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Notes on Integral Geometry for Manifolds of Curves

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To the memory of Friedrikh Karpelevich

ABSTRACT. The authors describe admissible families of curves, for which in the complex case there are Radon type inversion formulas for the (local) problem of integral geometry. Curves in such families admit compatible extensions up to rational curves. As special cases we obtain the complete description of admissible families of lines and conics. There is a connection with twistor constructions of explicit solutions of nonlinear differential equations.

These notes have a long and unusual history. In 1979 we started to work on the integral geometry for manifolds of curves. The impetus to these deliberations was the paper [**4**], where a general structure of Radon type inversion formulas for manifolds of complex curves was found. It turned out that every such formula is an integration over an appropriate cycle of a universal closed (1, 1)-form κ on the (infinite dimensional) manifold of all curves with densities (for the exact constructions, see Note 1 below). This form is the result of application of an explicit differential operator of order (1, 1) to the integrals of functions along curves with respect to densities. This result about the universal nature of local inversion formulas extends the notion of the operator κ developed by Gelfand, Graev and Shapiro for integral geometry on complex planes [**1**] (the Radon-John transform).

An important direction in integral geometry is the study of admissible families of submanifolds for which the problem of integral geometry admits local inversion formulas. In terms of the operator κ this problem reads as a problem about characteristic families on which κ can be computed (it does not use differentiations in transversal directions). This problem of nonlinear analysis was solved for families of complex lines in a quite general situation [**2, 3**]. In [**4**] all admissible 2-parameter families of complex curves with densities on 2-dimensional manifolds were described (locally). In particular, all densities on lines for which Radon type inversion formulas do exist were described.

In this article, we consider some similar problems for real curves. We start with a description of finite dimensional families of curves on which there exist densities

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yielding admissible families of curves with densities (in the complex version there is a Radon type inversion formula). We have found that such densities exist if and only if the curves can be simultaneously extended to rational curves and the family of extended curves is complete in a natural sense.

Our next problem is to describe complete subfamilies of complete families of rational curves. A typical example is the problem about 2-parameter subfamilies of quadrics in the projective plane with Radon type inversion formulas (the 5-parameter family of all nondegenerate quadrics is a complete family of rational curves). It turns out that such generic subfamilies are always defined by intersection and tangency conditions with some subvarieties. The role of such intersection and tangency conditions was already known in the case of admissible families of lines.

We give the general description of admissible subfamilies in the language of σ-processes (blowing up). Note that even in the case of lines the description of admissible subfamilies that are not generic is new.

It is natural to compare the role of manifolds of rational curves in integral geometry with the role of manifolds of rational curves in the Penrose twistor theory [**6**]. Probably, their appearance in such different problems reflects an important universal role of families of rational curves in explicitly solvable nonlinear problems.

Since the construction of self-dual solutions of the vacuum Einstein equation includes constructions of 4-parameter families of rational curves on 3-folds, we can apply to this problem our constructions of admissible subfamilies of large families of rational curves. Using this method, we can construct new solutions of the self-dual Einstein equation using subfamilies of conics in the projective space defined by conditions of tangency and intersection (for more details, see [**8**]–[**10**]).

In 1981 one of us (J.B.) emigrated from the U.S.S.R. Before his departure we prepared these notes to record some of the final results. The situation at that time was such that we could not even think about publishing them. Nevertheless, Mitya Leites translated them into English and later published them as preprints of Stockholm University. We thank him very much for his help. We also thank Hai Ying for the preparation of the TEX file of this paper.

We decided to publish a part of these rather old notes, since we believe that they can still be useful to mathematicians working in integral geometry. We only corrected a few typos and added a minimum of references reflecting the development of the subject. The most important progress was achieved by Goncharov, who gave a remarkable description of the algebraic admissible families of curves [**12**] and obtained also some substantial multidimensional generalizations [**13**]. Notice also an analytical approach to admissible complexes [**11**] parallel to the algebraic-geometrical approach of Note 3, and a version of the operator κ for families of submanifolds of dimension greater than 1 in [**14**].

NOTE 1
ADMISSIBLE FAMILIES OF CURVES

Admissible families of curves arise naturally in problems of integral geometry. In the simplest cases the object of the study is an integral transformation I that to a function f with compact support assigns its integrals If along straight lines. Since the problem is overdetermined in dimensions $n > 2$, the inversion formula is not unique. However, for complex lines a vast natural class of inversion formulas is obtained via integration of the universal differential form κIf along various cycles. The form κIf is made from If by means of a differential operator, see [**1**]. The family of lines that enables us to recover κIf from the restriction of If to the family is called *admissible*. This is a peculiar characteristic condition. It turns out that all generic admissible families of complex lines can be described [**2, 3**].

In [**4**] this situation is completely generalized to integration along curves. The universal form κIf can be extended, in a sense, from the manifold of lines to the manifold of curves, and in this way the notion of an admissible family of curves arises (see §1 below). It turns out that for the two-parameter family of curves on the plane the notion of admissibility coincides with the "infinitesimal Desargues property", see [**5**]. Our goal is to generalize this result to any finite dimensional family of curves.

Section 1 follows the lines of [**4**] with slight modifications (e.g., in the definition of admissible operator). Section 1 provides motivations for the geometric notion of an admissible family of curves.

Section 2 contains our main result — a *purely geometric description of admissible families of curves.* This description enables us to effectively describe these families in Note 2. We start with weakly admissible families that naturally arise in problems of integral geometry. It is known that an arbitrary 2-parameter family of curves on the plane has lots of remarkable properties. First, on each curve of the family a natural structure of a local projective line arises (see [**5**]). Weakly admissible families are just the families with this property (each 2-parameter family of curves on the plane is weakly admissible).

In what follows we will show that, as a rule, weak admissibility implies admissibility (a version of the classical Desargues theorem). In general, admissible families are distinguished among weakly admissible ones by a global projective structure on the curves (Main Theorem).

Section 3 contains some examples: in addition to two-parameter families on the plane considered in [**4**] we describe 4-parameter families on 3-dimensional manifolds. The latter example is related to Penrose's study of self-dual conformal metrics [**6, 8**].

As for the integral geometry, the situation is rather complicated. We consider here the real case and produce a well-defined differential form which, however, does not supply us with an inversion formula since in the real case there are no local inversion formulas for curves. A natural generalization to the complex case leads to inversion formulas, but through more cumbersome computations. Still another way of generalization to the complex case is to consider the problem of integral geometry for $\bar{\partial}$-cohomology [**7**].

§1. Admissible operators and admissible families

1.1. Notation. Let X be a manifold. On X, we will consider families K of smooth non-parameterized curves and families $\mathcal{K}$ of parameterized curves. In the latter case, the induced family of non-parameterized curves will be denoted by K. Our considerations are mostly local. For non-parameterized curves the elements of the tangent space $T_E K$ to K at the curve E (i.e., the variations δE of the curve E) are naturally identified with the sections of the normal bundle $N(E)$ to the curve E in X, while for parameterized curves the elements of $T_E\mathcal{K}$ are naturally identified with the sections of the restriction $TX|_E$ to E of the tangent bundle to X.

Let Π be the space of pairs (E, ψ) consisting of a curve E and a smooth density ψ on E. Let $K(\Pi)$ (resp. $\mathcal{K}(\Pi)$) be the corresponding family of curves. In Π, the curve may occur several times with different densities. So it is natural to assume that if $(E, \psi) \in \Pi$, then $(E, \lambda\psi) \in \Pi$ for all $\lambda \in \mathbb{R}$. If curves are parameterized, the tangent vectors to Π are identified with pairs $(\delta E, \delta\psi)$, where δE is a tangent vector to $\mathcal{K}(\Pi)$ (i.e., a section of $TX|_E$) and $\delta\psi$ is a density on $\mathbb{R}^1$. If curves are not parameterized, we shall consider the family $\tilde{\mathcal{K}}(\Pi)$ of all parameterizations of curves from $K(\Pi)$, lift densities to $\tilde{\mathcal{K}}(\Pi)$, and apply the above construction to the resulting family $\tilde{\Pi}$.

Let Π be the space of curves with densities. Let us consider the integral transformation $I : C_0^\infty(X) \longrightarrow C_0^\infty(\Pi)$ of the form

$$If(E, \psi) = \int_E f\psi. \tag{1}$$

We say that an operator $L : C_0^\infty(X) \longrightarrow C_0^\infty(\Pi)$ is *local* if $f|_E = 0$ implies that $Lf(E, \psi) = 0$.

1.2. Admissible operators. Denote by Π_x, $x \in X$, the subspace of pairs (E, ψ) such that $x \in E$. Consider a family of first-order differential operators $D : C^\infty(\Pi) \longrightarrow \Omega^1(\Pi_x)$ smoothly depending on the parameter $x \in X$.

An operator D is *admissible* if for almost every $x \in X$ we have

(i) $dDI = 0$, and

(ii) there exists a local operator $M : C_{0x}^\infty(X) \to C^\infty(\Pi)$, where $C_{0x}^\infty(X)$ is the subspace of functions $f \in C_0^\infty(X)$ that vanish at x, such that

$$DIf = dMf \quad \text{for any} \quad f \in C_{0x}^\infty(X).$$

LEMMA. *If* $\dim K > \dim X$ *and* $\dim X > 2$, *then* (i) *implies* (ii).

What does the notion of the admissibility of an operator mean? Its complex analogue provides us with a closed form DIf such that the integrals of this form along appropriate cycles return cf, where $c \neq 0$, i.e., we obtain an inversion formula [**4**].

In the real case the integrals along all cycles vanish; nevertheless, the notion of admissible operator is meaningful, though it does not provide us with any inversion formula. The transition to the complex case of the results stated below is straightforward.

Note that conditions (i), (ii) above are local, and in our investigation no integration along cycles is involved.

It turns out ([**4**]) that admissible operators are parameterized by functions $\mu_{E,x}$ on E depending on the fixed point $x \in E$. If curves are not parameterized, the admissible operators will be considered on $\tilde{\Pi}$.

PROPOSITION. *For any admissible operator D there exists a unique system of functions $\mu_{E,\psi,x}$ on $E \in K(\Pi)$ such that*

$$\operatorname{Res}\mu_{E,\psi,x} \cdot \psi|_{y=x} = c(x) \tag{2}$$

(i.e., Res *depends only on x and not on E or ψ) and D is of the form*

$$DF(\delta E, \delta\psi) = dF(\mu\delta E, d(\mu\psi)([\delta E, \delta\psi]), \tag{3}$$

where $\mu\psi$ is considered as a map from Π to the space of densities on $\mathbb{R}^1$ and $d(\mu\psi)$ is its differential.

