# Stacks in Representation Theory

**What is a continuous representation of an algebraic group ?**

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## 1 Introduction.

In this note I would like to introduce a new approach to (or rather a new language for) representation theory of groups. Namely, I propose to consider a (complex) representation of a group $G$ as a sheaf on some geometric object. This point of view necessarily leads to a conclusion that the standard approach to (continuous) representations of algebraic groups should be modified.

Let us start with a local or finite field $F$ and fix an algebraic group $\mathcal{G}$ defined over $F$. In the standard approach we consider the set $G = \mathcal{G}(F)$ of $F$-points of $\mathcal{G}$ as a topological group and study an appropriate category $Rep(G)$ of continuous representations of $G$.

The main goal of this note is to explain that this approach is philosophically inconsistent. In fact I will describe how to extend the category $Rep(G)$ to some larger category $\mathcal{M}(\mathcal{G}, F)$ that better corresponds to our intuitive understanding of representations of $G$.

We will see that this category can be naturally described as a product of categories $Rep(G_i)$ over all pure inner forms of the group $G$. On the level of simple objects this means that $Irr(\mathcal{M}(\mathcal{G}, F)) \approx \coprod Irr(G_i)$. This agrees with observation by several mathematicians (e.g. by D. Vogan [Vog]) that when we classify irreducible representations it is better to work with the union of sets $Irr(G_i)$ for several forms of the group $G$ than with one set $Irr(G)$.

### 1.1 Representations and sheaves on Stacks.

In order to describe the category $\mathcal{M}(G)$ I propose to consider representations as sheaves on algebraic stacks.

Stacks play a more and more important role in contemporary Mathematics. Since they are not yet the common language in representation theory I will recall some basic notions related to stacks.

Informally, stack is a “space” $X$ such that every point $x \in X$ is endowed with a group $G_x$ of automorphisms of inner degrees of freedom at this point.

We see that in order to consider stacks we should first fix a Geometric Environment, i.e. a category $\mathcal{S}$ of spaces on which we model our stacks. In fact the category $\mathcal{S}$ should be considered together with some Grothendieck topology. The standard term for such category $\mathcal{S}$ is “site”.

Usually one works with the following sites:

(i) Category of schemes over a field $F$ (with etale or smooth topology).

(ii) Category of smooth manifolds (with usual topology).

(iii) Category of locally compact Hausdorff topological spaces with usual topology (or some natural subcategory of it, e.g. the category of totally disconnected spaces).

(iv) Category $Sets$ of sets with discrete topology.

## 1.2 Groupoids.

Stacks modeled on the site $\mathcal{S} = Sets$ are **groupoids**. Let me remind that, by definition, groupoid is a category in which all morphisms are isomorphisms.

Groupoids represent rather elementary examples of stacks. However they exhibit many features of the general case. Also in this case the general ideas that I would like to explain are much easier to understand. For this reason I would like to discuss this case in some detail.

### 1.2.1 Examples of groupoids.

To every discrete group $G$ we assign the **basic groupoid** $BG = pt/G$ as follows:

An object of the category $BG$ is a $G$-torsor $T$ (i.e. a non-empty $G$-set on which $G$ acts transitively and free). The morphisms in this category are morphisms of $G$-sets.

More generally, given an action of the group $G$ on a set $Z$ we define the **quotient groupoid** $BG(Z) = Z/G$ as follows:

Object of $BG(Z)$ is a $G$-torsor $T$ equipped with a $G$-morphism $\nu : T \to Z$. Morphisms are morphisms of $G$-sets over $Z$.

### 1.2.2 Representations as sheaves on groupoids.

Given a groupoid $\mathcal{X}$ it is natural to think about it as a geometric object (some kind of a space). Then it is natural to consider sheaves on this space.

We define a **sheaf** $R$ (of complex vector spaces) on a groupoid $\mathcal{X}$ to be a functor $R : \mathcal{X} \to Vect$, where $Vect$ is the category of complex vector spaces (we will see later why this notion is natural). We denote by $Sh(\mathcal{X})$ the category of sheaves on $\mathcal{X}$.

**Claim.** The category $Sh(BG)$ is naturally equivalent to the category $Rep(G)$.

More generally, for every $G$-set $Z$ the category $Sh(BG(Z))$ is naturally equivalent to the category $Sh_G(Z)$ of $G$-equivariant sheaves on $Z$ (see 2.2).

This gives us a “geometric” description of the category $Rep(G)$. This construction, that is very elementary in case of groupoids, is the basis of the approach that I describe in this note.

## 1.3 Topological groupoids.

Let $G$ be a locally compact group. For technical reasons let us assume that it is totally disconnected. In this case we also can define the basic groupoid $BG$ and quotient groupoids $BG(Z)$ as stacks modeled on the site of locally compact spaces. For any stack $\mathcal{X}$ of this type we will define a category $Sh(\mathcal{X})$ of sheaves on $\mathcal{X}$. Using these constructions we can interpret the category $Rep(G)$ as the category of sheaves on the stack $BG$.

One of the ways to think about a stack modeled on the site of locally compact spaces is to interpret it as a topological groupoid. For example, using this interpretation it is easy to show that for a $G$-space $Z$ the category $Sh(BG(Z))$ of sheaves on the quotient stack $BG(Z) = Z/G$ is naturally equivalent to the category $Sh_G(Z)$ of $G$-equivariant sheaves on $Z$ (see 2.2).

