL(V)$ is diagonalizible, i.e. $V = \oplus V_\mu$, where $\mu \in L^*$ and on V_μ $\underline{A}$ acts by multiplication with character μ. We denote by $P(V) \subset L^*$ the set of weights of V, i.e. $P(V) = \{\mu \mid V_\mu \neq 0\}$. Since all sections κ are conjugate this set does not depend on κ.

We say that V is a highest weight module if there exists a weight $\lambda \in P(V)$ such that any other weight $\mu \in P(V)$ has a form $\mu = \lambda - \Sigma n_\gamma \cdot \gamma, \gamma \in \Sigma^+$, $n_\gamma \geq 0$. Such a weight λ, which is obviously unique, we call the highest weight of V.

Given a $\underline{G}$-module (ρ, V), we define a function $m_V : G \to \mathbb{R}$ by $m_V(g) = \log \| \rho(g) \|$ where $\| \ \|$ is the norm on $GL(V)$, defined with respect to some basis in V. Clearly m_V is a large scale map, defined canonically up to a large scale equivalence. If V is a highest weight module with the highest weight λ, then for any section $\kappa : \underline{A} \to \underline{G}$ the function $m_V(\kappa(\ell))$ on L^+ is equivalent to the function $\ell \mapsto \langle \lambda, \ell \rangle \log |c|$. Indeed, we can assume that the representation $\rho \cdot \kappa : \underline{A} \to GL(V)$ is diagonalized in the basis of V, used to construct a norm on $GL(V)$. Then for every $\ell \in L$ the matrix $\rho\kappa(\ell)$ has entries $\{c^{\langle \mu, \ell \rangle} \mid \mu \in P(V)\}$. If $\ell \in L^+$, then $\langle \mu, \ell \rangle \leq \langle \lambda, \ell \rangle$ for all $\mu \in P(V)$, and therefore $\| \rho\kappa(\ell) \| = |c|^{\langle \lambda, \ell \rangle}$ and $m_V(\kappa(\ell)) = \langle \lambda, \ell \rangle \log |c|$.

Step 3: Now define a function $m_V^\Gamma(g) = \inf \{m_V(g\gamma) \mid \gamma \in \Gamma\}$. It is well defined if $\{\| \rho(\gamma) \|, \gamma \in \Gamma\}$ is bounded from 0. This is not a restrictive condition, and we will consider only the G-modules V for which it holds. The function m_V^Γ is right Γ-invariant, and hence defines a map $m_V^\Gamma : X \to \mathbb{R}$. This is a large scale map defined canonically up to a large scale equivalence.

We say that a section $\kappa : \underline{A} \to \underline{G}$ is (Γ, V)-special if the functions m_V and m_V^Γ are equivalent on $\kappa(L^+)$. This is equivalent to the following condition:

(*) There exists a $C > 0$ such that

$$\| \rho(g) \| \leq C \| \rho(g\gamma) \| \quad \text{for all } g \in \kappa(L^+),\ \gamma \in \Gamma.$$

(in the example below, this inequality would hold for all $g \in \kappa(L)$, $\gamma \in \Gamma$).

Now suppose we have choosen a collection of G-modules (ρ_i, V_i), $i = 1, \ldots, \ell$, such that their highest weights $\lambda_1, \ldots, \lambda_\ell$ form a basis of $\underline{a}^* \supset L^*$. Then $\lambda_1, \ldots, \lambda_\ell$ define a coordinate system on $\underline{a}$, so we can define a map $m : X \to \underline{a}$ by condition $\langle\lambda_i, m(x)\rangle = m^{\Gamma}_{V_i}(x)/\log|c|$. This is a large scale map, which is defined by collection $\{V_i\}$ up to a large scale equivalence.

Let $\kappa : \underline{A} \to \underline{G}$ be a section special for (Γ, V_i), $i = 1, \ldots, \ell$. Then for any i $m^{\Gamma}_{V_i}(\kappa(\ell)) \sim m_{V_i}(\kappa(\ell)) \sim \langle\lambda_i, \ell\rangle \log|c|$ as functions on L^+. This means that on L^+ the map $\ell \mapsto mp\kappa(\ell)$ is equivalent to the standard embedding $L^+ \to \underline{a}$, i.e. that $\kappa : L \to G$ is a Siegel section of m.