The operator (3) is well defined. This follows from (2) since $\delta E(x) = 0$. The operator (3) is defined on the space of all parameterized curves with densities and can be descended to the space of all non-parameterized curves with densities.

Properties (i) and (ii) are quite straightforward.

The operator (3) may be restricted to Π whenever the vector $(\mu\delta E, d(\mu\psi))$ is tangent to Π. The admissible operator D is trivial if, in its presentation (2), $c(x) = 0$.

1.3. Admissible families of curves. A family of curves K is called *admissible* if on each curve E there exists a density ψ_E such that the family $\Pi(K) = \{(E, \psi_E)\}$ has a non-trivial admissible operator. Proposition 1.2 implies the following assertion.

COROLLARY. *The family of curves K is admissible if and only if there exist ψ_E and $\mu_{E,x}, p(E)$ such that for almost all E and x, where $x \in E$, we have*

(i) *$c(x) \neq 0$ in (2),*

(ii) *$\mu_{E,x}\delta E \in T_E K$ if $\delta E \in T_E K_x$, and*

(iii) *$d\psi[\mu\delta E] = d(\mu\psi)[\delta E] + p(E)(\psi)$.*

§2. Finite-dimensional admissible complexes

2.1. Admissible subspaces of sections. We begin with the study of families of curves satisfying conditions (i), (ii) of Corollary 1.3. Let us consider the tangent space to a fixed curve E.

Let E be a connected curve, N a vector bundle on E. The space of sections $W \subset \Gamma(E, N)$ is *admissible* if W generates almost all fibers, for almost all $x \in E$ the subspace W_x of sections vanishing at x has a non-zero dimension, and for almost all $x \in E$ there exists a function μ_x such that $\mu_x(y) = \frac{a}{x-y} + \dots$, where $a \neq 0$ and $\mu_x W_x \subset W$.

PROPOSITION. *Let W be a finite dimensional admissible subspace. The functions μ_x are related for various x by linear-fractional transformations, i.e., these functions define a morphism $\mu : E \longrightarrow \mathbb{P}^1$.*

The family μ_x for $x \in E$ is determined by W up to a transformation $\mu_x \mapsto \alpha(x)\mu_x + \beta(x)$.

2.2. Weakly admissible families of curves. A family of curves K (or a family of parameterized curves $\mathcal{K}$) is called *weakly admissible* if for each curve $E \in K$ (resp. $E \in \mathcal{K}$) the image of $T_E K$ in $\Gamma(N(E))$ (resp. $T_E\mathcal{K}$ in $\Gamma(TX|_E)$) under the natural embedding is an admissible subspace of sections.

In other words, weak admissibility means that conditions (i), (ii) of Corollary 1.3 (which do not involve densities ψ_E) are satisfied. Thus the question, "When is the weakly admissible family K admissible?" is reduced to the question, "When do densities ψ_E satisfying condition (iii) of Corollary 1.3 exist?"

As follows from Proposition 2.1, in the finite dimensional case the system of functions μ is completely determined by the conditions (i), (ii).

Let K be a weakly admissible finite dimensional family of curves. Proposition 2.1 implies that in each sufficiently small neighborhood $\Omega \subset X$ there is a natural structure of the locally projective line on the curves $E \in K$ (in general, the maps μ_x only cover $\mathbb{P}^1$). In Ω, consider the family $\mathcal{K}_p$ of curves from K with arbitrary projective parameterizations consistent with the local projective structures on these curves.

THEOREM. *A weakly admissible finite dimensional family of curves K is admissible in a sufficiently small neighborhood $\Omega \subset X$ if and only if the family $\mathcal{K}_p$ of parameterized curves is weakly admissible.*

Note that the possibility of constructing $\mathcal{K}_p$ from K assumes weak admissibility of K. The notion of weak admissibility of K (or $\mathcal{K}$) involves the first infinitesimal neighborhood of curves of K (resp. $\mathcal{K}$), while the weak admissibility of $\mathcal{K}_p$ involves the second infinitesimal neighborhood of curves of K, since the first one is already used in the definition of $\mathcal{K}_p$.

Consider the subset $V(E) = \bigcup_{x\in E} T_E(K_x)$ in $T_E K$ and let $L(V(E))$ be the linear envelope of $V(E)$.

THEOREM. *Let us assume that* $\dim X > 2$; *let K be a finite dimensional weakly admissible family of curves such that* $(\dim L(V(E)) - \dim T_E(K_x)) > 1$ *for almost all $x \in X$. Then K is admissible.*

This result is a local analogue of the Desargues theorem.

§3. Examples

3.1. Admissible 2-parameter families of curves on 2-dimensional manifolds. Let X be a manifold, $\dim X = 2$, and Θ a family of curves on X such that $\dim \Theta = 2$. To a point $x \in X$ there corresponds the set of curves $\theta \in \Theta$ such that $x \in \theta$. Let $\theta \in \Theta$ be a curve in X. To points $x \in \theta$ there correspond curves $\theta_x(s)$ from Θ such that $\theta_x(0) = \theta$. By assigning to points $x \in \theta$ tangent lines $\{c\dot{\theta}_x(0)\}$ at $s = 0$ we introduce a local projective line structure on θ.

When $\dim X = \dim \Theta = 2$, the families of curves are always weakly admissible and the above projective structure coincides with the general construction of the previous section.

In coordinates on Θ, the condition of admissibility is equivalent (according to [**4**]) to the existence for almost every $x \in \Theta$ and $\lambda \in T_\theta\Theta$ of the unique curve $\theta_x(s)$ such that $\theta_x(0) = \theta$ and $\dot{\theta}_x(0) = \lambda$, where $\pi_\theta(\lambda) = [\lambda, \ddot{\theta}_x(0)]$ is a third degree polynomial in λ and $[\cdot,\cdot]$ is the skew product of vectors on the plane.

Equivalently, this condition means that there is a diffeomorphism that straightens all curves θ_x on Θ up to third-order infinitesimals. This means, in turn, that the family θ_x may be realized as a family of geodesics for an affine connection.

For the initial family Θ of curves on X, the condition of admissibility coincides with the "infinitesimal Desargues property" [**5**]. In Note 2 we will show that an admissible 2-parameter generic subfamily in a 5-parameter family of second-order curves on the plane is a set of curves tangent to three fixed curves. On the other hand, the family of circles of fixed radius is an example of a non-admissible family.

3.2. Admissible 4-parameter families on 3-dimensional manifolds. The simplest example is the family of lines in $\mathbb{R}P^3$. Let X be a manifold, $\dim X = 3$, and K the manifold of curves on X, so that $\dim K = 4$. Let K_x be the family of curves of K passing through x ($x \in X$). In general, $\dim K_x = 2$ and there are functionally independent variations of curves $\delta E_1, \delta E_2 \in T_E K_x$. Suppose that this is the case and that Λ_x ($x \in E$) is the 2-dimensional plane in $T_E K$ tangent to K_x. Then $V_E = \bigcup_{x\in E} \Lambda_x$ is a cone in $T_E K$

It turns out that in this case weak admissibility coincides with admissibility (it is an infinitesimal version of the classical Desargues theorem, cf. Theorem 2.2.2) and is equivalent to the fact that V_E is a quadratic cone. Then the system of cones V_E, where $E \in K$, defines on K a conformal class of metrics with signature $(2, 2)$ (V_E are just the isotropic or "light" cones). This metric is automatically self-dual.

The converse is also true: any 4-dimensional manifold K equipped with a conformal self-dual metric of signature $(2, 2)$ can be realized via Penrose's scheme [**6**] as an admissible family of curves on a 3-dimensional manifold X so that all the structures are consistent. As we will show in Note 2, the family of second degree flat curves in $\mathbb{R}P^3$ tangent to four fixed surfaces is admissible.

The construction of the family of densities ψ_E satisfying condition (iii) of Corollary 1.3 is essentially equivalent to the choice of a metric in the given conformal class.

Note 2
The Geometric Structure of Admissible Families of Curves

This note is a continuation of Note 1, where we studied the *admissible families of curves* that arise in the integral geometry.

In Section 1 we give another definition of admissible families of curves. According to Note 1 the notions are equivalent.

In Section 2 the infinitesimal structure of the manifold K consisting of an admissible family of curves on a manifold X is described. Observe that this description is given in terms of K only and does not involve X; any manifold K endowed with this structure can be realized consistently with all structures as an admissible family of curves on a manifold.

This is a generalization of the assertion from Note 1 that *problems on 4-parameter admissible families of curves on 3-dimensional manifolds and problems on conformal self-dual metrics are equivalent.*

In Section 3 we move in the opposite direction: we start with a fixed manifold X with many compact rational curves, prove that the family of all such curves is admissible, and describe its admissible subfamilies. This is a generalization of results from [**1**] and [**2**] on admissible families of lines.

Our constructions allow one to give a lot of meaningful examples and, in particular, new examples of global self-dual conformal metrics on four-dimensional manifolds [**8, 9**]. Here the ground field is $\mathbb{C}$ and all manifolds, curves, bundles and so on are supposed to be holomorphic.

§1. Admissible families of curves

1.1. Admissible subspaces of sections of a vector bundle on the curve. Let C be a connected curve and E a vector bundle on C. A subspace $W \subset \Gamma(C, E)$ is *admissible* if W spans almost all fibers, $\dim W_x \neq 0$ for all $x \in C$, where W_x is a subspace of sections vanishing at x, and for almost all $x \in C$ there is a function $\mu_x(y) = \frac{a}{y-x} + \dots$, $y \in C$, such that $\mu_x W_x \subset W$ for some $a \neq 0$.

Proposition. *Let $W \subset \Gamma(C, E)$ be a finite dimensional subspace. Then the functions μ_x, $x \in C$, are related by linear-fractional transformations, i.e., they define a morphism $\mu : C \longrightarrow \mathbb{P}^1$.*

There exist a bundle $\tilde{E}$ on $\mathbb{P}^1$ and a bundle morphism $i : \mu^*\tilde{E} \longrightarrow E$ such that W is the image of $\Gamma(\mathbb{P}^1, \tilde{E})$ and i is an isomorphism on almost all fibers. The bundle $\tilde{E}$ and the morphism i are defined up to an isomorphism.