Technically, working with topological groupoids is a little difficult. Later we describe another way to define the category $Sh(\mathcal{X})$ of sheaves on $\mathcal{X}$ for stacks of this type that is technically simpler (see section 6).

## 1.4 Algebraic groups and stacks.

Let us consider more interesting case of an algebraic group $\mathcal{G}$ over a local (or finite) field $F$. By analogy with the discrete case we define the **basic stack** $B\mathcal{G} = pt/\mathcal{G}$. This is an algebraic stack over the field $F$ (i.e. it is modeled on the site $\mathcal{S}$ of schemes over $F$ (see section 5)).

For any algebraic stack $\mathcal{X}$ over $F$ we will construct the category $Sh(\mathcal{X})$ of sheaves of complex vector spaces on $\mathcal{X}$. The informal idea is that $F$-points $\mathcal{X}(F)$ of the stack $\mathcal{X}$ form a groupoid and a sheaf $R$ on $\mathcal{X}$ is just a sheaf $R$ on this groupoid.

For finite fields this works fine. For local fields we should take into account the topology of the groupoid $\mathcal{X}(F)$.

Given an algebraic group $\mathcal{G}$ over $F$ we define the category $\mathcal{M} = \mathcal{M}(\mathcal{G}, F)$ as the category $Sh(B\mathcal{G})$ of sheaves on the algebraic stack $B\mathcal{G}$. I call the objects of this category **stacky $G$-modules**.

### 1.4.1 Two competing definitions

Now starting with algebraic group $\mathcal{G}$ over $F$ we can consider two competing definitions of a representation.

**Definition 1**. Category $Rep(G)$ obtained from $\mathcal{G}$ by a chain of constructions

$$\mathcal{G} \quad\Longrightarrow\quad \text{group } G = \mathcal{G}(F) \quad\Longrightarrow\quad \text{groupoid } \mathcal{Y} = BG \quad\Longrightarrow\quad Sh(\mathcal{Y})$$

**Definition 2**. Category $\mathcal{M}(\mathcal{G}, F)$ obtained from $\mathcal{G}$ by a chain of constructions

$$\mathcal{G} \quad \Longrightarrow \quad \text{stack } \mathcal{X} = B\mathcal{G} \quad \Longrightarrow \quad \text{groupoid } \mathcal{X}(F) \quad \Longrightarrow \quad Sh(\mathcal{X}(F))$$

The subtle point is that the groupoids $\mathcal{Y}$ and $\mathcal{X}(F)$ are not always equivalent. So the category $\mathcal{M}(\mathcal{G}, F) = Sh(\mathcal{X}(F))$ might be not equivalent to the category $Rep(G) = Sh(\mathcal{Y})$.

The standard notion of a continuous representation of $G$ is based on definition 1. **The main goal** of this note is to convince the reader that from many points of view definition 2 is much more appropriate than the standard definition 1.

## 1.5 Vogan's picture.

One advantage of this definition is that it gives an explanation to representations that appear in Vogan's interpretation of the Langlands correspondence. Let me remind what is Vogan's suggestion (see [Vog], Conjecture 4.15 or [ABV], Theorem 1.18).

Vogan tried to describe the Langlands correspondence in the following explicit way. Let $F$ be a local field, $\mathcal{G}$ a reductive algebraic group over $F$ and $G$ the group of its $F$-points. Consider on one side the set $Irr(G)$ of equivalence classes of irreducible representations of $G$. On the other side consider the set $Lan(\mathcal{G}, F)$ of Langlands' parameters $\Phi$ defined in terms of the dual group $L(\mathcal{G})$ and the field $F$.

Vogan tried to construct a canonical bijection between these two sets. He realized that in many cases this can not work since the set $Irr(G)$ is just too small. But he also discovered that if we replace this set by the disjoint union $\coprod Irr(G_i)$ of corresponding sets for all pure inner forms $G_i$ of the group $G$ then this set has correct size (note that some of these forms might be isomorphic – then the "same" representation will appear in this list several times). In fact in many cases Vogan was able to describe a bijection of this set with the set of Langlands' parameters.

In the language I propose Vogan's conjecture can be formulated as follows.

First of all, instead of the group $\mathcal{G}$ we consider the algebraic stack $B\mathcal{G}$. Instead of representations of the group $G = \mathcal{G}(F)$ let us study the category $\mathcal{M} := Sh(B\mathcal{G})$ of sheaves on this stack (i.e. the category of stacky $G$-modules). Since for any pure inner form $\mathcal{G}'$ of the group $\mathcal{G}$ the stack $B\mathcal{G}'$ is equivalent to $B\mathcal{G}$, the category $\mathcal{M}$ depends only on the pure inner class of $\mathcal{G}$.

Our goal is to parameterize the set $Irr(\mathcal{M})$ of isomorphism classes of simple objects in $\mathcal{M}$. We will see in 6.2 that this set can be described as a disjoint union over all pure inner forms $\mathcal{G}_i$ of the group $\mathcal{G}$ of the sets $Irr(G_i)$, where $G_i = \mathcal{G}_i(F)$.