To summarize, in order to construct a model of X we have to choose a torus $\underline{A}$, root systems Σ, Σ^+ and a family of sections $\kappa : \underline{A} \to \underline{G}$; choose a collection of G-modules $V_1, \ldots, V_\ell$, whose highest weights form a basis of $\underline{a}^*$ and prove that there exist sections $\kappa_1, \ldots, \kappa_k$, which are special for (Γ, V_i), and a ball $B \subset G$ such that the Siegel domains $S_j = Bp\kappa_j(L^+)$ cover X.

Before we begin with the detailed analysis of examples, let us make some elementary observations.

Fact 1: Let k be a field, $\underline{G}$ a connected reductive k-group, $\underline{A} \subset \underline{G}$ a k-split torus, $\Sigma = \Sigma(\underline{A}, \underline{G})$. Fix a positive root system $\Sigma^+ \subset \Sigma$. Then there exists a family $\{V_i\}$ of highest weight G-modules whose highest weights λ_i span L^* as a group. In particular, one can choose a collection of G-modules $V_1, \ldots, V_\ell$, whose highest weights form a basis in $\underline{a}^* \supset L^*$.

Indeed, choose a Cartan subgroup $\underline{C} \subset \underline{G}$, containing $\underline{A}$. If $\underline{C}$ is k-split, then the family $\{V_i\}$ of all irreducible G-modules satisfies the above conditions for $\underline{C}$ and hence for $\underline{A}$. If $\underline{C}$ is not split, choose a finite extension k' of k over which $\underline{C}$ splits, find a family $\{V_i'\}$ of $\underline{G}$-modules, which are defined over k', whose highest weights span L^* and consider the G-modules V_i obtained from V_i' by restricting scalars from k' to k. These modules are reducible, but since $P(V_i) = P(V'_i)$ they are highest weight modules and their highest weights span L^*.

Fact 2: Let $\Gamma_1 \subset \Gamma$ be a subgroup of finite index, $X_1 = G/\Gamma_1$, $p_1 : X_1 \to X$ the natural projection. Then for any G-module V the functions m^{Γ}_V and $m^{\Gamma_1}_V$ on G are equivalent. If we fix a collection of $\underline{G}$-modules (ρ_i, V_i), $i = 1, \ldots, \ell$ and define the maps $m : X \to \underline{a}$, $m_1 : X_1 \to \underline{a}$ as above, then m_1 is equivalent to $mp_1 : X_1 \to \underline{a}$. The map m_1 is an $\underline{a}^+$-model iff m is an $\underline{a}^+$-model.

Indeed, write Γ as a finite union $\Gamma = \cup \Gamma_1\gamma_j$. Then clearly $\inf\{\|\rho(g\gamma)\|, \gamma \in \Gamma\}$ is comparable to $\inf\{\|\rho(g\gamma)\|, \gamma \in \Gamma_1\}$. This means that on G $m^{\Gamma}_V \sim m^{\Gamma_1}_V$, and on X_1 $m_1 \sim mp$. If $\kappa_i : L^+ \to G$ is a family of Siegel sections for X_1 such that the corresponding Siegel domains S_i cover X_1, then the same is true for X. Conversely, let $\kappa_i : L^+ \to G$ be a family of Siegel sections for X and B a ball such that the sets $S_i = B \cdot p \cdot \kappa_i(L^+)$ cover X, i.e. $G = \cup B \cdot \kappa_i(L^+)\Gamma$. Consider sections $\kappa_{ij} = \gamma_j\kappa_i\gamma_j^{-1} : L^+ \to G$ and a ball B_1 containing $B\gamma_j^{-1}$. Then κ_{ij} are Siegel sections for X_1, and $G = \cup B_1\kappa_{ij}(L^+)\Gamma_1$ i.e. Siegel domains S_{ij} cover X_1.