Regarding the relation between the μ_x for distinct x's, see Note 1. It is important that for generic points $x, y \in C$ we have $W_y = (\lambda_1\mu_x - \lambda_2)W_x$, where $\mu_x(y) = \frac{\lambda_2}{\lambda_1}$.

Conversely, if x is a generic point, we set $W(\lambda) = (\lambda_1\mu_x - \lambda_2)W_x$ for $\lambda = (\lambda_1, \lambda_2) \in \mathbb{C}^2 \setminus \{0\}$.

If $\mu_x(y) = \frac{\lambda_2}{\lambda_1}$, then $W(\lambda) = W_y$. The multiplication by $(\lambda_1\mu_x - \lambda_2)$ is an operator without kernel. Therefore, by setting

$$\tilde{E}_\lambda = W/W(\lambda),$$

we obtain the desired bundle.

1.2. $\mathcal{P}$-structure on the linear space. The above arguments make the following definition natural. Let W be a linear space. We say that a system $P(W)$ of ν-dimensional subspaces in W (i.e., $P(W) \subset \mathrm{Gr}_\nu W$) defines a *$\mathcal{P}$-structure on W* if there are a linear space L and maps $A_1, A_2 : L \longrightarrow W$ such that

$$P(W) = \{\mathrm{Im}\,(\lambda_1 A_1 + \lambda_2 A_2) \text{ for any } \lambda \in \mathbb{C}^2 \setminus \{0\}\} \quad \text{and} \quad \bigcap_{V \in P(W)} V = 0.$$

By definition, $P(W)$ is naturally isomorphic to $\mathbb{P}^1$.

LEMMA. *Let $P(W) \subset \mathrm{Gr}_\nu W$ define a $\mathcal{P}$-structure on W and let $E = E(P(W))$ be a bundle on $P(W)$ with fibers $E_V = W/V$. Then $\Gamma(P(W), E) = W$.*

This lemma implies that the triple $\{L, A_1, A_2\}$ is recovered from the $\mathcal{P}$-structure up to a natural transformation.

If W is an admissible subspace of sections, then on W a $\mathcal{P}$-structure arises: $P(W)$ is the Zariski closure of the set $\{W_x\} \subset \mathrm{Gr}_\nu W$.

1.3. Admissible families of curves. Let K be a family of non-parameterized curves $k \longrightarrow C(k)$, $k \in K$, on the manifold X. The family K is called a *covering* if $\bigcup_{k \in K} C(k)$ covers an open set in X. We will usually assume that our families are coverings.

There is a natural mapping $T_k K \longrightarrow \Gamma(C(k), N)$, where N is the normal bundle to $C(k)$. Up to a closed nowhere dense subset in K, this mapping is injective and we can identify $T_k K$ with a subspace of $\Gamma(C(k), N)$.

If $\mathcal{K}$ is a family of parameterized curves, then there is a natural mapping $T_k\mathcal{K} \longrightarrow \Gamma(C(k), TX|_{C(k)})$. Similarly, let us identify $T_k\mathcal{K}$ with its image under this mapping. The family K (resp. $\mathcal{K}$) is *weekly admissible* if subspaces of sections identified with $T_k K$ (resp. $T_k\mathcal{K}$) are admissible.

If K is a weakly admissible family, then Lemma 1.2 implies that on $C(k)$, $k \in K$, there is a natural structure of a local projective line. In a sufficiently small neighborhood Ω on X, consider a family $\mathcal{K}_\mathcal{P}$ of curves $C(k)$, $k \in K$, for all projective parameterizations consistent with their projective structures. The weakly admissible family K is *admissible* if $\mathcal{K}_\mathcal{P}$ is weakly admissible.

§2. Infinitesimal structure of admissible families of curves

2.1. Manifolds with $\mathcal{P}$-structures. We say that a manifold K is endowed with a *$\mathcal{P}$-structure of dimension ν* if this structure is introduced on every T_kK. In particular, if K is realized as a weakly admissible family of curves, it has a natural $\mathcal{P}$-structure corresponding to this realization.

Let K be a manifold with a $\mathcal{P}$-structure of dimension ν. Denote by $P(K)$ the manifold of pairs (k, V), where $k \in K$ and $V \in P(T_kK)$, i.e., V is a subspace of the $\mathcal{P}$-structure on T_kK. There is a canonical projection $\mathrm{pr} : P(K) \longrightarrow K$ with the fiber $\mathbb{P}^1$.

The (local) distribution $u \mapsto \mathcal{F}(u)$ of dimension ν, where $u \in P(K)$, on $P(K)$ is *consistent with the $\mathcal{P}$-structure on K* if $\mathrm{pr}_* \mathcal{F}(u) = V$ for any $u = (k, V)$, where $k \in K$ and $V \in T_kK$.

Let K be a weakly admissible family of curves on X and

$$C(K) = \{(x, k) \mid x \in X \text{ and } k \in K \text{ such that } x \in C(k)\}.$$

By Proposition 1.1 there is a natural mapping $\mu : C(K) \longrightarrow P(K)$. Let $\mathcal{F}(X, K)$ be a distribution on $C(K)$ tangent to the natural projection $C(K) \longrightarrow X$. Since μ is a diffeomorphism in a neighborhood of the generic point, $\mathcal{F}(X, K)$ can be transferred to $P(K)$.

The local distribution thus obtained is consistent with the $\mathcal{P}$-structure on K related to the realization of K as a family of curves. Obviously, $\mathcal{F}(X, K)$ is integrable.

THEOREM. *A weakly admissible family of curves K on the manifold X is admissible if and only if $\mathcal{F}(X, K)$ is algebraic along the fibers of the projection* $\mathrm{pr} : P(K) \longrightarrow K$.

Since $\mathcal{F}$ is algebraic, the local distribution $\mathcal{F}$ can be extended to a global distribution on $P(K)$. On the other hand, any global distribution is automatically algebraic along the fibers.

PROPOSITION. *If K is a manifold with a $\mathcal{P}$-structure of dimension $\nu > 1$ and $\dim K - \nu > 1$, then there exists at most one integrable distribution $\mathcal{F}$ on $P(K)$ consistent with the $\mathcal{P}$-structure on K. This distribution, if it exists, is algebraic along the fibers of* pr.

In particular this proposition implies the admissibility of weakly admissible families of curves provided $\dim K > \dim X > 2$ — the result formulated in Theorem 2.2.2 of Note 1.

Let us define a $(\mathcal{P}, \mathcal{F})$-structure on K to be a pair consisting of a $\mathcal{P}$-structure on K and a consistent integrable distribution $\mathcal{F}$ which is algebraic along the fibers of $\mathrm{pr} : P(K) \longrightarrow K$.

2.2. A realization of the manifold with a $(\mathcal{P}, \mathcal{F})$-structure as a family of curves. The $(\mathcal{P}, \mathcal{F})$-structure on K is an essential characterization of the pair (K, X), where K is an admissible family of curves on X.

PROPOSITION. *Let K be a manifold with a $(\mathcal{P}, \mathcal{F})$-structure. Then K may be realized as an admissible family of curves on $X = X(P(K), \mathcal{F})$ consistently with the $(\mathcal{P}, \mathcal{F})$-structure. This realization can be chosen so that $k \in K$ corresponds to a rational curve on X.*

This proposition is proved by a direct construction. Let us take a local quotient of $P(K)$ by $\mathcal{F}$. Since $\mathcal{F}$ is transversal to all compact fibers of projection pr : $P(K) \longrightarrow K$, the factorization can be continued to $P(\Omega)$, where Ω is a sufficiently small neighborhood in K.

Using Lemma 1.2, it is easy to show that any curve in $X(P(K), \mathcal{F})$ close to $C(k)$ is of the form $C(k')$ for some $k' \in K$.

2.3. A relation between X and $X(P(K), \mathcal{F})$. Let K be an admissible family on X. Then on K, there exists a $(\mathcal{P}, \mathcal{F})$-structure. From this structure using Proposition 2.2 we recover the manifold $X(P(K), \mathcal{F})$ by assigning to every point $k \in K$ a rational compact curve. How are X and $X(P(K), \mathcal{F})$ related with each other?

It is natural to believe that in good cases X is embedded in $X(P(K), \mathcal{F})$. The obstruction is that generally $\mu : C(K) \longrightarrow P(K)$ is not an inclusion. The pair (K, X) is said to be *irreducible* if μ is an inclusion. In the general case the reduction process based on the following lemma can be applied.

LEMMA. *Let $k \in K$ and $x, y \in C(k)$ be generic points. If $\mu(x, k) = \mu(y, k)$ and for $k' \in K$ close to k we have $x \in C(k')$, then $y \in C(k')$ and $\mu(x, k') = \mu(y, k')$.*

Thus, μ induces an equivalence at generic points of X. Let $\tilde{X}$ be the quotient of X modulo this equivalence. The curves from K are descended to $\tilde{X}$, and the pair (K, X) is irreducible.

If (K, X) is an irreducible covering pair, then to $C(K)$ there corresponds an open set $X(K) \in X(P(K), \mathcal{F})$.

PROPOSITION. $\dim X(K) = \dim X$, *and there exists a natural mapping $X(K) \longrightarrow X$ which is a local diffeomorphism almost everywhere.*

This implies that locally every irreducible covering family of curves corresponding to the manifold K with given $(P(K), \mathcal{F})$-structure can be obtained by the following construction.

On $X(P(K), \mathcal{F})$, the curves from K can be restricted to an arbitrary open part Ω, and to Ω we apply the mapping which is a local diffeomorphism almost everywhere. Further, if (K, X) is an irreducible covering pair, a small neighborhood $\Omega \subset X$ of a generic point of X is embedded into $X(P(K), \mathcal{F})$. The passage from Ω to $X(P(K), \mathcal{F})$ consists in constructing an extension of Ω together with the curves in such a way that on this extension all curves from K are compact and rational. Moreover, these are all the curves of this type on $X(P(K), \mathcal{F})$.

§3. A construction of admissible families of compact curves

In Section 2 we gave a general construction of admissible families of curves. In this section we produce a more effective description of global admissible families on the manifold X with plenty of compact curves.

We describe families such that $\mu : C(K) \longrightarrow P(K)$ is a diffeomorphism. This means that K is an irreducible family of compact rational curves. Under the natural assumption on X, the family $K(X)$ of all curves on X satisfies these conditions.

In the sequel we list the conditions that distinguish the admissible subfamilies in $K(X)$.

