DIMENSION OF COMMUTATIVE SUBRINGS IN $R_{n,k}$

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The ring $R_{n,k}$ was introduced in [1]. This ring is an algebra over a field K of characteristic 0 with generators $p_1, \ldots, p_n, q_1, \ldots, q_n, x_1, \ldots, x_k$ and defining relations

$$[p_i, p_j] = [q_i, q_j] = [p_i, x_j] = [q_i, x_j] = [x_i, x_j] = 0, \quad [p_i, q_j] = \delta_{ij}.$$

The purpose of the present article is to prove the following theorem.

THEOREM 1. If $A \subset R_{n,k}$ is a commutative subring (always considered over a field K), then $\dim A \le n + k$.

Here the term dimension is used as in [1]. In order to prove the theorem we will introduce another definition of dim A: Namely, we introduce the increasing filter $K = L^0 \subset L^1 \subset \ldots \subset L^t \subset \ldots$ in $R_{n,k}$, where L^t is the set of elements in $R_{n,k}$ that can be represented in the form of polynomials of degree $\le t$ in p_i, q_i, and x_j.

Let $a(t, A) = \dim_K(A \cap L^t)$, and set

$$\dim A = \overline{\lim_{t\to\infty}} \frac{\ln a(t, A)}{\ln t}.$$

It is easy to see that the dimension in our case is no less than the dimension as defined in [1], so we can prove Theorem 1 for our definition. As a matter of fact, we will prove a somewhat stronger theorem:

THEOREM 2. If $A \subset R_{n,k}$ is a commutative subring, then $a(t, A) \le C^t_{t+n+k}$.

Proof. Consider the associated graduated ring $gr = \sum_{t=0}^{\infty} gr^t$ for $R_{n,k}$ with respect to the filter L^t. Here $gr^t = L^t/L^{t-1}$. It is easy to show that $L^t \cdot L^s \subset L^{t+s}$ and $[L^t, L^s] \subset L^{t+s-2}$ ($[L^t, L^s]$ is the linear space generated by the elements [a, b], where $a \in L^t$, $b \in L^s$). Thus, in gr we can introduce a multiplication and Poisson brackets [,] so that $gr^s \cdot gr^{s+t} \subset gr^{s+t}$ and $[gr^s, gr^t] \subset gr^{s+t-2}$. It was shown in [1] that gr is a ring of polynomials in p_i, q_i, and x_j. With respect to [,], gr is a Lie algebra, where $[p_i,p_j] = [q_i,q_j] = 0$, $[p_i,q_j] = \delta_{ij}$, and x_i belongs to the center. It is easy to show that the Leibnitz formula $[ab, c] = a[b, c] + b[a, c]$ holds. With each element $r \in R_{n,k}$ we associate the homogeneous element [r] in gr as follows: If $r \in L^t$ and $r \notin L^{t-1}$, then for $[r] \in gr^t$ we take the image of r under the factorization $L^t \to gr^t$. Let [A] denote the linear space generated by the elements [a], where $a \in A$. Then [A] is a subring, its elements commute with respect to Poisson brackets (since $a_1a_2 = a_2a_1$ implies that $[[a_1], [a_2]] = 0$), and

$$a(t, [A]) = \dim\Big(\bigcup_{i\le t} gr^i \cap [A]\Big) = \sum_{i\le t} \dim(gr^i \cap [A])$$

$$= \sum_{i\le t} [\dim(L^i \cap A) - \dim(L^{i-1} \cap A)] = a(t, A).$$

It is therefore sufficient to show that if B is a subring in gr that is commutative with respect to Poisson brackets, then $a(t, B) \le C^t_{t+n+k}$.

We will treat gr as a ring of polynomials over the linear space M (conjugate to the space $\{p_i, q_i, x_j\}$).

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LEMMA 1. If P, Q $\in$ gr, $\alpha \in$ M, then the value of [P, Q] at the point α depends only on the linear terms dP(α) and dQ(α) at this point (for R $\in$ gr we let dR denote the differential form $dR = \sum \frac{\partial R}{\partial p_i} dp_i + \sum \frac{\partial R}{\partial q_i} dq_i + \sum \frac{\partial R}{\partial x_j} dx_j$). [P, Q]($\alpha$) is a skew-symmetric 2-form of vectors dP(α) and dQ(α) of the space M_α^* (which is the cotangent space at the point α).