Now we are ready to analyse the examples from 4.3:

Example 4.3.1. $G = G_0 \times G_0$, $\Gamma = \Delta G_0$. Choose a maximal split torus $\underline{A} \subset \underline{G}_0$ and a positive root system $\Sigma^+ \subset \Sigma = \Sigma(\underline{A}, \underline{G}_0)$. We fix one section $\kappa : \underline{A} \to \underline{G}$, $\kappa(a) = (a, e)$. To each $\underline{G}_0$-module (ρ_0, V_0) we assign a G-module $\rho(V)$ by $V = \mathrm{End}(V_0)$, $\rho(g_1, g_2)v = \rho_0(g_1)v\rho_0(g_2^{-1})$. If V_0 has highest weight λ_0, then V has the same highest weight λ_0.

In order to check, that the section κ is special for (Γ, V), let us note that the identity matrix $e \in V$ is Γ-invariant. This implies that

$$\| \rho(\ell\gamma) \| \geq \| \rho\,(\ell\gamma)\, e \|_V = \| \rho(\ell)\, e \|_V = \| \rho(\ell) \|, \quad \ell \in L,\ \gamma \in \Gamma$$

(here we have chosen a basis in V_0, consisting of eigenvectors for $\rho_0(\underline{A})$, and the corresponding "matrix" basis in V).

Usign fact 1 above, choose G_0-modules V_{0i} whose highest weights form a basis of $\underline{a}^*$, and use the corresponding G-modules V_i to construct a map $m : X \to \underline{a}$. Then κ is a Siegel section of the map.

By Cartan decomposition, there exists a ball $B_0 \subset G_0$ such that $G_0 = B_0 \cdot A^+ \cdot B_0$. Since A/L is compact, we can enlarge B_0 so that $B_0 \cdot L^+ \cdot B_0 = G_0$. This means that for the ball $B = B_0 \times B_0 \subset G$ the Siegel domain $S = Bp\kappa(L^+)$ covers X.

Example 4.3.2. Γ is commesurable to a maximal compact subgroup K of G (i.e. $\Gamma \cap K$ has finite index in both Γ and K). Usign fact 2 above, we can assume $\Gamma = K$. Choose a maximal split torus $\underline{A} \subset \underline{G}$ and a positive root system $\Sigma^+ \subset \Sigma = \Sigma(\underline{A}, \underline{G})$, and consider the standard section $\kappa : \underline{A} \to \underline{G}$.

Using fact 1 above, choose $\underline{G}$-modules (ρ_i, V_i) and define a map $m : X \to \underline{a}$. Since Γ is compact, κ is special for (Γ, V_i), i.e. it is a Siegel section.

By Cartan decomposition, there exists a ball $B \subset G$ such that $BA^+ \cdot K = G$. Hence for some larger ball B Siegel domain $S = Bp\kappa(L^+)$ covers X.

Example 4.3.3. $\Gamma = \{e\}$, $X = G$. In this case the standard measure m_X coincides with the Haar measure μ_X. Hence $\pi_X(t) = m_X(B(t)) = \mu_X(B(t))$. It is well known that this volume grows exponentially in t.

Remark: In the case when $\Gamma = \{e\}$, the natural model of a large scale space $X = G$ is given by $K \setminus G$. If F is Archimedean, this is just the symmetric space of G. For non-Archimedean F, consider the Bruhat-Tits building B of the group G (with a right action of G). If we fix a vertex $b \in B$, then the map $g \mapsto bg$ is a large scale equivalence of G with B, which gives a nice model of the large scale space G.

Note that for any subgroup $\Gamma \subset G$ the space B/Γ gives a nice model of the large scale space $X = G/\Gamma$. The building B is a union of images of natural simplicial maps $\underline{a}^+ \to B$, which shows that this model is closely related to the $\underline{a}^+$-models we are considering.