3.1. The critical set of an admissible pair (K, X)**.** Let (K, X) be an irreducible admissible pair, $\mu : C(K) \longrightarrow P(K)$ a diffeomorphism and $k \in K$. By Proposition 1.1, on the curve $C(k)$, there exists a subbundle $N_K \subset N$ such that the image of $T_k K$ in $\Gamma(C(k), N)$ coincides with $\Gamma(C(k), N_K)$. The bundle $S = N/N_K$ on $C(k)$ is supported at a finite number of points, which will be called *critical points*, and the dimension of the bundle at a critical point is called the *multiplicity* of the point, cf. [**1, 2**]. Denote by $\mathrm{Crit}(K, X) \subset X$ the critical set, i.e., the union of critical points of all curves $C(k)$, where $k \in K$.

STATEMENT. $\mathrm{Crit}(K, X)$ *is a nowhere dense analytic subset in* K.

3.2. Admissible families without critical points.

PROPOSITION. *Let X be a manifold, and C a compact rational curve on X with non-trivial normal bundle generated by its global sections. Then on X all curves close to C form an admissible family $K(X)$ of curves without critical points, provided this family is a covering one.*

The converse statement is, evidently, also true: *any admissible family of compact curves on X without critical points coincides with $K(X)$.* Observe that under the conditions of this proposition, X and $X(P(K), \mathcal{F})$ are diffeomorphic.

3.3. A description of admissible subfamilies in terms of critical points.

PROPOSITION. *Under the hypothesis of Proposition 3.2 the covering subfamily K in $K(X)$ is admissible if and only if* $\mathrm{codim} K$ *is equal to the sum of multiplicities of critical points on curves $C(k)$, $k \in K$.*

For the family of lines this statement is proved in [**3**].

3.4. A σ-construction of admissible families. Let K be an admissible family. We give two main constructions of admissible subfamilies.

(A) Let Y be a submanifold in X such that $\mathrm{codim} Y > 1$. Then the family of curves from K that intersect Y is admissible.

(B) Let Z be a submanifold in X such that $\mathrm{codim} Z = 1$. Then the family of curves from K that are tangent to Z with the fixed order of tangency l is admissible.

The construction (A) can be reformulated as follows.

(A′) Consider the σ-process (blowing up) $X_1 \longrightarrow X$ along Y, and take the curves from K that can be lifted to X_1 and have a nonempty intersection with the pullback of Y in X_1. More precisely, the σ-process must be performed in an open part of X such that Y is closed in this part.

The construction (A′) can be iterated and combined with (B) as follows. Consider the tower of σ-processes

$$\mathcal{A} : X_q \longrightarrow X_{q-1} \longrightarrow \dots \longrightarrow X_0 = X,$$

where $\sigma_i : X_i \longrightarrow X_{i-1}$ is the σ-process along the submanifold $Y_{i-1} \subset X_{i-1}$, where $i = 1, \dots, q$. Let $Z_1, \dots, Z_m$ be submanifolds in X_q of codimension 1 and $l_1, \dots, l_m$ some positive integers. Denote by $K(X; \mathcal{A}, Z_1, \dots, Z_m, l_1, \dots, l_m)$ the subfamily of curves in $K(X)$ that can be lifted to X_q, have nonempty intersections with the pullbacks of $Y_0, \dots, Y_{q-1}$, and for which the orders of tangency with $Z_1, \dots, Z_m$ are equal to $l_1, \dots, l_m$, respectively.

THEOREM. *If $K(X;\mathcal{A},Z_1,\dots,Z_m,l_1,\dots,l_m)$ is a covering family, then it is admissible.*

Conversely, under the hypothesis of Proposition 3.2 any admissible subfamily of curves in $K(X)$ in a neighborhood of a generic point is of the form $K(X;\mathcal{A},Z_1,\dots,Z_m,l_1,\dots,l_m)$.

Note that any generic admissible subfamily of curves in $K(X)$ is defined by the condition of being tangent to a collection of submanifolds $Z_1,\dots,Z_m$, each of codimension 1 in X. For the family of the form $K(X;\mathcal{A})$, i.e., in the absence of the Z_i, the manifolds X_q and $X(P(K),\mathcal{F})$ are diffeomorphic. In the general case the relation between X and $X(P(K),\mathcal{F})$ is more complicated.

3.5. Induced $(\mathcal{P},\mathcal{F})$-structures. The problem of describing admissible subfamilies of an admissible family of curves can be interpreted in terms of non-linear differential equations.

Let K be endowed with a $(\mathcal{P},\mathcal{F})$-structure. As a rule, this structure is not inherited by any submanifold $\tilde{K}\subset K$. Suppose a linear space W is endowed with a $\mathcal{P}$-structure $P(W)$; let $\tilde{W}$ be a subspace in W. Suppose that in general position $\dim \tilde{W}\cap V=\tilde{\nu}$ for $V\in P(W)$, and let $P(\tilde{W})$ be the Zariski closure in $\mathrm{Gr}_{\tilde{\nu}}\tilde{W}$ of the set of $\tilde{\nu}$-dimensional linear subspaces of the form $\tilde{W}\cap V$. We say that $\tilde{W}$ is a *characteristic subspace with respect to the $\mathcal{P}$-structure* $P(W)$ if and only if $P(\tilde{W})$ defines a $\mathcal{P}$-structure on $\tilde{W}$.

A submanifold $\tilde{K}\subset K$ is said to be *characteristic with respect to the $\mathcal{P}$-structure* $P(K)$ if the subspaces $T_k\tilde{K}$ are characteristic with respect to $P(T_k\tilde{K})$ for almost all $k\in\tilde{K}$. It is clear that *on $\tilde{K}$, the $(\mathcal{P},\mathcal{F})$-structure is induced if and only if $\tilde{K}$ is characteristic.*

The condition for possessing the characteristic property is a non-linear system of differential equations, cf. [**2, 3, 11**]. The construction in 3.2 based on σ-processes can be viewed as a procedure for solving this system explicitly. Namely, for $X=K(P(K)),\mathcal{F})$ we must take $\tilde{K}$ of the form $K(X;\mathcal{A},Z_1,\dots,Z_m,l_1,\dots,l_m)$.

§4. Examples

4.1. Admissible families of curves on a 2-dimensional manifold. Any 2-parameter family K of curves on a 2-dimensional manifold X is weakly admissible.

Let X satisfy the hypothesis of Proposition 3.2 and let the curves from K be compact and rational. Then any one-dimensional distribution $\mathcal{F}(X,K)$ is algebraic on the fibers of $\mathrm{pr}: P(K)\longrightarrow K$, and K is admissible. Observe that the degree 3 homogeneous function $\pi(\lambda)$ defined in Note 1, which determines admissibility, is now defined on $\mathbb{C}^2$; hence, $\pi(\lambda)$ is a polynomial.

If $X=\mathbb{C}P^3$ and K is the (5-parameter) family of all second order curves, then almost all admissible 2-parameter subfamilies have the following structure: they consist of curves containing q fixed points and tangent to m fixed curves, where $q+m=3$. Observe that all circles form an admissible 3-parameter family, since they are determined by the fact that they pass through two cyclic points.

4.2. Four-parameter families of curves on 3-dimensional manifolds. Let (X,K) be such a pair (a 3-dimensional manifold, a 4-parameter family). Then in the generic case weak admissibility implies admissibility. Two cases may occur:

a bundle $N_K \subset N$ such that variations of curves from K correspond to its sections is isomorphic to either $\mathcal{O}(1) \oplus \mathcal{O}(1)$ or $\mathcal{O}(2) \oplus \mathcal{O}$.

In the general position the first case occurs. It is related to self-dual metrics. This relation is based on considerations due to Penrose [4]. The notion of $\mathcal{P}$-structure on a 4-dimensional generic manifold is equivalent to the notion of conformal metric together with an orientation. Now, the isotopic (light) cone at a point $k \in K$ is the union of 2-subspaces $T_k K$. The existence of a consistent integrable distribution $\mathcal{F}$ on $P(K)$ is equivalent to the self-duality of this metric.

Using the results of 3.2, we can give several examples of global self-dual metrics. For instance, consider the 8-parameter family of flat second order curves in $\mathbb{C}P^3$ and the subfamily K of curves that intersect q given curves and are tangent to m surfaces, $q + m = 4$. The family K is admissible. Due to Penrose, a conformal metric on K is determined by the condition that two points on K corresponding to intersecting curves lie on an isotropic curve (t.e., the "conformal distance" between them equals zero).

Similarly, for an admissible family of curves in $\mathbb{C}P^3$, take the images of third degree mappings $\mathbb{C}P^1 \longrightarrow \mathbb{C}P^3$, and the subfamilies of this family obtained by a σ-process. We can also take admissible subfamilies in the family of graphs in $\mathbb{C}P^1 \times \mathbb{C}P^3$ of polynomial mappings $\mathbb{C}P^1 \longrightarrow \mathbb{C}P^3$ of fixed degree.

Note 3
Admissible Subfamilies of Rational Curves

This note contains the proofs of the basic statements of 2.3 in Note 2 about admissible subfamilies of admissible families of curves (the intersection-tangency conditions). These results give a very powerful tool that allows us to construct explicit examples of admissible families of curves starting with some large classical families of rational curves (e.g., a family of flat quadrics).

§0. Vocabulary

0.1. A *manifold* is a complex analytic manifold; it is assumed to be connected unless otherwise stated. A *submanifold* is an arbirary locally closed submanifold; a *map* or *morphism* is a holomorphic map of manifolds. Usually, our constructions are local; in particular, the coordinates are always local.

If $Y \subset X$ is a submanifold, denote by $\mathcal{I}_X$ the corresponding sheaf of ideals in the structure sheaf $\mathcal{O}_X$; the sheaf $\mathcal{I}_X$ is well defined on $X \setminus (\bar{Y} \setminus Y)$.

A map $\alpha : Z \longrightarrow X$ induces the map $\alpha^* : \Gamma(X, \mathcal{O}_X) \longrightarrow \Gamma(Z, \mathcal{O}_Z)$ and, more generally, the maps $\alpha^* : \Gamma(X, \Omega^i_X) \longrightarrow \Gamma(Z, \Omega^i_Z)$, where Ω^i is the sheaf of differential i-forms.

Any sheaf $\mathcal{I}$ of principal ideals in $\mathcal{O}_X$ is called a *divisor.* A divisor is called *nonsingular* if it is of the form $\mathcal{I}_T$, where T is a closed submanifold; we will usually make no distinction between $\mathcal{I}_T$ and T.