Proof. We have

$$\left.\begin{array}{l} P = P(\alpha) + P_1 + P_2 \\ Q = Q(\alpha) + Q_1 + Q_2 \end{array}\right\}, \text{ where } \begin{array}{l} P_1 \text{ and } Q_1 \text{ are linear terms at } \alpha, \\ P_2 \text{ and } Q_2 \text{ have a first-order zeroth.} \end{array}$$

[P, Q](α) does not depend on the constants P(α) and Q(α), nor on P_2 and Q_2, since P_2 (Q_2) can be represented in the form $P_2 = \sum Y_i Z_i$, where $Y_i(\alpha) = Z_i(\alpha) = 0$, and then for any R $\in$ gr

$$[P_2, R](\alpha) = \sum [Z_i, R](\alpha) Y_i(\alpha) + \sum [Y_i, R](\alpha) Z_i(\alpha) = 0.$$

The second part of the lemma is obvious.

Let T_α denote the form obtained on M_α^*. Then $T_\alpha(dp_i, dq_i) = - T_\alpha(dq_i, dp_i) = 1$, and T_α is equal to 0 on the remaining pairs of basis vectors dp_i, dq_i, dx_i. L_α^B denotes the subspace in M_α^* defined by the equation $L_\alpha^B = \{db(\alpha) | b \in B\}$. Then the form T_α degenerates on L_α^B, since $[b_1, b_2] = 0$ for $b_1, b_2 \in B$. Since the rank T_α is 2n, we have dim $L_\alpha^B \leq n + k$ (this is fundamental to the proof). Theorem 2 is now an immediate consequence of the following lemma.

LEMMA 2. Let R be a ring of polynomials over a linear space M(i.e., $R = K[Y_1, \ldots, Y_N]$, where Y_i are the coordinates in M), and let A be a subring of R, such that dim $L_\alpha^A \leq l$ for all $\alpha \in$ M. Then $a(t, A) \leq C_{t+l}^t$.

Proof. We can assume that dim$L_\alpha^A = l$ for some $\alpha \in$ M, since otherwise the lemma would be reduced to the case of a smaller l. We take α for the coordinate origin. dim $L_0^A = l$, so we can choose $g_1, \ldots, g_l \in A$, such that $dg_1(0), dg_2(0), \ldots, dg_l(0)$ are linearly independent. Changing coordinates and subtracting the appropriate constants from g_i, we find that $dg_i(0) = dY_i$, $g_i(0) = 0$ ($i \leq l$). The condition dim $L_\alpha^A \leq l$ is equivalent to the condition $da_1(\alpha) \wedge \ldots \wedge da_l(\alpha) \wedge da_{l+1}(\alpha) = 0$ for $a_1, \ldots, a_{l+1} \in A$, $\alpha \in M$ (here $\wedge$ is the outer product). Since this equation is true for all $\alpha \in$ M, it is true in the polynomial sense, i.e., the coefficient of $dY_{i_1} \wedge \ldots \wedge dY_{i_{l+1}}$ is a polynomial that is identically equal to 0 for any set $i_1, \ldots, i_{l+1}$. Let [r] with r $\in$ R denote a homogeneous polynomial composed of lower-order terms in r. Then

$$da_1 \wedge da_2 \wedge \ldots \wedge da_{l+1} = d[a_1] \wedge \ldots \wedge d[a_{l+1}]$$

$$+ \text{ terms with higher-degree coefficients.}$$

Thus, $d[a_1] \wedge \cdots \wedge d[a_{l+1}] = 0$ for $a_1, \ldots, a_{l+1} \in A$. Let $\bar{A}$ denote the linear space generated by [a] with a $\in$ A. It is easy to see that $a(t, \bar{A}) \geq a(t, A)$. If $\bar{a} \in \bar{A}$, then, since $Y_i = [g_i] \in \bar{A}$ ($i \leq l$),

$$dY_1 \wedge \ldots \wedge dY_l \wedge d\bar{a} = \sum_{i > l} \frac{\partial \bar{a}}{\partial Y_i} dY_1 \wedge \ldots \wedge dY_l \wedge dY_i = 0.$$

i.e., $\partial\bar{a}/\partial Y_i = 0$ for $i > l$. This means that $\bar{a}$ is a polynomial in $Y_1, \ldots, Y_l$. Since the number of polynomials of l variables of degree $\leq t$ is C_{t+l}^{t}, we have $a(t, \bar{A}) \leq C_{t+l}^{t}$, and a fortiori $a(t, A) \leq C_{t+l}^{t}$. This completes the proof of Theorems 1 and 2.

We can similarly prove

THEOREM 3. Let L be a finite dimensional Lie algebra over a field K of characteristic 0. U(L) is an algebra containing L, and A is a commutative subring of U(L). Then dim $A \leq n + k$, where k is the codimension of the orbits of common position in the representation conjugate to the associated representation, $2n + k = \dim L$.

LITERATURE CITED

1. I. M. Gel'fand and A. A. Kirillov, DAN SSSR, 167, No. 3 (1966), 503-505.