# INDUCED REPRESENTATIONS OF THE GROUP GL(n) OVER A $\mathfrak{P}$-ADIC FIELD

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1. Let F be a local non-Archimedean field and $G_n = GL(n, F)$. If $\beta = (n_1, \ldots, n_r)$ is the decomposition of the number n, and $\rho_1, \ldots, \rho_r$ representations* of the groups $G_{n_1}, \ldots, G_{n_r}$, then using the standard construction of induction we may construct a representation $i(\rho_1, \ldots, \rho_r)$ of the group $G_n$ (see [1, p. 11], or Paragraph 5 below). As H. Jacquet proved (see [1 and 2]), any irreducible representation $\omega$ of $G_n$ is imbedded in a representation of the form $i(\rho_1, \ldots, \rho_r)$, where all the $\rho_i$ are irreducible and cuspidal.† Therefore, henceforward we shall assume everywhere that the $\rho_i$ are irreducible and cuspidal.

2. THEOREM 1. The representation $\pi = i(\rho_1, \ldots, \rho_r)$ has finite length not exceeding $r!$.

THEOREM 2. Let $\pi = i(\rho_1, \ldots, \rho_r)$ and $\pi' = i(\rho'_1, \ldots, \rho'_s)$ be representations of the group $G_n$. Then the following conditions are equivalent:

(I) $\pi$ and $\pi'$ have a common subfactor-representation;

(II) the sets of composition factors of $\pi$ and $\pi'$ coincide;

(III) $\mathrm{Hom}(\pi, \pi') \neq 0$;

(IV) $r = s$, and the sets $(\rho_1, \ldots, \rho_r)$ and $(\rho'_1, \ldots, \rho'_s)$ may be obtained one from the other by some permutation.

Let $\rho$ and $\rho'$ be representations of the groups $G_m$ and $G_{m'}$, respectively. They are said to be $\nu$-connected if $m = m'$ and either $\rho \approx \nu\rho'$ or $\rho' \approx \nu\rho$, where $\nu$ is a character of the group $G_m$, defined by the formula $\nu(g) = |\det g|$ (here $|\ |$ is the standard norm in the field F).

THEOREM 3 (Irreducibility Criterion). The representation $i(\rho_1, \ldots, \rho_r)$ is irreducible if and only if no two of the representations $\rho_1, \ldots, \rho_r$ are $\nu$-connected.

3. We would like to describe the structure of subrepresentations of $\pi = i(\rho_1, \ldots, \rho_r)$. We shall give a complete description in the case when all the $\rho_i$ are distinct. In this case the set $\Omega$ of composition factors of $\pi$ is single. Therefore, by setting a correspondence between each subrepresentation $\tau \subset \pi$ and the set of its composition factors $\Omega(\tau) \subset \Omega$, we obtain an inclusion of the structure of the subrepresentations of $\pi$ in the structure of subsets of $\Omega$ (i.e., $\Omega(\tau + \tau') = \Omega(\tau) \cup \Omega(\tau')$, $\Omega(\tau \cap \tau') = \Omega(\tau) \cap \Omega(\tau')$).

Let $\sigma = (\rho_{\sigma 1}, \ldots, \rho_{\sigma r})$ be some ordering of the set $\{\rho_1, \ldots, \rho_r\}$. Set $\pi_\sigma = i(\rho_{\sigma 1}, \ldots, \rho_{\sigma r})$. It follows from Theorem 2 that $\Omega = \Omega(\pi_\sigma)$ does not depend on $\sigma$. Let $\Delta$ be the set of pairs of $\nu$-connected representations amongst the $\rho_1, \ldots, \rho_r$ (clearly, $|\Delta| \leqslant r - 1$). For each ordering $\sigma$ define a function $f_\sigma$ on $\Delta$, with $f_\sigma(\{\rho, \nu\rho\}) = 0$ if $\rho$ precedes $\nu\rho$ in the ordering $\sigma$, and $f_\sigma(\{\rho, \nu\rho\}) = 1$ if $\nu\rho$ precedes $\rho$.

Proposition 1. a) Each $\pi_\sigma$ contains a unique irreducible subrepresentation $\omega_\sigma$.

*We shall consider only algebraic representations, i.e., those for which the stabilizer of every vector is open.
†Cuspidality is understood in the sense of [2, 3, and 4]; in [1] such representations are called absolutely cuspidal.

