P-INVARIANT DISTRIBUTIONS ON GL(N) AND THE CLASSIFICATION OF UNITARY REPRESENTATIONS OF GL(N) (NON-ARCHIMEDEAN CASE)

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§0. INTRODUCTION

P-invariant Pairings

0.1. Let F be a non-archimedean local field, $G = GL(n,F)$, and $P \subset G$ the subgroup of all matrices with the last row equal to $(0,0,\ldots,0,1)$. Many results about representations of G were obtained by studying their restrictions to P (see [GK], [BZ1], [BZ2], [Z1]). In this paper we prove the following important technical result which clarifies the relations between representations of G and their restrictions to P.

<u>Theorem</u> A (see 5.1). Let (π,E) be a smooth irreducible representation of G in a (complex) vector space E, $\tilde{\pi} = (\tilde{\pi},\tilde{E})$ the contragredient representation. Then each P-invariant pairing $B: E \times \tilde{E} \to \mathbb{C}$ is proportional to the standard pairing.

0.2. Theorem A implies the following

<u>Theorem</u> (see 5.4). Each irreducible unitary representation of G remains irreducible when restricted to P.

H. Jacquet noticed that this theorem implies the following result about representations of G.

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<u>Corollary</u>. Any representation of G parabolically induced from an irreducible unitary representation of a Levi subgroup of G is always irreducible. In other words, in the case of GL all R-groups are trivial.

Jacquet's proof uses the explicit description, in terms of Mackey's construction, of the restriction of an induced representation to P. We give another proof in 8.2.

0.3. Using theorem 0.1 we prove that any nondegenerate unitarizable irreducible representation (π, E) of G is generic, i.e. the scalar product in E can be written as a standard integral in the Kirillov model of π (see 6.2).

In §6 we generalize this result to nonunitarizable representations. Namely, we prove that the scalar product between an irreducible nondegenerate G-module E and its contragredient $\tilde{E}$ can be written via an integral in their Kirillov models. (This integral does not converge, but there exists a natural regularization procedure for its evaluation, see 6.3-6.4.) This result gives an alternative proof of the uniqueness and the injectivity of the Kirillov model (see 6.5).

In the case of a degenerate irreducible representation (π, E) A. Zelevinsky described in [Z1, §8] a degenerate Kirillov model. If π is unitarizable, we also can write the scalar product in E via an integral (see 7.4, remark). If we had a regularization procedure for a degenerate Kirillov model we would prove an analogous result for any π.

An Algorithm for the Classification of Unitary Representations of GL(n)

0.4. Using theorem A we establish a unitarizability criterion for irreducible G-modules (see 7.4). It claims that an irreducible representation (π, E) of $GL(n)$ is unitarizable iff it is Hermitian and its derivatives $\pi^{(k)}$ satisfy some inequalities (these derivatives $\pi^{(k)}$ are representations of the groups $GL(m)$ with $m < n$, which describe the restriction of π to P, see 7.2).

This criterion gives an algorithm for the classification of irreducible unitary representations of $G = GL(n)$. More precisely, let us start from some classification of irreducible smooth representations of G (we use Zelevinsky's classification, which is based on the detailed study of derivatives of representations of G, see 7.5-7.8). Moreover, suppose we know the multiplicity matrix $m = (m_{ab})$, which describes the decomposition in the Grothendieck group of induced representations into irreducible ones. In terms of this matrix we can calculate all the derivatives for all irreducible representations of G. Now, using the unitarizability criterion, we can identify those irreducible representations of G which are unitarizable (see 7.9).

0.5. In [Z2] A. Zelevinsky described some polynomials $P_{ab}(q)$, analogous to the Kazhdan-Lusztig polynomials, and conjectured that $m_{ab} = P_{ab}(1)$. Later he proved that these polynomials can be expressed in terms of usual Kazhdan-Lusztig polynomials for symmetric groups (not published). Hence, if we believe Zelevinsky's conjecture, we have explicit formulae for m_{ab} in terms of Kazhdan-Lusztig polynomials, i.e. our algorithm becomes quite precise. This leads to a very interesting question about complexity of the set of unitarizable representations. The problem is that Kazhdan-Lusztig polynomials are given by some recursive formulae and apparently there are no explicit formulae for them. Thus it might happen that the description of unitarizable representations can not be given by explicit formulae and only by some inductive procedure. But maybe for the description of unitary representations we do not need the whole complexity of the Kazhdan-Lusztig polynomials (I even do not rule out the possibility that they can be described by simple-minded methods like those in section 8 without using Zelevinsky's conjecture). Then we can suppose that the classification of irreducible unitary representations for any reductive group ($\mathfrak{p}$-adic or real) can be given by reasonably explicit formulae, since the groups GL(n) are more simple but not much more simple than other groups.

In any case, algorithm 0.4 together with Zelevinsky's conjecture reduces this question to a pure combinatorial problem.

0.6. Our proof of theorem A is based on the following geometrical statement.

Theorem B. Any distribution E on G invariant under the adjoint action of the subgroup P is automatically invariant under the adjoint action of the whole group G.

We prove the implication Theorem B $\Rightarrow$ Theorem A using the technique of Gelfand-Kazhdan (see [GK]). Also the proof of theorem B is reminiscent of the proof in [GK]. But there is one essential difference - unlike [GK] we can not consider each G-orbit separately, since there exist G-orbits which have Ad(P)-invariant but not Ad(G)-invariant distributions*). Theorem B means that these distributions can not be extended from these orbits to the whole group G as Ad(P)-invariant distributions. In order to prove this we use the Fourier transform.

0.7. Let me illustrate the method of the proof of theorem B in the case of the group GL(2).

First of all, applying the localization principle 1.4, which is a formalization of Gelfand-Kazhdan's method, we can assume that E is concentrated on the closure of one G-orbit $\mathcal{O}_x = \mathrm{Ad}(G)x$. It is easy to check that a P-invariant distribution E on $\mathcal{O}_x$ corresponds to a distribution E' on the space $P\backslash G \simeq F^2 \setminus 0$, which is quasiinvariant under the action of the centralizer G_x of x in G, i.e. $\delta(g)E' = \nu(g)E'$, where $g \in G_x$, $\nu(g) = |\det g|$.

If x is semisimple, the distribution E' is proportional to

*) This is the reason why theorems A and B are false for a finite field F.

the Haar measure on F^2, i.e. the corresponding distribution E is G-invariant. But if x is unipotent, there exists a quasiinvariant distribution E', concentrated on a line in F^2. The corresponding distribution E is concentrated on the unipotent subgroup of P_2, which we identify with the affine line F. E is defined only on the subset $F \setminus 0$ of nontrival unipotent elements and is invariant under the action of the multiplicative group F^*. We claim that it can not be extended to an F^*-invariant distribution on F.

Indeed, for any F^*-invariant distribution Q on F, its Fourier transform $\hat{Q}$ is quasiinvariant. Using this it is easy to check that $\hat{Q}$ is proportional to a Haar measure on $\check{F}$, and hence Q is concentrated at 0.

In the general case, for $GL(n)$ with $n > 2$, the proof is analogous up to the last statement. This statement - that $\hat{Q}$ is proportional to a Haar measure - we deduce from theorem B for a group $GL(m)$ with $m < n$. This finishes the proof.

0.8. Let me give a brief description of the contents of the paper.

In chapter I (sections 1-4) we study invariant distributions. Section 1 contains a brief review of general properties of distributions. In section 2 we formulate theorem B and give several equivalent reformations which we use in the inductive proof. Section 3 contains the proof of the theorem. Some technical details, including the existence of orbital integrals in positive characteristic, are proved in section 4.

Chapter II (sections 5-9) gives applications to representation theory. Section 5 contains proofs of theorem A and related theorems A', A", and the proof of theorem 0.2. In section 6 we discuss corollaries of theorem A for Kirillov models. Section 7 describes an algorithm for the classification of irreducible unitary representations. In 7.1-7.4 we prove a unitarizability criterion for G-modules. In 7.5-7.9 we recall Zelevinsky's classification and formulate the algorithm.

In section 8 we discuss some miscellaneous results about irreducibility and unitarizability of G-modules. In 8.1-8.2 we prove some irreducibility criteria, based on unitarizability. In 8.3-8.7 we show that the algorithm describing unitary representations works essentially with discrete data. In other words, we show how to handle complementary series. In particular in 8.7 we establish some nice inequalities for unitarizable representations which are stronger than the inequalities in the unitarizability criterion 7.4.

In 8.8-8.9 we consider two examples of applications of the algorithm 7.9. Example 8.9 gives the classification of nondegenerate unitary representations of G. In 8.10 we formulate a conjecture that duality preserves unitarizability.

In section 9 we prove the unitarizability criterion 7.3 for P-modules which we use in section 7.

0.9. This paper arose from an attempt to answer the question by H. Jacquet and T. Shalika, whether each nondegenerate unitary irreducible representation is generic (i.e. is topologically irreducible when restricted to P). Relatively soon I understood that this can be proved using methods of [BZ2]. But these methods, even combined with [Z1], do not allow us to prove an analogous statement for the degenerate case (see 0.2). The only way to prove it which I see is to use theorems B and A.

Only much later I realized that the most interesting application of theorem A is an algorithm for the classification of unitary representations. I think that theorems A and B, criterion 7.4 and, with some modifications, algorithm 7.9 remain true for an archimedean field F. I even almost have a proof and I hope to overcome some technical problems which appear in the proof.

Remark. In [K] A.A. Kirillov tried to prove theorem 0.2 for the

Archimedean case, using essentially the same ideas. But his proof was incorrect and his means were absolutely insufficient for the proof.

0.10. I thank I.I. Piatetski-Shapiro, who turned my attention to the problem. I am very grateful to A. Zelevinsky for numerous fruitful discussions of representations of $\mathfrak{p}$-adic groups.

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CHAPTER I

THEOREMS ON INVARIANT DISTRIBUTIONS

1. PRELIMINARIES: GENERAL PROPERTIES OF DISTRIBUTIONS (SEE [BZ1, §1, 6])

Distributions On ℓ-Spaces

1.1. Let X be an ℓ-space, i.e. a Hausdorff topological space which has a basis consisting of open compact subsets. Denote by $S(X)$ the Schwartz space of X, i.e. the space of locally constant functions $f: X \to \mathbf{C}$ of compact support. Any linear functional E on $S(X)$ is called a distribution on X. We consider the weak topology on the space $S^*(X)$ of distributions on X.

Recall that the weak topology on the algebraic dual E^* of a vector space E is defined as the weakest topology, compatible with the linear structure, such that the set $e^\perp = \{ e^* \in E^* | \langle e^*, e\rangle = 0\}$ is open for each $e \in E$. For any linear subspace $L \subset F$ its orthogonal complement $L^\perp$ is closed and $(L^\perp)^\perp = L$. For any linear subspace $W \subset E^*$ the space $(W^\perp)^\perp$ coincides with the closure of W.

1.2. Let Z be a closed subset of X, $U = X\setminus Z$. We have natural exact sequences (see [BZ1; §1]):

$$(*) \qquad 0 \to S(U) \to S(X) \to S(Z) \to 0$$

$$(**) \qquad 0 \to S^*(Z) \xrightarrow{i} S^*(X) \xrightarrow{res} S^*(U) \to 0$$

(i=extension by zero; res = restriction of distributions $E \to E|_U$).

For any distribution $E \in S^*(X)$ there exists a minimal closed subset $\operatorname{supp} E \subset X$, called the support of E, such that $E|_{X\setminus \operatorname{supp} E} = 0$.

Using (**) we will identify $S^*(Z)$ with the subspace $S^*_Z(X) \subset S^*(X)$ consisting of distributions supported on Z. In particular, if Y is a locally closed subset of X (i.e. Y is open in its closure $\bar{Y}$) and E is a distribution supported on $\bar{Y}$ we will define the restriction $E|_Y$ by

$$E|_Y = (E|_{\bar{Y}})|_Y .$$

1.3. Let G be an ℓ-group and $\gamma: G \times X \to X$ a (left) continuous action of G on X. We denote by the same symbol γ the (left) actions of G on $S(X)$ and $S^*(X)$ given by

$$(\gamma(g)f)(x) = f(\gamma(g^{-1})x), \ \langle \gamma(g)E, f\rangle = \langle E, \gamma(g^{-1})f\rangle,$$
$$g \in G,\ x \in X,\ f \in S(X),\ E \in S^*(X).$$

Let χ be a character of G, i.e. a locally constant homomorphism $\chi: G \to \mathbb{C}^*$. We call a distribution $E \in S^*(X)$ χ-invariant under the action of G (or (G,χ)-invariant) if $\gamma(g)E = \chi(g)E$ for all $g \in G$. The space of (G,χ)-invariant distributions we denote by $S^*(X)^{G,\chi}$ (or simply $S^*(X)^G$ if $\chi = 1$).

Localization Principle

1.4. Let $q: X \to T$ be a continuous map of ℓ-spaces. Then $S(X)$ and hence $S^*(X)$ become $S(T)$-modules. For any $t \in T$ consider the fiber $X_t = q^{-1}(t)$ and identify the space $S^*(X_t)$ with the subspace $S^*_{X_t}(X) \subset S^*(X)$ of distributions concentrated on this fiber.

<u>Localization principle</u>. Let W be a closed subspace of $S^*(X)$ which is an $S(T)$-submodule. Then W is generated by distributions concentrated on fibers, i.e. the sum of subspaces $W^t = W \cap S^*(X_t)$, $t \in T$, is dense in W.

The following corollary is crucial for our proof.

<u>Corollary</u>. Let an ℓ-group G act on the space X preserving each fiber X_t, and let P be a subgroup of G. Suppose that for each $t \in T$ all P-invariant distributions on X_t are G-invariant, i.e. $S^*(X_t)^P = S^*(X_t)^G$. Then any P-invariant distribution on X

is G-invariant, i.e. $S^*(X)^P = S^*(X)^G$.

Indeed, $S^*(X)^P$ is a closed $S(T)$- submodule of $S^*(X)$ and hence it is generated by subspaces $S^*(X_t)^P$. Since $S^*(X_t)^P = S^*(X_t)^G \subset S^*(X)^G$ we have $S^*(X)^P \subset S^*(X)^G$, i.e. $S^*(X)^P = S^*(X)^G$.

Proof of localization principle. Let M be an $S(T)$-module. We say that M is unital if $M = S(T)\cdot M$. For any point $t \in T$ put

$$J_t = \{ f \in S(T) | f(t) = 0\}, \quad M_t = M/J_t \cdot M .$$

The space M_t is called the fiber of M at the point t. For any $m \in M$ we denote by m_t its image in M_t.

