SCHUBERT CELLS AND FLAG SPACE COHOMOLOGIES

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0. Let G be a linear semisimple algebraic group on C; we assume that it is connected and simply connected. Let B be some Borel subgroup of the group G, X = G/B the fundamental projective space of the group G. There are two ways of approaching the study of homology properties of the space X. The first (see Sec. 1) sees the homology spaces $H_*(X, Z)$ endowed with the basis $\{s_w\}$ ($w \in W$), where W is the Weyl group of the group G. The second (see Sec. 2) sees the cohomology ring $H^*(X, Q)$ identified with the factor ring $\bar{P} = P/J$ of the ring of polynomials P on the Lie algebra $\mathfrak{h}$ of the maximal torus $H \subset G$ by the ideal $J \subset P$, generated by the W-invariant polynomials without a free term. In this note we establish the relation between these two approaches (Theorems 4, 6) and also describe the the action of the group W on $H_*(X, Z)$ through the basis $\{s_w\}$ (Theorem 8). In Sec. 5 we explain the geometrical ideal behind the operations introduced in Sec. 3 in terms of the ring of correspondences of the space X (Theorem 9).

1. Homologies of X. Fix a Borel subgroup $B \subset G$. It is known [1] that X = G/B is a complete nonsingular algebraic manifold. Let $H \subset B$ be the maximal torus of G, N a unipotent radical of B, and N_- the unipotent subgroup "opposed" to N. Let $\mathfrak{g}$ be the Lie algebra of the group G, $\mathfrak{h}$ the subalgebra corresponding to H, $\mathfrak{h}^*$ the dual space of $\mathfrak{h}$, Δ, the system of roots of $\mathfrak{h}$ in $\mathfrak{g}$, Δ_+ the set of all positive roots ordered according to our choice of B, and let Σ be the set of simple roots. For each root $\gamma \in \Delta$ we denote by $H_\gamma \in \mathfrak{h}$ the element such that $\chi(H_\gamma) = 2(\chi, \gamma)/(\gamma, \gamma)$ for all $\chi \in \mathfrak{h}^*$. Here (,) is the scalar product on $\mathfrak{h}^*$, defined by the Killing form of the algebra $\mathfrak{g}$. If $\gamma \in \Delta_+$, then σ_γ is the element of the Weyl group W of the group G which is the reflection in $\mathfrak{h}^*$ relative to the plane orthogonal to γ. For each $w \in W$ denote by $l(w)$ the least number of factors in the decomposition $w = \sigma_{\alpha_1} \cdots \sigma_{\alpha_l}$ $(\alpha_i \in \Sigma)$. The decomposition for which $l = l(w)$ is called the reduced decomposition. If we put $N_w = N \cap w N_- w^{-1}$, then N_w is the unipotent subgroup of dimension $l(w)$.

Choose a maximal compact subgroup $K \subset G$ such that $T = K \cap H$ is a maximal torus in K. Then (see [1]) the natural map $K/T \to X$ is a homeomorphism. W acts on K/T, by acting in fact on $H_*(X)$ and $H^*(X)$.

THEOREM 1. (See [1]). Let $x_0 \in X$ be the image of $e \in G$. The open-closed submanifolds $X_w = Nwx_0$ ($w \in W$) give a partition of X into N-orbits. The natural map $N_w \to X_w (n \to nwx_0)$ is an isomorphism of algebraic manifolds.

Suppose $\bar{X}_w$ is the closure of X_w in X, $[\bar{X}_w] \in H_{2l(w)}(\bar{X}_w, Z)$ the fundamental cycle of the complex algebraic manifold $\bar{X}_w$ and $s_w \in H_{2l(w)}(X, Z)$ the image of $[\bar{X}_w]$ under the map induced by the natural inclusion $\bar{X}_w \hookrightarrow X$.

THEOREM 2. (Bott [2]). The elements s_w form a free basis in $H_*(X, Z)$.

2. Cohomologies of X. Let* $\chi \in \mathfrak{h}^*_Z$ and $\theta = \exp \chi$ be the character of the group H. If we extend θ on B by putting $\theta(n) = 1$ for $n \in N$, then we specify a one-dimensional representation of the group B. Since $G \to X$ is a principal bundle space with group B, θ defines a one-dimensional vector fiber E_χ on X. Suppose $c_\chi \in H^2(X, Z)$ is the first Chern class of E_χ. If we put $\alpha_1(\chi) = c_\chi$, we get the isomorphism $\alpha_1 : \mathfrak{h}^*_Z \to H^2(X, Z)$ see [3]. Let $P = S(\mathfrak{h}^*_Z \otimes Q)$ be the algebra of polynomial functions on $\mathfrak{h}^*_Z$ with rational coefficients. The isomorphism α_1 extends naturally to the homomorphism of rings α: $P \to H^*(X, Q)$.

*$\mathfrak{h}^*_Z$ is the lattice in $\mathfrak{h}^*$, generated by the fundamental.

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Extend the action of W from $\mathfrak{h}^*$ to P, and denote by I the subring of W-invariant elenents in P. Let

THEOREM 3. (See [3]). 1) The homomorphism α is compatible with the action of W on P and $H^*(X, Q)$. 2) $\operatorname{Ker}\alpha = J$ and $\bar{\alpha}$: $P/J \to H^*(X, Q)$ is an isomorphism.

