CHEVALLEY'S THEOREM FOR COMPLEX CRYSTALLOGRAPHIC COXETER GROUPS

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1. We consider a complex crystallographic group W, i.e., a discrete group of affine transformations of a complex affine space V such that the quotient X = V/W is compact. We assume that W is generated by affine reflections (W is a crr-group). A ccr-group W is called a Coxeter group (or ccc-group) if the group dW of linear parts of W is a Coxeter group (i.e., can be expressed by real matrices in some basis). In what follows, we restrict ourselves to the case when the group W is irreducible (as an affine group).

The purpose of this paper is to describe the structure of the analytic space X = V/W for a ccc-group W. It turns out that X is a rational variety (with singularities), or more precisely, a "weighted" projective space. This is an analog for ccc-groups of the classical theory of Chevalley on invariants [1]. An analogous result is apparently true for any ccc-groups.

2. A classification of ccr-groups is given in [3]. Let S be an affine system of roots on the real affine space V(R) (see [2]). We will assume that the finite root system of dS is irreducible and reduced, and the minimal constant function c in the lattice of functions generated by S is equal to 1 (see [2, Sec. 6]). Let $\tau \in \mathbf{C}, \operatorname{Im} \tau > 0$, V the complexification of the space V(R). For each pair $\alpha \in S$, $k \in \mathbf{Z}$ we consider in V the hyperplane $\pi(\alpha, k) = \{z \in V \mid \tau\alpha(z) = k\}$ the group of motions of V generated by the second-order reflections in the hyperplanes $\pi(\alpha, k)$, $\alpha \in S, k \in \mathbf{Z}$ is denoted by W(S, τ). As is shown in [3], W(S, τ) is an irreducible ccc-group, and every (irreducible) ccc-group is isomorphic to some group W(S, τ).

We define the number p = p(S) as follows: p = 2 for systems of the type $B_l^\vee$, $C_l^\vee$, $F_4^\vee$, p = 3 for $G_2^\vee$, and p = 1 for the remaining systems (in particular, for all systems of type S(R)) (see [2, Sec. 5]). Let $\Gamma_0(p)$ be a group of transformations of the upper halfplane Im τ > 0 having the form $\tau \to (a\tau + b)/(c\tau + d)$, where $a, b, c, d \in \mathbf{Z}$, $ad - bc = 1$ and $c \in p\mathbf{Z}$. Then $W(S, \tau) \approx W(S, \tau')$, if $\tau \sim \tau' \pmod{\Gamma_0(p)}$, and $W(S(R)^\vee, \tau) \approx W(S(R^\vee)^\vee, \tau')$, if $\tau' \sim -1/p\tau \pmod{\Gamma_0(p)}$; this exhausts all the isomorphisms among the groups W(S, τ) (see [3]).

3. Let W = W(S, τ) be a ccc-group acting in an l-dimensional space V, and let the numbers $n_0, \ldots, n_l$ index the corresponding vertices of the Dynkin diagram of the system $S^\vee$, dual to S (see [2, Application 1]).

<u>THEOREM.</u> The analytic space X = V/W is isomorphic to a "weighted" projective space of the type $n_0, \ldots, n_l$, i.e., to the quotient space of $\mathbf{C}^{l+1} \setminus \{0\}$ by the action of the group given by the formula $(z_0, \ldots, z_l) \mapsto (t^{n_0} z_0, \ldots, t^{n_l} z_l)$, $t \in \mathbf{C}^*$.

In the following paragraphs we present a scheme of proof of this theorem based on the theory of θ-functions and automorphic forms.

It is possible to obtain the MacDonald identies [2] and certain similar identities by analogous methods. At the final stage (see Sec. 9) we limit ourselves for economy of space to the case S = S(R). We remark that this case is treated (from another point of view) in [4]. However, the proof given there rests on the erroneous assertion that all elliptic curves are isomorphic as real algebraic varieties (see [4, p. 29]).