Lemma (see the proof in [BZ1; 1.13, 1.14, 2.36])

(i) Subquotients of a unital $S(T)$-module are unital. The functor $M \to M_t$ is exact. If $m \in M$ and $m \neq 0$ then for some point $t \in T$ $m_t \neq 0$.

(ii) $S(X)$ is a unital $S(T)$-module and the natural morphism $S(X)_t \to S(X_t)$ is an isomorphism.

We will prove the following result:

(*) Let M be a unital $S(T)$-module and $W \subset M^*$ be a closed $S(T)$-submodule. Then W is generated by subspaces $W^t = W \cap (M_t)^*$ for $t \in T$. The localization principle is a particular case of (*) for $M = S(X)$, since $S(X)_t = S(X_t)$.

Put $L = W^{\perp} \subset M$, $N = M/L$. It is clear that L and N are $S(T)$-modules. Since W is closed it is isomorphic to N^*. Moreover, for each t, $W^t = (N_t)^* \subset N^*$. Consider the space $W' = \oplus W^t = \oplus (N_t)^* \subset$ $\subset N^*$ and its orthogonal complement in N. If $n \in W'^{\perp}$ then for any t, $n \in (N_t)^{*\perp}$, i.e. $n_t = 0$. Statement (i) of the lemma implies that $n = 0$, i.e. $W'^{\perp} = 0$. Therefore the closure of W' coincides with $W'^{\perp\perp} = 0^{\perp} = N^* = W$. This proves (*).

Frobenius Reciprocity

1.5. Let an ℓ-group G act on an ℓ-space X and let χ be a character of G. Sometimes we can reduce the study of χ-invariant distributions on X to the study of distributions on a smaller space. Namely, suppose we could find a continuous G-equivariant map $p: X \to Z$, where Z is a homogeneous G-space. For simplicity assume that we have a quasiinvariant measure μ on Z, i.e. $\mu \in S^*(Z)^{G,\nu}$ for some character ν. Fix such a measure $\mu \neq 0$ and fix a point $z_0 \in Z$.

Put $X_0 = p^{-1}(z_0) \subset X$, $H = \mathrm{Stab}(z_0, G) \subset G$.

Lemma. There exists a canonical isomorphism $\Psi_\mu: S^*(X_0)^{H,\chi\nu^{-1}} \to S^*(X)^{G,\chi}$. If $E_0 \in S^*(X_0)^{H,\chi\nu^{-1}}$, then $\operatorname{supp} \Psi_\mu(E_0) = G \operatorname{supp}(E_0)$. The morphism Ψ_μ can be written explicitly:

$$\langle \Psi_\mu(E_0), f\rangle = \int_Z (\chi\nu^{-1})(g_z)\cdot\langle E_0, \gamma(g_z) f\rangle \, d\mu(z),$$

where $f \in S(X)$ and $g_z \in G$ is an element such that $g_z(z) = z_0$. This lemma is an easy consequence of Frobenius reciprocity (see [BZ1; 2.21-2.36]).

Remark 1. If Z does not have a quasiinvariant measure one can nevertheless prove an analogue of the lemma. Namely, consider the character $\nu = \Delta_G|_H \cdot \Delta_H^{-1}$ of the group H (here Δ is the module of a group). Then there exists an isomorphism

$$\Psi: \quad S^*(X_0)^{H,\chi\nu^{-1}} \to S^*(X)^{G,\chi}$$

(see [BZ1, 2.21-2.36]).

Remark 2. Frobenius reciprocity in particular implies that all G-invariant distributions on Z are proportional.

§2. REFORMULATIONS OF THE MAIN THEOREM

2.1. We fix a nonarchimedean local field F and put $G = G_n = GL(n,F)$,

$$P = P_n = \{ g = (g_{ij}) \in G_n | g_{ni} = \delta_{ni} \text{ for all } i\}.$$

Denote by Ad the adjoint action of G on itself.

Our aim is the following

<u>Theorem</u> B. Let E be a distribution on G invariant under the adjoint action of the subgroup P. Then it is invariant under the adjoint action of the whole group G.

Statements $X(n)$, $Y(n)$ and $Y^*(n)$

2.2. Let $X = X_n = Mat(n,F)$ be the algebra of $n \times n$ matrices. We define the adjoint action of G on X by $Ad(g)x = gxg^{-1}$. In the proof of the theorem we can assume that $\operatorname{supp} E \subset G$ is closed in X (for instance, we can multiply E by some locally constant compactly supported function of $\det(g)$). Hence we can consider E as a distribution on X. Therefore we should prove for each n the following

<u>Statement X(n)</u>. $S^*(X_n)^{P_n} = S^*(X_n)^{G_n}$.

We will prove the statement by induction on n. In the proof we will use some reformulations of $X(n)$, which are interesting by themselves.

2.3. Denote by $A = A_n$ the space $Mat(1,n;F)$ of row-vectors of length n and fix a standard basis $e_1,\dots,e_n$ in A. Let δ be the standard action of G on A given by $\delta(g)a = ag^{-1}$.

Denote by ν the character of the group $G = G_n$ given by $\nu(g) = |\det g|$, where $|\ |$ is the standard norm on the field F. Fix a Haar measure μ on A. It is clear that $\mu \in S^*(A)^{G,\nu}$.

Consider the ℓ-space $Y = Y_n = A_n \times X_n$ and the action $\gamma = \delta \times Ad$ of G on Y. The measure μ gives a canonical morphism $\mu: S^*(X) \to S^*(Y)$ by $\mu(E) = \mu \otimes E$. It is clear that $\mu(S^*(X)^G) \subset S^*(Y)^{G,\nu}$.

We claim that statement $X(n)$ implies (and in fact is equivalent to) the following.

Statement $Y(n)$. The morphism $\mu: S^*(X_n)^G \to S^*(Y_n)^{G,\nu}$ is an isomorphism.

Put $A' = A\setminus 0$, $Y' = A' \times X \subset Y$. Consider the morphism $\mu': S^*(X)^G \to S^*(Y')^{G,\nu}$, given by $\mu'(E) = \mu'\otimes E$, where $\mu' = \mu|_{A'}$. Since G acts transitively on A' and $\mathrm{Stab}(e_n,G)$ coincides with P_n, we can apply Frobenius reciprocity (see 1.5). It gives an isomorphism $\Psi_\mu: S^*(X)^P \simeq S^*(Y')^{G,\nu}$. The explicit formula for Ψ_μ given in 1.5 shows that $\mu' = \Psi_\mu \circ i$, where $i: S^*(X)^G \to S^*(X)^P$ is the natural imbedding. Hence statement $X(n)$ implies the following statement.

$Y'(n)$. $\mu': S^*(X)^G \to S^*(Y')^{G,\nu}$ is an isomorphism.

In order to prove the implication $X(n) \Rightarrow Y(n)$ it remains to prove that $S^*(Y')^{G,\nu} \approx S^*(Y)^{G,\nu}$. Since $Y' = Y\setminus X$, where $X = 0 \times X \subset Y$, we have an exact sequence $0 \to S^*(X) \to S^*(Y) \xrightarrow{res} S^*(Y') \to 0$ and hence the morphism $res: S^*(Y)^{G,\nu} \to S^*(Y')^{G,\nu}$.

Fix an element z in the center of G such that*) $\nu(z) \neq 1$ and define an endomorphism α of $S^*(Y)$ by $\alpha(E) = \gamma(z)E - E$. Since z acts trivially on X, $\gamma(z)$ is the identity on $S^*(X)$, i.e. $\alpha(S^*(X)) = 0$. Hence we can consider α as a morphism $\alpha: S^*(Y') \to S^*(Y)$. It is clear that on ν-invariant distributions operators $\alpha\circ res$ and $res\circ\alpha$ are multiplications by the nonzero constant $\nu(z) - 1$. Hence res gives an isomorphism $res: S^*(Y)^{G,\nu} \simeq S^*(Y')^{G,\nu}$.

2.4. Let $A^* = A_n^*$ be the dual space of $A = A_n$. It can be described as a space $\mathrm{Mat}(n,1;F)$ of column-vectors of length n. The action δ^* of G on A^* is given by $\delta^*(g)a^* = ga^*$.

Consider the ℓ-space $Y^* = Y_n^* = X_n \times A_n^*$ and the action $\gamma^* = \mathrm{Ad} \times \delta^*$ of G on Y^*. We identify X with a closed subset

*) This trick does not work for a finite field F.

$X \times 0 \subset Y^*$ and denote by $i: S^*(X) \to S^*(Y^*)$ the natural inclusion.

We claim that the statement $Y(n)$ implies (and in fact is equivalent to) the following

Statement $Y^*(n)$. The morphism $i: S^*(X_n)^{G_n} \to S^*(Y_n^*)^{G_n}$ is an isomorphism. In other words, each G-invariant distribution on Y^* is concentrated on X.

In order to prove this implication we fix a nontrivial additive character ψ of F and consider the Fourier transform $\Phi: S(Y) \to S(Y^*)$ given by $\Phi(f)(x,a^*) = \int_A f(a,x)\psi(\langle a^*,a\rangle)d\mu(a)$.

The usual theory of Fourier transform implies that Φ is an isomorphism. Let $\Phi^*: S^*(Y^*) \to S^*(Y)$ be the dual isomorphism. It is easy to check that Φ^* gives an isomorphism $S^*(Y^*)^G \to S^*(Y)^{G,\nu}$ and that the morphism $\Phi^* \circ i: S^*(X) \to S^*(Y^*) \to S^*(Y)$ coincides with the morphism μ, defined in 2.3. Hence the statement $Y(n)$ implies that $i: S^*(X)^G \to X^*(Y^*)^G$ is an isomorphism, i.e. the statement $Y^*(n)$.

§3. PROOF OF THE STATEMENT $X(n)$

The Geometric Structure of G- and P-orbits on X.

3.1 Consider the following invariants of the matrix $x \in X_n$:

t_x = characteristic polynomial of x (deg $t_x = n$)

K_x = [span of $e_n, e_n x, \ldots, e_n x^n$] $\subset A$

$k_x = \dim K_x$ = minimal k such that $e_n x^k$ is a linear combination of $e_n, e_n x, \ldots, e_n x^{k-1}$

τ_x = characteristic polynomial of the operator x on K_x (i.e. $\tau_x = \lambda^k + a_1\lambda^{k-1} + \ldots + a_k$, where $k = k_x$, and $e_n x^k + \Sigma a_i e_n x^{k-i} = 0$).

By definition τ_x is the minimal monic polynomial such that $e_n \tau_x(x) = 0$. We call the matrix x P-regular if $\tau_x(x) = 0$. It is clear, that t_x is constant along G-orbits, k_x and τ_x are constant along P-orbits. Besides, the function $x \mapsto k_x$ is upper semicontinuous.

Geometric lemma (see the proof in 4.1-4.2).

a) For any polynomial t the set $X_t = \{x \in X | t_x = t\}$ contains a finite number of G-orbits.

b) Each G-orbit $\mathcal{O}$ contains a finite number of P-orbits.

c) Each G-orbit $\mathcal{O}$ contains a unique P-orbit $\mathcal{O}_P$ open and dense in $\mathcal{O}$. Namely $\mathcal{O}_P \doteq \{x \in \mathcal{O} | x$ is P-regular$\}$.

Proof of the Statement X(n)

3.2. We will prove X(n) by induction on n, i.e. we assume X(m) to be true for $m < n$. We fix a P_n-invariant distribution $E \in S^*(X)^{P_n}$ and prove that E is G_n-invariant. Put $S = \operatorname{supp} E \subset X$.

Let T be the space of polynomials of degree n and $q: X \to T$ the characteristic map $q: x \mapsto t_x$. Using the localization principle 1.4 we can (and will) assume that $S \subset X_t$ for some t. Then by 3.1 a,b, S contains a finite number of P-orbits. We will proceed by induction on the number of P-orbits in S.

Key lemma. S contains an open P-orbit $\mathcal{O}_P$ which consists of P-regular elements.

Let us deduce X(n) from the lemma. Consider the G-orbit $\mathcal{O} = \operatorname{Ad}(G)\mathcal{O}_P$. Since $\mathcal{O}_P$ consists of P-regular elements it is open and dense in $\mathcal{O}$, i.e. $\overline{\mathcal{O}} = \overline{\mathcal{O}}_P \subset S$ (see 3.1c). We will use the following statement which we will prove in 4.3.

Statement. For any G-orbit $\mathcal{O} \subset X$ there exists a G-invariant distribution $\mu_{\mathcal{O}}$ such that $\operatorname{supp} \mu_{\mathcal{O}} = \overline{\mathcal{O}}$.

Consider the restrictions of the distributions E and $\mu_{\mathcal{O}}$ on the P-orbit $\mathcal{O}_P$ (this makes sence since $\mathcal{O}_P$ is open in $\operatorname{supp} E = S$ and $\operatorname{supp} \mu_{\mathcal{O}} = \overline{\mathcal{O}} \subset S$). They both are P-invariant and nonzero. Hence for some $c \in \mathbb{C}^*$, $E|_{\mathcal{O}_P} = c \cdot \mu_{\mathcal{O}}|_{\mathcal{O}_P}$ (see 1.5). This means that the distribution $E_o = E - c\mu_{\mathcal{O}}$ restricts to zero on $\mathcal{O}_P$. The distribution E_o is P-invariant and $\operatorname{supp} E_o \subset S \setminus \mathcal{O}_P$ contains strictly fewer

P-orbits than $S = \mathrm{Supp}\, E$. By induction E_o is G-invariant and therefore E is G-invariant.

3.3. Proof of the key lemma.

For each $i = 1, 2, \ldots, n$ put $X_i = \{x \in X \mid k_x = i\}$, $\overline{X}_i = \{x \in X \mid k_x \leq i\}$. The sets $\overline{X}_i$ are closed and X_i is open in $\overline{X}_i$.

Let k be the minimal index such that $S \subset \overline{X}_k$. Consider the distribution $E' = E|_{X_k}$ and put $S' = \mathrm{Supp}\, E' = S \cap X_k$. Then S' is a nonempty open subset of S. Since S' contains a finite number of P-orbits it contains an open P-orbit O_p. Hence it is sufficient to prove the following statement:

(*) Any P-invariant distribution E' on X_k has support $S' = \mathrm{supp}\, E'$ consisting of P-regular elements, provided it contains a finite number of P-orbits.

