2) Does there exist a nontrivial countably additive two-valued measure μ on D^{Δ} such that $\mu(1_q) = 0$, for all $q \in \Delta, \mu(1_{\Delta}) = 1$? (Ulam [3]).

Remarks. 1. We obtain Mackey's theorem [2] from Theorem 6 by taking K = R. The hypotheses of Theorem 6 are satisfied, e.g., by the fields of rational, complex, and p-adic numbers, the skew field of quaternions, etc., in addition to R.

2. In contrast to [5], we do not require that K be complete in Theorem 6 (see [5, Theorem 6]).

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INTEGRATION OF DIFFERENTIAL FORMS ON SUPERMANIFOLDS

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Due mainly to the requirements of theoretical physics, the subject of "super" analysis has recently been vigorously developed. Many differential-geometric concepts can be easily generalized to the case of supermanifolds (see [1] and the references there). The notion of integration, however, has given rise to difficulties. The complexity of this generalization stems from the fact that differential forms cannot be integrated on supermanifolds. The first advance was made by Berezin [2, 3], who introduced the notion of density which can be integrated. In [4], densities are shown to be contained in a complex of differential forms for which an analog of Stokes' theorem is derived.

However, it would be desirable to learn how to integrate differential forms. We show below that differential forms can be integrated provided they are generalized correctly. We use the principal definitions and results of [4].

Let $\mathcal{U}$ be a superdomain with coordinates x = (u, ξ). Let $\Omega(\mathcal{U})$ denote the commutative superalgebra over $C^\infty(\mathcal{U})$, with generators dx = (du, dξ), where, moreover, $p(du_i) = \bar{1}$, $p(d\xi_j) = \bar{0}$. $\Omega(\mathcal{U})$ is called the algebra of differential forms on $\mathcal{U}$. (This definition differs from the one in [4] since in [4] the superalgebra is noncommutative; however, this difference is not essential since one algebra is obtained from the other by changing the grading and the signs of products under multiplication.)

The elements of the superalgebra Ω can be viewed as functions of even variables u_i and $d\xi_j$ and odd variables ξ_j and du_j which are polynomial in $d\xi_j$. Such functions cannot be integrated. One can, however, consider functions of u, dξ, ξ and du which are rapidly decreasing in the even variables (e.g., we can consider the function $\exp(-[(d\xi_1)^2 + \ldots + (d\xi_q)^2])$ on $\mathcal{R}^{0,q}$). We show that "forms" of such a type can be integrated.

We turn to some precise definitions. Let $\mathcal{U}$ be a superdomain with coordinates $x = (u_1, \ldots, u_p, \xi_1, \ldots, \xi_q)$. Let $\hat{\mathcal{U}}$ denote the superdomain $\mathcal{U} \times \mathcal{R}^{q,p}$ with coordinates $(x, \hat{x}) = (u_1, \ldots, u_p, \xi_1, \ldots, \xi_q, \hat{u}_1, \ldots, \hat{u}_p, \hat{\xi}_1, \ldots, \hat{\xi}_q)$, where $p(u_i) = p(\hat{\xi}_j) = \bar{0}$, $p(\hat{u}_i) = p(\xi_j) = \bar{1}$, $1 \leqslant i \leqslant p$, $1 \leqslant j \leqslant q$. (Informally speaking, $\hat{u}_i = du_i$, $\hat{\xi}_j = d\xi_j$.)

We call the commutative superalgebra $\hat{\Omega}(\mathcal{U}) = C^\infty(\hat{\mathcal{U}})$ the algebra of pseudodifferential forms. If y = (v, η) is another coordinate system on $\mathcal{U}$, then the change of coordinates from $\hat{x}$ to $\hat{y}$ is given by the formulas

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$$\hat{y}_i = \sum_j \hat{x}_j \frac{\partial y_i}{\partial x_j}. \tag{*}$$

Using these formulas $\hat{\mathcal{U}}$ is defined invariantly in terms of $\mathcal{U}$.

If $\mathcal{M}$ is a (p, q)-dimensional supermanifold, then by taking an open covering consisting of superdomains $\mathcal{U}_k$ and gluing them together in the standard way using (*) for the supermanifold $\hat{\mathcal{U}}_k$, we get a supermanifold $\hat{\mathcal{M}}$; it is easy to check that $\hat{\mathcal{M}}$ does not depend on the choice of the covering $\{\mathcal{U}_k\}$. It is clear that $\hat{\mathcal{M}}$ depends functorially on $\mathcal{M}$, i.e., to each morphism $\mathcal{M} \to \mathcal{N}$ there corresponds a morphism $\hat{\mathcal{M}} \to \hat{\mathcal{N}}$. Moreover, we have a canonical imbedding $i: \mathcal{M} \to \hat{\mathcal{M}}$ (it is given by the equations $i^*(\hat{x}_i) = 0$).

