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Sobolev norms of automorphic functionals and Fourier coefficients of cusp forms

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Abstract. We propose a new approach to the study of eigenfunctions of the Laplace–Beltrami operator on a Riemann surface of curvature -1. It is based on Frobenius reciprocity from the theory of automorphic functions. We determine Sobolev class of arising automorphic functionals and discuss some applications. © Académie des Sciences/Elsevier, Paris

Normes de Sobolev des fonctionnelles automorphes et coefficients de Fourier des formes cuspidales

Résumé. *Nous proposons une nouvelle approche pour étudier les fonctions propres de l'opérateur Laplace–Beltrami sur une surface de Riemann de courbure* -1*. Cette approche repose sur la réciprocité de Frobenius en théorie des fonctions automorphes. Nous déterminons la classe de Sobolev de certaines fonctionnelles automorphes qui apparaissent, et donnons quelques applications.* © Académie des Sciences/Elsevier, Paris

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Soient $G = \mathrm{SL}(2,\mathbb{R})$, $K = \mathrm{SO}(2,\mathbb{R})$, Γ un réseau de $\mathrm{SL}(2,\mathbb{R})$ et $X = \Gamma\backslash\mathrm{SL}(2,\mathbb{R})$. Soit $\mathfrak{h}$ le plan hyperbolique. Considérons la surface de Riemann $Y = \Gamma\backslash\mathfrak{h}$ et l'opérateur Laplace–Beltrami Δ sur Y. On identifie Y à X/K et les fonctions de $\mathrm{L}^2(Y)$ aux fonctions de $\mathrm{L}^2(X)$ K-invariantes. À toute fonction propre ϕ de l'opérateur Δ sur Y, on associe le sous-espace fermé G-invariant $L_\phi \subset \mathrm{L}^2(X)$ engendré par ϕ sous l'action de G. Il est bien connu que $(\pi, L) = (\pi_\phi, L_\phi)$ est une représentation unitaire irréductible de G. Soit $\mu = \frac{1-\lambda^2}{4}$ la valeur propre de ϕ. Alors (π, L) peut être réalisée comme la représentation naturelle π_λ du groupe G dans l'espace L_λ des fonctions homogènes (paires) de degré $\lambda - 1$ sur $\mathbb{R}^2\backslash 0$. Ainsi, les fonctions propre ϕ de valeur propre μ sur la surface de Riemann Y, correspondent aux G-morphismes $\nu_\phi : L_\lambda \to \mathrm{L}^2(X)$. Soit $V_\lambda \subset L_\lambda$ le sous-espace des vecteurs lisses. Il est facile de voir que $\mathrm{Mor}_G(L_\lambda, \mathrm{L}^2(X)) \simeq \mathrm{Mor}_G(V_\lambda, \mathrm{C}^\infty(X))$. Par ailleurs, le théorème

Note présentée par Michel DUFLO.

de dualité de Frobenius fournit l'isomorphisme $\mathrm{Mor}_G(V_\lambda, \mathrm{C}^\infty(X)) \simeq \mathrm{Mor}_\Gamma(V_\lambda, \mathbb{C})$. Finalement, à toute fonction propre ϕ est associée une fonctionnelle I_ϕ sur V_λ. En tant qu'espace vectoriel, V_λ est isomorphe à l'espace $\mathrm{C}^\infty_{\mathrm{pair}}(\mathrm{S}^1)$ des fonctions paires lisses sur S^1. Ainsi, I_ϕ peut être considérée comme une distribution sur S^1. Nous posons la question naturelle suivante :

Question : Quelle est la classe L^2 de Sobolev de la fonctionnelle I ?

Réponse : La fonctionnelle I a la même classe de Sobolev que la fonction δ sur S^1. De plus, pour tout s on peut déterminer explicitement des constantes C, $c > 0$ telles que $c\|\delta\|_s \leq \|I\|_s \leq C\|\delta\|_s$.

Nous présentons également une application de ce résultat à la théorie des fonctions automorphes.