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b) The following conditions are equivalent: (I) $\pi_\sigma \approx \pi_{\sigma'}$, (II) $\omega_\sigma \approx \omega_{\sigma'}$, (III) $f_\sigma = f_{\sigma'}$.

c) Let $\mathfrak{P}(\Delta)$ be the set of functions $f\colon \Delta \to \{0, 1\}$. Define a map $\alpha\colon \Omega \to \mathfrak{P}(\Delta)$, such that $\alpha(\omega_\sigma) = f_\sigma$. Then $\alpha$ is well-defined and gives a bijection between $\Omega$ and $\mathfrak{P}(\Delta)$; in particular, $|\Omega| = 2^{|\Delta|} \leqslant 2^{r-1}$.

Proposition 2. Let $\sigma$ be some ordering. Then for each $\delta \in \Delta$ there exists a subrepresentation $\tau_{\sigma,\delta} \subset \pi_\sigma$ such that $\alpha(\Omega(\tau_{\sigma,\delta})) = \{f \in \mathfrak{P}(\Delta) \mid f(\delta) = f_\sigma(\delta)\}$. The subrepresentations $\tau_{\sigma,\delta}$ generate the structure of the subrepresentations in $\pi_\sigma$.

4. In the case when several of the representations $\rho_1, \ldots, \rho_r$ coincide, the structure of the subrepresentations of $i(\rho_1, \ldots, \rho_r)$ is considerably more complex. For example, the representation $i(1, \nu, 1)$ of the group $G_3$ decomposes into the direct sum of two distinct irreducible subrepresentations; but for the representation $i(1, \nu, 1, \nu)$ of $G_4$, the structure of the subrepresentations is infinite.

5. In the formulation of further results we shall need several definitions. Denote by Alg G the category of algebraic representations of the topological group G. Let G' and U be closed subgroups of G, for which G' normalizes U and $G' \cap U = \{e\}$; let $\theta$ be a character of the group U which normalizes G'. Define functors $I_{U,\theta}$ and $i_{U,\theta}$ from Alg G' into Alg G.

Let $\rho \in \operatorname{Alg} G'$ act in the space V. Define $I_{U,\theta}(\rho)$ (or explicitly $I_{U,\theta}(G, G', \rho)$) as a representation of the group G by right translations in the space of functions $f\colon G \to V$, satisfying the conditions:

1) $f(hug) = \operatorname{mod}_U^{1/2}(h)\cdot\theta(u)\cdot\rho(h)f(g)$ (here $h \in G'$, $u \in U$, $g \in G$, and $\operatorname{mod}_U(h)$ is the modulus of the automorphism $u \to huh^{-1}$ of the group U, see [5]).

2. There exists a neighborhood $N_f$ of the identity in G such that $f(gx) = f(g)$ for all $g \in G$, $x \in N_f$.

Denote by $i_{U,\theta}(\rho)$ the subrepresentation in $I_{U,\theta}(\rho)$, which acts on the subspace of functions which are finite with respect to the modulus of the subgroup G'U.

Example. Let $G = G_n$, $\beta = (n_1, \ldots, n_r)$ the decomposition of the number n, $P_\beta \subset G$ the corresponding parabolic subgroup, U the unipotent radical of $P_\beta$, and $G' = G_{n_1} \times \ldots \times G_{n_r}$ the Levi subgroup. Then if $\rho_i \in \operatorname{Alg} G_{n_i}$, $i(\rho_1, \ldots, \rho_r) = i_{U,1}(\rho_1 \otimes \ldots \otimes \rho_r) = I_{U,1}(\rho_1 \otimes \ldots \otimes \rho_r)$.

6. Let $P = P_n \subset G_n$ be the subgroup of matrices whose last line is of the form (0, 0, . . ., 0, 1). Our aim is to study the restriction of the representation $i(\rho_1, \ldots, \rho_r)$ to P. Let U be the group of unipotent upper triangular matrices, $M \subset U$ the unipotent radical of P and $\theta$ a character of U defined by the formula $\theta((u_{ij})) = \psi(\Sigma u_{i,i+1})$, where $\psi$ is a nontrivial additive character of the field F. Define functors $\Phi^+\colon \operatorname{Alg} P_{n-1} \to \operatorname{Alg} P_n$ and $\Psi^+\colon \operatorname{Alg} G_{n-1} \to \operatorname{Alg} P_n$, setting $\Phi^+ = i_{M,\theta}$, $\Psi^+ = i_{M,1}$. Note that these functors take irreducible representations into irreducible representations.

The following theorem describes the composition factors of the restriction of $i(\rho_1, \ldots, \rho_r)$ to P. It is useful in the computation of zeros and poles of the Gel'fand–Kazhdan $\Gamma$-function (see [3]).