We call the superalgebra $C^\infty(\hat{\mathcal{M}})$ the algebra of pseudodifferential forms on the supermanifold $\mathcal{M}$ and write it as $\hat{\Omega}(\mathcal{M})$. Essentially, $\hat{\mathcal{M}} = \operatorname{Spec} \Omega(\mathcal{M})$ ([5]), where $\Omega(\mathcal{M})$ is the commutative superalgebra of differential forms on $\mathcal{M}$.

We list some properties of pseudodifferential forms.

1. There is a natural imbedding $\Omega(\mathcal{M}) \to \hat{\Omega}(\mathcal{M})$.

2. If $\varphi: \mathcal{M} \to \mathcal{N}$ is a morphism of supermanifolds, then the homomorphism $\varphi^*: \Omega(\mathcal{N}) \to \Omega(\mathcal{M})$ of the algebras of differential forms extends uniquely to a homomorphism $\hat{\varphi}^*: \hat{\Omega}(\mathcal{N}) \to \hat{\Omega}(\mathcal{M})$; $\hat{\varphi}^*$ corresponds to a morphism $\hat{\varphi}: \hat{\mathcal{M}} \to \hat{\mathcal{N}}$.

3. There is a differential d in the algebra $\Omega(\mathcal{M})$ such that $p(d) = \bar{1}$. This differential extends uniquely to a differential $\hat{d}$ on the algebra $\hat{\Omega}(\mathcal{M})$ of pseudodifferential forms. The differential $\hat{d}$ satisfies the Leibniz formula

$$\hat{d}(\omega_1\omega_2) = (\hat{d}\omega_1)\,\omega_2 + (-1)^{p(\omega_1)}\,\omega_1(\hat{d}\omega_2),\ \omega_1, \omega_2 \in \hat{\Omega}(\mathcal{M})$$

and the condition $\hat{d}^2 = 0$.

In the local coordinates $(x, \hat{x})$ on $\hat{\mathcal{M}}$, it has the form $\hat{d} = \sum \hat{x}_i\, \partial/\partial x_i$.

4. Every vector field X on $\mathcal{M}$, i.e., every derivation of the algebra $C^\infty(\mathcal{M})$, extends uniquely to a derivation L_X of $\Omega(\mathcal{M})$, which commutes (in the "super" sense) with the differential $\hat{d}$; L_X is called the Lie derivative corresponding to the field X.

Let us show how to integrate pseudodifferential forms. There exists on $\hat{\mathcal{M}}$ a distinguished volume form D which in the local coordinates $(x, \hat{x})$ can be written in the form $D(x, \hat{x})$ (see [4]); it follows from the rule for changing variables (*) that this volume form does not depend on the choice of coordinate system x on $\mathcal{M}$.

The form $\omega \in \hat{\Omega}(\mathcal{M}) = C^\infty(\hat{\mathcal{M}})$ is said to be finite if the function ω is finite on $\hat{\mathcal{M}}$; the algebra of such forms is written $\hat{\Omega}_c(\mathcal{M})$. We fix an orientation (see [4]) on $\hat{\mathcal{M}}$. We define the integral of the form $\omega \in \hat{\Omega}_c(\mathcal{M})$ by the formula

$$\int_{\mathcal{M}} \omega = \int_{\hat{\mathcal{M}}} \omega D. \tag{**}$$

The integral on the right is understood in the sense of [4]; it is easy to see that it does not depend on the choice of coordinate system. If $x = (u, \xi)$ is a global coordinate system on $\mathcal{M}$, the integral can be written down explicitly,

$$\int_{\mathcal{M}} \omega = \int_{\hat{M}} \omega_{1\ldots1}(u; \hat{\xi})\, du\, d\hat{\xi},$$

where $\omega_{1\ldots1}$ is the coefficient of $\xi_1 \ldots \xi_q \hat{u}_1 \ldots \hat{u}_p$ in the expression $\omega = \sum \omega_{\mu,\nu}(u, \hat{\xi})\, \xi^\mu \hat{u}^\nu$, and $\hat{M}$ is the underlying manifold of $\hat{\mathcal{M}}$. The integral (**) has the following important properties:

1. If $\omega \in \hat{\Omega}_c(\mathcal{M})$, then $\int d\omega = 0$.

2. For any vector field X on $\mathcal{M}$ $\int L_X \omega = 0$.

Remarks. 1. On any supermanifold $\mathcal{M}$, we can extend the space $C^\infty(\mathcal{M})$ in the usual way to the space of distributions on $\mathcal{M}$. In particular, we can extend the space $\hat{\Omega}(\mathcal{M}) = C^\infty(\hat{\mathcal{M}})$ to a space of pseudodifferential distribution forms on $\mathcal{M}$. Consider the subspace $\Sigma(\mathcal{M})$, of this space consisting of distributions on $\hat{\mathcal{M}}$, with support on the subsupermanifold $\mathcal{M} \subset \hat{\mathcal{M}}$

and which are smooth along this subsupermanifold. It can be verified that this space coincides with the space of integral forms on $\mathcal{M}$, introduced in [4].

2. Let $\mathcal{F}$ be a closed superdomain (see [4]); let its boundary $\partial\mathcal{F}$ be given by an even equation. Then it is easy to define the integral of the form $\omega \in \hat{\Omega}_c(\mathcal{M})$ over $\mathcal{F}$ and to prove the analog of Stokes' theorem. It would be interesting to generalize Stokes' theorem to a superdomain whose boundary is given by an odd equation.

3. For forms in $\hat{\Omega}(\mathcal{M})$ there is an analog of the Poincaré lemma. Consider an open contractible superdomain $\mathcal{V}$ in $\mathcal{M}$. Then if $\omega \in \hat{\Omega}(\mathcal{V})$ and $d\omega = 0$, there exist $c \in \mathbf{R}$ and $\varphi \in \hat{\Omega}(\mathcal{V})$, such that $\omega = d\varphi + c$.

4. The choice of orientation on $\hat{\mathcal{M}}$ can be described in terms of $\mathcal{M}$. In order to do this, we must cover $\mathcal{M}$ by local coordinate systems in such a way that the values of the Jacobians (see [4]) of the coordinate transitions are positive at all points.

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STATE DENSITY OF SELF-ADJOINT ELLIPTIC OPERATORS WITH STOCHASTIC COEFFICIENTS

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The state density (normed distribution function) for a Schrödinger operator with a stochastic potential has been considered in numerous papers (see [1] and the references given there), and has been studied in [2] for the case of general elliptic operators with almost periodic coefficients. In this note, we study analogous questions for general elliptic operators with stochastic coefficients.

I. Let A be a differential operator

$$A = \sum_{\alpha \leqslant m} a_\alpha(x) D^\alpha, \tag{1}$$

where $x \in \mathbf{R}^n, \alpha$ is a multiindex. The choice of the coefficients $\mathcal{A}(x) = \{a_\alpha(x), |\alpha| \leqslant m\}$ determines a homogeneous measurable field defined on some probability space Ω. We may assume that the points of Ω consist of all possible realizations $\mathcal{A}(\cdot) = \{\alpha_\alpha(\cdot)\}$ (see [4, Chap. 2, Sec. 8] or [1, Chap. 3]).

We also assume that for almost all (a.a.) realizations $\mathcal{A}(\cdot)$ (in the measure induced on Ω by the stochastic field) the following conditions hold:

1) $a_\alpha(x) \in C^\infty(\mathbf{R}^n)$, and for any multiindex β there exists a constant c_β not depending on $\mathcal{A}$, such that $|D^\beta a_\alpha(x)| \leqslant c_\beta$ for any α and a.a. $\mathcal{A}$.

2) the condition of uniform ellipticity, i.e., there exists an $\varepsilon > 0$ not depending on $\mathcal{A}$, such that for a.a. $\mathcal{A}$ we have $\operatorname{Re} a_m(x,\xi) \geqslant \varepsilon |\xi|^m$, where $a_m(x,\xi) = \sum_{|\alpha|=m} a_\alpha(x)\xi^\alpha$.

We shall consider a sequence $\{\Omega_k\}$ in $\mathbf{R}^n$, which is regular (see [5, Sec. 2]) and satisfies the following condition (see [2, Sec. 3]):

3) $\mu(\Omega_k) \to \infty$ and there exists a sequence h_k, $h_k \to \infty$ as $k \to \infty$ such that

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