1. Introduction

Let $\mathfrak{h}$ be the upper half plane with the hyperbolic metric of constant curvature -1. The group of motions G of $\mathfrak{h}$ is isomorphic to $\mathrm{PSL}(2,\mathbb{R}) = \mathrm{SL}(2,\mathbb{R})/\{\pm 1\}$. Fix a discrete subgroup Γ of $\mathrm{SL}(2,\mathbb{R})$ and consider the Riemann surface $Y = \Gamma\backslash\mathfrak{h}$; we will assume that Y is compact. Denote by Δ the Laplace–Beltrami operator acting in the space of functions on Y. We denote by $0 = \mu_0 < \mu_1 \leq \mu_2 \leq \cdots$ its eigenvalues in $\mathrm{L}^2(Y)$ and by ϕ_i the corresponding eigenfunctions; we normalize these eigenfunctions so that $\|\phi_i\|_{\mathrm{L}^2} = 1$.

The study of eigenfunctions ϕ and their corresponding eigenvalues is important in many areas of representation theory, number theory and geometry.

In this Note, we present a new approach to the study of eigenfunctions ϕ based on the study of Sobolev norms of corresponding automorphic functionals.

We state our main result and give applications to some questions from the theory of automorphic functions. Complete proofs and other applications will appear elsewhere.

1.1. Automorphic representations

Let $G = \mathrm{SL}(2,\mathbb{R})$, $K = \mathrm{SO}(2,\mathbb{R})$; we will identify $\mathfrak{h}$ with $G\backslash K$.

Fix a cocompact lattice Γ in G and denote by X the compact quotient $X = \Gamma\backslash G$. (Our results below hold in the general case of cofinite Γ (i.e. $\mathrm{vol}(X) < \infty$). For the sake of simplicity, we mostly restrict ourselves to the cocompact case.) The group G acts on X and hence on the space of functions on X.

We will identify the Riemann surface $Y = \Gamma\backslash\mathfrak{h}$ with X/K. This induces the imbedding $\mathrm{L}^2(Y) \subset \mathrm{L}^2(X)$, the image consisting of all K-invariant functions. For any eigenfunctions ϕ of the Laplace operator on Y, we may consider the closed G-invariant subspace $L_\phi \subset \mathrm{L}^2(X)$ generated by ϕ under the action of G. It is known that $(\pi, L) = (\pi_\phi, L_\phi)$ is an irreductible unitary representation of G (*see* [3]).

Conversely, fix an irreducible unitary representation (π, L) of the group G and a K-fixed unit vector $v_0 \in L$. Then any G-morphism $\nu : L \to \mathrm{L}^2(X)$ defines an eigenfunction $\phi = \nu(v_0)$ of the Laplace operator on Y; this function is normalized if ν is an isometric imbedding.

Thus eigenfunctions ϕ correspond to tuples (π, L, v_0, ν).

All irreducible unitary representations of G with K-fixed vector are classified: these are representations of principal and complementary series, and trivial representation. For the sake of simplicity, we consider only representations of principal series.

Such a representation (π, L) can be realized as follows. Fix a purely imaginary number λ and consider the natural representation π_λ of the group G in the space L_λ of (even) homogeneous functions on $\mathbb{R}^2\backslash 0$ of degree $\lambda - 1$. Thus, vectors in L_λ are just locally L^2 functions f on $\mathbb{R}^2\backslash 0$

satisfying $f(ax, ay) = |a|^{\lambda-1} f(x, y)$ for all $a \in \mathbb{R}^*$. The representation π is induced by the natural action of G on (x, y).

Note that the value of such a function is determined by its values on the unit circle S^1; hence we may identify the space L_λ with the space $\mathrm{L}^2(\mathrm{S}^1)_{\text{even}}$ of even functions on S^1. The G-invariant scalar product in L_λ is given by $Q_0(f, g) = \frac{1}{2\pi}\int_{\mathrm{S}^1} f\overline{g}\mathrm{d}\theta$. The K-fixed unit vector v_0 corresponds to the constant function 1 on S^1. The eigenfunction ϕ of the Laplace operator which corresponds to a representation (π_λ, L_λ) will have the eigenvalue $\mu = \frac{1-\lambda^2}{4}$.

Thus, we see that eigenfunctions ϕ on the Riemann surface Y with the given eigenvalue μ correspond to G-morphisms $\nu_\phi : L_\lambda \to \mathrm{L}^2(X)$ (namely $\phi = \nu_\phi(v_0)$). Normalization $||\phi|| = 1$ means ν preserves the scalar product.