Then the conjecture 4.15 in [Vog] is essentially the statement that the set $Irr(\mathcal{M})$ is in natural bijection with the set $Lan(\mathcal{G}, F)$ of Langlands' parameters.

**Remarks**. **1.** In fact Vogan formulated his conjectures only for pure inner forms of quasi-split groups. Later they were generalized by Kaletha to other forms (see [Kal]).

**2.** It would be interesting to understand whether one has a **canonical** bijection between these sets. The constructions proposed in [Vog], [ABV] are not quite canonical – they depend on choice of some "Whittaker data".

I think that in fact they are not in canonical bijection. Namely, I think that the set of Langlands' parameters $Lan(\mathcal{G}, F)$ should be slightly modified – and after this the resulting set $Lan'(\mathcal{G}, F)$ will be in a canonical bijection with the set $Irr(\mathcal{M})$.

Possible suggestions for this modified set of Langlands' parameters can be found in [BG] and sources listed there.

## 1.6 About this note.

In section 2 I remind basic facts about sheaves that are relevant in Representation Theory.

In particular, I discuss the notion of an equivariant sheaf on a topological $G$-space $Z$. This notion plays central role in any geometric approach to Representation Theory. This is also an important computational tool – in fact later on when I talk about explicit description of some object I mean a description in the language of equivariant sheaves.

In section 3 I describe a striking example that shows that the standard definition of representations is not a good one.

In subsequent sections I make various comments on the notions we discussed above, indicate how to formulate precise technical definitions that describe these notions and describe some technical tools that help to make computations with them.

In section 4 I discuss the case of groupoids. In section 5 I shortly describe how to give a technical definition of a stack.

In section 6 I discuss a technical definition of the category $Sh(\mathcal{X})$ of sheaves of complex vector spaces on a stack $\mathcal{X}$ and describe how to make computations with them. In particular, I explain how one can describe the category $Sh(\mathcal{X})$ in terms of equivariant sheaves on topological spaces.

In section 7 I explain the algebro-geometric structure that describes the relation between groupoids $\mathcal{X}(F)$ for different fields $F$.

This note is an expanded version of the lecture that I gave at the Fourth Conference of Tsinghua Sanya International Mathematics Forum (TSIMF) in December 2013. I would like to thank TSIMF organizers for the invitation.

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# 2 Sheaves and equivariant sheaves.

## 2.1 Sheaves relevant in representation theory.

Let $F$ be a finite or local field, $\mathcal{G}$ an algebraic group over $F$. We consider the group $G = \mathcal{G}(F)$ of its $F$-points as a topological group.

In order to study representations of the group $G$ we usually place this group in some geometric environment. Namely we fix a site $\mathcal{S}$ appropriate for the group $G$ and for every space $Z \in \mathcal{S}$ we consider some category $Sh(Z)$ of sheaves of complex vector spaces on $Z$. Let us describe this in more detail.

(i) Let $F$ be a finite field. We consider the site $\mathcal{S}$ of finite discrete sets. For a space $Z \in \mathcal{S}$ we denote by $Sh(Z)$ the category of all sheaves of complex vector spaces on $Z$.

(ii) Let $F$ be a local non-Archimedean field. We consider the site $\mathcal{S}$ that consists of Hausdorff locally compact totally disconnected spaces $Z$ with countable base of open subsets ($l$-spaces in terminology of [BZ]).

Given an $l$-space $Z$ we consider the category $Sh(Z)$ of all sheaves of complex vector spaces on $Z$ (in [BZ] they were called $l$-sheaves).

(iii) Let $F$ be a local Archimedean field, i.e. $F = \mathbf{R}$ or $F = \mathbf{C}$. In this case I see several candidates for the site $\mathcal{S}$.

We can consider $\mathcal{S}$ to be the category of smooth manifolds. For a manifold $Z$ the category $Sh(Z)$ is the category of sheaves of $\mathcal{O}_Z$-modules, where $\mathcal{O}_Z$ is the sheaf of smooth complex valued functions on $Z$.

Another possibility is to work with the site of Nash manifolds and some sheaves of Schwartz functions (or distributions) on these manifolds (see [AG]).

I do not know what is the correct approach to this case. So in what follows I mostly deal with finite and local non-Archimedean fields.

**Remarks**. **1**. In this note for simplicity I discuss only representations over the field $k = \mathbf{C}$ of complex numbers. In case when $F$ is a finite or non-Archimedean field we can consider other fields $k$ of coefficients.

If $F$ is a local non-Archimedean field and $k$ is an extension of $F$ we can try to include the theory of locally analytic representations by considering the site $\mathcal{S}$ of analytic manifolds over $F$ and categories $Sh(Z)$ of sheaves of modules over the sheaf $\mathcal{O}_Z$ of locally analytic functions on $Z$. It seems that the stacky language that I will introduce might be applicable and useful in this theory.

**2**. The structures I described are important when we are trying to specify the category $Rep(G)$ of representations of the group $G$ that we would like to study.

In case when the group $G$ is defined over a finite or local non-Archimedean field $F$ there is a consensus what is the "correct" category of representations $Rep(G)$. Namely, in case of a finite field we consider the category of all representations and in case of a local non-Archimedean field we consider the category of smooth representations (smooth means that every vector has open stabilizer in the group $G$).