We can study P-invariant distributions on X_k using Frobenius reciprocity 1.5. Consider the natural map $\pi\colon X_k \to A^{k-1}$ where $A^{k-1} = \{(a_1,\ldots,a_{k-1}) \mid a_i \in A = \mathrm{Mat}(1,n;F)\}$ given by $\pi(x) = (e_n x, e_n x^2, \ldots, e_n x^{k-1})$. This map is P-equivariant and its image Z is an open subset of A^{k-1} given by $Z = \{(a_1,\ldots,a_{k-1}) \mid e_n, a_1, \ldots, a_{k-1}$ are linearly independent$\}$. Z is a homogeneous P-space and it has a quasiinvariant measure $\mu_Z = \mu^{k-1}$, which is ν^{k-1}-invariant with respect to P.

Put $z = (e_{n-1}, \ldots, e_{n-k+1}) \in Z$, $X = \pi^{-1}(z)$, $H = \mathrm{Stab}(z,P)$. Then by Frobenius reciprocity the distribution E' corresponds to an (H, ν^{1-k})-invariant distribution E'' on X such that $S' = \mathrm{Ad}(P)S''$, where $S'' = \mathrm{Supp}\, E''$. Hence we should prove:

(**) Any (H,ν^{1-k})-invariant distribution E'' on X has support S'' consisting of P-regular elements, provided it contains a finite number of H-orbits.

3.4. Let us describe H and X in detail. Put $m = n-k$ and let

us write the $n\times n$ matrix x in a block form $\begin{pmatrix} A_x & B_x \\ C_x & D_x \end{pmatrix}$, where A_x, B_x, C_x and D_x are matrices of sizes $m \times m$, $m \times k$, $k \times m$ and $k \times k$.
By definition

$$H = \{ x \in G_n | e_i x = e_i \text{ for } m < i \le n\} = \{ x \in G_n | C_x = 0,\ D_x = 1_k\}$$

$$X = \{ x \in X_n | e_i x = e_{i-1} \text{ for } m+1 < i \le n \text{ and } e_{m+1} x \in \operatorname{span}(e_n, \dots, e_{m+1})\}$$

$$= \{x \in X_n | C_x = 0 \text{ and } D_x \in W\},$$

where W is the set of $k \times k$ matrices of the form

$$\left(\begin{array}{c|c} ** \dots & * \\ \hline 1_{k-1} & 0 \end{array}\right).$$

Note that each matrix $w \in W$ is completely defined by its characteristic polynomial τ (the coefficients of the upper row coincide with minus coefficients of τ). We will denote it by w_τ.

The function $x \mapsto \tau_x$ is continuous on X (indeed τ_x is the characteristic polynomial of D_x) and constant on H-orbits. Since S'' consists of a finite number of H-orbits, τ_x assumes only a finite number of values on S''. Fix one of these values τ and put $X_\tau = \{ x \in X | \tau_x = \tau\} = \{ x \in X | D_x = w_\tau\}$. Then $X_\tau \cap S''$ is open in S'', so we can restrict E'' to X_τ. Hence in the proof of (**) we can (and will) assume that E'' is an (H,ν^{1-k})-invariant distribution on X_τ.

Put $U = V = \mathrm{Mat}(m,k;F)$. We identify X_τ with $X_m \times V$ by $(x,v) \to \begin{pmatrix} x, & v \\ 0, & \omega_\tau \end{pmatrix}$. For any $u \in U$ we denote by u the matrix $\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \in H$. All these matrices form a subgroup $U \subset H$. We identify the group G_m with a subgroup of H by $g \to \begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix}$. Then H is a semidirect product of G_m and U. The action of H on X_τ is given by

$$Ad(g)(x,\nu) = (gxg^{-1}, g\nu),\ g \in G_m,\ Ad(u)(x,\nu) = (x, \nu - xu + u\omega_\tau),$$

$u \in U$. Let $u_1,\ldots,u_k$ be the columns of the matrix $u \in U$. Put

$$U^+ = \{u \in U \mid u_1 = 0\}$$

$$X_\tau^+ = \{(x,\nu) \in X_\tau \mid \nu_1 = \nu_2 = \ldots = \nu_{k-1} = 0\}$$

$$X_\tau^o = \{(x,\nu) \in X_\tau \mid \nu = 0\}.$$

Lemma (see the proof in 3.6). The natural map $\kappa: U^+ \times X_\tau^+ \to X_\tau$, given by $(u,x) \mapsto Ad(u)x$ is a homeomorphism.

Using κ we will identify X_τ and $U^+ \times X_\tau^+$. Since U^+ acts only on the first factor in $U^+ \times X_\tau^+$ and E'' is U^+-invariant, it can be written as $E'' = \mu^+ \otimes E^+$, where μ^+ is a Haar measure on U^+ and $E^+ \in S^*(X_\tau^+)$. The measure μ^+ is ν^{1-k} invariant with respect to G_m, hence E^+ is G_m-invariant.

Now let us note that as G_m-spaces X_τ^+ is isomorphic to the space Y_m^*, introduced in 2.4. Since we have assumed that the statements $X(m)$ and hence $Y^*(m)$ are true, the support S^+ of the distribution E^+ is concentrated on $X_\tau^o = \{(x,0)\}$. Hence the subset S'' of X_τ satisfies the following conditions.

(i) S'' is $Ad(U)$-invariant

(ii) $S'' \subset Ad(U^+)X_\tau^o$.

3.5. Now let us prove that conditions (i), (ii) imply that S'' consists of P-regular elements, i.e. that for any $x \in S''$, $\tau(x) = 0$.

Consider the set $R = \tau(S'')$ and prove that $R = \{0\}$. Since the map $x \to \tau(x)$ commutes with the adjoint action we have

(i)' R is $Ad(U)$-invariant

(ii)' $R \subset Ad(U^+)X_m$, where $X_m = \begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix} \supset \tau(X_\tau^o)$.

(we use the fact that $\tau(\omega_\tau) = 0$).

The action of U on R is given by $Ad(u)(x,\nu) = (x,\nu + xu)$, where $(x,\nu) = \begin{pmatrix} x & \nu \\ 0 & 0 \end{pmatrix} \in R$. If $u \in U^+$, i.e. $u_1 = 0$, then $(xu)_1 = 0$. Hence (ii)' implies that $\nu_1 = 0$ for all $(x,\nu) \in R$. By (ii)' it is sufficient to prove that any element $(x,0) \in R$ is equal to 0.

Let $u \in U$. Then $Ad(u)(x,0) = (x,xu) \in R$ and, as we have proved, $(x\cdot u)_1 = 0$. Since it is true for all u we have $x = 0$, Q.E.D.

3.6. Proof of the lemma 3.4.

We have $\kappa(u,(x,\nu)) = (x,\nu')$, where $\nu' = \nu - xu + u w_\tau$. Let us write this for each column:

$$\nu_1' = u_2$$

$$\nu_2' = -xu_2 + u_3$$

$$\cdots$$

$$\nu_{k-1}' = -xu_{k-1} + u_k$$

$$\nu_k' = \nu_k - xu_k + \Sigma a_i u_i.$$

It is clear that for any $\nu_1',\ldots,\nu_k'$ there exist unique $u_2,\ldots,u_k$, ν_k which satisfy this system of equations.

§4. PROOFS OF SOME LEMMAS

Description of G-Orbits in X.

4.1. Let $C = F[\lambda]$ be the algebra of polynomials in one variable. Each element $x \in X$ defines on A a C-module structure by $\lambda \mapsto x$. This gives a one-to-one correspondence

$$\{\text{G-orbits on } X\} \leftrightarrow \{\text{n-dimensional C-modules } M \text{ up to isomorphism}\}.$$

The centralizer G_x of x in G corresponds to the group $Aut_C M$.

Fix a monic polynomial $t \in C$ and its decomposition

$t = \tau_1 \cdot \ldots \cdot \tau_r$, where τ_i are irreducible monic polynomials, not necessarily distinct. Let $x \in X$ be any matrix, annihilated by t, i.e. $t(x) = 0$. Define the t-invariant $\nu = \nu_x$ of x by $\nu = (\nu_1, \ldots, \nu_r)$, where $0 < \nu_1 \leq \nu_2 \leq \ldots \leq \nu_r = n$ are given by

$$\nu_i = \dim \operatorname{Ker}(\tau_1(x) \cdot \ldots \cdot \tau_i(x)).$$

Lemma. Let $x, y \in X$ be annihilated by t. Then they lie on the same G-orbit iff $\nu_x = \nu_y$.

Indeed, let M be the C-module corresponding to x. We can decompose $M \simeq \bigoplus_\alpha M_\alpha$, were $M_\alpha = C/(f_\alpha^{r(\alpha)})$, f_α are irreducible polynomials, $r(\alpha) > 0$. Since $t(x) = 0$, the polynomial f_α appears in the sequence $\tau_1, \ldots, \tau_r$ at least $r(\alpha)$ times. Denote by b_i the multiplicity of τ_i in the sequence $\tau_1, \tau_2, \ldots, \tau_i$. Then it is clear that $\nu_i - \nu_{i-1} = \dim_F(C/(\tau_i)) \cdot \#\{\alpha \mid f_\alpha = \tau_i,\ r(\alpha) \geq b_i\}$. It is easy to see that this formula enables us to reconstruct f_α and $r(\alpha)$, and hence the C-module M up to isomorphism, from the invariants $\nu_1, \ldots, \nu_r$. This proves the lemma. If t is a polynomial of degree n and $x \in X_t$, i.e. $t_x = t$, then $t(x) = 0$ and the lemma implies 3.1a).

Description of P-Orbits in X.

4.2. Consider a G-orbit $\mathcal{O} = \operatorname{Ad}(G)x$ in X and denote by M the corresponding G-module. We have $P\backslash\mathcal{O} \approx P\backslash G/G_x \approx (A\backslash 0)/G_x \approx (M\backslash 0)/\operatorname{Aut}_C M$ (as topological spaces). Hence we can reformulate 3.1b),c) as statements about $\operatorname{Aut}_C M$ orbits in $M\backslash 0$.

Decompose $M = \bigoplus M_\alpha$, $M_\alpha = C/(f_\alpha^{r(\alpha)})$. Assign to each vector $\xi = \Sigma\xi_\alpha \in M$ invariants $\mu_\alpha = \min\{i \mid f_\alpha(\lambda)^i \xi_\alpha = 0\}$. It is clear that these invariants completely determine ξ up to the action of $\operatorname{Aut}_C M$. This proves 3.1b).

Denote by t the minimal polynomial $t_x^{\min}$ of x, i.e. the polynomial of minimal degree such that $t(x) = 0$, and put $\bar{C} = C/(t)$.

Put $M^o = \{\xi \in M | \operatorname{Ann}(\xi,\overline{C}) = 0\} = \{\xi \in M | \operatorname{Ann}(\xi,C) = \operatorname{Ann}(M,C)\}$.

Statement 3.1c) geometrically means that:

(*) M^o is an open dense subset of M and the group $\operatorname{Aut}_C M$ acts transivively on M^o.

By definition $M \setminus M^o = \cup M_J$, where J runs through nonzero ideals of $\overline{C}$, $M_J = \operatorname{Ann}(J,M)$. Since there are finitely many different ideals J and for any $J \neq 0$ M_J is a proper linear subspace of M, M^o is open and dense in M.

Any vector $\xi \in M^o$ defines an inclusion $\overline{C} \xrightarrow{\sim} \overline{C}\xi \subset M$. Since $\overline{C}$ is an injective $\overline{C}$-module*), $\overline{C}\xi$ is a direct summand of M. The Krull-Schmidt theorem implies that any two vectors $\xi,\xi' \in M^o$ are conjugate under $\operatorname{Aut}_C M$. This prove 3.1c).

Existence of Orbital Integrals

4.3. Proof of the Statement 3.2

In case $\operatorname{char} F = 0$ a more general result was proved by Ranga Rao and P. Deligne (see [R]). Using specific properties of $GL(n)$ we will adjust the proof for arbitrary characteristic. Let $x \in X'$, $\mathcal{O} = \operatorname{Ad}(G)x$. We would like to prove the existence of G-invariant distribution $\mu_{\mathcal{O}}$ such that $\operatorname{supp} \mu_{\mathcal{O}} = \overline{\mathcal{O}}$.

Denote by t the minimal polynomial of x.

(i) Consider at first the case when t is irreducible. Then $\overline{C} = C/_{(t)}$ is a field and $G_x \approx GL(n/\dim \overline{C}, \overline{C})$ is a unimodular group. Hence on $\mathcal{O}$ there exists a Haar measure $\mu_{\mathcal{O}}$. Besides, in this case $\mathcal{O}$ is closed, since any element $y \in \overline{\mathcal{O}}$ satisfies the equation $t(y) = 0$ and by lemma 4.1 is conjugate to x.

(ii) Now consider a general case. Fix a decomposition $t = \tau_1 \cdot \tau_2 \cdot \ldots \cdot \tau_r$ as in 4.1 and consider the t-invariant $\nu = (\nu_1,\ldots,\nu_r)$ of x (see 4.1). Since t is the minimal polynomial of x we have

*) Indeed, consider the $\overline{C}$-module $L = \operatorname{Hom}_F(\overline{C},F)$. Since $\operatorname{Ann}(L,\overline{C}) = 0$ there exists an inclusion $\overline{C} \to L$. Since $\dim L = \dim \overline{C}$, this inclusion is an isomorphism. Hence L is a projective $\overline{C}$-module, i.e. $\overline{C}$ is an injective $\overline{C}$-module.

$0 < \nu_1 < \nu_2 < \ldots < \nu_r = n.$

By a ν-flag we mean a sequence $\phi = (L_1 \subset L_2 \subset \ldots \subset L_r)$ of subspaces such that $\dim L_i = \nu_i$. The set Φ_ν of all ν-flags is a compact topological space and the natural action of G on Φ_ν is transitive. To any matrix $y \in \mathcal{O}$ we assign a ν-flag $\phi_y = (L_1, \ldots L_r)$, where $L_i = \ker(\tau_1(y) \cdot \ldots \cdot \tau_i(y))$. Consider the space $\Sigma_\nu = X \times \Phi_\nu$ with the natural action of G and put $\mathcal{Q} = \{(y, \phi_y) \in \Sigma_\nu | y \in \mathcal{O}\}$. The natural projection $pr: \Sigma_\nu \to X$ is a proper map and hence it defines the morphism of distributions $pr_*: S^*(\Sigma_\nu) \to S^*(X)$ by $\langle pr_*(E), f\rangle = \langle E, pr^*(f)\rangle$. Therefore it is sufficient to construct a G-invariant distribution $\mu_{\mathcal{Q}}$ such that $\mathrm{Supp}(\mu_{\mathcal{Q}}) = \bar{\mathcal{Q}}$.