We will identify $H^*(X, Q)$ with $\bar{P} = P/J$.

3. The Integration Formula. In Secs. 1 and 2 we explained the two ways of describing the homological structure of the space X. In the first case we introduced the basis $\{s_W\}$ into $H_*(X, Z)$, and in the second we identified $H^*(X, Q)$ with the ring $\bar{P} = P/J$. We now establish the relation between these two approaches.

Let $\{P_w\}_{w\in W}$ be the basis in $\bar{P}$, dual to the basis s_W from the Schubert cells.

THEOREM 4. Let $w_0 \in W$ be the unique element of maximal length $l(w_0) = \dim N = r$. Then $P_{w_0} = \rho^r/r!$ (mod J) (ρ is the half-sum of the positive roots).

To calculate the elements P_W for other $w \in W$ we construct for each root $\gamma \in \Delta$ the operator A_γ: $P \to P$, by putting $A_\gamma f = (f - \sigma_\gamma f)/\gamma \in P$ for $f \in P$. The operators A_γ have the following properties.

THEOREM 5. 1) $A_\gamma(J) \subset J$. 2) Let $w = \sigma_{\alpha_1} \cdots \sigma_{\alpha_l}$ $(\alpha_i \in \Sigma)$. If $l(w) < l$, then $A_{\alpha_1} \cdots A_{\alpha_l} = 0$. If $l(w) = l$, then the operator $A_w = A_{\alpha_1} \cdots A_{\alpha_l}$ depends only on w but not on the reduced decomposition of w.

Property 1) ensures that the operators A_W specify the operators $\bar{A}_W$: $\bar{P} \to \bar{P}$.

THEOREM 6. $P_w = \bar{A}_{w^{-1}w_0} P_{w_0}$.

B. Kostant had previously calculated the elements P_W by another method.

The elements s_W can be described similarly. To do so, we consider a ring S of differential operators on $\mathfrak{h}$ with constant rational coefficients, and we introduce and nondegenerate product $P \times S \to Q$, by putting $\langle D, f\rangle = Df(0)$. Each element s_W specifies a linear functional on P to which there corresponds a differential operator $D_W \in S$ of degree $l(w)$. Let F_W: $S \to S$ be the linear transformation conjugate to A_W with respect to the product $\langle\ ,\rangle$.

THEOREM 6'. $D_W = F_W$.

Let $w_1, w_2 \in W$, $\gamma \in \Delta_+$. If $w_1 = \sigma_\gamma w_2$ and $l(w_1) = l(w_2) + 1$, then we will write $w_1 \xrightarrow{\gamma} w_2$.

THEOREM 7. 1) Let $w \in W$, $l(w) = l$ and $\chi_1, \ldots, \chi_l \in \mathfrak{h}^*_Z$. Then $D_w(\chi_1 \cdots \chi_l) = \sum \chi_1(H_{\gamma_1}) \cdots \chi_l(H_{\gamma_l})$, where the sum runs over all chains $e \xrightarrow{\gamma_1} w_1 \xrightarrow{\gamma_2} \ldots \xrightarrow{\gamma_l} w_l = w^{-1}$. 2) Let $\chi \in \mathfrak{h}^*_Z$. Then the commutator of D_W with the multiplication operator on χ is specified by the formula $[D_w, \chi] = \sum w'\chi(H_\gamma) D_{w'}$ (the sum runs over all $w' \xrightarrow{\gamma} w$).

4. Action of the Weyl Group. Finally, it is easy to describe the action of W on $H^*(X, Q)$, if we use the isomorphism $\bar{\alpha}$: $P/J \to H^*(X, Q)$ from Theorem 3. The question of how to describe the action of W on the basis $\{s_W\}$ in $H_*(X, Z)$ is interesting, however.

THEOREM 8. Let $\alpha \in \Sigma$, $w \in W$. Then $\sigma_\alpha s_w = -s_w$, if $l(w\sigma_\alpha) = l(w) - 1$, and $\sigma_\alpha s_w = -s_w + \sum w'\alpha(H_\gamma) s_{w'}$ (the sum runs over $w' \xrightarrow{\gamma} w\sigma_\alpha$) if $l(w\sigma_\alpha) = l(w) + 1$.

5. The Ring of Correspondences. In paragraph 3 we constructed operators $\bar{A}_W$: $H^*(X, Q) \to H^*(X, Q)$. Now we shall give a geometrical realization of these operators. Suppose $Y = X \times X$. For each $w \in W$ we put $Y_W = \{(gB, gwB)\} \subset Y$. The sets Y_W specify a partition of Y into orbits with respect to the action of G. Moreover, $\dim Y_W = \dim X + l(w)$. Each manifold Y_W specifies an operator y_W: $H^*(X, Q) \to H^*(X, Q)$, which lowers the degree by $2l(w)$; to apply the construction described in [4] one must be the Poincaré duality on Y).

THEOREM 9. 1) $y_W = \bar{A}_W$. 2) The operators $y_W (w \in W)$ and the multiplication operators on $\alpha(\alpha \in \Sigma)$ generate the algebra of all operators in $H^*(X, Q)$.

We wish to note that, as the example of GL_n indicates, the results of Secs. 4 and 5 are intimately related with the finite group representation theory of Chevalley.

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