The situation for real groups is different – the correct choice of the appropriate category $Rep(G)$ is a very non-trivial question. In case of reductive groups this was done by

Casselman-Wallach (see [Cass],[W], [BK]). For general real algebraic group I do not know a good candidate for this category.

In this note I am not going to discuss this tricky question. For this reason I will mostly deal with finite and local non-Archimedean fields.

## 2.2 Equivariant sheaves.

From now on we assume that all topological spaces we consider are $l$-spaces (locally compact, totally disconnected with countable base of open subsets).

Let $G$ be a topological group and $Z$ a $G$-space, i.e. $Z$ is equipped with a continuous action $a : G \times Z \to Z$. We denote by $Sh_G(Z)$ the category of $G$-equivariant sheaves (of complex vector spaces) on $Z$. This category will play central role in what follows.

Recall, that a $G$-equivariant sheaf is a sheaf $R$ on $Z$ equipped with an isomorphism $\alpha : a^*(R) \to pr_Z^*(R)$ of two liftings of $R$ to the space $G \times Z$ satisfying some natural cocycle condition (this condition is that after the lifting to the space $G \times G \times Z$ two morphisms $\nu, \mu : (m \cdot a)^*(R) \to pr^*(R)$ of sheaves naturally constructed from $\alpha$ should coincide (see details of the definition and discussion in [BL], Part1, section 0).

**Remark**. In case of real groups we can work with the site $\mathcal{S}$ of smooth manifolds and with sheaves of $\mathcal{O}$-modules. In this case the definition of equivariant sheaves formally looks exactly the same, but has quite different geometric meaning. The reason is that the pullback functor $a^*$ in the category of $\mathcal{O}$-modules is quite different from the pullback functor in the category of sheaves.

Let me remind two standard facts about equivariant sheaves (here we assume that $G$ is an $l$-group).

**Fact 1**. Let $Z$ be a point. Then the category $Sh_G(Z)$ is equivalent to the category $Rep(G)$ of **smooth** representations of $G$.

**Fact 2**. Suppose that $Z$ is a quotient space of the group $G$. Fix a point $z \in Z$ and denote by $H$ its stabilizer in $G$. Then we have natural equivalences of categories $Sh_G(Z) \approx Sh_H(z) \approx Rep(H)$.

# 3 An example.

In this section I describe a striking example that illustrates what is wrong with the standard approach.

## 3.1 Heuristics.

My example is based on the following heuristic geometric principle.

Let $a : \mathcal{G} \times \mathcal{Z} \to \mathcal{Z}$ be a transitive action of an algebraic group $\mathcal{G}$ on an algebraic variety $\mathcal{Z}$. Passing to $F$-points we get a continuous action $a : G \times Z \to Z$.

This action is usually not transitive. In many cases we can write $Z$ as a union of open orbits $Z = \bigsqcup Z_i$, $i = 1, ..., n$.

**Heuristic Geometric Principle**.

1. The space $Z$ is "good", i.e. it is easy to describe.

2. Every individual orbit $Z_i$ might be a "bad" space, that means that it is difficult to describe.

**Illustration**. Consider the space $V$ of real symmetric $8 \times 8$ matrices and try to give explicit descriptions of subsets $Z, Z_3 \subset V$ that describe non-degenerate quadratic forms and quadratic forms of signature 3 on $\mathbf{R}^8$ respectively.

## 3.2 An example – representations of orthogonal groups.

Fix an $n$-dimensional vector space $V$ over $F$.

The group $G = GL(V)$ acts on the space $Z$ of non-degenerate quadratic forms.

Fix a form $Q \in Z$ and denote by $H$ its stabilizer in $G$ (i.e. $H$ is the orthogonal group $O(Q)$). Let us denote by $Z_Q$ the $G$-orbit of $Q$ in $Z$. Then we have an equivalence of categories $Rep(H) \approx Sh_G(Z_Q)$.

According to the heuristic geometric principle the space $Z_Q$ might be (and often is) a "bad" space. This means that the category $Rep(H) \approx Sh_G(Z_Q)$ might be a "bad" category (i.e. very difficult to describe). In other words, the "natural" problem of classification of irreducible representations of the orthogonal group $H = O(Q)$ turns out to be not that natural.

However we see that the bad category $Rep(H) \approx Sh_G(Z_Q)$ can be naturally extended to a larger category $\mathcal{M} := Sh_G(Z)$ of all $G$-equivariant sheaves on the good algebraic space $Z$. We can expect (and this is really the case) that this larger category $\mathcal{M}$ is a "good" category.

Note that we have a natural decomposition $\mathcal{M} \approx \prod Sh_G(Z_i)$, where $Z_i$ are $G$-orbits in $Z$. In particular the set $Irr(\mathcal{M})$ of isomorphism classes of simple objects in $\mathcal{M}$ is a disjoint union of sets $Irr(Sh_G(Z_i))$.

It seems reasonable to assume that the classification of simple objects of the category $\mathcal{M}$ might be relatively simple problem, but then to sort out which of them are related to the orbit $Z_Q$ might turn out to be much more difficult problem (and it is not clear whether this problem is a meaningful one).