(iii) Consider the natural projection $pr_2: \Sigma_\nu \to \Phi_\nu$. In order to construct the G-invariant distribution $\mu_{\mathcal{Q}}$ we will use the Frobenius reciprocity (see 1.5).

Fix the ν-flag $\phi = \phi_x = (L_1 \subset \ldots \subset L_r)$ and put $P_\phi = \mathrm{Stab}(\phi, G)$, $\chi = \Delta_{P_\phi} \cdot \Delta_G^{-1} = \Delta_{P_\phi}$, $\mathcal{Q}_\phi = \{y \in \mathcal{O} | \phi_y = \phi\} = pr_2^{-1}(\phi) \cap \mathcal{O}$. According to 1.5, Remark 1, it is sufficient to construct a (P_ϕ, χ)-invariant distribution $\mu_\phi \in S^*(X)$ such that $\mathrm{Supp}(\mu_\phi) = \bar{\mathcal{Q}}_\phi$.

(iv) Put $\mathcal{P} = \{x \in X | xL_i \subset L_i \text{ for all } i\}$. It is clear that $P_\phi = \mathcal{P}^* = \mathcal{P} \cap G$ and $\mathcal{Q}_\phi \subset \mathcal{P}$. Put $\mathcal{U} = \{x \in X | xL_i \subset L_{i-1} \text{ for all } i\}$, $\mathcal{M} = \mathcal{P}/\mathcal{U}$. It is clear that $\mathcal{U}$ is a nilpotent two-sided ideal of $\mathcal{P}$ and the algebra $\mathcal{M}$ is isomorphic to $X_{m_1} \times \ldots \times X_{m_r}$, where $m_i = \nu_i - \nu_{i-1}$. For any $z \in \mathcal{P}$ we denote by $\pi(z) = (z_1, \ldots, z_r)$ the corresponding element of $\mathcal{U}$.

Put $U = 1 + \mathcal{U} \subset P_\phi$. It is the unipotent radical of P_ϕ and the quotient $M = P_\phi/U$ is isomorphic to the group of invertible elements of $\mathcal{M}$.

Let $y \in \mathcal{Q}_\phi$, i.e. $\mathrm{Ker}(\tau_1(y) \cdot \ldots \cdot \tau_i(y)) = L_i$. Then the element $\pi(y) = (y_1, \ldots, y_r)$ satisfies $\tau_i(y_i) = 0$ for all i. Lemma 4.1 implies that $\pi(y)$ lies on the orbit $\mathcal{O}_M$ of the element $\pi(x)$ under the action of the group $M = G_{m_1} \times \ldots \times G_{m_r}$. Since $\mathcal{O}_M$ is closed, $\pi(\overline{\mathcal{Q}}_\phi) \subset \mathcal{O}_M$. The morphism $\pi: \overline{\mathcal{Q}}_\phi \to \mathcal{O}_M$ is P_ϕ-equivariant and, again using Frobenius reciprocity and the existence of a P_ϕ-invariant measure on $\mathcal{O}_M$, we can reduce the problem to the construction of the (H,χ)-invariant distribution μ such that $\operatorname{Supp}\mu = Q$, where $H = \operatorname{Stab}(\pi(x), P_\phi) = M_x \cdot U$, $Q = \pi^{-1}(\pi(x)) \cap \overline{\mathcal{Q}}_\phi$.

(v) The main observation is that:

$$(*) \quad \pi^{-1}(\pi(x)) \cap \overline{\mathcal{Q}}_\phi = \pi^{-1}(\pi(x)) = x + \mathcal{U}.$$

Now if we denote by μ the Haar measure on $\mathcal{U}$ and consider it as a distribution on $x + \mathcal{U}$, we see that it is $\mathcal{U}$-invariant and (M_x,χ)-invariant. (Indeed, the Haar measure on $\mathcal{U}$ is (M,χ)-invariant, since $\chi = \Delta(P_\phi)|_M = \Delta_M \cdot \operatorname{mod}(U) = \operatorname{mod} U = \operatorname{mod} \mathcal{U}$.)

In order to prove $(*)$ it is sufficient to prove that for almost all $y \in x + \mathcal{U}$ we have $y \in \mathcal{O}$ and $\phi_y = \phi$. Since $y_i = x_i$, we have $\tau_i(y_i) = 0$, i.e. $\tau_i(y)L_i \subset L_{i-1}$. Hence $\operatorname{Ker}(\tau_1(y)\cdot \ldots \cdot\tau_i(y))$ contains L_i. Since for $y = x$ this kernel coincides with L_i, for almost all y (more precisely, for y from some Zariski open subset of $x + \mathcal{U}$) we have equalities $\operatorname{Ker}(\tau_1(y)\cdot \ldots \cdot\tau_i(y)) = L_i$ for all i. This means that $\phi_y = \phi$, and lemma 4.1 implies that y is conjugate to x, i.e. $y \in \mathcal{O}$. Q.E.D.

4.4 <u>Remark</u>. The distribution $\mu_{\mathcal{O}}$ we have constructed is positive, i.e. $\langle\mu_{\mathcal{O}},f\rangle \geq 0$ for positive functions f. Hence it defines a measure on X. In other words, for any continuous function f on X with compact support the integral $\int_{\mathcal{O}} f(x)\,d\mu_{\mathcal{O}}(x)$ converges absolutely.

CHAPTER II

APPLICATIONS TO REPRESENTATION THEORY

In this chapter we study representations of the group $G = GL(n,F)$ and their restrictions to the subgroup P. We use the notations of [BZ1], [BZ2]. In particular the notation (π,H,E) means a representation π of a group H in a complex vector space E. The representation π is called <u>smooth</u> (algebraic in the terminology of [BZ1], [BZ2]) if the stabilizer of each vector $\xi \in E$ is open in H. The category of smooth representations of H we denote by $\mathrm{Alg}(H)$.

§5. P-INVARIANT PAIRINGS OF G-MODULES

Proof of Theorem A.

5.1. <u>Theorem A</u>. Let (π,G,E) be a smooth irreducible representation, $(\tilde{\pi},G,\tilde{E})$ the contragredient representation, $B_o\colon E \times \tilde{E} \to \mathbb{C}$ the canonical pairing $B_o(\xi,\tilde{\xi}) = \langle\xi,\tilde{\xi}\rangle$. Then any P-invariant pairing $B\colon E \times \tilde{E} \to \mathbb{C}$ is G-invariant and hence is proportional to B_o.

<u>Proof</u>. It is known that π and $\tilde{\pi}$ are admissible (see [BZ1,3.25]), so $(\pi \otimes \tilde{\pi},\ G \times G,\ E \otimes \tilde{E})$ is admissible and irreducible. Consider the regular representation $(\mathrm{Reg},\ G \times G,\ S(G))$ given by $\mathrm{Reg}(g_1,g_2)f(g) = f(g_1^{-1}gg_2)$, $f \in S(G)$, $g_1,g_2,g \in G$. We will use the following standard lemma (see 5.6).

<u>Lemma</u>. For any admissible representation (π,G,E) there exists a nonzero morphism of $G \times G$-modules.

$$\pi_\mu\colon S(G) \to E \otimes \tilde{E}$$

If π is irreducible, π_μ is an epimorphism.

Pairings $B\colon E \times \tilde{E} \to \mathbb{C}$ correspond to morphisms $E \otimes \tilde{E} \to \mathbb{C}$, i.e. to elements of $(E \otimes \tilde{E})^*$. The morphism $\pi_\mu^*\colon (E \otimes \tilde{E})^* \to S^*(G)$, adjoint to π_μ, is $G \times G$-invariant. In particular, if a pairing B

is g-invariant for some $g \in G$, then the corresponding distribution $\mathcal{E}_B = \pi_\mu^*(B)$ is $\mathrm{Ad}(g)$-invariant.

Since π is irreducible, π_μ is onto and π_μ^* is a monomorphism. Hence $\mathcal{E}_B$ is $\mathrm{Ad}(g)$-invariant iff B is g-invariant.

Thus we have: B is P-invariant $\Rightarrow$ $\mathcal{E}_B$ is $\mathrm{Ad}(P)$-invariant $\Rightarrow$ $\mathcal{E}_B$ is $\mathrm{Ad}(G)$-invariant (by Theorem B) $\Rightarrow$ B is G-invariant. Since π is irreducible and admissible, B is proportional to the standard pairing, Q.E.D.

5.2. Let us discuss a slightly different version of theorem A.

Let (π,G,E) be a smooth representation. Fix a Haar measure μ on G and for any $f \in S(G)$ define an operator $\pi(f) = \int_G f(g)\pi(g)d\mu(g)$ $: E \to E$ and put $U_f = \mathrm{Ker}\,\pi(f)$. Consider the weakest topology on E for which all subspaces U_f are open (the weak topology). Denote by $\hat{E}$ the completion of E in this topology and by $(\hat{\pi},G,\hat{E})$ the natural representation. It is easy to check that

(i) E is a dense G-submodule of $\hat{E}$ and it coincides with the smooth part of $\hat{E}$.

(ii) For admissible π, $(\hat{\pi},\hat{E})$ is canonically isomorphic to the representation $(\tilde{\pi}^*,\tilde{E}^*)$ dual to the contragredient representation $(\tilde{\pi},\tilde{E})$.

One can also define $(\hat{\pi},\hat{E})$ by $\hat{E} = \mathrm{Hom}_G(S(G),E)$, $(\hat{\pi}(g)\alpha)(f) = \alpha(\mathrm{Reg}(1,g^{-1})f)$. Namely, for any $\hat{\xi} \in \hat{E}$ the morphism $\alpha_{\hat{\xi}}: S(G) \to \hat{E}$, given by $\alpha_{\hat{\xi}}(f) = \hat{\pi}(f)\hat{\xi}$, maps $S(G)$ into the subspace $E \subset \hat{E}$ of smooth vectors.

<u>Theorem A'</u>. Let (π,G,E) be a smooth irreducible representation. Then any P-equivariant morphism $\beta: E \to \hat{E}$ is G-equivariant and, hence, is proportional to the standard inclusion.

This theorem is just a reformulation of theorem A. It can also be proved directly. Indeed, for any $\beta: E \to \hat{E}$, $f \in S(G)$ the operator $\hat{\pi}(f)\circ\beta: E \to \hat{E}$ has a finite dimensional image, which lies in E. Hence for any β we can define the distribution $\mathcal{E}_\beta \in S^*(G)$ by $\mathcal{E}_\beta(f) = \mathrm{tr}(\hat{\pi}(f)\circ\beta)$.

Thus we have constructed a morphism of $G \times G$-modules $\mathrm{Hom}(E,\hat{E}) \to$ $\to S^*(G)$. Using the formula $\mathrm{tr}(\hat{\pi}(f)\circ\beta\circ\pi(f')) = \mathrm{tr}(\hat{\pi}(f' * f)\circ\beta)$ it is easy to show that, in case of irreducible π, this is a monomorphism. The rest of the proof is the same as in 5.1.

The same arguments enable us to prove the following generalization which may be of some use.

Theorem A". Let (π,G,E) be an admissible representation. Suppose E has only one irreducible G-submodule V, i.e. any proper G-submodule of E contains V. Then for any P-equivariant morphism $\beta: E \to \hat{E}$ its restriction to V is proportional to the standard inclusion $V \to E \to \hat{E}$.

Corollaries of Theorem A.

5.3. Proposition. Let (π,G,E) be a smooth irreducible representation, and let B_o be a nonzero G-invariant bilinear (or Hermitian) form on E. Then any P-invariant bilinear (respectively, Hermitian) form B on E is proportional to B_o.

Proof. Since π is admissible and irreducible B_o defines an isomorphism $E \simeq \tilde{E}$ (respectively, $\bar{E} \simeq \tilde{E}$, where $\bar{E}$ is the space complex conjugate to E). The form B defines a P-invariant pairing of E with $E \approx \tilde{E}$ (respectively, of E with $\bar{E} \approx \tilde{E}$).

Theorem A then implies that B is proportional to B_o.

5.4. Theorem. Let (σ,G,H) be a unitary topologically irreducible representation. Then its restriction to P is also topologically irreducible.

Proof. It is sufficient to check that any continuous P-equivariant morphism $\alpha: H \to H$ is a scalar operator.

Let (π,G,E) be the smooth part of (σ,H). It is known (see

[BZ1, 4.21]) that π is irreducible and E is dense in H. Consider the Hermitian form B_α given by $B_\alpha(\xi,\eta) = <\alpha\xi,\eta>$, where $<,>$ is the scalar product in H. The form B_α is P-invariant and by proposition 5.3. on E we have an equality $B_\alpha(\xi,\eta) = c<\xi,\eta>$, $c \in \mathbb{C}$. Since E is dense in H, it implies $\alpha = c\cdot 1_H$, Q.E.D.

5.5. Remark. We can prove a more precise result.

Theorem. Let (σ,G,H) be a continuous representation of G in a complete topological vector space H, (π,G,E) its smooth part. Suppose π is irreducible. Then any continuous P-equivariant morphism $\alpha: H \to H$ is a scalar operator.

Indeed, we have a natural inclusion $H \to \hat{E}$ (see 5.2). By theorem A' $\alpha|_E = c\cdot 1_E$. Since α is continuous, it is a scalar.

5.6. Proof of lemma 5.1.

Let G,H be arbitrary ℓ-groups, (π,G,E) and (ρ,H,V) smooth representations. Define the representation $(H(\pi,\rho),\ G\times H,\ \mathrm{Hom}(E,V))$ by $\mathrm{Hom}(E,V) = \mathrm{Hom}_{\mathbb{C}}(E,V)$, $H(\pi,\rho)(g,h)(\alpha) = \rho(h)\circ\alpha\circ\pi(g^{-1})$, $g \in G$, $h \in H$, $\alpha \in \mathrm{Hom}(E,V)$.

We have a natural imbedding $i: \tilde{\pi}\otimes\rho \to H(\pi,\rho)$ given by $i(\tilde{\xi}\otimes\eta)\xi = <\tilde{\xi},\xi>\cdot\eta$. Denote by $(h(\pi,\rho),\ G\times H,\ h(E,V))$ the smooth part of $H(\pi,\rho)$. Then i is an inclusion $i: \tilde{\pi}\otimes\rho \to h(\pi,\rho)$.

Suppose π is admissible. Then for any open compact subgroup $N \subset G$ the space E^N of N-invariant vectors is finite dimensional and $\tilde{E}^N = (E^N)^*$. This implies that $h(E,V)^N = \mathrm{Hom}(E^N,V) \approx (E^N)^*\otimes V \approx$ $\approx (E\otimes V)^N$, i.e. i give an isomorphism $i = \tilde{\pi}\otimes\rho \xrightarrow{\sim} h(\pi,\rho)$.