Example 4.3.4: $F = \mathbb{R}$, $\Gamma \subset G$ is an arithmetic subgroup. By the definition of an arithmetic group, there exists a connected reductive $\mathbb{Q}$-group $\underline{G}$, a subgroup $\Gamma_0 = \underline{G}(\mathbb{Z}) \subset \underline{G}(\mathbb{R})$, and an epimorphic homomorphism with compact kernel $p : \underline{G}(\mathbb{R}) \to G$, such that $p(\Gamma_0)$ is commensurable with Γ. Using fact 2 above, we can reduce our analysis to the case $G = \underline{G}(\mathbb{R})$, $\Gamma = \Gamma_0$.

Choose a maximal $\mathbb{Q}$-split torus $\underline{A} \subset \underline{G}$ and a positive root system $\Sigma^+ \subset \Sigma = \Sigma(\underline{G}, \underline{A})$. For our sections $\kappa : \underline{A} \to \underline{G}$ we will choose all the $G(\mathbb{Q})$-conjugates of the standard embedding $\underline{A} \to \underline{G}$.

We claim that any such section κ is special for (Γ, V) if (ρ, V) is a $\underline{G}$-module defined over $\mathbb{Q}$. Indeed, in this case we can choose a basis in $V(\mathbb{Q})$, consisting of eigenvectors for $\kappa(\underline{A})$. If we denote by $V(Z)$ the lattice spanned by this basis then there exists a subgroup of finite index, $\Gamma_1 \subset \Gamma$, which preserves this lattice, and, without loss of generality, we can assume that $\Gamma_1 = \Gamma$ (see fact 2). Then for any $\gamma \in \Gamma$ $\rho(\gamma)$ is a nondegenerate matrix with integral entries, and hence for any $g \in \kappa(L)$ $\| \rho(g) \| \leq \| \rho(g\gamma) \|$.

Using fact 1, choose $\underline{G}$-modules (ρ_i, V_i) defined over $\mathbb{Q}$, and using them construct a map $m : X \to \underline{a}$.

Consider a pair $(\underline{P}, \underline{A}_1)$, where $\underline{P}$ is a minimal $\mathbb{Q}$-rational parabolic subgroup, $\underline{A}_1 \subset \underline{P}$ a maximal $\mathbb{Q}$-split torus. Then we can find a section $\kappa : \underline{A} \to \underline{G}$ which will identify $(\underline{A}, \Sigma^+)$ with $(\underline{A}_1, \Sigma(\underline{P}))$.

In [Bor], a Siegel domain S for P is defined as $S = KA^+ \Omega$, where K is the maximal compact subgroup of G, Ω a compact subset of P. It is also shown in [Bor] that the set $\{aga^{-1} \mid a \in A^+, g \in \Omega\}$ is relatively compact in G. Hence for some ball B $B\kappa(L^+) \supset S$, i.e. Siegel domain $S(\kappa) = Bp\kappa(L^+) \subset X$ which we use contains the image of the Siegel domain S defined in [Bor]. It is shown in [Bor] that one can choose a finite number of parabolic subgroupes P_j, $j = 1, \ldots, k$, and the corresponding Siegel domains S_j, so that their images cover X. If we consider the corresponding sections $\kappa_j : \underline{A} \to \underline{G}$, and choose a large enough ball $B \subset G$, then Siegel domains $S(\kappa_j) = Bp\kappa_j(L^+)$ will cover X, which shows that $m : X \to \underline{a}$ is an $\underline{a}^+$-model.

Example 4.3.5. $\Gamma = U$, a maximal unipotent subgroup of G. Fix a maximal split torus $\underline{A} \subset \underline{G}$, normalizing U, and set $\Sigma = \Sigma(\underline{A}, \underline{G})$; $\Sigma^+ = \Sigma(\underline{A}, U)$.

Each $w \in W = \mathrm{Norm}(\underline{A}, G)/\mathrm{Cent}(\underline{A}, G)$ defines a homomorphism $w : \underline{A} \to \underline{A} \subset \underline{G}$, and we consider these homomorphisms as sections. These sections are (Γ, V)-special for any $\underline{G}$-module (ρ, V). Indeed, choose a basis in V, consisting of $\underline{A}$ eigenvectors. Then in this basis $\rho(a)$ is diagonal for $a \in A$, and $\rho(\gamma)$ is unipotent and uppertriangular for $\gamma \in \Gamma$, which implies that $\| \rho(a) \| \leq \| \rho(a\gamma) \|$ for all $a \in A$, $\gamma \in \Gamma$.