The example we are considering suggests a certain pattern that seems to work also in the general case. Namely, we see that if $G$ is a group of points of an algebraic group then the category $Rep(G)$ of its representations might be a bad category, but we can include it as a direct factor into some larger good category $\mathcal{M}(G)$.

I had this example in mind for some time until I realized how one can define this larger category using sheaves on stacks.

# 4 Some remarks about groupoids

## 4.1 Equivalence of groupoids

We know that if two objects of some category are isomorphic then it is better to consider them as two realizations of the same geometric structure. Similarly, if two groupoids $\mathcal{X}$ and $\mathcal{Y}$ are equivalent (as categories) we can assume that they represent two realizations of the same geometric structure.

A subtle point here is that the equivalences between these groupoids form a groupoid. This means that if we fix an equivalence $Q : \mathcal{X} \to \mathcal{Y}$ then this equivalence itself has automorphisms, and it is not immediately clear how we should think about them.

Similarly, if we would like to show that two groupoids are **canonically** equivalent we have to construct an equivalence between them and show that this equivalence is defined up to a canonical isomorphism.

**Example**. Consider an action $a : G \times Z \to Z$.

Let us define a groupoid $BG_0(Z)$ as follows:

$Ob(BG_0(Z)) = Z, Mor(BG_0(Z)) = G \times Z$, where morphism $(g, z)$ is a morphism from the object $z$ to the object $gz$.

The groupoid $BG_0(pt)$ we denote by $BG_0$.

**Claim**. The groupoid $BG_0(Z)$ is canonically equivalent to the groupoid $BG(Z)$.

The groupoid $BG_0(Z)$ might be considered as a "matrix" version of the groupoid $BG(Z)$. It is better suited for computations.

## 4.2 Theory of groups and theory of groupoids.

I would like to explain that the theories describing groups and groupoids are essentially equivalent.

**Proposition**. **1**. Every groupoid $\mathcal{X}$ is canonically decomposed as a disjoint union of connected groupoids (groupoid is connected if all its objects are isomorphic).

**2**. A connected groupoid $\mathcal{Y}$ is equivalent to the basic groupoid for some group $G$.

This result shows that any question about groupoids can be reduced to a question in group theory. In other words, the difference between theories of groups and groupoids is in their emphasis.

In my opinion the relation between the theory of groupoids and the group theory is very similar to the relation between linear algebra and matrix calculus. While these two theories are basically equivalent, clearly linear algebra is much more intuitive. So I expect that eventually the stacky approach will become a standard tool in representation theory.

### 4.2.1 Equivalence between groups and connected groupoids.

The group $G$ corresponding to a connected groupoid $\mathcal{Y}$ is not defined canonically. It depends on a choice of an object $Y \in \mathcal{Y}$.

Namely, given an object $Y$ we can define a group $G = G_Y$ by $G := Aut(Y)$. Then we get canonical equivalence of categories $Q = Q_Y : \mathcal{Y} \to BG$, defined by $Q(X) = Mor(Y, X)$.

If we pick another object $Y'$ we get a different group $G'$ and a different equivalence $Q' : \mathcal{Y} \to BG'$.

Note that any choice of an isomorphism $\nu : Y \to Y'$ defines natural isomorphisms $G \simeq G'$ and $Q \simeq Q'$. However there is no preferred choice for such an isomorphism $\nu$.

## 4.3 Examples of groupoids.

The next three constructions show that in Mathematics we usually encounter groupoids and not groups.

### 4.3.1 Multiplicative groupoid of a category.

**Construction I.** Starting with any category $C$ we construct the **multiplicative groupoid** $C^* = Iso(C)$ that has the same collection of objects as category $C$ and isomorphisms of $C$ as morphisms.

**Example 1.** $C = Finsets$ – the category of finite sets.

In this case the groupoid $Iso(C)$ is essentially the collection of all symmetric groups $S_n$.

**Example 2.** $C = Vect_k$ – the category of finite dimensional vector spaces over a field $k$.

In this case the groupoid $Iso(C)$ describes the collection of groups $GL(n, k)$ for all $n$.

### 4.3.2 Poincaré groupoid.

**Construction II.** Poincaré groupoid $Poin(X)$ of a topological space $X$.

Objects of $Poin(X)$ are points of $X$. Morphisms from $x$ to $y$ are homotopy classes of paths from $x$ to $y$.

If the space $X$ is path connected then the groupoid $Poin(X)$ is connected. For any point $x \in X$ the group $Aut_{Poin(X)}(x)$ is the fundamental group $\pi_1(X, x)$.

This shows that the Poincaré groupoid is more basic notion than the fundamental group.

### 4.3.3 Galois groupoid.

**Construction III.** Galois groupoid $Gal(F)$ of a field $F$.

Objects of the groupoid $Gal(F)$ are field extensions $F \to \Omega$ such that $\Omega$ is an algebraic closure of $F$. Morphisms are isomorphisms of field extensions.

The groupoid $Gal(F)$ is connected. If we fix an algebraic closure $\Omega$ then by definition the group $Aut_{Gal(F)}(\Omega)$ is the absolute Galois group $Gal(\Omega/F)$.

Again we see that the notion of Galois groupoid is more basic than the notion of Galois group.