We say that nonsingular divisors $T_1, \dots, T_k$ *have normal crossings* if in a neighborhood of every point $x \in X$ coordinates $x_1, \dots, x_n$ may be chosen so that

$$U \cap T_i = \{(x_1, \dots, x_n) \subset U \mid x_i = 0\} \text{ for } i = 1, \dots, k \text{ (here } n = \dim X).$$

0.2. A subset K of a manifold X is called *analytic* if in a neighborhood of every point $x_0 \in K$ the set K is defined by a system of equations $f_i = 0$, $i \in I$, I is a finite set, and the f_i are analytic functions. If K is given by k (independent) functions, we say that $\operatorname{codim} K \leq k$ (in a neighborhood of x_0). The *tangent space* to K at x is

$$T_x(K) = \{\xi \in T_x(X) \mid df(\xi) = 0 \text{ for any function } f \text{ such that } f|_K = 0\}.$$

STATEMENT. $\operatorname{codim} T_x(K) \leq \operatorname{codim} K$. *The equality is attained if and only if K is nonsingular (i.e., a submanifold) in a neighborhood of x.*

0.3. A *curve* in X is a map $\varphi : E \longrightarrow X$, where E is a 1-dimensional manifold and φ is a diffeomorphism of E onto a submanifold of X.

A *family of curves* is a collection $(H, E(H), \pi, \varphi)$, where H and $E(H)$ are manifolds, $\pi : E(H) \longrightarrow H$ is a fibration, and $\varphi : E(H) \longrightarrow X$ are maps such that $E(h) = \pi^{-1}(h)$ is a 1-dimensional manifold and $\varphi(h) = \varphi|_{E(h)}$ is a curve in X for every $h \in H$.

Clearly, coordinates $u_1, \ldots, u_N$ on H considered as functions on $E(H)$ can be extended to a coordinate system $(u_1, \ldots, u_N; t)$ on $E(H)$; such a system will be called *standard*.

A *germ of a curve* is a nondegenerate germ of a map $\alpha : (E, e) \longrightarrow X$ at a point $e \in E$ of a curve E. A *family of germs of curves* is a collection

$$(H,\ E(H),\ \pi : E(H) \longrightarrow H,\ \ e = e(H),\ \ \alpha : E(H) \longrightarrow X),$$

where $e(H)$ is a submanifold in $E(H)$ such that $\pi : e(H) \longrightarrow H$ is a diffeomorphism, α is the germ of the map along $e(H)$ and $\alpha(h) = \alpha|_{E(h)}$ is a germ of a curve for any $h \in H$.

Set $e(h) = e \cap E(h)$. Given a family of germs of curves $\alpha : (E(H), e) \longrightarrow X$, coordinates $(u_1, \ldots, u_N; t)$ on $E(H)$ such that $t|_{e(H)} = 0$ are called *standard coordinates.*

§1. Families of curves given by intersection conditions

1.1. Given a submanifold $Y \subset X$, $\dim X = n$, $\operatorname{codim} Y = d$, we say that a map $\sigma : Z \longrightarrow X$ is a *σ-map with center in Y* if

a) σ determines a diffeomorphism $Z \setminus \sigma^{-1}(Y) \simeq X \setminus \bar{Y}$, and

b) for a domain $U \subset X$ with coordinates $x_1, \ldots, x_n$ such that

$$Y \cap U = \{(x_1, \ldots, x_n) \in U \mid x_1 = \cdots = x_d = 0\}$$

and a submanifold

$$Z_{U,d} = \{(x_1, \ldots, x_n) \in U, (t_1, \ldots, t_d) \in \mathbb{P}^{d-1} \mid t_i x_j = t_j x_i \text{ for } 1 \leq i, j \leq d\} \subset U \times \mathbb{P}$$

there exists a diffeomorphism $\sigma^{-1}(U) \simeq Z_{U,d}$ compatible with the projection on U.

It is well known that a σ-map always exists and is uniquely determined (up to a canonical diffeomorphism) by X and Y.

Set $T_\sigma = \sigma^{-1}(Y)$. Clearly, T_σ is a nonsingular divisor in Z and $\mathcal{I}_{T_\sigma} = \sigma^*(\mathcal{I}_Y)$. Moreover, $\sigma^*(\Omega^n) = \mathcal{I}_{T_\sigma}^{d-1} \Omega^n Z$, since σ is an open map for $d = 1$.

Given a closed subset $F \subset X$, define its *exact preimage* $\sigma^-(F) \subset Z$ to be $\sigma^{-1}(Z \setminus Y)$.

Let T be a nonsingular divisor in X. If $T \supset Y$, then $\sigma^-(T)$ is a nonsingular divisor in Z and the *preimage* of T is $\sigma^*(T) = \sigma^-(T) + T_\sigma$. If $T \cap Y = \emptyset$, then $\sigma^*(T) = \sigma^-(T)$ is a nonsingular divisor in Z.

For nonsingular divisors $T_1, \dots, T_k$ in X with normal crossings such that either $T_i \supset Y$ or $T_i \cap Y = \emptyset$ for each i, the divisors T_σ, $\sigma^-(T_1), \dots, \sigma^-(T_k)$ also have normal crossings in Z.

1.2. For a manifold X, a *σ-series of length m in X* is a collection $A = (X_i, Y_i, \sigma_i)$ of manifolds $X_0 = X, X_1, \dots, X_m$, submanifolds $Y_i \subset X_i$, and maps $\sigma_i : X_i \longrightarrow X_{i-1}$ for $i \neq 0$ such that

a) σ_i is a σ-map with center in Y_{i-1}, and

b) $\sigma_i(Y_i) \subset Y_{i-1}$, and the morphism $\sigma_i : Y_i \longrightarrow Y_{i-1}$ is a submersion onto an everywhere dense subset.

The *support* of a σ-series A is the closure of Y_0 in X.

Set

$$T_i^{(i)} = T_{\sigma_i} \subset X_i \quad (i = 1, \dots, m).$$

For $k = i+1, \dots, m$, define the *exact preimages* $T_i^{(k)} \subset X_k$ by setting

$$T_i^{(k)} = \sigma^-(T_i^{(k-1)}).$$

A σ-series A is called nonsingular if

c) for any $k > i$, either $Y_{k+1} \subset T_i^{(k)}$ or Y_{k+1} does not intersect with $T_i^{(k)}$.

Clearly, any σ-series may be made into a nonsingular one if we delete from each Y_i a nowhere dense subset and accordingly modify the X_k for $k \geq i$.

For a nonsingular σ-series, all the divisors $T_i^{(k)} \subset X_k$ are nonsingular and have normal crossings.

Set $T_i = T_i^{(m)} \subset X_m$.

Let $A = (X_i, Y_i, \sigma_i)$ be a nonsingular σ-series of length m. Suppose $d_k = \operatorname{codim} Y_k = 1$ for some $k < m$. Then

$$d_k = d_{k+1} = \cdots = d_m = 1,$$

and the maps $\sigma_k, \dots, \sigma_m$ are open embeddings. Therefore, we can abbreviate the notation for A to a σ-series $\tilde{A}$ of length k by setting

$$\tilde{X}_i = X_i,\ \tilde{Y}_i = Y_i,\ \tilde{\sigma}_i = \sigma_i \quad \text{for } i < k$$
$$\tilde{X}_k = X_m,\ \tilde{Y}_k = Y_k \cap \tilde{\sigma}_k(X_m), \quad \text{where } \tilde{\sigma}_k = \sigma_k \dots \sigma_m.$$

A nonsingular σ-series of length m will be called *complete* if $d_m = 1$. The above description implies that no complete series admits a nontrivial continuation; moreover, σ_m is an open embedding and $\sigma_m : T_m \longrightarrow Y_m$ is a diffeomorphism. Therefore, T_m does not intersect T_i for $i < m$, since A is nonsingular.

1.3. To every nonsingular σ-series A, assign the matrix (a_{ij}) defined as follows: for every $k = 1, \dots, m$ set

$$I_k = \{i \mid 0 < i < k \text{ and } T_i^{(k)} \cap T_k^{(k)} \neq \emptyset\} = \{i \mid 0 < i < k \text{ and } Y_k \subset T_i^{(k-1)}\}.$$

and let

$$a_{ik} = 0 \text{ for } i > k,$$
$$a_{kk} = 1,$$
$$a_{ik} = \textstyle\sum_{j \in I_k} a_{ij} \text{ for } i < k.$$

A geometric interpretation of (a_{ij}) is as follows. For the sheaf of ideals $\mathcal{I}_{Y_i} \subset \mathcal{O}_{X_{i-1}}$ denote by $\mathcal{I}_i \subset \mathcal{O}_{X_m}$ its preimage $(\sigma_i \dots \sigma_m)^*(\mathcal{I}_{Y_i})$. The sheaf $\mathcal{I}_i$ determines a divisor Z_i in X_m equal to the preimage $(\sigma_{i+1} \dots \sigma_m)^*(T_i^{(i)})$ of the divisor $T_i^{(i)} \subset X_i$.

Using induction in m and formulas for $\sigma^*(T)$ from 1.1, it is easy to prove that $Z_i = \sum_k a_{ik} T_k$.

Denote $p_k = \mathrm{card} I_k$. Clearly, $p_1 = 0$ and $p_k > 0$ for $k > 1$, since $I_k \in k-1$. Set

$$\mathrm{codim} A = \sum_i a_{im}(d_i - p_i), \text{ where } d_i = \mathrm{codim}\ Y_i.$$

The meaning of this definition will be clarified in 1.6 and 1.7.

LEMMA. $\mathrm{codim} A = 1 + \sum_i a_{im}(d_i - 1)$.

PROOF. Let us prove by induction that $\sum_i a_{ik}(p_i - 1) = -1$. For $k = m$ we get the desired formula.

For $k = 1$ the formula is true. Suppose it is proved for all $j < k$. Then

$$\sum_i a_{ik}(p_i - 1) = p_k - 1 + \sum_{j \in I_k} \sum_i a_{ij}(p_i - 1) = p_k - 1 + \sum_{j \in I_k} (-1) = -1.$$

□

1.4. Given a submanifold $Y \subset X$ and a germ of a curve $\alpha : (E, e) \longrightarrow X$, we say that α *intersects* Y *at least* l *times (exactly* l *times)* if $\alpha^*(\mathcal{I}_Y) \subset \mathcal{I}_e^l$ (resp. $\alpha^*(\mathcal{I}_Y) = \mathcal{I}_e^l$). Here we implicitly assume that $\alpha(e) \notin \bar{Y} \setminus Y$, since otherwise $\alpha^*(\mathcal{I}_Y)$ is not defined.