Now suppose G is a unimodular ℓ-group and fix a Haar measure on G. Define a morphism of representations

π_μ: (Reg, $G \times G$, $S(G)) \to (H(\pi,\pi)$, $G \times G$, $H(E,E))$ by $\pi_\mu(f) = \int_G f(g)\pi(g)d\mu(g)$. Since Reg is a smooth representation its image belongs to $h(\pi,\pi)$. If π is admissible we obtain a nontrivial morphism π_μ: $S(G) \to \tilde{E} \times E$ of $G \times G$-modules. If π is irreducible, then $\tilde{\pi} \otimes \pi$ is irreducible and hence π_μ is an epimorphism. Starting from $(\tilde{\pi},\tilde{E})$ instead of (π,E) we obtain lemma 5.1.

§6. SCALAR PRODUCT IN THE KIRILLOV MODEL

Kirillov Model

6.1. Let $U \subset P$ be the subgroup of unipotent upper triangular matrices. Fix a nonzero additive character $\psi: F \to \mathbb{C}^*$ and define the nondegenerate character θ of U by $\theta(u) = \psi(u_{12}+\ldots+u_{n-1,n})$, where $u = (u_{ij})$.

Consider the smooth induced representation $(\tau,P,\hat{S}) = \mathrm{Ind}_U^P(\theta)$. By definition $\hat{S} = \{ f: P \to \mathbb{C} \mid f(up) = \theta(u)f(p)$ for $u \in U$, $p \in P$ and f is smooth under the right action of P $\}$. Denote by (τ°,P,S) the subrepresentation in the subspace of functions with compact support modulo U.

Analogously define representations $(\tau',P,\hat{S}')$ and $(\tau^{\circ\prime},P,S')$ by replacing θ by θ^{-1}.

Note that τ and τ' are isomorphic. The isomorphism ε is given by $\varepsilon f(p) = f(\varepsilon p)$, where $\varepsilon = (\delta_{ij}\cdot(-1)^{n-i}) \in P$.

Let (π,P,E) be a smooth representation. A <u>Kirillov model</u> for π is by definition a nonzero morphism of P-modules $K: E \to \hat{S}$. A P-module (π,E) which has a Kirillov model is called <u>nondegenerate</u>.

The following facts are proved in [BZ1,§5].

(i) There are the natural isomorphisms $\tau \simeq \widetilde{\tau^{\circ\prime}}$ and $\tau' \simeq \widetilde{\tau^\circ}$. Corresponding pairings are given by integration over $U\backslash P$.

(ii) Any proper P-submodule of $\hat{S}$ contains S. In particular the P-module S is irreducible and any morphism $i: S \to \hat{S}$ of P-modules is proportional to the standard inclusion.

(iii) Let (π,P,E) be a smooth representation, $K: F \to \hat{S}$ its

Kirillov model. Then there exists a morphism of P-modules $i_K: S \to E$ such that $K \circ i_K = 1_S$.

6.2. Fix a right invariant Haar measure $\mu_{U\backslash P}$ on $U\backslash P$ and define the P-invariant scalar product $< , >$ on S by

$$(*) \qquad <f,h> = \int_{U\backslash P} f(p)\cdot\bar{h}(p)\,d\mu_{U\backslash P}(p), \quad f,\ h \in S.$$

Let $L^2(S)$ be the completion of S with respect to $<,>$ and $L^2_{sm}(S)$ its smooth part (with respect to the natural action of P). It is clear that $L^2_{sm}(S)$ is naturally imbedded in $\hat{S}$ (it consists of functions $f \in \hat{S}$ such that $\int_{U\backslash P} |f|^2 d\mu$ converges).

Theorem. Let (π, G, E) be a smooth irreducible representation and $K: E \to \hat{S}$ its Kirillov model (i.e. K is a nonzero morphism of P-modules). Suppose π is unitarizable, i.e. there exists a G-invariant positive definite Hermitian form B_0 on E. Then $K(E) \subset L^2_{sm}(S)$ and for some constant $c \in \mathbb{R}^{+*}$

$$<K(\xi),\ K(\eta)> \ \equiv\ c \cdot B_0(\xi,\eta).$$

Proof. Consider an inclusion $i_K: S \to E$ (see 6.1(iii)) and put $E^+ = i_K(S)$. Statements 6.1 (i) & (ii) imply that $<,>$ is the only P-invariant Hermitian form on S. It means that we can normalize B_0 such that $B_0(i_K(f), i_K(h)) \equiv <f,h>$.

Let H be the completion of E with respect to the norm given by B_0. Then by theorem 5.4 E^+ is dense in H. Hence i_K can be extended to the isomorphism $i_K^{(2)} = L^2(S) \simeq H$.

Consider the inverse map $K': H \to L^2(S)$. Then $K'(E) \subset L^2_{sm}(S) \subset \hat{S}$. Using the uniqueness of the Kirillov model for an irreducible G-module (see [G-K], [Sh], [BZ1] or 6.5) we see that $K = K'$, Q.E.D.

Regularity of the Ψ-Function at $s = 0$.

6.3. We want to generalize theorem 6.2 for nonunitary representations. Let $f \in \hat{S}$, $f' \in \hat{S}'$ (see 6.1). Define formally the function $\Psi(s,f,f')$ of the complex variable s by

(*) $$\Psi(s;f,f') = \int_{U\backslash P} f(p)f'(p)\nu(p)^s d\mu_{U\backslash P} ,$$

where $\nu(p) = |\det p|$.

If $f \in S$ or $f' \in S'$ this integral converges and is regular as a function of s. Moreover $\Psi(0;f,f')$ gives the canonical pairings $\hat{S}$ with S' and S with $\hat{S}'$.

Let (π,G,E), (π',G,E') be two admissible representations of finite length with Kirillov models $K: E \to \hat{S}$, $K': E' \to \hat{S}'$. Define formally for $\xi \in E$, $\xi' \in E'$ the function $\Psi(s;\xi, \xi')$ by $\Psi(s;\xi,\xi') = \Psi(s;K(\xi), K'(\xi'))$.

<u>Statement</u>. For any $\xi \in E$, $\xi' \in E'$ the integral (*) defining the function $\Psi(s;K(\xi),K'(\xi'))$ is absolutely convergent for $\operatorname{Re} s >> 0$. Furthermore, $\Psi(s;\xi,\xi')$ is a rational function of q^s, where q is the cardinality of the residue field of F. There exists a nonzero polynomial $P(q^s)$ which depends only on π and π' such that $P(q^s)\Psi(s,\xi,\xi')$ is a polynomial of $q^{\pm s}$ for any $\xi \in E$, $\xi' \in E'$.

This statement is standard (see e.g. [JPS]).

6.4. <u>Theorem</u>. Let (π,G,E) be a smooth irreducible representation, $(\tilde{\pi},G,\tilde{E})$ the contragredient representation, $K: E \to \hat{S}$, $K': \tilde{E} \to \hat{S}'$ their Kirillov models.

Then

(i) For any $\xi \in E$, $\tilde{\xi} \in \tilde{E}$ the function $\Psi(s,\xi,\tilde{\xi})$ is regular at $s = 0$.

(ii) There exists $c \in \mathbb{C}^*$ such that $\Psi(s,\xi,\tilde{\xi}) \equiv c \cdot \langle\xi,\tilde{\xi}\rangle$.

<u>Proof</u>. Let k be the maximal order of the poles of all functions $\Psi(s,\xi,\tilde{\xi})$ at $s = 0$. Define a pairing $B: E \times \tilde{E} \to \mathbb{C}$ by

$B(\xi,\tilde{\xi}) = (s^k \cdot \Psi(s,\xi,\tilde{\xi}))|_{s=0}$. It is clear that B is a P-invariant nonzero pairing. By theorem A there exists $c \in \mathbb{C}^*$ such that $B(\xi,\tilde{\xi}) \equiv c \cdot \langle\xi,\tilde{\xi}\rangle$.

Choose a vector $\xi \in E$ such that $K(\xi) \in S$ and $K(\xi) \neq 0$ (it is possible because of 6.1(ii)), and then choose a vector $\tilde{\xi} \in \tilde{E}$ such that $\langle\xi,\tilde{\xi}\rangle \neq 0$. Since the function $\Psi(s,\xi,\tilde{\xi})$ is regular everywhere and $(s^k \cdot \Psi(s,\xi,\tilde{\xi}))|_{s=0} = c \cdot \langle\tilde{\xi},\xi\rangle \neq 0$ we see that $k \leq 0$. Further, again using 6.1(ii) we can find $\xi \in E$, $\tilde{\xi} \in \tilde{E}$ such that $\Psi(s;\xi,\tilde{\xi}) \neq 0$, which gives $k \geq 0$. Hence $k = 0$, Q.E.D.

6.5. Corollary. Let (π,G,E) be a smooth irreducible nondegenerate representation. Then its Kirillov model $K: E \to \hat{S}$ is uniquely defined up to a scalar and it is an inclusion.

Proof. The contragredient representation $(\tilde{\pi},G,\tilde{E})$ also has a Kirillov model $K': \tilde{E} \to \hat{S}'$. This can be proved either by using the Gelfand-Kazhdan approach as in [GK] or [BZ1], or by using more simple results about pairings of representations of the group P (see [BZ2,§3]).

Consider the formula from theorem 6.4

$$\Psi(0;\xi,\tilde{\xi}) = c \cdot \langle\xi,\tilde{\xi}\rangle , \quad c \in \mathbb{C}^*.$$

Let $\xi \in E$, $K(\xi) \in \hat{S}$. The function $K(\xi)$ is completely determined by its scalar products with all functions $f' \in S'$. Since $S' \subset K'(\tilde{E})$, we see that $K(\xi)$ is determined by the constant c, i.e. all morphisms $K: E \to \hat{S}$ are proportional.

If $K(\xi) = 0$ we have $\langle\xi,\tilde{\xi}\rangle = 0$ for all $\tilde{\xi}$, i.e. $\xi = 0$.

This corollary gives an alternative proof of the theorem by Gelfand-Kazhdan (see [GK], [Sh], [BZ1]) and of the conjecture by Gelfand-Kazhdan, proved in [BZ2], [JS].

§7. CLASSIFICATION OF UNITARY IRREDUCIBLE REPRESENTATIONS OF $G = GL(n,F)$ VIA MULTIPLICITIES

Criteria for Unitarizability

7.1. <u>Lemma</u>. Let (π, G, E) be a smooth irreducible representation. Suppose π is Hermitian and $\pi|_P$ is unitarizable. Then π is unitarizable.

<u>Proof</u>. Let B_o be a G-invariant Hermitian form on E and B a P-invariant positive definite form on E. By corollary 5.3 B is G-invariant and proportional to B_o, i.e. π is unitarizable.

<u>Remark</u>. It is sufficient to assume that $\pi|_P$ is semiunitarizable, i.e. that the form B is positive semidefinite and nonzero. Since B is proportional to B_o, it is nondegenerate and hence positive definite.

For a given representation (π, E) it is usually easy to determine whether there exists a G-invariant Hermitian form on E, but it is very difficult to determine whether this form is positive definite. The lemma above allows us to restrict the problem to P. In the next subsections we will formulate an inductive unitarizability criterion for P-modules and deduce from it a unitarizability criterion for G-modules. Using this criterion we will describe an algorithm which classifies all unitary representations of G in terms of multiplicities of induced representations.

7.2. We need some constructions and results from [BZ2].

First define exact functors

$$\Phi^-: \mathrm{Alg}(P_n) \to \mathrm{Alg}(P_{n-1}) \text{ and } \Psi^-: \mathrm{Alg}(P_n) \to \mathrm{Alg}(G_{n-1})$$

as in [BZ2, §3] (see also 8.2). For any smooth representation (π, P, E) we define its <u>derivatives</u> $\underline{\pi^{(k)}} \in \mathrm{Alg}\, G_{n-k}$, $k = 1,2,\dots,n$, by $\pi^{(k)} = \Psi^-(\Phi^-)^{k-1}\pi$.

The highest number h for which $\pi^{(h)} \neq 0$ is called the depth of π and the representation $\pi^{(h)}$ is called the highest derivative of π. We say that π is homogenous (of depth h) if the depth of any nonzero P-submodule of π is equal to h.

For any representation (π, E) of the group G we define the derivatives $\pi^{(k)}$, $k = 0,1,\ldots,n$, by $\pi^{(0)} = \pi$, $\pi^{(k)} = (\pi|_P)^{(k)}$.

For the classification of unitary representations it is convenient to introduce the shifted derivatives $\pi^{[k]} = \nu^{1/2} \cdot \pi^{(k)}$, where $\nu^{1/2}$ is the character of G given by $\nu^{1/2}(g) = |\det g|^{1/2}$. Henceforth we consider multiplication by $\nu^{1/2}$ as an autoequivalence of the category $\mathrm{Alg}(G)$.

7.3. Let us identify the group F^* with the center of the group G_m, $m > 0$. For any irreducible representation (ω, G_m, L) we denote by χ_ω its central character, given by $\omega(\lambda) = \chi_\omega(\lambda) \cdot 1_L$, and by $e(\omega)$ the real number given by $|\chi_\omega(\lambda)| \equiv |\lambda|^{e(\omega)}$, where $|\lambda|$ is the standard norm on F^*. We call the number $e(\omega)$ the central exponent of ω.

For any smooth representation (π, G_m, E) we denote by $e(\pi)$ the set of central exponents of all irreducible subquotients of π. The set $e(\pi) \subset \mathbb{R}$ we call central exponents of π. For example, if π is unitarizable $e(\pi) = \{0\}$.

Unitarizability criterion for P-modules.

Let (π, P_n, E) be a smooth representation, homogeneous of depth h. Suppose π is of finite length. Then π is unitarizable if and only if

(i) $\pi^{[h]}$ is a unitarizable representation of G_{n-h}

(ii) For any $k < h$ $e(\pi^{[k]}) > 0$, i.e. all central exponents of $\pi^{[k]}$ are strictly positive.

We will prove this criterion in §9. We say that a P-module π is P-positive if $e(\pi^{[h]}) = 0$ and for any $k < h$ $e(\pi^{[k]}) > 0$. Then the condition (ii) can be written as (ii)' π is a P-positive representation.

We will use a version of this criterion for semiunitarizable representations, i.e. representations which have a nonzero invariant positive semidefinite Hermitian form.

Proposition. Let (π,P,E) be a smooth representation of depth h, such that $\pi^{[h]}$ is semiunitarizable and π is P-nonnegative, i.e. $e(\pi^{[k]}) \geq 0$ for $k < h$. Then π is semiunitarizable. We will prove this in §9.