Note that the constructions of Poincaré and Galois groupoids are very similar.

# 5 What is a stack ?

Let us fix some site $\mathcal{S}$. I would like to describe the notion of a stack $\mathcal{X}$ modeled on $\mathcal{S}$. I assume two features of this notion.

1. For every two stacks $\mathcal{X}, \mathcal{Y}$ the collection of morphisms from $\mathcal{X}$ to $\mathcal{Y}$ forms a groupoid $Mor(\mathcal{X}, \mathcal{Y})$.
2. Every object $S \in \mathcal{S}$ is a stack.

The natural idea is to characterize a stack $\mathcal{X}$ by the collection of groupoids $\mathcal{X}(S) := Mor(S, \mathcal{X})$ for all objects $S \in \mathcal{S}$.

In fact usually it is enough to know the groupoids $\mathcal{X}(S)$ for objects $S$ in some subcategory $\mathcal{B} \subset \mathcal{S}$ provided it is large enough. For example, if $\mathcal{S}$ is the category of schemes we can restrict everything to the subcategory $\mathcal{B}$ of affine schemes.

## 5.1 Informal technical definition of a stack.

Fix a large subcategory $\mathcal{B} \subset \mathcal{S}$. We define a stack $\mathcal{X}$ over the site $\mathcal{S}$ to be the following collection of data:

(i) To every object $S \in \mathcal{B}$ we assign a groupoid $\mathcal{X}(S)$

(ii) To every morphism $\nu : S \to S'$ in $\mathcal{B}$ we assign a functor $\mathcal{X}(S') \to \mathcal{X}(S)$

(iii) To every composition of morphisms in $\mathcal{B}$ we assign an isomorphism of the corresponding functors.

This data should satisfy a variety of compatibility conditions . These include

(i) Compatibility conditions for isomorphisms we have chosen.

(ii) Descent properties for morphisms and for objects with respect to the Grothendieck topology on the site $\mathcal{S}$.

(iii) Some finiteness conditions.

(iv) We also usually assume that the stack $\mathcal{X}$ is dominated by some object $Z \in \mathcal{S}$ (for example the quotient stack $Z/G$ that we will describe bellow is dominated by an object $Z$).

A relatively elementary exposition of stacks one can find in [Fan]. More detailed and more sophisticated exposition see in [Vis].

## 5.2 Stacks modeled on the site $Sets$

Consider the site $\mathcal{S} = Sets$. To describe a stack $\mathcal{Y}$ modeled on $\mathcal{S}$ we have to assign to every set $S \in \mathcal{S}$ a groupoid $\mathcal{Y}(S)$.

**Example.** Given a group $G \in \mathcal{S}$ we construct the basic stack $BG$ as follows:

For any set $S$ the objects of the groupoid $BG(S)$ are principle $G$-bundles $P$ over $S$ and morphisms are $G$-morphisms over $S$.

Let me remind that principle $G$-bundle over $S$ is a $G$-set $P$ equipped with a $G$-morphism $p : P \to S$, where $G$ acts trivially on $S$, that locally on $S$ is isomorphic to a trivial $G$-bundle $pr : G \times U \to U$.

More generally, if $Z$ is a $G$-space we define the quotient stack $BG(Z)$ as follows:

For every set $S$ the objects of the groupoid $BG(Z)(S)$ are principle $G$-bundles over $S$ equipped with a $G$-morphism to $Z$ and morphisms are morphisms of $G$-bundles over $Z$.

Note that in case of sets we can restrict everything to a subcategory $B \subset \mathcal{S}$ that contains just one object $pt$ – this is a big enough subcategory. This shows that every stack $\mathcal{Y}$ modeled on the site $\mathcal{S}$ can be completely described by the groupoid $\mathcal{X} = \mathcal{Y}(pt)$. This explains why the stacks in this case can be considered as groupoids.

## 5.3 Stacks modeled on the site $S$ of $l$-spaces.

Consider the site $\mathcal{S}$ of $l$-spaces. Let $G \in \mathcal{S}$ be an $l$-group. We define the basic stack $BG$ as follows:

For every $S \in \mathcal{S}$ the objects of the groupoid $BG(S)$ are principal $G$-bundles over $S$ and morphisms are morphisms of $G$-bundles. In this case again a $G$-bundle is a morphism of $G$-spaces $p : P \to S$ that locally in $S$ is isomorphic to the trivial $G$-bundle $pr : G \times U \to U$.

Similarly one defines a stack $BG(Z) = Z/G$ for a $G$-space $Z$.

In case of the site of $l$-spaces we can restrict everything to the subcategory $B \subset \mathcal{S}$ consisting of compact spaces. Using this fact it is easy to check that the basic stack $BG$ can be described in terms of a topological groupoid $\mathcal{X} = BG(pt)$.

Up to equivalence this groupoid can be explicitly described as follows: $\mathcal{X}$ has one object $X$ and its automorphism group $Aut(X)$ is the topological group $G$.

## 5.4 Stacks modeled on the site of $F$-schemes.

### 5.4.1 Basic stack $B\mathcal{G}$.

Let us fix a field $F$ and consider the site $\mathcal{S}$ of schemes over $F$. Let $\mathcal{G} \in \mathcal{S}$ be an algebraic group defined over $F$. We define the basic stack $B\mathcal{G}$ in the same way as before.