Choose G-modules (ρ_i, V_i), as in fact 1, and consider the corresponding map $m : X \to \underline{a}$. Iwasawa decomposition of G shows, that for some ball $B \subset G$,

$G = BLU$. Since L is a union of wL^+, $w \in W$, we see that Siegel domains $S_w = Bpw(L^+)$ cover X, and hence $m : X \to \underline{a}$ is an $\underline{a}^+$-model.

Remark 1: In this case, one can easily describe a better model for X. Namely, consider $\underline{a}$ as before, and construct an $\underline{a}$-model $m' : X \to a$, which corresponds to the trivial reflection group in $\underline{a}$, as follows:

$$\langle \lambda_i, m'(g) \rangle = \log \| \rho(g)\, v_i \| / \log | c | ,$$

where $v_i \in V_i$ is a U-invariant vector of weight λ_i.

The corresponding section $L \to \underline{G}$ is the standard embedding. Thus m' is a large scale equivalence of X and a. The map m can be obtained from m' using the composition $X \overset{\sim}{\to} \underline{a} \to \underline{a}/W \sim \underline{a}^+$.

Remark 2: Suppose we are studying the G-module $H = \mathrm{Ind}_U^G(\psi)$, where $\psi : U \to \mathbb{C}^*$ is a nondegenerate character of U. Then it is easy to check that a weight w on $\underline{a} \sim X$ is ψ-summable provided that it is summable on $\underline{a}^+$. Thus the correct large scale model of the pair (X, ψ) should be $\underline{a}^+$ and not $\underline{a}$.

Example 4.3.6. Γ is an open subgroup of finite index of the fixed point group G^σ for some involution $\sigma : \underline{G} \to \underline{G}$.

Using fact 2 above, we can assume that $\Gamma = G^\sigma$. Let $\underline{A} \subset \underline{G}$ be a maximal split torus such that $\sigma(a) = a^{-1}$ for $a \in A$. We will assume the following facts:

(i) $\Sigma = \Sigma(\underline{A}, \underline{G})$ is a root system, and elements $w \in W(\Sigma)$ are realized by inner automorphism of G.

(ii) There exists a ball $B \subset G$ such that $G = BA\Gamma$.

For $F = \mathbb{R}$ these facts are proven in [Ro], but probably they are true for any local field F with $\mathrm{char}(F) \neq 2$. To every $\underline{G}$-module (ρ_0, V_0) assign a $\underline{G}$-module (ρ, V) by $V = \mathrm{End}(V_0)$, $\rho(g)(v) = \rho_0(g) v \rho_0(\sigma(g)^{-1})$. If V_0 has highest weight λ_0, then V has highest weight $\lambda = 2\lambda_0$. Let us show that there exists a $C > 0$ such that $\| \rho(a) \| \leq \| \rho(a\gamma) \|$ for all $a \in A$, $\gamma \in \Gamma$. Indeed, the identity matrix $e \in V$ is Γ-invariant, so $\| \rho(a\gamma) \| \geq \| \rho(a\gamma)e \|_V = \| \rho(a)e \|_V = \| \rho(a) \|$, where we compute the norms with respect to a basis in V_0 consisting of eigenvectors for $\underline{A}$.

Choose $\underline{G}$-modules (ρ_{oi}, V_{oi}), as in fact 1, and using the corresponding modules (ρ_i, V_i) construct a map $m : X \to \underline{a}$.

Fix a positive root system $\Sigma^+ \subset \Sigma$ and consider sections $w : \underline{A} \to \underline{A} \subset \underline{G}$ defined by elements $w \in W = W(\Sigma)$. Then all of them are Siegel sections. As follows from (ii), for large enough ball

$$B \qquad G = BL\Gamma = \bigcup_w Bw(L^+)\,\Gamma,$$

which means the Siegel domains S_w cover X.