Let $\sigma : Z \longrightarrow X$ be a σ-map with center in Y. A *lift* of α is a map $\tilde{\alpha} : (E, e) \longrightarrow Z$ such that $\sigma\tilde{\alpha} = \alpha$. If $\alpha(E) \not\subset Y$, i.e., the multiplicity l of the intersection of α with Y is finite, then such a lift exists and is unique. It is given by the formula

$$\tilde{\alpha}^*(x_i) = \alpha^*(x_i), \quad \tilde{\alpha}^*(t_i) = \alpha^*(t_i)/t^l,$$

where $x_1, \dots, x_n; t_1, \dots, t_d$ are the coordinates in $U \times \mathbb{P}^{d-1}$ considered in 1.1 and t is the standard coordinate on E. Clearly, $\tilde{\alpha}$ intersects T exactly l times.

We say that a family of curves $\alpha : (E(H), e) \longrightarrow X$ *intersects* Y *(exactly)* l *times* if so do all the germs $\alpha(h)$, $h \in H$. A *lift* of α is a family of germs $\tilde{\alpha} : (E(H), e) \longrightarrow Z$ such that $\sigma\tilde{\alpha} = \alpha$. If α intersects Y exactly l times, then such a lift exists and is unique.

1.5. Let $A = (X_i, Y_i, \sigma_i)$ be a nonsingular σ-series in X of length m, and $\alpha : (E, e) \longrightarrow X$ a germ of a curve. Suppose that α admits a collection of lifts $\alpha_i : (E, e) \longrightarrow X_i$ for $i = 0, 1, \dots, m$ such that

$$\alpha_0 = \alpha; \ \alpha_{i-1} = \sigma_i \alpha_i \text{ for } i > 0.$$

If $\alpha(E) \not\subset Y$ — this is the only case we are interested in — then the α_i are uniquely determined.

Denote by l_i the multiplicity of the intersection of the germ of the curve α_i with the divisor $T_i^{(i)}$; clearly, l_i is equal to the multiplicity of the intersection of α_{i-1} with the divisor Y_i. The set $l_1, \dots, l_m$ is called the *set of the multiplicities of the intersection of* α *with* A.

Denote by λ_k the multiplicity of the intersection of α_m with T_k. As follows from 1.3,

$$l_i = \sum_k a_{ik} \lambda_k.$$

Since $\det(a_{ij}) \neq 0$, all the λ_k are uniquely determined from the l_i.

For a complete σ-series A and an integer $L > 0$ we say that a *germ* α *intersects* A *with multiplicity (exactly equal to)* L if there exist lifts α_i to α and the multiplicity of the intersection of α_m with T_m is equal to L. Since T_m does not intersect T_k for $k < m$, this is equivalent to

$$\lambda_1 = \cdots = \lambda_{m-1} = 0; \quad \lambda_m = L \quad (\text{i.e., } l_i = la_{im}).$$

We say that a *family of germs of curves* $\alpha : (E(H), e) \longrightarrow X$ *intersects* A *exactly* L *times* if so do all the germs $\alpha(h)$, $h \in H$.

1.6. Let A be a nonsingular σ-series in X of length m and $l_1, \ldots, l_m$ a set of positive integers; let $\alpha : (E(H), e) \longrightarrow X$ be a family of germs of curves. Set

$$H_\alpha(A; l_1, \ldots, l_m) = \{h \in H \mid \text{the set of the multiplicities of the intersection of } \alpha(h) \text{ with } A \text{ is equal to } (l_1, \ldots, l_m)\}.$$

STATEMENT. a) *$H_\alpha(A; l_1, \ldots, l_m)$ is an analytic subset in H.*

b) *In a neighborhood of every point $h \in H_\alpha(A; l_1, \ldots, l_m)$ there exists a family of germs of curves $\beta : (E(H), e) \longrightarrow X$ such that $H_\beta(A; l_1, \ldots, l_m) = H$ and $\alpha(h) = \beta(h)$ for all $h \in H_\alpha(A; l_1, \ldots, l_m)$.*

c) $\operatorname{codim} H_\alpha(A; l_1, \ldots, l_m) \leq \sum l_i(d_i - p_i)$.

PROOF. a) Let $m = 1$, i.e., A consists of one σ-map $\sigma : X_1 \longrightarrow X$ with center in $Y_1 = Y$, and

$$H_\alpha(A; l_1) = H_\alpha(Y; l_1) = \{h \in H \mid \alpha(h) \text{ intersects } Y \text{ exactly } l_1 \text{ times}\}.$$

Let $h_0 \in H_\alpha(Y; l_1)$. Introduce coordinates $x_1, \ldots, x_n$ in a neighborhood of the point $\alpha(e(h_0)) \in X$ so that Y is defined by the equations $x_1 = \cdots = x_d = 0$; let t be the coordinate on $E(H)$.

On H, consider the functions

$$f_j^p = \left.\frac{\partial^p \alpha^*(x_j)}{\partial t^p}\right|_{t=0} \quad \text{for } 0 \leq p < l_1 \text{ and } 1 \leq j \leq d,$$

and define a map $\beta : (E(H), e) \longrightarrow X$ by setting

$$\beta^*(x_j) = \begin{cases} \alpha^*(x_j) & \text{for } j > d, \\ \alpha^*(x_j) - \sum\limits_{0 \leq p \leq l-1} f_j^p \frac{t^p}{p!} & \text{for } j \leq d. \end{cases}$$

Clearly, in a neighborhood of h_0 the set $H_\alpha(Y; l_1)$ is singled out by the equations

$$\{f_j^p(h) = 0 \mid 0 \leq p < l_1 \text{ and } 1 \leq j \leq d\},$$

and $H_\beta(Y; l_1) = H$. Moreover, $\alpha(h) = \beta(h)$ for all $h \in H_\alpha(Y; l_1)$.

Let us use induction on m. Deleting X_m, Y_m, and σ_m from A, we obtain a σ-series A' of length $m-1$. Let β' be a family of germs for which $H_{\beta'}(A'; l_1, \ldots, l_{m-1}) = H$ and which coincides with α on $H_\alpha(A'; l_1, \ldots, l_{m-1})$.

By 1.4, the lifts $\beta'_1, \ldots, \beta'_{m-1}$ are families of curves; set

$$\gamma = \beta'_{m-1} : (E(H), e) \longrightarrow X_{m-1}.$$

Clearly,

$$H_\alpha(A; l_1, \ldots, l_m) = H_\alpha(A'; l_1, \ldots, l_{m-1}) \cap H_\gamma(Y_m; l_m).$$

So, a) follows by induction.

b) Let $\delta : (E(H), e) \longrightarrow X_{m-1}$ be a family of germs of curves that intersect Y_m exactly l_m times and coincide with γ on $H_\gamma(Y_m; l_m)$ (see the proof of part a)).

Clearly, $\beta = \sigma_1 \dots \sigma_{m-1}\delta : (E(H), e) \longrightarrow X$ satisfies the conditions of b).

c) To prove this statement, it suffices, by b), to verify that $\operatorname{codim} H_\gamma(Y_m; l_m) \leq l_m(d_m - p_m)$.

On X_{m-1}, introduce the coordinates $x_1, \dots, x_n$ such that Y_m is given by the equations $x_1 = \dots = x_{d_m} = 0$ and each equation $x_j = 0$ for $j = 1, \dots, p_m$ cuts off one of the divisors $T_i^{(m-1)}$, where $i \in I_m$. This can be done, since the divisors have normal crossings.

Set $f_j^p = \left.\frac{\partial^p \gamma^*(x_j)}{\partial t^p}\right|_{t=0}$. Then $H_\gamma(Y_m; l_m)$ is given by equations

$$f_j^p(h) = 0 \text{ for } 0 \leq p < l_m \text{ and } 1 \leq j \leq d_m.$$

Therefore, it suffices to verify that $f_j^p(h) = 0$ for $0 \leq p < l_m$ and $1 \leq j \leq p_m$, so there remain exactly $l_m(d_m - p_m)$ equations.

Let λ_k^γ for $k = 1, \dots, m-1$ be the multiplicity of the intersection of γ with $T_k^{(m-1)}$. We have to verify that $\lambda_k^\gamma \geq l_m$ for $k \in I_m$. From 1.3 it follows that $l_i \leq \sum_{k<m} a_{ik}\lambda_k^\gamma$; hence, the λ_k^γ are uniquely determined from $l_1, \dots, l_{m-1}$. Therefore, if there exists at least one germ ξ that intersects A with multiplicities $l_1, \dots, l_{m-1}, l_m$, then $\lambda_k^\gamma = \lambda_k^\xi \geq l_m$ for $k \in I_m$, because $T_k^{(m-1)} \supset Y_m$. □

1.7. Let $A = (X_i, Y_i, \sigma_i)$ be a complete σ-series of length m in X. In what follows we assume that $\operatorname{supp} A = \bar{Y}_1$ is compact: since all our arguments are local, this can be achieved in a small neighborhood of Y_1.

We will say that *a curve* $\varphi : E \longrightarrow X$ *intersects* A *exactly* $L > 0$ *times* if for each point $c \in E$ this is true for the germ $\varphi : (E, c) \longrightarrow X$ and $\overline{\varphi(E)} \cap \operatorname{supp}^A = \varphi(c)$.

Given a family of curves $\varphi : E(H) \longrightarrow X$, set

$$H_\varphi(A; L) = \{h \in H \mid \varphi(h) \text{ intersects } A \text{ exactly } L \text{ times}\}.$$

COROLLARY. *$H_\varphi(A; L)$ is an analytic subset of H, and* $\operatorname{codim} H_\varphi(A; L) \leq L \cdot \operatorname{codim} A - 1$.

PROOF. For every $h \in H_\varphi(A; L)$, denote by $c(h)$ a point at which $\varphi(h)$ intersects A exactly L times. Clearly, at $c(h)$ the multiplicity of intersection of $\varphi(h)$ and A is equal to exactly $l = l_1 = La_{1m}$, see sect. 1.5.

Let $h_0 \in H_\varphi(A; L)$ and $c_0 = c(h_0)$. In a neighborhood of h_0, construct a holomorphic map $h \mapsto e(h) \in E(h)$ so that $e(h) = c(h)$ for $h \in H_\varphi(A; l)$.