1.2. Automorphic functionals

Let $V_\lambda \subset L_\lambda$ be the subspace of smooth vectors. It is easy to see that we have the following isomorphism $\mathrm{Mor}_G(L_\lambda, \mathrm{L}^2(X)) \simeq \mathrm{Mor}_G(V_\lambda, \mathrm{C}^\infty(X))$. The last space can be described using the following Frobenius duality theorem ([3], [6]):

Proposition. – $\mathrm{Mor}_G(V_\lambda, \mathrm{C}^\infty(X)) \simeq \mathrm{Mor}_\Gamma(V_\lambda, \mathbb{C})$.

Namely, to every G-morphism $\nu : V_\lambda \to \mathrm{C}^\infty(\Gamma\backslash G)$ we assign a Γ-invariant functional I on the space V_λ given by $I(v) = \nu(v)(e)$ (here e is the identity in G). Given I, we can recover ν as $\nu(v)(g) = I(\pi(g)v)$.

Thus eigenfunctions ϕ of the Laplace operator on Y with eigenvalue μ correspond to Γ-invariant functionals I on the space V_λ.

All this is well known (*see* [3]). What is new in our approach is that we are trying to get an information about the eigenfuntion ϕ by looking at the analytic properties of the corresponding functional I_ϕ.

For example, we can consider I as a distribution on S^1, i.e. as a functional on $\mathrm{C}^\infty(\mathrm{S}^1)$. This fonctional is continuous, i.e. it is continuous with respect to some seminorms. Then we may ask:

Question: What is the L^2 Sobolev class of the functional I? In other words, for which real s is the functional I continuous with respect to the L^2-type Sobolev norm S_s, and how do we estimate the norm $||I||_s$ of I with respect to this Sobolev norm? (For the definition of Sobolev norms and the normalization we use see 1.3.)

We get a very simple answer:

Theorem. – *The functional I has the same Sobolev class as the δ-function on* S^1. *Moreover, for every s one can write explicit constants c, $C > 0$ such that* $c||\delta||_s \leq ||I||_s \leq C||\delta||_s$.

In particular, this shows that the functional I is bounded with respect to a Sobolev norm S_s if and only if $s > 1/2$.

Our main result is in fact a more general statement which shows that I and the δ-function have comparable norms with respect to any Hermitian norm on the space V_λ such that the representation of G is continuous with respect to this norm (*see* Section 2).

1.3. Sobolev norms

Let (π, V) be a smooth representation of $G = \mathrm{SL}(2, \mathbb{R})$ equipped with an invariant positive definite Hermitian form P; we denote by $||\ ||_P$ the corresponding norm on V.

Definition. – For every real number s, we define the Sobolev Hermitian scalar product Q_s and the corresponding Sobolev norm S_s on V as follows. Fix the standard K-invariant scalar product on $\mathfrak{g} = \mathrm{sl}(2,\mathbb{R})(X \mapsto \mathrm{tr}XX^t)$. Let X_1, X_2, X_3 be an orthonormal basis of $\mathfrak{g} = \mathrm{sl}(2,\mathbb{R})$. Consider the element Δ of the universal enveloping algebra of $\mathfrak{g}$ given by $\Delta = -\sum_i X_i^2$; this element is K-invariant and generates an essentially self adjoint operator on V. We define the Sobolev norm S_s on V by $\mathrm{S}_s(v) = ||(\Delta+1)^{s/2}v||_P$, or equivalently, by $Q_s(v) = P((\Delta+1)^{s/2}v)$.

1.4. Invariant norms on representations and L^p-norms on X

Let N_∞ denote the supremum norm on the space of functions on X, i.e. $\mathrm{N}_\infty(f) = \sup_{x\in X}|f(x)|$. Theorem 1.2 can be interpreted as an estimate $\mathrm{N}_\infty(\nu(v)) \ll \mathrm{S}_s(v)$ which holds for every $s > 1/2$.

In fact, we can prove a better bound on N_∞ in terms of L^1 Sobolev norm $\mathrm{S}_{1,s}$. Namely, we claim that for any $s > 1/2$ we have $\mathrm{N}_\infty(\nu(v)) \ll \mathrm{S}_{1,s}(v)$. Note that $\mathrm{S}_{1,s}$ is a *nonhermitian* norm on V and it is almost G-invariant when s is close to 1/2.

Thus we see that for the function $\nu(v)$ on the space X we have explicit estimates of two norms – the supremum norm N_∞ and the L^2 norm N_2. Using the interpolation theorem for Banach norms on the space V, we can prove, for $p \geq 2$, the following estimate for the L^p norm N_p of the function $\nu(v)$ in terms of Sobolev norms of the vector v : $\mathrm{N}_p(v) \ll \mathrm{S}_{q,s}(v)$, where q and s are chosen such that $1/q + 1/p = 1$ and $s > 1/2 - 1/p$. Note that when s is close to $1/2 - 1/p$, the norm $\mathrm{S}_{q,s}$ on V is close to a G-invariant norm. We hope to return to this subject elsewhere.

