MODULES OVER A RING OF DIFFERENTIAL OPERATORS. STUDY OF THE FUNDAMENTAL SOLUTIONS OF EQUATIONS WITH CONSTANT COEFFICIENTS

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In this paper we study modules over the ring D of differential operators with polynomial coefficients on the space $\mathbf{R}^N$.

An example of such a module is the space S' of generalized functions on $\mathbf{R}^N$.

To each D-module M with a finite number of generators there corresponds its carrier $\Delta(M)$ which is an algebraic submanifold in $\mathbf{C}^N \times \mathbf{C}^{N*}$. In particular, to each generalized function $\mathscr{E} \in S'$ there corresponds the manifold $\Delta(D(\mathscr{E}))$, where $D(\mathscr{E})$ is the submodule of S' generated by the function $\mathscr{E}$.

The first chapter is devoted to the study of the space $S_0' \subset S'$, which consists of generalized functions $\mathscr{E}$, for which $\dim(\Delta(D(\mathscr{E}))) \leqslant N$.

The main result of this chapter is the proof of the following theorem.

THEOREM A. Let $\mathscr{E} \in S_0'$. We set $\Delta' = \Delta(D(\mathscr{E})) \setminus \mathbf{C}^N \times 0$ and denote by $\tilde{\Delta}$ and $\tilde{\Delta}_{\mathbf{R}}$ the projections of the sets Δ' and $\Delta' \cap \mathbf{R}^N \times \mathbf{R}^{N*}$ onto $\mathbf{C}^N$. Then

a) $\dim_{\mathbf{C}} \tilde{\Delta} < \dim_{\mathbf{R}} \tilde{\Delta}_{\mathbf{R}} < N$.

b) $\mathscr{E}$ is a real analytic function of the set $\tilde{\Delta}_{\mathbf{R}}$.

c) The function $\mathscr{E}$ has a continuation as a multivalued analytic function to the region $\mathbf{C}^N \setminus \tilde{\Delta}$.

d) The distinct branches of the function $\mathscr{E}$ generate a finite-dimensional linear space.

The proof of Theorem A is based on the construction over the set $\mathbf{C}^N \setminus \tilde{\Delta}$ of a certain algebraic bundle with an integrable connection, while the function $\mathscr{E}$ is a coordinate of its flat section.

In the first chapter it is also shown that the space S_0' is a D-module and is invariant under Fourier transform.

In the second chapter the following theorem is proved.

THEOREM B. Let f be a generalized function on the line lying in the space S', and let P be a polynomial on $\mathbf{R}^N$ with real coefficients. Then the generalized function $f(P) \in S_0'$ on $\mathbf{R}^N$.

In §7 of Chapter 2 we use Theorem B to study the fundamental solutions of equations with constant coefficients. In particular, we prove the following theorem.

THEOREM C. Any linear differential operator L with constant coefficients has a fundamental solution $\mathscr{E}_L$, lying in S_0'.

COROLLARY. Assertions a), b), c), and d) of Theorem A hold for the function $\mathscr{E}_L$.

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CHAPTER 1

Modules Over the Ring of Differential Operators and the Spaces S_j^i

1. The Carrier of a Module Over a Noncommutative Ring. The ring of differential operators with polynomial coefficients is naturally considered as an example of a noncommutative filtered algebra. We begin by studying a certain class of such algebras.

Let D be an algebra with identity over the field **C** and let $0 = D^{-1} \subset D^0 \subset \ldots \subset D^n \subset \ldots$ be its filtration by subspaces with the following conditions satisfied:

A0. $\bigcup_{n=0}^{\infty} D^n = D,$

A1. $D^m \cdot D^n \subset D^{m+n},$

A2. $[D^m, D^n] \subset D^{m+n-1},$

A3. $1 \in D^0.$

For $\mathcal{D} \in D$ we set $\deg \mathcal{D} = \min\{n \mid \mathcal{D} \in D^n\}$, $\deg 0 = -\infty$. We introduce the notation $\Sigma^{(n)} = D^n/D^{n-1}$ and $\Sigma^{(n)} = D^n/D^{n-1}$ и $\Sigma = \bigoplus_{n=0}^{\infty} \Sigma^{(n)}$.

In Σ a graded ring structure is introduced in a natural way (see [10]); Σ is then a commutative ring with identity.

We will assume that the following conditions are satisfied:

A4. Σ is a ring without zero divisors.

A5. Σ is a finitely generated algebra over **C**.