For this, consider standard coordinates $(u_1, \dots, u_N; t)$ on $E(H)$ and a function f in a neighborhood of $\varphi(c_0) \in X$ such that $f|_{Y_1} = 0$ and $\frac{\partial^l \varphi^*(f)}{\partial^l t}|_{h_0, c_0} \neq 0$. Set $g = \frac{\partial^{l-1}\varphi^*(f)}{\partial^{l-1} t}$. Then $g = 0$ and $\frac{\partial g}{\partial t} \neq 0$ at (h_0, c_0); hence, the equation $g(h, e(h)) = 0$ determines a holomorphic map $h \mapsto e(h)$.

If $h \in H_\varphi(A; L)$ is close to h_0, then $c(h)$ is close to c_0 since $\overline{E(h_0)} \cap \operatorname{supp} A = \varphi(c_0)$. Clearly, $g(h, c(h)) = 0$; therefore, $c(h) = e(h)$.

(ii) Let there be given a family of gems of curves $\alpha : (E(\hat{H}), e) \longrightarrow X$, where

$$\hat{H} = E(H) = \{(h, e) \mid h \in H, e \in E(h)\},$$
$$E(\hat{H}) = \{\hat{h} = (h, e) \in \hat{H}, c \in E(h)\},$$

where $\alpha(\hat{h}, c) = \varphi(h, c)$, $e(\hat{h}) = e$.

It follows from 1.5, 1.6 that $\hat{H}_\alpha(A;L)$ is an analytic subset of H whose codimension satisfies the inequality $\operatorname{codim}\hat{H}_\alpha(A;L) \leq L \cdot \operatorname{codim}A$.

Let us identify H with a submanifold of H of codimension 1 consisting of the points of the form $(h;e(h))$, where $e(h)$ is determined in (i). Then it follows from (i) that $\hat{H}_\alpha(A;L) \subset H$ and, moreover, $\hat{H}_\alpha(A;L) \doteq H_\varphi(A;L)$, implying

$$\operatorname{codim}_H H_\varphi(A;L) \leq L \cdot \operatorname{codim}A - 1.$$

□

§2. Critical points

2.1. Given a family of curves $\varphi : E(H) \longrightarrow X$ and points $h \in H$, $c \in E(h)$, we call the point $(h,c) \in E(H)$ *critical* if all the forms from $\varphi^*(\Omega^n X)$, where $n = \dim X$, vanish at this point, i.e., for a germ ω of an n-form on X at $\varphi(h,c)$ the form $\varphi^*(\omega)$ vanishes on $T_{(h,c)}E(H)$.

We will also say that c is a *critical point on the curve $E(h)$ with respect to φ*.

On $E(H)$, select coordinates $z_1, \dots, z_{N+1}$ and express every form from $\varphi^*(\Omega^n X)$ in these coordinates; let $\mathcal{I}_\varphi$ be the ideal of functions on $E(H)$ generated by the coefficients of these forms. Restricting onto $E(h)$, we get an ideal in the ring of functions on $E(h)$. The multiplicity of this ideal at c will be denoted $\operatorname{cr}_\varphi(h,c)$. We call c *critical* if $\operatorname{cr}_\varphi(h,c) > 0$.

A point $h \in H$ is *critical* (with respect to φ) if all $c \in E(h)$ are critical or, equivalently, $\mathcal{I}_\varphi|_{E(H)} = 0$.

LEMMA. *Let $\operatorname{cr}_\varphi(h_0,c_0) = k > 0$.*

a) *For h close to h_0 the sum of multiplicities of all the critical points $c \in E(h)$ close to c_0 does not exceed k.*

b) *Suppose for every h close to h_0 there exists a point $c(h) \in E(h)$ of multiplicity k close to c_0 (such a point, if any exist, is unique due to a)). Then the map $h \mapsto c(h)$ is holomorphic.*

PROOF. Let $(u_1, \dots, u_N; t)$ be standard coordinates on $E(H)$, and $f \in \mathcal{I}_\varphi$ a function with zero of multiplicity k at (h_0,c_0). Then on $E(H)$ the multiplicity of c does not exceed the multiplicity of zero of $f|_{E(h)}$ at c, implying a).

Set $g = \frac{\partial^{k-1}(f)}{\partial^{k-1}t}$. Clearly, $g(h,c(h)) = 0$ for all h. Since $\frac{\partial g}{\partial t}(h_0,c_0) \neq 0$, we get b) due to the implicit function theorem. □

2.2. A family of germs of curves $\alpha : (E(H),e) \longrightarrow X$ is said to be *critical* if $e(h) \in E(h)$ is critical with respect to α for all $h \in H$.

STATEMENT. *Let A be a nonsingular σ-series of length m in X, and also let $\alpha : (E(H),e) \longrightarrow X$ be a family of germs of curves.*

a) *Suppose that all the germs $\alpha(h)$ intersect A with multiplicities $l_1, \dots, l_m$. Then $\operatorname{cr}_\alpha(h,e(h)) \geq \sum l_i$ for all $h \in H$.*

b) *For a complete σ-series A, an α that intersects A strictly L times, and a submersion $\gamma_m = \alpha_m|_{e(H)} : e(H) \longrightarrow T_m$ we have $\operatorname{cr}_\alpha(h,e(h)) = L \cdot \operatorname{codim}A - 1$.*

Proof. (i) By 1.4 there exists a set of lifts $\alpha_i : (E(H),e) \longrightarrow X_i$, $0 \leq i \leq m$. Let $\mathcal{I}_i = \mathcal{I}_{T_i^{(i)}} \subset \mathcal{O}_{X_i}$. Then

$$\sigma_i^*(\Omega^n X_{i-1}) = \mathcal{I}_i^{d_i-1}\Omega^n X_i,$$

see 1.1. By definition $\alpha_i^*(\mathcal{I}_i) = \mathcal{I}_e^{l_i}$. Therefore,

$$\alpha_{i-1}^*(\Omega^n X_{i-1}) = \mathcal{I}_r^{l_i(d_i-1)}\alpha_i^*\Omega^n X_i,$$

i.e.,

$$\mathrm{cr}_{\alpha_{i-1}}(h, e(h)) = \mathrm{cr}_\alpha(h, e(h)) + l_i(d_i - 1).$$

Thus, it suffices to estimate $\mathrm{cr}_{\alpha_m}(h, e(h))$.

Introduce coordinates $x_1, \dots, x_n$ on X_m so that T_m is singled out by the equation $x_n = 0$, and let $u_1, \dots, u_N, t$ be standard coordinates on $E(H)$. Then $\alpha_m^*(x_n) = t^l f$, where f is an invertible function. Hence

$$\alpha_m^*(dx_1 \wedge \cdots \wedge dx_n) = l_m t^{l_m-1} f \alpha_m^*(dx_1 \wedge \cdots \wedge dx_{n-1}) \wedge dt + t^{l_m}\omega,$$

where ω is a form. Therefore, $\mathrm{cr}_{\alpha_m}(h, e(h)) \geq l_m - 1$, implying a).

b) In this case $l_i = La_{im}$; in particular, $l_m = L$. Further, by hypothesis the form $\alpha_m^*(dx_1 \wedge \cdots \wedge dx_{n-1})$ is nondegenerate on $e(H)$. So $\mathrm{cr}_{\alpha_m}(h, e(h)) = L - 1$ and, therefore,

$$\mathrm{cr}_{\alpha_m}(h, e(h)) = \sum l_i(d_i - 1) + (L - 1) = L(\sum a_{im}(d_i - 1) + 1) - 1.$$

By Lemma 1.3 this number is equal to $L \cdot \mathrm{codim} A - 1$.

2.3.

STATEMENT. *Let $\alpha(E(H), e) \longrightarrow X$ be a critical family of germs of curves of multiplicity exactly equal to k, $0 < k < \infty$. Then for every generic point $h \in H$ there exist a neighborhood H_0, a complete σ-series A, and an integer $L > 0$ such that $k = L \cdot \mathrm{codim} A - 1$ and $\alpha(h)$ intersects A exactly L times for any $h \in H_0$.*

PROOF. To construct A, consider $\gamma = \alpha|_{e(H)} : e(H) \longrightarrow X$. Since $e(H)$ consists of critical points of $E(H)$, it follows that $\gamma^*(\Omega^n X) = 0$, i.e., $\mathrm{rk}\gamma < n$.

Replacing H by a neighborhood of a generic point, we can assume that γ is of constant rank $r < n$. By the theorem on maps of constant rank, there exists a submanifold Y_1 of dimension r in X such that $\gamma(e(H)) \subset Y_1$ and $\gamma : (e(H)) \longrightarrow Y_1$ is a submersion (all this is only true in a neighborhood of a point of H). At generic points α intersects Y_1 exactly l_1 times, where $l_1 > 0$.

Consider a σ-map $\sigma_1 : X_1 \longrightarrow X$ with center in Y_1. By 1.4 there exists a lift $\alpha_1 : (E(H), e) \longrightarrow X_1$ of α such that $\alpha_1(E(H)) \subset T_1^{(1)} = T_{\sigma_1}$.

Using arguments similar to those above, we construct a submanifold $Y_2 \subset X_1$, a σ-map $\sigma_2 : X_2 \longrightarrow X_1$ with center Y_2, and a lift $\alpha_2 : (E(H), e) \longrightarrow X_2$ so that $\gamma_1 = \alpha_1|_{e(H)} : (e(H)) \longrightarrow Y_2$ is a submersion and α_1 intersects Y_2 exactly l_2 times (in a neighborhood of a generic point in H).

We similarly construct Y_i, σ_i, X_i for $i = 3, 4, \dots$ by deleting at each step a nowhere dense set from Y_i in order to get a nonsingular σ-series.

Statement 2.2 implies that for any q we have

$$k = \mathrm{cr}_{\alpha_m}(h, e(h)) \geq \sum_{1 \leq i \leq q} l_i(d_i - 1),$$

where $d_i = \mathrm{codim} Y_i$. Therefore, $d_m = 1$ for some m.