4. Let C be the Weyl chamber of the system S with vertices $x_0, \ldots, x_l$ (x_0 is a special point for $S^\vee$), $\alpha_0, \ldots, \alpha_l$ the corresponding basis of S, and let σ_i be the reflection corresponding to the root α_i (see [2]). It is easy to construct quadratic functions $U_0, \ldots, U_l$ on V such that

$$\sigma_j(U_i) = U_i + \delta_{ij}\alpha_j \quad (j = 0, \ldots, l), \qquad U_i(x_0) = 0.$$

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Let Λ be the additive semigroup generated by the functions U_i, $P = U_0 + \ldots + U_l \in \Lambda$. For each function $U \in \Lambda$ we put

$$M(U) = \min U(x),\ x \in V(\mathbf{R}), \text{ and } N(U) = \tilde{U}/\tilde{U}_0 \in \mathbf{Z},$$

where $\tilde{U}$ is the quadratic part of the function U. We put

$$n_i = N(U_i), \quad g = N(P) = n_0 + \ldots + n_l.$$

Let W_S be the affine Weyl group for the system S, $T \subset W$ the subgroup of translations in the direction $\tau^{-1}V(\mathbf{R})$. Then $W = W_S \times T$.

5. A 1-cocycle $\gamma = \{\gamma_w\}$ of W with values in the group $\mathcal{O}^*(V)$ of invertible holomorphic functions on V is said to be even if for every reflection $\sigma \in W$ $\gamma_\sigma(z) = 1$ if $\sigma(z) = z$. Let $\tilde{H}^1(W, \mathcal{O}^*(V))$ be the subgroup of classes of even cocycles.

Proposition. The group $\tilde{H}^1(W, \mathcal{O}^*(V)) \approx \mathbf{Z}$. A generator is given by the class of the cocycle $\gamma: \gamma_w = 1$ for $w \in T$, $\gamma_w = \exp[\nu(U_0 - w(U_0))]$ for $w \in W_S$ (here and below, $\nu = 2\pi i\tau$).

For $k \geqslant 0$ we consider the cocycles $\gamma_w^k = (\gamma_w)^k$ and $\delta_w^k = (\det w)\,\gamma_w^{k+g}$ (here $\det w = \det(dw) = \pm 1$). We denote by A_k and B_k the corresponding spaces of θ-functions ($A_k = \{f \in \mathcal{O}(V) \mid wf = (\gamma_w)^k$ for all $w \in W\}$ and similarly for B_k). The theorem follows in the standard way from the next lemma.

LEMMA. There exist functions $f_i \in A_{n_i}$, such that the ring $A = \oplus A_k$ is isomorphic to $\mathbf{C}[f_0, \ldots, f_l]$.

6. To each function $U \in \Lambda$ we associate the θ-function $\Sigma_U \in B_{N(U)}$, defined by the series

$$\Sigma_U = \exp[-\nu(N(U+P)\,U_0 + M(U+P))] \sum (\det w)\ \exp[\nu w(U+P)]$$

(the sum is over $w \in W_S$).

We introduce the Siegel inner product in the space B_k by putting

$$\| f \|^2 = \int | f(z) \exp(k\nu U_0(\varkappa(z))) |^2\, d\mu(z),$$

where $\varkappa: V \to V(\mathbf{R})$ is the projection along $\tau^{-1}V(\mathbf{R})$, the integral is over V/W, and μ is Lebesgue measure normalized by the condition $\mu(V/W) = 1$. Just as in the classical theory (see [5, Chap. III]), it is proved that the functions Σ_U with $N(U) = k$ form an orthogonal basis in B_k, and $\| \Sigma_U \|^2 = \text{const}\,(\operatorname{Im}\tau)^{l/2}$.

We note that the map $\theta \mapsto \Sigma_0 \cdot \theta$ defines an isomorphism between A_k and B_k; in particular, $B_0 = \mathbf{C} \cdot \Sigma_0$.

7. Consider the derivation $D_i: A \mapsto \mathcal{O}(V)$, where for $i = 1, \ldots, l$ D_i is the derivative along the vector τ^{-1} grad α_i, and $D_0(f) = kf$ for all $f \in A_k$. If $f_i \in A_{n_i}$, then we denote the determinant of the matrix $D_i f_j$ by $J(f_0, \ldots, f_l)$. It is easy to see that $J(f_0, \ldots, f_l) \in B_0$. We prove that $J(f_0, \ldots, f_l) \neq 0$ for some choice of f_i; it follows that the functions $f_i \in A$ are algebraically independent. Then a simple calculation of the dimensions of A_k (see Sec. 6) shows that $A = \mathbf{C}[f_0, \ldots, f_l]$.

8. Consider the space of multilinear forms $B_{n_0} \times B_{n_1} \times \ldots \times B_{n_l} \to \mathbf{C}$ with inner product induced by the Siegel inner product on the spaces B_k. We define the form L by the formula $L(b_0, \ldots, b_l) = J(b_0/\Sigma_0, \ldots, b_l/\Sigma_0)/\Sigma_0$ and put $F(\tau) = \| L \|^2$. We need to prove that $F(\tau) \neq 0$ for all τ, i.e., $L \not\equiv 0$.