7.4. Unitarizability criterion for G-modules.

Let (π,G,E) be a smooth irreducible representation. Then π is unitarizable if and only if

(i) π is Hermitian

(ii) The highest shifted derivative $\pi^{[h]}$ is a unitarizable representation of G_{n-h}

(iii) π is P-positive, i.e. $e(\pi^{[k]}) > 0$ for $k < h$.

Indeed, according to [Z1,6.8] the representation $\pi|_P$ is homogeneous of depth h (and of finite length). Then criterion 7.3 and lemma 7.1 establish the criterion.

Remark. Let (π,G,E) be an irreducible unitarizable representation. Then using 5.3 and the results of §8 one can reprove results of A. Zelevinsky: $\pi|_P$ is homogeneous and its highest derivative $\pi^{[h]}$ is irreducible (and unitarizable). Moreover the considerations of §8 essentially prove that the scalar product in E can be written as an integral in its degenerate Kirillov model (see [Z1,5.2]). It would be interesting to apply an analogous approach to nonunitarizable representations. For nondegenerate π it is done in 6.3-6.5. For degenerate π I could not do it since I do not know an analogue of the regularization procedure, described in 6.3.

Zelevinsky's Classification of Irreducible smooth G-Modules

7.5. Let R_n $(n = 0, 1, 2, \ldots)$ be the Grothendieck group of the category of smooth G_n-modules of finite length (here $G_0 = \{e\}$, $R_0 = \mathbb{Z}$). The induction functor defines a bilinear morphism $R_n \times R_m \to R_{n+m}$, $(\pi, \rho) \mapsto \pi \times \rho$.

We put $R = \oplus_{n=0}^{\infty} R_n$. Then $\times$ defines on R the structure of a commutative algebra.

For any $\alpha \in \mathbb{C}$ we denote by ν^α an automorphism of the ring R, given by $\nu^\alpha(\pi) = \nu^\alpha \cdot \pi$.

Define a morphism $D: R \to R$ by $D(\pi) = \pi^{(0)} + \pi^{(1)} + \ldots + \pi^{(n)}$ for $\pi \in R_n$ (see 7.2). Then D is a ring homomorphism, i.e. $(\pi \times \rho)^{(k)} = \Sigma_{i+j=k}\, \pi^{(i)} \times \rho^{(j)}$, and D commutes with ν^α ([Z1, §3]).

We will use another homomorphism $D^{[\ ]} = \nu^{1/2} \circ D$ of the ring R, based on shifted derivatives, i.e. $D^{[\ ]}(\pi) = \Sigma \pi^{[k]}$ (see 7.2).

Denote by $\mathrm{Irr} = \cup \mathrm{Irr}_n$ the subset of R, corresponding to irreducible representations. This subset defines on R the structure of an ordered group. By definition the multiplication $\times$ and morphisms ν^α, D, $D^{[\ ]}$ are positive operators.

7.6. We would like to parametrize the set Irr of irreducible representations in terms of cuspidal representations. So denote by $\mathcal{C} = \cup \mathcal{C}_n$, $n > 0$, the subset of cuspidal representations in Irr.

The subset $\Delta \subset \mathcal{C}_d$ of the form $\Delta = (\rho, \nu\rho, \nu^2\rho, \ldots, \nu^{\ell-1}\rho)$, $\ell > 0$, we call a <u>segment</u>; ℓ is called the <u>length</u> of Δ and the representation $\nu^{(\ell-1)/2}\rho$ the center of Δ. The number d is called the depth of Δ.

The set of all segments $\Delta \subset \mathcal{C}$ we denote by $\mathcal{S}$.

<u>Statement</u> ([Z1, §3]). Let $\Delta = (\rho, \nu\rho, \ldots, \nu^{\ell-1}\rho) \subset \mathcal{C}_d$ be a segment. Then the representation $\rho \times \nu\rho \times \ldots \times \nu^{\ell-1}\rho$ contains a unique irreducible constituent $\langle\Delta\rangle$ of the depth $d = \mathrm{depth}\ (\Delta)$.

We define the segments Δ^- and Δ' by $\Delta^- = (\rho, \nu\rho, \ldots, \nu^{\ell-2}\rho)$, $\Delta' = \nu^{1/2} \cdot \Delta^-$. If $\ell = 1$ we put $\Delta^- = \Delta' = \emptyset$. From [Z1, §3] we can deduce:

$$\nu^{\alpha}\langle\Delta\rangle = \langle\nu^{\alpha}\Delta\rangle$$

$$D\langle\Delta\rangle = \langle\Delta\rangle + \langle\Delta^{-}\rangle, \qquad D^{[\ ]}(\langle\Delta\rangle) = \langle\nu^{1/2}\Delta\rangle + \langle\Delta'\rangle,$$

where $\langle\Delta\rangle \in \mathrm{Irr}_{\ell d}$, $\langle\Delta'\rangle \in \mathrm{Irr}_{(\ell-1)d}$ and we put $\langle\Delta'\rangle = 1 \in R_0$ if $\Delta' = \emptyset$.

Note that the operation $\Delta \to \Delta'$ preserves the center of a segment.

7.7. Denote by $\mathcal{O}$ the set of finite multisets in S. In other words, an element $a \in \mathcal{O}$ is a sequence of segments $\Delta_1, \Delta_2, \ldots, \Delta_r \in S$ up to permutation.

For any $a = (\Delta_1, \ldots, \Delta_r) \in \mathcal{O}$ we put depth $(a) = \Sigma$ depth (Δ_i) $a^{-} = (\Delta_1^{-}, \ldots, \Delta_r^{-})$, $a' = (\Delta_1', \ldots, \Delta_r')$ (if for some i $\Delta_i' = \emptyset$ we simply throw it away).

Define an element $\pi(a) \in R$ by

$$\pi(a) = \langle\Delta_1\rangle \times \ldots \times \langle\Delta_r\rangle .$$

Consider multisets $a_1 = a$, $a_2 = a_1'$, $a_3 = a_2'$, ... and put h_i = depth (a_i). Formulas in 7.5 and 7.6 imply that h_i = depth $\pi(a_i)$ and the highest shifted derivative $\pi(a_i)^{[h]}$ is isomorphic to $\pi(a_{i+1})$. For large i, $a_i = \emptyset$, i.e. $\pi(a_i) = 1 \in R_0$. Hence there exists a unique irreducible constituent $\langle a\rangle$ of $\pi(a)$ such that $\langle a\rangle^{[h_1][h_2]\ldots[h_i]} = 1$.

Theorem ([Z1,§6, 8.1]).

a) depth $\langle a\rangle$ = depth a. The highest derivative $\langle a\rangle^{(h)}$ and the highest shifted derivative $\langle a\rangle^{[h]}$ are isomorphic to $\langle a^{-}\rangle$ and $\langle a'\rangle$.

b) The map $a \mapsto \langle a\rangle$ gives a one-to-one correspondence $\mathcal{O} \to \mathrm{Irr}$.

c) Elements $\pi(a)$ for $a \in \mathcal{O}$ form a basis of the ring R. In other words R is the polynomial ring over $\mathbb{Z}$ in variables $\Delta \in S$.

Define the matrix $m = (m_{ab} | a,b \in O)$ by

$$\langle a\rangle = \sum m_{ab}\pi(b)$$

This matrix is invertible (even unipotent, see [Z1]) and the inverse matrix describes the decomposition of representations $\pi(a)$ into irreducible components. Because of this we call m <u>the multiplicity matrix</u>.

7.8. It remains to describe central exponents and the Hermitian duality in this classification.

If $\pi \in R_n$, $\sigma \in R_m$ are positive elements, then $e(\pi\times\sigma) = e(\pi) + e(\sigma)$, $e(\nu^{\alpha}\pi) = e(\pi) + n\alpha$.

For $\Delta = (\rho, \nu\rho, \ldots, \nu^{\ell-1}\rho) \subset C_d$ we put $e(\Delta) = \ell \cdot e(\text{center } \Delta) = \ell \cdot e(\rho) + d\ell(\ell-1)/2$. Then $e(\langle\Delta\rangle) = e(\Delta)$.

For $a = (\Delta_1, \ldots, \Delta_r)$ we put $e(a) = \Sigma e(\Delta_i)$. Then $e(\langle a\rangle) = e(a)$.

Denote by $+$ the Hermitian conjugation, i.e. the ring homomorphism $+: R \to R$ given by $\pi \mapsto \pi^+ = \bar{\tilde{\pi}}$ the Hermitian contragredient of π. Then we have

$$(\nu^{\alpha}\pi)^+ = \nu^{-\bar{\alpha}}(\pi^+), \quad e(\pi^+) = -e(\pi).$$

The morphism $+$ preserves Irr and C. Moreover, if $\rho \in C_d$ we have $\rho^+ = \nu^{-2e(\rho)/d} \cdot \rho$.

The morphism $+$ acts on S and O since $(\rho^+, (\nu\rho)^+, \ldots, (\nu^{\ell-1}\rho)^+) = (\rho^o, \nu\rho^o, \ldots, \nu^{\ell-1}\rho^o)$ with $\rho^o = \nu^{1-\ell}\rho^+$.

Note that center $\Delta^+ = (\text{center } \Delta)^+$.

<u>Statement</u> ([Z1,7.10])

$$\langle\Delta\rangle^+ = \langle\Delta^+\rangle, \quad \langle a\rangle^+ = \langle a^+\rangle$$

$$(a')^+ = (a^+)'$$

Algorithm for Description of Unitary Representations Via Multiplicities

7.9. Assume we know the multiplicity matrix $m = (m_{ab})$, i.e. we know all multiplicites. Let us describe an algorithm which enables us to find out whether a given irreducible representation $\langle a \rangle$ is unitarizable. We can rewrite the criterion 7.4 in the following way.

Criterion. Let $a \in \mathcal{O}$. Then the representation $\langle a \rangle \in \mathrm{Irr}$ is unitarizable if and only if

(i) $a^+ = a$

(ii) Let $h = \mathrm{depth}(a)$. Then the representation $\langle a \rangle^{[h]} = \langle a' \rangle$ is unitarizable.

(iii) For any $k < h$ in the expression $\langle a \rangle^{[k]} = \Sigma_b m^k_{ab}\ \pi(b)$ all coefficients m^k_{ab} with $e(b) \leq 0$ vanish.

Condition (i) can be checked straight-forwardly. In (iii) we can express all coefficients m^k_{ab} via the multiplicity matrix m, using the formula

$$D^{[\ ]}(\langle \Delta_1 \rangle \times \langle \Delta_2 \rangle \times \ldots \times \langle \Delta_r \rangle) = \prod(\langle \nu^{1/2}\Delta_i \rangle + \langle \Delta_i' \rangle).$$

So it remains to check (ii). But this is the same problem of unitarizability for a smaller group. Hence after several steps we can find out whether the representation $\langle a \rangle$ is unitarizable or not.

§8. UNITARIZABILITY AND IRREDUCIBILITY: MISCELLANEOUS RESULTS

Some Criteria of Irreducibility

8.1. The algorithm 7.9 shows that knowledge of multiplicities enables us to describe all unitary representations. Conversely it turns out that the study of unitary representations enables us to say something about multiplicities. More precisely, it allows us to give some criteria of irreducibility.

Proposition. Let (π, G, E) be a representation of finite length. Suppose that $\pi \approx \pi^+$ in the Grothendieck group R and the highest shifted derivative $\pi^{[h]}$ is irreducible and unitarizable. Then

a) If π is P-positive, i.e. $e(\pi^{[k]}) > 0$ for $1 \leq k < h$, then π is irreducible (and unitarizable).

b) If π is P-nonnegative, but not P-positive, i.e. $e(\pi^{[k]}) \geq 0$ and for some $k < h$ $e(\pi^{[k]}) \not\ni 0$, then π is reducible.

Remark. If $\pi^{[h]}$ is reducible, then π is also reducible (see [Z1,8.1]).

Proof.

a) Let $\omega \in$ Irr. By 7.8 ω and ω^+ have the same depth, say d, and their highest shifted derivatives are Hermitian dual, i.e. $e(\omega^{+[d]}) = -e(\omega^{[d]})$. Consider an irreducible constituent ω of the representation π. Then ω^+ is a constituent of $\pi^+ \approx \pi$. If $d = \mathrm{depth}(\omega)$ is less than $h = \mathrm{depth}(\pi)$ then $e(\omega^{[d]})$, $e(\omega^{+[d]}) \in \in e(\pi^{[d]})$ should be both positive, which is impossible. Hence any constituent ω has depth h and since $\pi^{[h]}$ is irreducible, π has only one constituent, i.e. π is irreducible. By criterion 7.4 π is unitarizable.

b) By 7.3 $\pi|_P$ is semiunitarizable. If π were irreducible, it would be unitarizable (see Remark 7.1), and therefore P-positive, contradicting the condition of the proposition.

8.2. We call a G-module π of finite length G-positive, if for any $k < h = \mathrm{depth}(\pi)$, including $k = 0$,

$$e(\pi^{[k]}) > e(\pi^{[h]}) .$$

The formula $(\pi \times \sigma)^{[k]} = \Sigma\pi^{[i]} \times \sigma^{[j]}$, $i + j = k$ implies that $\pi \times \sigma$ is G-positive iff both π and σ are G-positive. Criterion 7.4 and proposition 8.1) imply

Criterion. Let π be a G-module of finite length, $h = \mathrm{depth}(\pi)$. Then π is irreducible and unitarizable iff $\pi = \pi+$ in R, $\pi^{[h]}$ is irreducible and unitarizable and π is G-positive.

Corollary.

a) If π, σ are irreducible and unitarizable, then $\pi \times \sigma$ is irreducible and unitarizable.

b) If π, σ are irreducible and Hermitian and $\pi \times \sigma$ is unitarizable, then π and σ both are unitarizable.

Proof.

a) By induction, the highest shifted derivative $(\pi \times \sigma)^{[h+d]} = \pi^{[h]} \times \sigma^{[d]}$ is irreducible and unitarizable (here $h = \text{depth}(\pi)$, $d = \text{depth}(\sigma)$). Since $\pi \times \sigma$ is G-positive as a product of G-positive modules, $\pi \times \sigma$ is irreducible and unitarizable.

b) By induction the highest shifted derivatives $\pi^{[h]}$ and $\sigma^{[d]}$ are irreducible and unitarizable. The representation $\pi \times \sigma$ is unitarizable and hence G-nonnegative. Therefore π and σ are G-nonnegative. By proposition 8.1b) they are G-positive and hence unitarizable.