1.5. Triple products

We consider an application of the above approach to a particular problem in the theory of automorphic functions related to the theory of Rankin–Selberg L-functions (*see* [8] for more details and historic remarks). To state the problem, let us fix an automorphic function ϕ and consider the new function ϕ^2 on Y (it is *not* an automorphic function since it is not an eigenfunction). Since ϕ^2 is in $\mathrm{L}^2(Y)$, we may consider its spectral decomposition:

$$\phi^2 = \sum_i c_i\phi_i,$$

with respect to the basis $\{\phi_i\}$, $\Delta\phi_i = \lambda_i\phi_i$. Here $c_i = \langle\phi^2, \phi_i\rangle$.

CLAIM. – *The c_i's have an exponential decay.*

We introduce new (normalized) coefficients: $b_i = |c_i|^2 \exp\left(\frac{\pi}{2}|\lambda_i|\right)$. Our main result is

THEOREM. – *We have the following inequality:*

$$\sum_{|\lambda_i|\leq T} b_i \leq C(\ln T)^3,$$

for some constant $C > 0$, as $T \to \infty$.

This was conjectured earlier by P. Sarnak.

1.6. Fourier coefficients of cusp forms

Similar estimates hold when Γ is non-cocompact but of finite covolume, e.g., $\Gamma = \mathrm{SL}(2,\mathbb{Z})$. Namely, let Γ be a non-uniform lattice in $\mathrm{SL}(2,\mathbb{R})$ with a cusp at ∞. We assume that the generator of the corresponding unipotent subgroup is given by $\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}$. Let ϕ be a cusp form. We have then the following Fourier decomposition (*see* [4]): $\phi(x+iy) = \sum a_n y^{\frac{1}{2}} K_{\frac{\lambda}{2}}(2\pi|n|y)e^{2\pi inx}$, where $K_{\frac{\lambda}{2}}(y)$

is the K-Bessel function. In order to study the coefficients a_n, Rankin and Selberg introduced the following L-function: $L(s) = \sum_{n>0} \frac{|a_n|^2}{n^s}$ (*see* [10]). It is known as the L-function $L(s, \phi \otimes \phi)$ associated to a pair of cusp forms (*see* [2]). They discovered that $L(s)$ has the following integral representation. Let $E(z,s)$ be the Eisenstein series associated to the cusp at ∞. As before we can define coefficients $c_s = \langle \phi^2, E(z,s)\rangle = \int_{\Gamma\backslash\mathfrak{h}} \phi^2, \overline{E}(z,s)\mathrm{d}z$. We then have $L(s) = g(s)c_s$, where $g(s)$ is explicitly given (it is a product of Γ-functions). Our theorem above implies nontrivial bounds on c_s and hence on these L-functions *without the assumption of arithmeticity* of Γ. For example we have the following

COROLLARY 1. – $\int_T^{T+1} \left|L\left(\frac{1}{2} + i\tau\right)\right| \mathrm{d}\tau \ll T \ln T$.

This in turn implies a bound on the Fourier coefficients a_n of ϕ by standard methods of analytic number theory. Namely we have the following

COROLLARY 2. – *We have* $|a_n| \ll n^{\frac{1}{3}+\varepsilon}$ *for any* $\varepsilon > 0$.

1.7. Remarks

1. The interest in triple scalar products as above stems mostly from their connection to the theory of automorphic L-functions. Namey, as we mentioned earlier, the scalar product of ϕ^2 with the Eisenstein series has been considered by Rankin and Selberg for this reason.

P. Garret discovered that this is a special case of a more general construction. He considered triple products of three automorphic functions and showed that this is (up to an explicit factor) the value for the triple L-function $L(s, \phi_1 \otimes \phi_2 \otimes \phi_3)$ at $\frac{1}{2}$ (*see* [1], [2]). Our result above (Theorem 1.5) could be interpreted as the mean value Lindelöf conjecture for these L-functions (*see* [9]) for the spectral significance of Lindelöf conjectures). We note that our proof does not use any *arithmetical* information.

