MEROMORPHIC PROPERTY OF THE FUNCTIONS $P^{\lambda}$

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In this note the following problem is solved:

Let $P(x_1, \ldots, x_n)$ be a polynomial with real coefficients and $\varphi(x)$ be a fundamental function on $\mathbf{R}^n$ (i.e., an infinitely differentiable function with compact support). Consider the integral

$$I(\lambda) = \int_{\mathbf{R}^n} |P(x)|^{\lambda} \varphi(x)\, dx_1 \cdot \ldots \cdot dx_n.$$

The integral converges absolutely for $\operatorname{Re} \lambda > 0$ and is an analytic function of $\lambda$. We wish to prove that $I(\lambda)$ can be analytically continued on the whole of the complex plane as a meromorphic function of $\lambda$.

The problem was posed by I. M. Gel'fand at the Amsterdam congress. It has been solved for various classes of polynomials (see [1]). I. M. Gel'fand conjectured that one might be able to solve this problem for an arbitrary polynomial, using H. Hironaka's results on the resolution of singularities. Indeed, as we show in this note, the solution of the $P^{\lambda}$ problem follows easily from Hironaka's results [2]. (Similar methods are applied in [1], Ch. 3, §4.)

Let X be an n-dimensional algebraic manifold, defined by equations with real coefficients and let $X_{\mathbf{R}}$ and $X_{\mathbf{C}}$ be the sets of its real and complex points respectively. We shall always assume below that $X_{\mathbf{C}}$ is nonsingular and irreducible and that $X_{\mathbf{R}}$ is not empty. We shall say that a rational function P on X with real coefficients is a regular function, if P is regular on $X_{\mathbf{C}}$. We shall denote the set of zeros of such a function by Z(P). $Z(P) = \{x \in X_{\mathbf{C}} \mid P(x) = 0\}$.

<u>LEMMA 1.</u> If P is a regular function on X, then there exists a manifold $\hat{X}$ and a mapping given by functions with real coefficients, $\pi : \hat{X} \to X$, such that the following conditions are satisfied:

1. $\hat{X}_{\mathbf{C}}$ is nonsingular and irreducible.

2. $\pi$ is a proper mapping, i.e., the preimage of a compactum is a compactum.

3. Exterior to the zeros of P the mapping $\pi$ is an isomorphism: $\pi : \hat{X}_{\mathbf{C}} \backslash Z(\hat{P}) \xrightarrow{\sim} X_{\mathbf{C}} \backslash Z(P)$, where $\hat{P} = \pi^*(P)$.

4. For any point $x \in \hat{X}_{\mathbf{C}}$ there exist rational functions with real coefficients $y_1, y_2, \ldots, y_n$ and Q such that

a) $y_1(x) = \ldots = y_n(x) = 0$, $Q(x) \neq 0$.

b) The differentials $dy_1, \ldots, dy_n$ are linearly independent at the point x (i.e., $y_1, \ldots, y_n$ are local parameters in a neighborhood of x).

c) $P = y_1^{a_1} \cdot \ldots \cdot y_n^{a_n} \cdot Q$, where $a_1, \ldots, a_n$ are nonnegative integers.

This follows from H. Hironaka's results [see [2], Ch. 0, §5, Corollary 3).

<u>LEMMA 2.</u> Let $\varphi(x_1, \ldots, x_n, \mu)$ be a fundamental function on $\mathbf{R}^n$ which is a meromorphic function of the parameter $\mu$. Then the function

$$I(\lambda_1, \ldots, \lambda_n, \mu) = \int_{\mathbf{R}^n} |x_1|^{\lambda_1} \cdot \ldots \cdot |x_n|^{\lambda_n} \cdot \varphi(x, \mu)\, dx_1 \ldots dx_n$$

can be analytically continued to all values of $\lambda_1, \ldots, \lambda_n$ as a meromorphic function of $\lambda_1, \ldots, \lambda_n, \mu$, and its poles that differ from those already existing in the function $\varphi(u)$ can lie on hypersurfaces of the form $\lambda_i + s = 0$, where s are odd natural numbers.

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By repeated integration this assertion reduces to the case n = 1, which is discussed in [1], Ch. 1, §3.