Namely, for an affine $F$-scheme $S$ an object of the groupoid $B\mathcal{G}(S)$ is a principal $G$-bundle $P$ over $S$. Morphisms are morphisms of $G$-bundles.

In this case the notion of a principal $G$-bundle is more subtle. Namely, we should consider morphisms $p : P \to S$ of $G$-schemes that should be locally trivial in etale topology. This is more sophisticated notion.

### 5.4.2 Torsors.

Let us consider in more detail the case of a point, i.e. assume that $S = Spec(F)$. In this case we have to consider a morphism $p : T \to Spec(F)$, where $T$ is an $F$-scheme with an action of the group $G$ that is locally trivial in etale topology (such $T$ is called a $G$**-torsor**).

The condition of local triviality is equivalent to the statement that after the base change from $S = Spec(F)$ to $S' = Spec(F')$, where $F'$ is an algebraic closure of $F$, the torsor $T'$ will be isomorphic as a $G'$-space to the trivial $G'$ torsor $G'$.

Here is a typical example of torsors.

**Example.** Let $K$ be a finite separable field extension of $F$. Multiplicative groups $K^*$ and $F^*$ we consider as $F$-points of algebraic groups $\mathcal{K}^*, \mathcal{F}^*$ defined over $F$. Then we have the norm morphism of algebraic groups $N : \mathcal{K}^* \to \mathcal{F}^*$. We denote by $G$ the kernel of this morphism. This is an algebraic group defined over $F$.

For any point $a \in F^*$ we consider the variety $\mathcal{T}_a = N^{-1}(a) \subset \mathcal{K}^*$. This variety is defined over $F$ and is a $G$-torsor.

Two torsors $\mathcal{T}_a, \mathcal{T}_b$ are isomorphic iff the quotient $a/b$ lies in the image of $K^*$ under the norm map $N : K^* \to F^*$.

### 5.4.3 Quotient stack $B\mathcal{G}(\mathcal{Z})$.

Let $\mathcal{G}$ act on a scheme $\mathcal{Z} \in \mathcal{S}$. We define the quotient stack $\mathcal{X} = B\mathcal{G}(\mathcal{Z}) = \mathcal{Z}/\mathcal{G}$ as follows:

For an affine scheme $S \in \mathcal{S}$ the category $B\mathcal{G}(Z)(S)$ consists of principal $G$-bundles $p : P \to S$ equipped with a $G$-morphism $P \to Z$ and morphisms are morphisms of principal $G$-bundles over $Z$.

# 6 Sheaves on stacks.

## 6.1 Technical definition of sheaves on stacks.

Let us fix a site $\mathcal{S}$. In situations we consider we have the notion of sheaves (of complex vector spaces) on spaces $Z \in \mathcal{S}$. In other word, to every space $Z \in \mathcal{S}$ we assign a category $Sh(Z)$, every morphism $\nu : Z \to W$ in $\mathcal{S}$ induces a functor $\nu^* : Sh(W) \to Sh(Z)$ with a compatibility isomorphisms for the products of two morphisms.

**Basic examples. 1**. Let $S$ be the site of $l$-spaces. To every space $Z \in \mathcal{S}$ we assign the category $Sh(Z)$ of $l$-sheaves (i.e sheaves of complex vector spaces) on $Z$; the functor $\nu^*$ is the usual pullback functor.

**2**. Fix a local non-Archimedean field $F$ and consider the site $\mathcal{S}$ of $F$-schemes of finite type. For every scheme $Z \in \mathcal{S}$ we consider the set $Z(F)$ of its $F$-points as an $l$-space and we set $Sh(Z) := Sh(Z(F))$ (i.e. a sheaf $R$ on $Z$ is just a sheaf of complex vector spaces on the $l$-space $Z(F)$).

We would like to extend these categories of sheaves to stacks modeled on the site $\mathcal{S}$. In other words, given a stack $\mathcal{X}$ we would like in some natural way to assign to it a category $Sh(\mathcal{X})$ so that for a space $Z \in \mathcal{S}$ considered as a stack it will be the category $Sh(Z)$. Let me indicate some general strategy how to construct the category $Sh(\mathcal{X})$.

Suppose we already have some notion of sheaves on stacks. Fix a sheaf $R$ on a stack $\mathcal{X}$. Then for any space $S \in \mathcal{S}$ and any point $p \in \mathcal{X}(S) = Mor(S, \mathcal{X})$ we get a sheaf $R_p = p^*(F) \in Sh(S)$. We also get a family of isomorphisms connecting these sheaves. Now we can try to use these sheaves and isomorphisms to characterize the sheaf $R$ as follows.

**Informal definition**. A sheaf $R$ on the stack $\mathcal{X}$ is a collection of sheaves $R_p \in Sh(S)$ for all spaces $S \in \mathcal{S}$ and all morphisms $p : S \to \mathcal{X}$ and a collection of isomorphisms satisfying correct compatibility relations.