2. Fourier coefficients of cusp forms have also been extensively investigated. The upper bound $|a_n| \ll n^{\frac{1}{2}}$ is due to Hencke and follows from the fact that ϕ is bounded (it is sometimes called the standard (or convexity) bound). The Petersson–Ramanujan Conjecture claims that $|a_n| \ll n^{\varepsilon}$ and it is expected for congruence subgroups. The best known bound, $|a_n| \ll n^{\frac{5}{28}+\varepsilon}$ for congruence subgroups, is due to Bump–Duke–Hoffstein–Iwaniec. However, for nonarithmetic subgroups, there was no improvement over the Hecke bound and it was suspected that the Hecke bound might be of true order. Recently Sarnak in [8] gave the first improvement over the Hecke bound for a general Γ (he treated the case of $\mathrm{SL}(2,\mathbb{C})$ and $\mathrm{SL}(2,\mathbb{R})$ case done in [7]). He also suggested that the Petersson–Ramanujan Conjecture might be true in this general setting.

3. For $\Gamma = \mathrm{SL}(2,\mathbb{Z})$, an analog of Theorem 1.5 (in a slightly weaker form) was proved by M. Jutila [5] using nontrivial arithmetic information.

2. Proofs

We will discuss the method behind the proof of Theorem 1.2. The proof of Theorem 1.5 is based on this fact and some other considerations involving invariant (non-hermitian) norms on representations and analytic continuation of representations of $\mathrm{SL}(2,\mathbb{R})$ to a domain in $\mathrm{SL}(2,\mathbb{C})$. We will discuss these elsewhere.

2.1. Relative traces

In order to prove the theorem, consider the following general problem. Suppose we are given a representation (π, G, V) of a locally compact group G on a topological vector space V. Suppose we are given a morphism of representations $\nu : V \to \mathrm{C}(X)$, where X is a G-space and $\mathrm{C}(X)$ the space of continuous functions on X. Then, each point $x \in X$ defines a continuous functional I_x on V by $I_x(v) = \nu(v)(x)$. We wish to establish some bounds on the norm of the functional I_x. More precisely, suppose we are given a norm N on V (we always assume that π is continuous with respect to this norm). We would like to give a priori estimate of the norm $||I||_N$ of the functional I with respect to N, where $||I||_N := \sup \frac{|I(v)|}{N(v)}$.

Of course, for this, we have to know something about the morphism ν. Let us assume that the action of G on X is transitive and that a G-invariant measure μ_X is fixed such that the image of V lies inside $\mathrm{L}^2(X, \mu_X)$. Then the scalar product in $\mathrm{L}^2(X)$ defines a Hermitian form P on V.

We are supposed to bound $||I||_N$ in terms of the norm N and the Hermitian form P on the space V. It turns out, that in the case when the norm N is obtained from a Hermitian scalar product Q on V, we sometimes can give a reasonable bound for $||I||_N$. Namely, we claim that $||I||_N$ can be estimated in terms of the relative trace $\mathrm{tr}(P|Q)$ of Hermitian forms P and Q.

More precisely, let $\mathrm{H}(V)$ be the space of continuous Hermitian forms and let $\mathrm{H}(V)^+ \subset \mathrm{H}(V)$ be the subset of nonnegative Hermitian forms. For any pair of forms P, $Q \in \mathrm{H}^+(V)$, one can define a number $\mathrm{tr}(P|Q)$, the relative trace of P with respect to Q, taking values in $\mathbb{R}_+ \cup \infty$. (For example, if Q is positive definite and $P \ll Q$, the form P can be represented in a Hilbert space completion of the space V by some selfadjoint operator A_P; in this case we have $\mathrm{tr}(P|Q) = \mathrm{tr}A_P$).

This number may be usually effectively computed. It turns out that one case on give tight estimates of the norm $||I||_N$ in terms of this number. Namely, we have the following general result.

THEOREM 2.2. – *We have an estimate $||I_x||_N^2 \leq C \cdot \mathrm{tr}(P|Q)$, where C is an effectively computable constant. If X is compact this estimate is tight, i.e. $||I_x||_N^2 \geq c \cdot \mathrm{tr}(P|Q)$ for a constant $c > 0$.*

Specifying this theorem to the case of an irreducible representation of principal series, where P is the invariant L^2 Hermitian form and $Q = Q_s$ is the s-Sobolev Hermitian form, we see that $\mathrm{tr}(P|Q) = ||\delta||_s^2$. This proves Theorem 1.2.

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