Let $A = (X_i, Y_i, \sigma_i)$ be the σ-series of length m thus obtained. The map $\gamma_m = \alpha_m|_{e(H)}$ determines a submersion $e(H) \longrightarrow T_m$, since $\sigma_m : T_m \longrightarrow Y_m$ is a diffeomorphism and $\sigma_m\gamma_m = \gamma_{m-1} : e(H) \longrightarrow Y_m$ is a submersion. Besides,

α intersects A exactly $L = l_m$ times. It follows from Statement 2.2b) that $k = L \cdot \operatorname{codim} A - 1$. □

2.4. Denote by $\mathcal{E}$ the set of all curves in X. Every family of curves $\varphi : E(H) \longrightarrow X$ determines a map $H \longrightarrow \mathcal{E}$. Considering $\mathcal{E}$ as an infinite dimensional manifold, it is natural to view the map $H \longrightarrow \mathcal{E}$ as being holomorphic. In particular, for any point $h \in H$ it should determine the tangent map $t\varphi : T_h(H) \longrightarrow T_E\mathcal{E}$, where $E = E(h)$ and $T_E\mathcal{E}$ is the tangent space to $\mathcal{E}$ at E.

Without going into details related to subtleties of infinite dimensional manifolds, we will explicitly describe $T_E\mathcal{E}$ and $t\varphi$ and show how to express in these terms the critical points on E.

a) Given a curve $E \subset X$, denote by $N_E(X)$ the normal bundle to E in X, and let $T_E = \Gamma(E, N_E(X))$.

b) Given a family of curves $\varphi : E(H) \longrightarrow X$, set $E = E(h)$ for $h \in H$. Let us construct $t\varphi$.

The differential of φ determines a map $t' : N_E(E(H)) \longrightarrow N_E(X)$ of normal bundles over E. The projection $\pi : E(H) \longrightarrow H$ determines an isomorphism of bundles $\pi' : N_E(E(H)) \longrightarrow N_h(H)$, i.e., $N_E(E(H))$ is the trivial bundle with the fiber $N_h(H) = T_h(H)$. Denote by $t\varphi$ the through map

$$t\varphi : T_h(H) \longrightarrow \Gamma(E, T_h(H)) = N_E(E(H)) \longrightarrow \Gamma(E, N_E(X)) = T_E.$$

c) Let $N = N_E(X)$, $k = \dim N = \dim X - 1$, and let $L = \Lambda^k N$ be the determinant line bundle over E. For any $\xi_1, \dots, \xi_k \in T_E$ denote by $\operatorname{cr}(\xi_1, \dots, \xi_k; c)$ the order of the zero at c of the section $\xi_1 \wedge \cdots \wedge \xi_k$ of L. For a subspace $W \subset T_E$ set

$$\operatorname{cr}_W(E, c) = \min_{\xi \in W} \operatorname{cr}(\xi_1, \dots, \xi_k; c).$$

STATEMENT. *Let $\varphi : E(H) \longrightarrow X$ be a family of curves, $h \in H$, $E = E(h)$, $W = t\varphi(T_h(H)) \subset T_E$. Then $\operatorname{cr}_\varphi(h, c) = \operatorname{cr}_W(E, c)$.*

The proof is a direct consequence of the definitions.

2.5. Let us use the above definitions to refine Statement 2.2.

STATEMENT. *Let $\alpha : (E(H), e) \longrightarrow X$ be a family of germs of curves, A a nonsingular σ-series of length m in X, and $\hat{K} = H_\alpha(A, L) = \{h \in H \mid \alpha(h)$ intersects A exactly L times$\}$. For $h \in \hat{K}$ denote*

$$W = T_h(\hat{K}), \quad E = E(h), \quad e = e(h).$$

Then

$$\operatorname{cr}_{t\alpha W}(E, e) \geq L \cdot \operatorname{codim} A - 1.$$

PROOF. Replace α by a family β that intersects A exactly L times and coincides with α on $\hat{K}$. This can be done due to 1.6.

Clearly, the maps $t\alpha, t\beta : T_h(H) \longrightarrow T_E$ coincide on W. Therefore,

$$\operatorname{cr}_{t\alpha W}(E, e) = \operatorname{cr}_{t\beta W}(E, e) \geq \operatorname{cr}_{t\beta T_h(H)}(E, e) = \operatorname{cr}_\beta(E, e).$$

By Statement 2.2, $\operatorname{cr}_{t\alpha W}(E, e) \geq L \cdot \operatorname{codim} A - 1$. □

§3. Admissible complexes

In this section we fix a family of curves $\varphi : E(H) \longrightarrow X$.

3.1. To every submanifold $K \subset H$ there corresponds a subfamily $\varphi(K)$: $E(K) \longrightarrow X$.

Set $\mathrm{cr}_K(h,c) = \mathrm{cr}_{\varphi(K)}(h,c)$ for every $(h,c) \in E(K)$, and

$$\mathrm{cr}_K(h) = \sum_{c \in E(h)} \mathrm{cr}_{K}(h,c).$$

We say that $h \in K$ is *critical (with respect to K)* if $\mathrm{cr}_K(h,c) = \infty$.

A submanifold $K \subset H$ not all of whose points are critical with respect to K, i.e., such that $\varphi(K)^*(\Omega^n) \neq 0$, is called a *complex of curves.*

a) A family of curves $\varphi : E(H) \longrightarrow X$ is called *perfect* if it is a complex and $\mathrm{cr}_K(h) \leq \mathrm{codim} K$ in H for each complex $K \subset H$ and each noncritical point $h \in K$.

b) Given a perfect family of curves $\varphi : E(H) \longrightarrow X$, a complex K is called *admissible* if $\mathrm{cr}_K(h) = \mathrm{codim} K$ for generic points $h \in K$.

Note that the definitions a) and b) are local in H.

3.2. Let us reformulate the definition of a perfect family in terms of tangent spaces. Let $h \in H$, $E = E(h)$, $t\varphi : T_h(H) \longrightarrow T_E$. For every subspace $W \subset T_h(H)$ set

$$\mathrm{cr}_W(h,c) = \mathrm{cr}_{t\varphi(W)}(E,c) \quad \text{for } c \in E,$$
$$\mathrm{cr}_W(h) = \sum_c \mathrm{cr}_W(h,c).$$

Clearly,

$$\mathrm{cr}_K(h,c) = \mathrm{cr}_{T_h(K)}(h,c) \quad \text{and} \quad \mathrm{cr}_K(h) = \sum_c \mathrm{cr}_{T_h(K)}(h).$$

We call W *saturated* if $\mathrm{cr}_W(h,c) < \infty$.

A family $\varphi : E(H) \longrightarrow X$ is called *perfect* if $\mathrm{cr}_W(h) \leq \mathrm{codim} W$ for any $h \in H$ and a saturated $W \subset T_h(H)$ (assuming such pairs (h, W) exist). A subspace W for which the equality is attained is called *admissible.*

A submanifold $K \subset H$ is called a *complex* (resp. an *admissible complex*) if $T_h(K) \subset T_h(H)$ is saturated (resp. admissible) for a generic point $h \in K$.

3.3. Given a complete σ-series $A_1, \ldots, A_r$ such that the supports of $A_1, \ldots, A_r$ are compact and disjoint, and positive integers $L_1, \ldots, L_r$, set

$$H_\varphi(A_1, \ldots, A_r; L_1, \ldots, L_r) = H_\varphi(A_1; L_1) \cap \cdots \cap H_\varphi(A_r; L_r)$$

for every family of curves $\varphi : E(H) \longrightarrow X$. By 1.7 this is an analytic subset in H, and $\mathrm{codim}\varphi(A_1, \ldots, A_r; L_1, \ldots, L_r) \leq \sum(L_i \cdot \mathrm{codim} A_i - 1)$.

THEOREM (Main Theorem). *For a perfect family of curves $\varphi : E(H) \longrightarrow X$, set*

$$\hat{K} = H_\varphi(A_1, \ldots, A_r; L_1, \ldots, L_r); \quad C = \sum(L_i \mathrm{codim} A_i - 1).$$

a) *If K is a complex, then it is admissible and* $\mathrm{codim} K = C$. *Moreover, if $W = T_h(K)$ is saturated at some point $h \in K$, then $\hat{K}$ is nonsingular in a neighborhood of h and is an admissible complex.*

b) *Any admissible complex K in a neighborhood of a generic point is of the form $K = H_\varphi(A_1, \ldots, A_r; L_1, \ldots, L_r)$, where the A_i form a complete σ-series with compact disjoint supports and with $L_i > 0$.*

PROOF. a) Let $c_i(h)$ be the intersection point of h and A_i. Using the same arguments as in 1.7, we can construct a family of germs $\alpha_i : (E(H), e) \longrightarrow X$ such that $\alpha_i = \varphi$ and $e(h) = c_i(h)$ for $h \in K$. Then it follows from 2.2 that $\mathrm{cr}_W(h) \geq L_i \mathrm{codim} A_i - 1$. Since all the points $c_i(h)$ are distinct, $\mathrm{cr}_W(h) \geq C$. If W is saturated, then $\mathrm{codim} W \geq \mathrm{cr}_W(h) \geq C$, since φ is perfect.

On the other hand, from 1.7 it follows that $\mathrm{codim}\hat{K} \leq C$. If K is a manifold this implies $\mathrm{codim}\hat{K} = C = \mathrm{codim} W = \mathrm{cr}_W(h)$, i.e., K is an admissible complex. If it is not known that K is a manifold but it is known that $W = T_h(K)$ is saturated, then the inequality $\mathrm{codim} W \geq C \geq \mathrm{codim}\hat{K}$ implies that $\hat{K}$ is nonsingular in a neighborhood of h (see Statement 0.2). Hence, $\hat{K}$ is an admissible complex.

b) Let K be an admissible complex. There are different critical points $c_1(h), \ldots, c_r(h)$ of multiplicities $k_1, \ldots, k_r$ at a generic point $h \in K$, and neither r nor $k_1, \ldots, k_r$ depend on h. By 2.1, the $c_i(h)$ holomorphically depend on h.

By Statement 2.3, if $H = K$, then for every generic point $h \in K$ there exist a neighborhood K_0, a complete σ-series $A_1, \ldots, A_r$, and positive integers $L_1, \ldots, L_r$ such that

$$k_i = L_i \cdot \mathrm{codim} A_i - 1 \quad \text{and} \quad K_0 \subset \hat{K} = H_\varphi(A_1, \ldots, A_r; L_1, \ldots, L_r).$$

Clearly, $T_h(\hat{K}) \supset T_h(K)$ is saturated and, due to a), $\hat{K}$ is an admissible complex, $\mathrm{codim}\hat{K} = C = \sum k_i$. Since K is an admissible complex, $\mathrm{codim} K = \mathrm{cr}_K(h) = \sum k_i = \mathrm{codim}\hat{K}$ and, therefore, K_0 is a neighborhood of h. □

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