Remark: Set $W' = \text{Norm}(\underline{A}, \Gamma)/\text{Cent}(\underline{A}, \Gamma) \subset W$. I think that the following facts are true:

(i) W' is a reflection group in $\underline{a}$.

(ii) There exists a large scale equivalence $m' : X \xrightarrow{\sim} \underline{a}/W'$ such that m is equivalent to the composition

$$X \to \underline{a}/W' \to \underline{a}/W \sim \underline{a}^+ .$$

For $F = \mathbb{R}$ these facts can be deduced from [Ro].

4.7. Proofs for 4.4

In this section we consider adelic groups. We fix a global field F and denote by $\mathbb{A}$ the corresponding ring of adeles. We also fix a reductive F-group $\underline{G}$, denote by G the corresponding adelic group $\underline{G}(\mathbb{A})$, and by r_a the algebraic radial function on G (see 4.4).

In order to describe the homogeneous spaces $X = G/\Gamma$ which have polynomial growth, we will use the same strategy as in 4.6, with the following modifications.

Step 1: Let $\underline{A}$ be a split F-torus, $A = \underline{A}(\mathbb{A})$, $L = \text{Hom}(G_m, \underline{A})$, $\underline{a} = L \otimes R$. We fix an element $c \in A^*$ such that $|c| > 1$ and $|c_p| \geq 1$ for all places p. Then we identify L with a subgroup of A using an embedding $i : L \to A$, $\ell \mapsto \ell(c)$. The inverse homomorphism $j : A \to \underline{a}$ is given by the condition $\langle \lambda, j(a) \rangle = \log|\lambda(a)|/\log|c|$, for $\lambda \in L^*$. Since $j \circ i : L \to \underline{a}$ is the natural embedding, $A/L\underline{A}(F)$ is compact, and $j(\underline{A}(F)) = 0$, we see that i and j give mutually inverse large scale equivalences of $L \sim \underline{a}$ and $A/\underline{A}(F)$. We choose Σ, Σ^+, as before, and set $A^+ = j^{-1}(\underline{a}^+)$.

Step 2: We fix a system of sections $\kappa : \underline{A} \to \underline{G}$, all conjugate under $\underline{G}(F)$, and define highest weight $\underline{G}$-modules as in 4.6.

We define the norm $\| \ \|$ on $GL(n, \mathbb{A})$ by $\|g\| = \Pi \|g_p\|$, for $g = (g_p)$. For every finite-dimensional vector space V over F we define a norm on $GL(V(\mathbb{A}))$ using some isomorphism $V \xrightarrow{\sim} F^n$.

For every $\underline{G}$-module (ρ, V) we define a large scale map $m_V : G \dot{\to} \mathbb{R}$ by $m_V(g) = \log\|\rho(g)\|$. If V has highest weight λ, then for any section $\kappa : \underline{A} \to G$ the function $m_V(\kappa(\ell))$ on L^+ is equivalent to $\langle \lambda, \ell \rangle \log|c|$. Indeed, in a basis of eigenvectors for $\kappa(\underline{A})$ we have for each $\ell \in L^+$ $\|\rho(\kappa(\ell))\| = \Pi \|\rho(\kappa(\ell))_p\| = \Pi |c_p|^{\langle \lambda, \ell \rangle} = |c|^{\langle \lambda, \ell \rangle}$ (here we used that $|c_p| \geq 1$ for all places p, see 4.6).

Step 3: We define functions $m_V^\Gamma : V \to \mathbb{R}$ and a map $m : X \to \underline{a}$ using $\underline{G}$-modules (ρ_i, V_i) as in 4.6. We call a section $\kappa : \underline{A} \to \underline{G}$ (Γ, V)-special if $\|\rho(g\gamma)\| \gg \|\rho(g)\|$ for $g \in \kappa(L^+)$, $\gamma \in \Gamma$; and we call it a Siegel section if it is (Γ, V_i)-special for all i.