9. Let S be a system of type S(R). Consider the function $H(\tau) = F(\tau)\,(\operatorname{Im}\tau)^{l/2} \times |\eta(\tau)^{-l}|^2$, where η is the Dedekind function (see [2]). Using the invariance of the Siegel inner product under the isomorphisms $W(S, \tau) \approx W(S, \tau')$ and properties of the functions Σ_U (see Para. 6), it can be shown that the function $H(\tau)$ is invariant under the group $\Gamma_0(1)$, is a sum of squares of the moduli of analytic functions, and admits the estimate $H(\tau) | \exp(\nu/3) |^2 \to 0$ as $\operatorname{Im}\tau \to +\infty$. These properties imply that $H(\tau)$ does not vanish anywhere, for $H(\tau) \not\equiv 0$.

We remark that the estimate for $H(\tau)$ is derived separately for each root system. This is a very crude estimate; for example, for the systems A_l and C_l, $H(\tau) = \text{const}$.

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PRODUCT FORMULAS FOR THE NILSEN NUMBERS OF BUNDLE MAPS

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Two fixed points x_1, x_2 of a continuous map $g: X \to X$ are said to be Nilsen equivalent if there exists a path τ between them which is homotopic rel x_1, x_2 to the image $g(\tau)$. The Nilsen number $N(g)$ of a continuous map g of a compact connected CW complex is the number of those classes of Nilsen-equivalent fixed points whose mapping index is nonzero. Thus the Nilsen number is a lower bound, and in many cases turns out to be a precise lower bound, for the number of geometrically distinct fixed points of mappings homotopic to g.

Like the Lefschetz number $L(g)$, the number $N(g)$ has a whole series of useful properties. As is known from [1, 2, 8], the Lefschetz numbers of bundle maps have the multiplicative property. However, this is not in general true for the Nilsen numbers. Nevertheless, in certain special cases (see [1-4]), the Nilsen numbers also have the multiplicative property; in proving the corresponding formulas, one assumes homological conditions of orientability of the bundle or the Yang conditions, which are purely homotopic in character. We hold to the point of view that the conditions under which the product formula holds for the homotopically defined number $N(g)$ must also be homotopic.

Theorem 1 gives a counterexample which shows that the second part of the Brown—Fadell theorem [3], and in addition some propositions of Fadell [6], are incorrect without the additional assumption that the bundle is orientable. Theorem 2 states conditions under which there is a general product formula for the Nilsen numbers.

Let $\mathscr{F} = (F, E, B, p)$ be a locally trivial bundle of compact CW spaces, $g: E \to E$ and $g': B \to B$ continuous maps such that $g' \circ p = p \circ g$. Each path τ from the point b_1 to a point b_2 in the base space B induces a homotopy equivalence

$$h_\tau: F_{b_1} \to F_{b_2}$$

of the fibers $F_{b_1} = p^{-1}(b_1)$ and $F_{b_2} = p^{-1}(b_2)$. If a path τ is given from the point $g'(b)$ to the point b in the base space of the bundle then it is possible to define a map g_b of the fiber F_b into itself by the formula $g_b = h_\tau \circ g|_{F_b}: F_b \to F_b$.

<u>LEMMA 1</u>. The Nilsen number $N(g_b)$ is relatively independent of the choice of the point b, i.e., the set of numbers $N(g_b)$ for all classes of homotopic paths between the points $g'(b)$ and b coincides with the set of numbers $N(g_{b'})$, where $b' \in B$.

If the product formula is valid for the Nilsen numbers, i.e., if $N(g) = N(g') N(g_b)$, and if $N(g') \neq 0$, then the Nilsen number $N(g_b)$ is also independent of the choice of the homotopy class of the path τ. The same thing holds if the generalized product formula

$$P(\mathscr{F}, g) N(g_b) N(g') = Q(\mathscr{F}, g) N(g)$$

is valid, where the nonzero numbers P and Q depend only on the homotopy type of the bundle $\mathscr{F}$ and map g.

<u>THEOREM 1</u>. Let F and $h: F \to F$ be a CW complex and a homotopy equivalence such that

1) the map $h^n (n > 0)$ is homotopic to the identity map id F;

2) $N(h) \neq N(\mathrm{id}\, F)$.

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