THEOREM 1. Let X be a manifold; P be a regular function on X; $\omega$ be a regular differential form of the highest dimension on X (i.e., a form defined over the real numbers and regular on $X_{\mathbf{R}}$); and $\varphi$ be a fundamental function on $X_{\mathbf{R}}$. Let $\omega$ be nondegenerate on $X_{\mathbf{C}} \setminus Z(P)$. Then the integral $I(\lambda) = I(X, P, \varphi, \omega; \lambda) = \int_{X_{\mathbf{R}}} |P|^{\lambda} \cdot \varphi(x) \cdot |\omega|$. defined for Re $\lambda > 0$, can be analytically continued as a meromorphic function of $\lambda$ on the whole of the complex plane, its poles belonging to a finite number of arithmetic progressions of the form $a\lambda + b + s = 0$, where $a > 0$ and $b \geq 0$ are fixed integer numbers and s runs through all the odd natural numbers (here $|\omega|$ is the measure on $X_{\mathbf{R}}$ corresponding to the form $\omega$; for example, on a straight line the measure corresponding to the form xdx is $|x|dx$).

Proof. By Lemma 1 we find the mapping $\pi: \hat{X} \to X$. We define $\hat{P} = \pi^*(P)$, $\hat{\varphi} = \pi^*(\varphi)$, $\hat{\omega} = \pi^*(\omega)$. Then, as can be seen easily, $\hat{X}$, $\hat{P}$, $\hat{\varphi}$, $\hat{\omega}$ satisfy the condition of the theorem and $I(X, P, \varphi, \omega; \lambda) = I(\hat{X}, \hat{P}, \hat{\varphi}, \hat{\omega}; \hat{\lambda})$. Therefore, it is sufficient to prove the theorem for $\hat{X}$. Let $x \in \hat{X}_{\mathbf{R}}$. Consider the local parameters $y_1, \ldots, y_n$ mentioned in Lemma 1. In some neighborhood of x we have $\hat{P} = y_1^{a_1} \cdot \ldots \cdot y_n^{a_n} \cdot Q$ and $\hat{\omega} = S(y)\, dy_1 \cdot \ldots \cdot dy_n$. At the same time, since $S(y) = 0$ only on $Z(\hat{P})$, one can find, by Hilbert's theorem on zeros, a number N such that $\hat{P}^N$ is divisible by S(y). I.e., $S(y) = y_1^{b_1} \cdot \ldots \cdot y_n^{b_n} \cdot Q_1$, where $Q_1(x) \neq 0$. By the resolution of the identity one has the representation $\hat{\varphi} = \sum \varphi_j$, where the support of each function $\varphi_j$ is concentrated in such a small neighborhood U of some point $x_j$ that: a) Q(y) and $Q_1(y)$ do not vanish on U; b) $y_1, \ldots, y_n$ yield an isomorphism of U with some domain in $\mathbf{R}^n$ ($y_1, y_2, \ldots, y_n$, Q, $Q_1$, and U depend on $x_j$). Then,

$$I(\lambda) = \sum I_j(\lambda), \quad I_j(\lambda) = \int_{\mathbf{R}^n} |y_1|^{a_1\lambda + b_1} \cdot \ldots \cdot |y_n|^{a_n\lambda + b_n} \cdot |Q|^{\lambda} | \cdot |Q_1| \cdot \varphi_j \, dy_1 \cdot \ldots \cdot dy_n,$$

and the theorem follows from Lemma 2.

THEOREM 2. Let X be a manifold; $P_1, \ldots, P_k$ be regular functions on X; $\omega$ be a regular form; and $\varphi$ be a fundamental function on $X_{\mathbf{R}}$; then the integral

$$I(\lambda_1, \ldots, \lambda_k) = \int_{X_{\mathbf{R}}} |P_1|^{\lambda_1} \cdot \ldots \cdot |P_k|^{\lambda_k} \cdot \varphi \cdot |\omega|$$

can be continued as a meromorphic function on the whole space of the complex variables $\lambda_1, \ldots, \lambda_k$; at the same time the poles can be situated on a finite number of series of hypersurfaces of the form $a_1\lambda_1 + \ldots + a_k\lambda_k + b + s = 0$, where $a_1, \ldots, a_k$, and b are fixed nonnegative integers and s runs through all the odd natural numbers.

This can be proved in the same way as Theorem 1. Using the resolution of the identity, one must make a reduction to an affine neighborhood U in which $\omega = S(y)dy_1 \cdot \ldots \cdot dy_n$ ($dy_1 \cdot \ldots \cdot dy_n$ is nondegenerate), and then apply Lemma 1 to the function $P = P_1 \cdot \ldots \cdot P_k \cdot S$.

Remark 1. These theorems can easily be extended to the case when $P_1, \ldots, P_k$ are any rational functions.

Remark 2. Analogous theorems can be proved for manifolds over any locally compact field.

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## LITERATURE CITED


1. I. M. Gel'fand and G. E. Shilov, Generalized Functions, New York (1964).
2. H. Hironaka, "On the resolution of singularities," Ann. Math. (2) 79, 109-203 (1964); 79, 205-326 (1964).