Boundary of the Complementary Series

8.3. Let σ be a smooth irreducible representation of G_n. For any $\alpha \in \mathbb{R}$ denote by $\pi(\alpha) = \pi(\sigma,\alpha)$ the representation of G_{2n}

$$\pi(\sigma,\alpha) = \nu^{\alpha}\sigma \times \nu^{-\alpha}\sigma^{+} = (\nu^{\alpha}\sigma) \times (\nu^{\alpha}\sigma)^{+}.$$

Let $h = \text{depth}(\sigma)$. Define inductively an interval $I(\sigma) \subset \mathbb{R}$ by $I(\sigma) = \{\alpha \mid \alpha \in I(\sigma^{[h]}),\ \nu^{\alpha}\sigma \text{ and } (\nu^{\alpha}\sigma)^{+} \text{ are G-positive}\}$. The latter condition can be written as

$$(*) \qquad \text{For any } 0 \leq k < h \quad e(\sigma^{[k]}) - e(\sigma^{[h]}) > (h-k)\alpha \quad \text{and}$$

$$e(\sigma^{+[k]}) - e(\sigma^{+[h]}) > -(h-k)\alpha \ .$$

Proposition. If $\alpha \in I(\sigma)$ $\pi(\alpha)$ is irreducible and unitarizable. If α lies on the boundary of $I(\sigma)$ $\pi(\alpha)$ is reducible. If α lies outside of the closure of $I(\sigma)$, $\pi(\alpha)$ is not unitarizable.

Indeed, if $\alpha \in I(\sigma)$ then by induction the highest shifted derivative $\pi(\alpha)^{[2h]} = \pi(\sigma^{[h]},\alpha)$ is irreducible and unitarizable and

$\pi(\alpha)$ is G-positive. By criterion 8.2 π is irreducible and unitarizable.

If α lies on the boundary of $I(\sigma)$, then either $\pi(\alpha)^{[2h]}$ is reducible, or it is irreducible and unitarizable and $\pi(\alpha)$ is G-nonnegative but not G-positive. By proposition 8.1b) $\pi(\alpha)$ is reducible.

If α lies outside of the closure of $I(\sigma)$ then either $\pi(\alpha)^{[2h]}$ is not unitarizable or $\pi(\alpha)$ is not G-nonnegative. In both cases 7.4 implies that $\pi(\alpha)$ is not unitarizable.

Representations $\pi(\alpha)$ for $\alpha \in I(\sigma)$ we call a (generalized) complementary series. Note, that if σ is unitarizable, $I(\sigma)$ is not empty, namely $I(\sigma) \ni 0$, and $I(\sigma)$ is symmetric with respect to 0.

Proposition 8.3 together with corollary 8.2b) allows us to describe all complementary series.

Remark. The length of the interval $I(\sigma)$ is always less than or equal to 1 (if $\sigma \neq 1 \in R_o$). Indeed, if the highest shifted derivative $\sigma^{[h]}$ is not $1 \in R_o$, then $I(\sigma) \subset I(\sigma^{[h]})$ and we can use induction. Now, let $\sigma^{[h]} = 1$, i.e. σ is nondegenerate. Applying condition (*) for $k = 0$ we see that for $\alpha \in I(\sigma)$

$$h\alpha < e(\sigma^{[o]}) = e(\nu^{1/2}\sigma) = h/2 + e(\sigma)$$
$$-h\alpha < e(\sigma^{+[o]}) = h/2 + e(\sigma^{+}) = h/2 - e(\sigma), \quad \text{i.e.} \quad |\alpha - e(\sigma)/h| < 1/2.$$

Reduction to the Discrete Data

8.4. We want to show that the algorithm described in 7.9 essentially works with discrete data.

Let $\Delta_1, \Delta_2 \subset C$ be two segments. We say that Δ_1 and Δ_2 are linked if $\Delta_1 \not\subset \Delta_2$, $\Delta_2 \not\subset \Delta_1$ and the set theoretic union $\Delta_1 \cup \Delta_2 \subset C$ is also a segment.

Statement ([Z1, §8]). Let $a_1 = (\Delta_1^1, \ldots, \Delta_r^1)$, $a_2 = (\Delta_1^2, \ldots, \Delta_s^2)$. Suppose that for any i, j the segments Δ_i^1 and Δ_j^2 are not linked.

Then $\langle a_1\rangle \times \langle a_2\rangle = \langle a_1 + a_2\rangle$, where $a_1 + a_2$ is the union of multisets.

Consider the action of the group $\mathbb{R}$ on C given by $\alpha:\rho \to \nu^{\alpha}\rho$. Orbits of this action we call $\mathbb{R}$-lines in C, and orbits of the subgroup $\mathbb{Z} \subset \mathbb{R}$ we call $\mathbb{Z}$-lines. We say that a multiset a is concentrated on a line $\Pi \subset C$ if all segments of a lie in Π. Then we see that any irreducible representation can be written as $\langle a_1\rangle \times \langle a_2\rangle \times\ldots\times\langle a_k\rangle$, where $a_1,\ldots,a_k$ are concentrated on different $\mathbb{Z}$-lines $\Pi_1,\ldots,\Pi_k$. Hence all problems about multiplicities m_{ab} can be reduced to one line.

From now on we fix a unitary cuspidal irreducible representation $\rho \in C_d$ and consider only segments Δ and multisets a, concentrated on the $\mathbb{R}$-line $\Pi_\rho = \{\nu^{\alpha}\rho,\ \alpha \in \mathbb{R}\}$. We will identify in a natural way $\mathbb{R}$ and Π_ρ, $\alpha \leftrightarrow \rho_\alpha = \nu^{\alpha}\rho$. In particular $\rho_\alpha^{+} = \rho_{-\alpha}$.

Let a be a multiset which we suspect to be unitary. Using the statement above we can consider only the case when a is concentrated on a subset

$$\Pi_\alpha = \{\pm\alpha + \mathbb{Z}\} \subset \mathbb{R} = \Pi_\rho, \quad \text{where} \quad 0 \le \alpha \le 1/2 .$$

We consider 2 cases.

Case 1 (rigid case): $\alpha = 0$ or $\alpha = 1/2$, i.e. Π_ρ is a $\mathbb{Z}$-line.

Case 2 (nonrigid case): $0 < \alpha < 1/2$.

8.5. Let us show that in the "nonrigid" case we can exclude the parameter α.

We can decompose a into the union of multisets a_α and $a_{-\alpha}$ concentrated on $\alpha + \mathbb{Z}$ and $-\alpha + \mathbb{Z}$. By statement 8.4 $\langle a\rangle = \langle a_\alpha\rangle \times \langle a_{-\alpha}\rangle$.

Let us denote by b the multiset $\nu^{-\alpha}a_\alpha$ concentrated on the $\mathbb{Z}$-line $\mathbb{Z}$ and put $\sigma = \langle b\rangle$. Then $\langle a\rangle = \pi(\sigma,\alpha)$ (see 8.3). The representation $\pi(\sigma,\alpha)$ is irreducible and unitarizable for all α in the interval $I(\sigma)$, described in 8.3, and it is reducible when α lies on the boundary of $I(\sigma)$. According to the statement 8.4 both boundary points of $I(\sigma)$ are half-integers. Hence for $\alpha \in (0,1/2)$ either all representations $\pi(\sigma,\alpha)$ are unitarizable, or all of them are not unitarizable.

Let us rewrite condition 8.3 (*) for the unitarizability of $\pi(\sigma,\alpha)$.

$$(*) \quad (h-k)\alpha < e(\sigma^{[k]}) - e(\sigma^{[h]}), \quad -(h-k)\alpha < e(\sigma^{+[k]}) - e(\sigma^{+[h]}) .$$

Since it should be true for all $\alpha \in (0,1/2)$ it is equivalent to the condition

$$(**) \quad e(\sigma^{[k]}) - e(\sigma^{[h]}) \geq (h-k)/2, \quad e(\sigma^{+[k]}) - e(\sigma^{+[h]}) \geq 0 .$$

Hence we have proved the following inductive Criterion.

Criterion. Let $b \in \mathcal{O}$ be a multiset of depth h concentrated on $\mathbb{Z}$-line $\mathbb{Z} \subset \Pi_\rho$, and let $\alpha \in (0,1/2)$. Then the representation $\nu^{\alpha}\langle b\rangle \times \nu^{-\alpha}\langle b^+\rangle = \langle \nu^{\alpha}b + \nu^{-\alpha}b^+\rangle$ is unitarizable if and only if $\langle \nu^{\alpha}b^{[h]} + \nu^{-\alpha}b^{[h]+}\rangle$ is unitarizable for all $\alpha \in (0,1/2)$ and (**) holds for $\sigma = \langle b\rangle$.

This criterion does not depend on α. It allows us to formulate an algorithm, dealing only with discrete data, for the classification of unitarizable representations in the nonrigid case. In the rigid case we automatically work with discrete data.

Some Conjectures.

8.6. In 8.5 the length of $I(\sigma)$ is less than or equal to 1. Hence if $I(\sigma) \supset (0,1/2)$ we have 3 possibilities: $I(\sigma) = (0,1)$, $I(\sigma) = (-1/2,1/2)$ or $I(\sigma) = (0,1/2)$. For instance, if σ is unitarizable, $I(\sigma) = (-1/2,1/2)$; if $\nu^{1/2}\sigma$ is unitarizable, $I(\sigma) = (0,1)$ (see corollary 8.2 a)).

Conjecture.

a) if $I(\sigma) = (-1/2,1/2)$ then σ is unitarizable. Respectively, if $I(\sigma) = (0,1)$ then $\nu^{1/2}\sigma$ is unitarizable.

b) If $I(\sigma) = (0,1/2)$, then $\sigma = \sigma_1 \times \sigma_2$, where σ_1 and σ_2 are irreducible representations, $I(\sigma_1) = (-1/2,1/2)$, $I(\sigma_2) = (0,1)$.

This conjecture, if true, reduces the studying of the non-rigid case 2 to the rigid case 1.

8.7. Now one remark about the rigid case.

Lemma. Let b be a multiset concentrated on one $\mathbb{Z}$-line $\mathbb{Z}$ or $1/2 + \mathbb{Z}$ and $\sigma = \langle b \rangle$. Suppose that σ is unitarizable. Then for all $0 \le k < h = \mathrm{depth}(\sigma)$ we have

$$(*) \qquad e(\sigma^{[k]}) \ \ge \ (h-k)/2 \ .$$

This inequality refines the condition $e(\sigma^{[k]}) > 0$ of 7.4.

Proof. Since $\sigma \times \sigma$ is irreducible and unitarizable by 8.2a), $I(\sigma) = (-1/2, 1/2)$. Thus for any $\alpha \in (-1/2, 1/2)$ we have (see 8.3(*))

$$e(\sigma^{[k]}) \ = \ e(\sigma^{[k]}) - e(\sigma^{[h]}) > \alpha(h-k).$$

This gives the inequality (*).

I should confess that inequality (*) reminds me of the inequalities for Kazhdan-Lusztig polynomials.

Unitarizability and Duality

8.8. Let us apply the unitarizability criterion to the case when $\langle a \rangle = \pi(a)$. According to [Z1,4.2] this is true if and only if any two segments Δ_1, Δ_2 of a are not linked. For simplicity consider the case when a is concentrated on one $\mathbb{R}$-line Π_ρ (see 8.4).

Lemma. $\pi(a)$ is irreducible and unitarizable if and only if $a^+ = a$ and for any segment $\Delta \in a$ we have $|\mathrm{center}(\Delta)| < 1/2$ (we consider the representation $\mathrm{center}(\Delta)$ as a real number on $\Pi_\rho \simeq \mathbb{R}$).

Proof. Suppose $\pi(a)$ is irreducible and unitarizable and $\Delta \in a$. Then the highest shifted derivative $\pi(a')$ is also irreducible and unitarizable. If $\Delta' \neq \emptyset$, then by induction $\mathrm{center}\ (\Delta) = \mathrm{center}(\Delta') > -1/2$. If $\Delta' = \emptyset$, i.e. Δ consists of one element, then the condition that $\langle \Delta \rangle$ is G-positive, i.e.

$e(\Delta^{[0]}) = e(\nu^{1/2}\Delta) > 0$, implies that $\overline{\text{center}}(\Delta) > -1/2$. Since Δ, $\Delta^{+} \in a$, we have $|\text{center}(\Delta)| < 1/2$. Conversely, suppose that $a^{+} = a$ and $|\text{center}(\Delta)| < 1/2$ for all $\Delta \in a$. Then all representations $\langle\Delta\rangle$ are G-positive and applying criterion 8.2 we can prove by induction that $\pi(a)$ is irreducible and unitarizable.

8.9. In [Z1, 9] A. Zelevinsky described an automorphism t of the ring R, which he called duality. On generators $\langle\Delta\rangle$ it is given by

$$\langle\Delta = (\rho, \nu\rho, \dots, \nu^{\ell-1}\rho)\rangle \mapsto \langle\Delta\rangle^{t} = \langle\{\rho\}, \{\nu\rho\}, \dots, \{\nu^{\ell-1}\rho\}\rangle,$$

where on the right side we consider the multiset of ℓ one-point segments.

One can show that t maps Irr into itself. The representations $\langle\Delta\rangle^{t}$ play a very important role - they are the so-called square integrable representations.

Lemma. Let $a = (\Delta_1, \dots, \Delta_r) \in \mathcal{O}$. Then the representation $\pi(a)^{t} = \langle\Delta_1\rangle^{t} \times \dots \times \langle\Delta_r\rangle^{t}$ is irreducible and unitarizable if and only if $a^{+} = a$ and for any segment $\Delta \in a$ we have $|\text{center}(\Delta)| < 1/2$.

Proof. According to [Z1, 9.6] for $\Delta = (\rho, \nu\rho, \dots, \nu^{\ell-1}\rho)$ we have

$$D(\langle\Delta\rangle^{t}) = \Sigma\langle\Delta_i\rangle^{t},$$

where

$$\Delta_i = (\nu^{i}\rho, \nu^{i+1}\rho, \dots, \nu^{\ell-1}\rho), \quad i = 0, 1, \dots, \ell$$

and we assume $\langle\Delta_\ell\rangle = 1$. Hence $\langle\Delta\rangle^{t}$ is G-positive if and only if $e(\nu^{1/2}\Delta) > 0$, i.e. center(Δ) $-1/2$. Now criterion 8.2 implies the lemma.

Remark. This lemma gives a classification of nondegenerate irreducible unitarizable representations.