Now consider

Example 4.4.1. $\Gamma = G(F)$. Choose a maximal F-split torus $\underline{A} \subset \underline{G}$ and a positive root system $\Sigma^+ \subset \Sigma = \Sigma(\underline{A}, \underline{G})$. We claim that the standard section $\kappa : \underline{A} \to \underline{G}$ is (Γ, V)-special for any highest weight $\underline{G}$-module V, i.e. $\| \rho(g\gamma) \| \gg \| \rho(g) \|$ for all $g \in L^+, \gamma \in \Gamma$.

Indeed, choose a basis of V consisting of eigenvectors for A, and for each $g \in G$ denote by $\rho(g)_{ij}$ the matrix entries in this basis. We can choose an ordering of the basis such that $\rho(g)_{1,1} = \lambda(g)$ for $g \in A$, where λ is the highest weight of V.

Choose an idex j such that the entry $b = \rho(\gamma)_{1,j} \in F$ is nonzero. Then $\rho(\ell\gamma)_{1,j} = b \cdot c^{\langle\lambda,\ell\rangle}$, and therefore $\| \rho(\ell\gamma) \| \geq | c^{\langle\lambda,\ell\rangle} | = | c |^{\langle\lambda,\ell\rangle} = \| \rho(\ell) \|$ for $\ell \in L^+$.

As follows from [Bor], there exists a ball $B \subset G$ such that $BA^+\Gamma = G$. Since $A/L\underline{A}(F)$ is compact and $\underline{A}(F) \subset \Gamma$, we can enlarge B so that $BL^+\Gamma = G$, i.e. so that the Siegel domain $S = B \cdot p(L^+)$ coincides with X. Thus $m : X \to \underline{a}^+$ is a large scale equivalence.

Remark: The decomposition $G = BL^+\Gamma$ implies that the space $X = G/\Gamma$ is compactly generated in the sense of 4.5, and that the resulting natural large scale on X is equivalent to the algebraic one.

Example 4.4.2. Let $\underline{P} \subset \underline{G}$ be a parabolic subgroup, $\underline{P} = \underline{MU}$ its Levi decomposition, $\Gamma = \underline{P}(F) \cdot \underline{U}(\mathbb{A}) \subset G$.

This example is a mixture of examples 4.4.1. and 4.3.5. We leave it to the reader to define a map $m : X \to \underline{a}$ and to show that it is an $\underline{a}^+$-model. In fact, in this case one can construct a large scale equivalence $m' : X \to \underline{a}/W_M$, where W_M is the Weyl group of M. In order to do so one can either use the method described in remark 1 to example 4.3.5, or to prove directly that as a large scale space X is equivalent to $M/\underline{M}(F) = P/\underline{P}(F) \cdot \underline{U}(\mathbb{A})$, using the fact that G/P is compact.

Example 4.4.3. $G = G_0 \times G_0$, $\Gamma = \Delta G_0$. This is a negative statement, so let us consider just the simplest case: $F = \mathbb{Q}$, $G_0 = SL(2)$. We can write $G = G(\mathbb{R}) \times G(f)$, $X = X(\mathbb{R}) \times X(f)$, where $(\mathbb{R})$ stands for the real component, (f) for the product of all the p-adic components. Let us ignore the real component, which we have already analyzed in example 4.3.1, and consider the group $G(f)$ and its homogeneous space $X(f)$.

Let N be the set of natural numbers with a distance function $d(n, m) = \log n + \log m - 2 \log(n, m)$ where (n, m) is the greatest common divisor of m and n. Using Cartan decomposition for p-adic groups it is easy to check that the natural embedding $N \to X(f)$, $n \to \mathrm{diag}(n, n^{-1})$ is a large scale equivalence, which provides a concrete model for the large scale space $X(f)$.

This model shows that the space $X(f)$ has exponential growth.

On the other hand, using the same model we can describe summable weights which are close to minimal. (Namely the weight $w(n) = n(\log n)^d$ is summable for $d > 1$ and is not summable for $d \leq 1$. This shows that as a large scale space $X(f)$ has a very regular structure, without being a space of polynomial growth.

We will leave it to the reader to construct an explicit model for the homogeneous space X in example 4.4.4.

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