THEOREM 4. For each subset J of the set of indices {1, . . ., r}, set $\tau_J = (\Phi^+)^{m-1} \circ \Psi^+ i(\rho_j \mid j \neq J) \in \operatorname{Alg} P$ (here $m = \Sigma n_j$, $j \in J$). Then for $i(\rho_1, \ldots, \rho_r)$ there exists a filtration by P-subrepresentations whose set of factors coincides with $\{\tau_J \mid J \neq \phi\}$.

7. Set $\tau = \tau_n = I_{U,\theta}(P, \{e\}, 1)$. The representation $\pi \in \operatorname{Alg} P$ is called nonsingular if $\operatorname{Hom}(\pi, \tau) \neq 0$ (see [2]). If in these circumstances $\pi$ coincides with $\tau$, we say that $\pi$ admits the Kirillov model; it is easily shown that in this case all morphisms from $\pi$ into $\tau$ are proportional.

THEOREM 5. The restriction of $i(\rho_1, \ldots, \rho_r)$ to P is nonsingular. It admits the Kirillov model if and only if for any pair of indices i, j, where $i < j$, $\rho_j \not\approx \nu\cdot\rho_i$.

COROLLARY. The restriction of a nonsingular irreducible representation of $G_n$ to P admits the Kirillov model.

This proposition was stated in [3] as a hypothesis.

8. Notes. a) The statement of Theorem 1 without the estimate of length, and the implications (I) $\Leftarrow$ (II) $\Leftarrow$ (III) $\Leftrightarrow$ (IV) in Theorem 2, are not new.

b) Theorem 3 gives an estimate of the width of the "critical interval" for complementary series of $G_n$ (see [4]).

c) Our proof of Theorem 3 is based on Theorem 4; therefore, as distinct from the proofs of Theorems 1 and 2, it does not relate to other groups.

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# THE PROBLEM OF INTEGRAL GEOMETRY ON THE GROUP $P_n(k)$ AND ITS APPLICATION TO THE THEORY OF REPRESENTATIONS


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Let k be a local non-Archimedean field, $O$ the ring of integers of k, $P_0$ the principal maximal prime ideal of this ring, and q = Card $O/P_0$. Denote by GL(n, k) the group of all nonsingular matrices of order n with elements in k, by $P = P_n(k)$ the subgroup of matrices $\|g_{ij}\| \in GL(n, k)$ such that $g_{n1} = \ldots = g_{n,n-1} = 0$, by $U \subset P$ the upper unipotent subgroup of GL(n, k), and by Z the commutator group of U. Clearly, Z is the subgroup of matrices $z = \|z_{ij}\| \in U$ such that $z_{i,i+1} = 0$ for i = 1, . . ., n − 1. If H is a subgroup or factor space in the group P, denote by S(H) the space of finite, locally constant complex-valued functions on H. The functions $f \in S(H)$ are called Schwarz–Bruhart functions on H.

Define a linear map from the space S(P) into the space of functions on P × P by the following formula:

$$\varphi(g_1, g_2) = \int_Z f(g_1^{-1} z g_2)\, dz, \qquad f \in S(P), \tag{1}$$

where dz is an invariant measure on Z, normalized by the condition $\int_{Z_0} dz = 1$ and $Z_0 \subset Z$ is the subgroup of integral matrices. It follows immediately from (1) that $\varphi(z_1 g_1, z_2 g_2) = \varphi(g_1, g_2)$ for any $z_1, z_2 \in Z$, and consequently $\varphi$ may be considered as a function on $Z \setminus P \times Z \setminus P$.

In this article we shall obtain the converse of (1) and give an application of this formula to the study of regular representations of GL(n, k).

1. Consider the generalized function $|t|^{\lambda-1}/\Gamma(\lambda)$ on k $(\lambda \in \mathbb{C})$, where $\Gamma(\lambda) = (1 - q^{\lambda-1})/(1 - q^{-\lambda})$.* It follows from the results in [1] that the generalized function $|t|^{\lambda-1}/\Gamma(\lambda)$, considered as an analytic function in λ, is an entire function. In view of this,

$$\left.\frac{|t|^{\lambda-1}}{\Gamma(\lambda)}\right|_{\lambda=0} = \delta(t), \quad \left.\frac{|t|^{\lambda-1}}{\Gamma(\lambda)}\right|_{\lambda=-l+1} = \frac{1-q^{l-1}}{1-q^{-l}}|t|^{-l}, \quad l = 2, 3, 4, \ldots, \tag{2}$$

*The function Γ(λ) may also be defined from the equation $\widetilde{|t|^{\lambda-1}} = \Gamma(\lambda)|t|^{-\lambda}$, where $\widetilde{|t|^{\lambda-1}}$ is the Fourier transform of the function $|t|^{\lambda-1}$ (see [1]).


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