**Claim.1**. Let $\mathcal{S}$ be the site of sets, $\mathcal{Y}$ a stack modeled on $\mathcal{S}$. Then the category $Sh(\mathcal{Y})$ is naturally equivalent to the category $Sh(\mathcal{X}) := Funct(\mathcal{X}, Vect)$, where $\mathcal{X}$ is the groupoid $\mathcal{Y}(pt)$ that characterizes stack $\mathcal{Y}$.

**2**. Let $\mathcal{S}$ be the site of $l$-spaces. Consider an action of an $l$-group $G$ on an $l$-space $Z$ and denote by $BG(Z)$ the quotient stack, Then the category $Sh(BG(Z))$ is naturally equivalent to the category $Sh_G(Z)$ of $G$-equivariant sheaves on $Z$.

**Remark**. In some cases the pullback functor is defined only partially (for some good morphisms). This happens, for example, if we consider the site of manifolds and sheaves of distributions on them. In this case pullback functor is well defined only for submersions.

In these cases we often can define appropriate notion of a sheaf for a stack $\mathcal{X}$ that is dominated by an object $Z \in \mathcal{S}$, i.e. has a morphism $p : Z \to \mathcal{X}$, provided that this morphism $p$ is good. The quotient stacks that we are interested in usually have this property.

## 6.2 How to describe sheaves on an algebraic stack.

Let $\mathcal{X}$ be an algebraic stack over $F$. I would like to give a convenient description of sheaves on the stack $\mathcal{X}$ in terms of equivariant sheaves. I will do this for the case of a quotient stack $\mathcal{X} \approx \mathcal{Z}/\mathcal{G}$.

Let $T_1, ...T_r$ be representatives of isomorphisms classes of $\mathcal{G}$-torsors. They are described by elements of $H^1(Gal(F), \mathcal{G}(\bar{F}))$ (for simplicity we assume that this set is finite; this is always the case when $char(F) = 0$).

For every index $i$ consider the group $\mathcal{G}_i = Aut_{\mathcal{G}}(T_i)$ – this gives us the collection of all pure inner forms of the group $G$. Also we consider a $\mathcal{G}_i$-scheme $\mathcal{Z}_i = Mor_{\mathcal{G}}(T_i, \mathcal{Z})$.

Passing to $F$ points we construct a topological group $G_i = \mathcal{G}_i(F)$ and a topological $G_i$-space $Z_i = \mathcal{Z}_i(F)$.

**Claim**. The category $Sh(\mathcal{X})$ of sheaves on the stack $\mathcal{X}$ is naturally equivalent to $\prod Sh_{G_i}(Z_i)$ (product of categories of equivariant sheaves).

In particular we see that the collection of simple objects of the category $Sh(\mathcal{X})$ is a disjoint union of collections of simple $G_i$-equivariant sheaves on $Z_i$.

**Remark**. In case when $\mathcal{Z} = pt$ we see that $Irr(\mathcal{M}(\mathcal{G}, F)) = \coprod Irr(G_i)$. This means that the set $Irr(\mathcal{M}(\mathcal{G}, F))$ of equivalence classes of irreducible stacky $G$-modules can be naturally described as the disjoint union $\coprod Irr(G_i)$ taken over all pure inner forms $\mathcal{G}_i$ of the group $\mathcal{G}$.

### 6.3 Reduction to the case of the group $GL(n)$.

Let me present one more description of the category of sheaves on a quotient stack $\mathcal{X} = \mathcal{Z}/\mathcal{G}$ that is often convenient in computations.

**Construction**. Suppose that $\mathcal{G}$ is a linear algebraic group. Then we can imbed it into a group $\mathcal{P}$ isomorphic to $GL(n)$.

Using this we can realize our quotient stack $\mathcal{X} = \mathcal{Z}/\mathcal{G}$ as the quotient stack $\mathcal{W}/\mathcal{P}$, where $\mathcal{W} = \mathcal{P} \times_{\mathcal{G}} \mathcal{Z}$.

The group $\mathcal{P}$ has only one pure inner form (this is Hilbert 90 theorem). This implies that the category $Sh(\mathcal{X})$ can be realized as the category $Sh_P(W)$ of $P$-equivariant sheaves on $W$, where $P = \mathcal{P}(F)$ and $W = \mathcal{W}(F)$.

**Remark**. The example in section 3 is a special case of this construction.

## 7 Algebro-geometric structure of the groupoids $\mathcal{X}(F)$.

Let $\mathcal{X}$ be an algebraic stack. Am important role in the study of sheaves on the stack $\mathcal{X}$ plays the fact that the collection of groupoids $\mathcal{X}(F)$ for different fields $F$ has an algebro-geometric structure.

**Proposition**. Let $L \supset F$ be a finite Galois field extension and $\Gamma = Gal(L/F)$ its Galois group. Then the group $\Gamma$ acts on the groupoid $\mathcal{X}(L)$ and the **fixed point groupoid** $\mathcal{X}(L)^\Gamma$ is naturally equivalent to the groupoid $\mathcal{X}(F)$.

Here the fixed point groupoid $\mathcal{X}(L)^\Gamma$ is defined in a standard categorical manner. Its object is an object $X$ of the groupoid $\mathcal{X}(L)$ equipped with a collection of isomorphisms $\alpha_\gamma : X \to \gamma(X)$ satisfying natural compatibility conditions.

This proposition is just a reformulation of the descent property for the stack $\mathcal{X}$.

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