8.10. Lemmas 8.8 and 8.9 make reasonable the following.

Conjecture. Duality t: Irr $\to$ Irr maps unitarizable representations into unitarizable representations,

§9. PROOF OF CRITERION AND PROPOSITION 7.3

9.1. Let (π,P,E) be a smooth representation. We call π <u>Φ^--degenerate</u> if $\Phi^-(\pi) = 0$. We call π <u>Φ^--homogeneous</u> if for any nonzero subrepresentation ρ $\Phi^-(\rho) \neq 0$.

Criterion and proposition 7.3 inductively follows from the following.

<u>Proposition</u>. Let (π, P_{m+1},E) be a smooth representation of finite length.

a) Suppose π is Φ^--degenerate. Then π is (semi) unitarizable iff $\nu^{1/2} \cdot \Psi^-(\pi) \in \text{Alg } G_m$ is (semi) unitarizable.

b) Suppose π is Φ^--homogeneous. Then π is unitarizable iff

(i) $\Phi^-(\pi) \in \text{Alg } P_m$ is unitarizable.

(ii) $e(\nu^{1/2}\Psi^-(\pi)) > 0$.

c) Suppose π is Φ^--nondegenerate, $\Phi^-(\pi)$ is semiunitarizable and $e(\nu^{1/2}\Psi^-(\pi)) \geq 0$. Then π is semiunitarizable.

9.2. In the proof of proposition 9.1 we will use the geometric realization of the representation (π,E), described in [BZ1,§5].

The group P_{m+1} is the semidirect product of the subgroups*)
$G_m = \{(p_{ij}) \mid p_{ij} = \delta_{ij}$ for $i > m$ or $j > m\}$ and
$V = V_m = \{(p_{ij}) \mid p_{ij} = \delta_{ij}$ for $j \leq m\}$. Let us identify V_m with the linear space $\text{Mat}(m,1;F)$ of column-vectors of length m and denote by W the dual space $W = \text{Mat}(1,m;F) = V^*$ of row-vectors. For any $v \in V$ we denote by ψ_v the character of W, given by $\psi_v(w) = \psi\langle v,w\rangle$, where ψ is a fixed nontrivial additive character of F. We denote by δ the natural action of G_m on W, given by $\delta(g)w = wg^{-1}$.

*) V is the unipotent radical of P and G is a Levi component of P.

<u>Statement</u>. Let (π, P_{m+1}, E) be a smooth representation. Then there exists a sheaf F on W with an action δ of the group G_m and an isomorphism $i: E \tilde{\to} S(F)$ of the space E with the space $S(F)$ of compactly supported sections of F such that

$$i(\pi(g)\xi) = \nu(g)^{1/2}\delta(g)(i(\xi))$$

$$i(\pi(v)\xi) = \psi_v \cdot i(\xi) \qquad g \in G_m, \ v \in V, \ \xi \in E.$$

The triple (F,δ,i) is uniquely defined by (π,E) up to a canonical isomorphism.

This statement is a variant of Mackey's construction. It is proved in [BZ1, 5]. More precisely, in [BZ1] the factor $\nu^{1/2}$ is omitted, so we should apply [BZ1] to $\nu^{1/2}\pi$.

We will identify E with $S(F)$ using the isomorphism i. Put

$$E_o = S_o(F) = \{\phi \in S(F) \,|\, \mathrm{supp}\,\phi \subset W\backslash 0\}, \ \pi_o = \pi|_{E_o}$$

For any point $w \in W$ we denote by F_w the stalk of the sheaf F at w. Consider two points $0 \in W$ and $e = (0,\dots,1) \in W$. It is clear, that

$$\mathrm{Stab}(0,G_m) = G_m, \quad \mathrm{Stab}(e,G_m) = P_m.$$

By definition we have

$$\Psi^-(\pi) = (\delta, G_m, F_o), \quad \Phi^-(\pi) = (\delta, P_m, F_e)$$

(this coincides with the definition in [BZ2, 3]).

From these formulae we see that:

(i) π is Φ^--degenerate $\Leftrightarrow$ F is concentrated at $0 \Leftrightarrow V$ acts trivially on E.

(ii) π is Φ^--homogeneous $\Leftrightarrow$ F has no nonzero section concentrated at 0.

9.3. Proof of the proposition 9.1.

(i) Suppose π is Φ^{-}-degenerate. Then V acts trivially on E and $\pi|_{G_m} = \nu^{1/2}\Psi^{-}(\pi)$. This implies 9.1a).

(ii) Suppose that the representation $\Phi^{-}(\pi) = (\delta, P_m, F_e)$ is Hermitian, i.e. the space F_e has a P_m-invariant Hermitian form B_e. Since G_m acts transitively on $W\smallsetminus 0$ and $P_m = \mathrm{Stab}(e, G_m)$, we can define a G-invariant system of Hermitian forms B_w for all $w \in W\smallsetminus 0$.

Now fix a Haar measure μ on W and define the Hermitian form B_o on $E_o = S_o(F)$ by

$$B_o(\phi,\phi') = \int_W B_w(\phi_w,\phi'_w)\,d\mu(w).$$

Since μ is (G_m,ν)-invariant, the formulae of statement 9.2 describing the representation π imply that the form B_o is G-invariant with respect to π_o. It is clear that B_o is positive definite if and only if B_e is positive definite.

Statement. The correspondence $B_e \mapsto B_o$ is a one-to-one correspondence between P_m-invariant Hermitian forms on (δ, F_e) and P_{m+1}-invariant Hermitian forms on (π_o, E_o).

For the (easy) proof see [BZ2, §3].

(iii) Let us fix a positive definite Hermitian form B_e on F_e and the corresponding form B_o on E_o. We want to find out when we can extend B_o to the positive definite P_{m+1}-invariant form B on (π, E). The answer is given by the following.

Analytic criterion. The form B_o can be extended to the P-invariant positive definite Hermitian form B on E iff for any $\phi \in E = S(F)$ the integral $I_\phi = \int_{W\smallsetminus 0} B_w(\phi_w,\phi_w)\,d\mu(w)$ converges. Indeed, if all these integrals converge we can define B by $B(\phi,\eta) = \int_W B_w(\phi_w,\eta_w)\,d\mu(w)$. Conversely, suppose we can extend B_o to B. Consider any open compact subgroup $W^o \subset W$ and denote by V^o the dual subgroup of V (i.e. $V^o = \{\nu \in V \mid \psi_\nu(W^o) = 1\}$). Then V^o is an open compact subgroup of V and we can define an operator

$A : E \to E$ by

$$A = \int_{V^o} \pi(v)\, d\mu_V(v) \ / \ \text{measure}(V^o) \ .$$

This operator is a projector and in the geometric realization it is given by $A(\phi) = \chi \cdot \phi$, where $\phi \in E = S(F)$ and χ is the characteristic function of W^o.

Since the form B is P_{m+1}-invariant, A is an orthogonal projector with respect to this form. Hence for any $\phi \in E$ we have

$$\|\phi\|_B^2 \geq \|(1-A)\phi\|_B^2 =$$

$$= \|(1-\chi)\phi\|_B^2 = \|(1-\chi)\phi\|_{B_o}^2 = \int_{w \in W \setminus W^o} \|\phi_w\|^2 \, d\mu(w)$$

This implies that the integral I_ϕ converges.

9.4. In order to finish the proof of proposition 9.1b) it remains to check that integrals I_ϕ converge for all $\phi \in E$ iff $e(\nu^{1/2}\Psi^-(\pi)) > 0$.

(i) Denote by $\mathfrak{p}$ a generator of the maximal ideal of the ring of integers of F (i.e. $|\mathfrak{p}| = q^{-1}$). We will identify $\mathfrak{p}$ with the central element $\mathfrak{p} \cdot 1_m \in G_m$.

Consider the quotient representation $(\pi' = \pi/\pi_o, G_m, E' = E/E_o)$. Since π has finite length, π' also has finite length. The operator $\pi'(\mathfrak{p})$ is then finite, i.e. it generates a finite dimensional algebra of operators.

Denote by $\mu_1, \ldots, \mu_r$ the eigenvalues of $\pi'(\mathfrak{p})$. By definition $|\mu_i| = q^{-e(\omega)}$ for each i, where $e(\omega)$ is the central exponent of an irreducible subquotient of $\pi' = \nu^{1/2}\Psi^-(\pi)$. Hence we can rewrite condition $e(\nu^{1/2}\Psi^-(\pi)) > 0$ as

(*) $$|\mu_i| < 1 \ \text{ for all } \ i.$$

(ii) Consider the space C of functions $f : W \setminus 0 \to \mathbb{C}$ such that supp f lies in a compact subset of W and f is locally constant on $W \setminus 0$, and define the representation (δ, G_m, C) by

$$\delta(g)f(w) = \nu(g)f(\delta(g^{-1})w) .$$

The restriction of δ on the subspace

$$C_o = \{ f \in C | f = 0 \text{ in a neighborhood of } 0\}$$

we denote by δ_o.

The Hermitian form B_e defines a pairing $\beta: E \otimes \bar{E} \to C$ by $\beta(\phi,\eta)(w) = B_w(\phi_w,\eta_w)$, which is a morphism of representations $\beta: \pi \otimes \bar{\pi} \to \delta$. It is clear that $\beta(\phi,\eta) \in C_o$ if $\phi \in E_o$ or $\eta \in E_o$.

Denote by $C_\beta \subset C$ the image of β. This space of functions satisfies the following conditions

(**) (α) $C_\beta \supset C_o$

(β) C_β is the linear span of positive functions $f \in C_\beta$.

(γ) Put $C' = C_\beta/C_o$ and denote by A the action of the operator $\delta(\mathfrak{p})$ on this quotient space. Then A generates a finite dimensional algebra of operators and all its eigenvalues are of the form $\mu_i\bar{\mu}_i$.

(δ) For any $\mu = \mu_1,\ldots,\mu_r$ there exists a positive function $f \in C_\beta$ such that $f \notin C_o$ and $A(f) = \mu\bar{\mu}f (\mathrm{mod}\ C_o)$.

Condition (α) follows from the fact that E, and hence C_β, is invariant with respect to multiplication on locally constant functions on W; (β) follows from the polarization formula.

The pairing β defines an epimorphism $\beta': E' \times \bar{E}' \to C'$ of G_m-modules which proves (γ). If $\phi \in E \setminus E_o$ is a vector, such that $\pi(\mathfrak{p})\phi = \mu\phi (\mathrm{mod}\ E_o)$, then the function $f = \beta(\phi,\phi)$ satisfies condition (δ).

(iii) <u>Lemma</u>. Let $C_\beta \subset C$ be a subspace, satisfying (α) - (δ). Then

a) if $|\mu_i| < 1$ for all i, then for any $f \in C_\beta$ the integral $I_f = \int_W f(w)d\mu(w)$ converges.

b) If $|\mu| \geq 1$ for one of the $\mu = \mu_i$, then I_f does not converge for the corresponding function $f \in C_\beta$.

This lemma implies that the conditions of criterion 9.3 (iii) are satisfied iff $|\mu_i| < 1$ for all i, that proves 9.1b).

<u>Proof of the lemma</u>. Consider some norm $\| \ \|$ on W, put $W_n = \{ w \in W \mid \|w\| = q^{-n}\}$ and define the function $f(n)$, $n \in \mathbb{Z}$, by

$$f(n) = \int_{W_n} f(w)\, d\mu(w).$$

This reduces the lemma to the analogous lemma about functions on $\mathbb{Z}$, where

$$C = \{ f: \mathbb{Z} \to \mathbb{C} \mid f(n) = 0 \text{ if } -n \text{ is large} \}.$$
$$C_o = \{ f \in C, \ \text{supp}\,(f) \text{ is finite}\}$$
$$I_f = \Sigma_{-\infty}^{\infty} f(n)$$
$$(\delta(p)f)(n) = f(n+1)$$

(we use the fact that $\delta(p)$ preserves integrals).

Condition (**) (γ) implies that $f(n)$ is an exponential polynomial for large n, i.e. $f(n) = \Sigma P_k(n) \cdot \lambda_k^n$, where λ_i are of the form $\mu_i \bar{\mu}_i$. This implies a).

If $|\mu| \geq 1$ and $Af = \mu\bar{\mu} f (\text{mod } C_o)$, then $f = c \cdot (\mu\bar{\mu})^n$ for large n. Since $f \notin C_o$, the constant c is not equal to 0, and hence I_f does not converge; this proves b).

9.5. <u>Proof of 9.1 c)</u>. Let B_e be a nonzero positive semidefinite form on $\Phi^-(\pi)$. As in 9.3, 9.4 we define the pairing $\beta: E \otimes \bar{E} \to C$ and put $C_\beta = \text{Im}\, \rho$. This space satisfies conditions (**) (α) (β) (γ) of 9.4.

<u>Lemma</u>. If $|\mu_i| \leq 1$ for all i then there exists a nonzero positive G_m-equivariant functional $I: C_\beta \to \mathbb{C}$ (here positive means $I(f) \geq 0$ for $f \geq 0$).

The formula $B(\phi,\eta) = I(\beta(\phi,\eta))$ defines a nonzero positive semidefinite P-invariant Hermitian form on E, that proves 9.1c).

<u>Proof of the lemma</u>. Choose some norm $\kappa = \| \ \|$ on W.

For any $s > 0$ the integral $I(\kappa,s;f) = \int_W f(w)\kappa(w)^s d\mu(w)$ converges and defines a positive functional $I(\kappa,s)$ on C_β.

The function $I(\kappa,s;f)$ is a rational function in q^s, and the order of the pole of this function at $s = 0$ is bounded by some number k, which does not depend on f (it does not exceed the degree of the minimal polynomial of the operator A). Let us choose minimal possible k and put

$$I(\kappa;f) = \lim_{s\to 0} s^k I(\kappa,s;f)$$

$I(\kappa)$ is a nonzero positive functional on C_β. In order to prove that it is G_m-equivariant it is sufficient to check that $I(\kappa)$ does not depend on κ.

If $\kappa \geq \kappa'$, then $I(\kappa,s) \geq I(\kappa',s)$ and hence $I(\kappa) \geq I(\kappa')$. Besides, $I(\lambda\kappa;s) = \lambda^s I(\kappa,s)$, i.e. $I(\lambda\kappa) = I(\kappa)$. Since for any other norm κ' we have $\lambda\kappa \geq \kappa' \geq \lambda^{-1}\kappa$ for some $\lambda > 0$, we have

$$I(\kappa) = I(\lambda\kappa) \geq I(\kappa') \geq I(\lambda^{-1}\kappa) = I(\kappa),$$

i.e., $I(\kappa) = I(\kappa')$. This proves the lemma.

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