Definition 1.1. 1. If $\mathcal{D} \in D$ then by $\sigma(\mathcal{D})$ we denote the element in $\Sigma^{(n)}$ (where $n = \deg \mathcal{D}$), which is the image of $\mathcal{D}$ under the mapping $D^n \to \Sigma^{(n)}$; $\sigma(0) = 0$.

2. If L is a linear subspace of D, then by $\sigma(L)$ we denote the linear subspace of Σ, generated by the elements $\sigma(\mathcal{D})$, where $\mathcal{D} \in L$.

Elements of the ring D we call operators; if $\mathcal{D} \in D$, then the element $\sigma(\mathcal{D}) \in \Sigma$ we will call the symbol of the operator $\mathcal{D}$.

It is easy to verify the following lemma.

LEMMA 1.1. 1. If $\mathcal{D}_1, \mathcal{D}_2 \in D$, then

$$\sigma(\mathcal{D}_1 \mathcal{D}_2) = \sigma(\mathcal{D}_1) \cdot \sigma(\mathcal{D}_2) \text{ and } \deg(\mathcal{D}_1 \mathcal{D}_2) = \deg \mathcal{D}_1 + \deg \mathcal{D}_2.$$

2. If L is a left ideal in D, then $\sigma(L)$ is an ideal in Σ.

3. If L is a finite-dimensional subspace of D, then $\dim L = \dim \sigma(L)$.

Definition 1.2. We denote by W the affine variety corresponding to the ring Σ (see [6]). As a set, W coincides with the set of maximal ideals of the ring Σ.

Example. Let V be a linear space over **C**, D_V the ring of differential operators with polynomial coefficients on V, D_V^n the space of operators of degree not greater than n. It is easy to verify that the ring D_V satisfies conditions A0–A5, while Σ is canonically isomorphic to the ring of polynomial functions on $W = V \times V^*$, and σ is the usual symbol of the operator.

Definition 1.3. 1. If M is a D-module,* $e_1, \ldots, e_s \in M$, then by $D(e_1, \ldots, e_s)$ we denote the D-submodule of M generated by $e_1, \ldots, e_s$, and by $D^n(e_1, \ldots, e_s)$ the linear subspace of M (over C), generated by the elements $\mathcal{D} e_i$, where $\mathcal{D} \in D^n$.

2. The D-filtration $\{M^n\}$ in the D-module M we call that filtration of M by subspaces $0 = M^{-1} \subset M^0 \subset \ldots \subset M^n \subset \ldots$, for which $D^m \cdot M^n \subset M^{m+n}$ and $\bigcup_{n=0}^{\infty} M^n = M$.

3. Two D-filtrations $\{M^n\}$ and $\{\widetilde{M}^n\}$ of the D-module M will be called equivalent if there exists a number k, such that $M^n \subset \widetilde{M}^{n+k}$ and $\widetilde{M}^n \subset M^{n+k}$ for all n.†

LEMMA 1.2. If $e_1, \ldots, e_q$ and $f_1, \ldots, f_s$ are two systems of generators of the D-module M, then the D-filtrations $\{M^n\} = \{D^n(e_1, \ldots, e_q)\}$ and $\{\widetilde{M}^n\} = \{D^n(f_1, \ldots, f_s)\}$ are equivalent.

Proof. We choose k such that $e_1, \ldots, e_q \in \widetilde{M}^k$ and $f_1, \ldots, f_s \in M^k$. It is clear that $M^n \subset \widetilde{M}^{n+k}$ and $\widetilde{M}^n \subset M^{n+k}$ for all n.

Definition 1.4. If $e_1, \ldots, e_s$ is any system of generators of the D-module M, then the D-filtration $\{M^n\} = \{D^n(e_1, \ldots, e_s)\}$ will be called standard. It follows from Lemma 1.2 that all standard filtrations are equivalent.

PROPOSITION 1.3. Let M be a finitely generated D-module, and let L be a D-submodule of M. Then

1) L is a finitely generated D-module.

2) If $\{M^n\}$ is the standard filtration of M, then the D-filtration $\{\widetilde{L}^n\} = \{L \cap M^n\}$ is equivalent to the standard filtration $\{L^n\}$.

COROLLARY. D is a Noetherian ring (see [10]).

Proof of the Proposition. We first consider the case in which M is a free module with basis $e_1, \ldots, e_s$. We put $M^n = D^n(e_1, \ldots, e_s)$ and consider the Σ-module $M_\Sigma = \bigoplus_{n=0}^{\infty} M_\Sigma^{(n)}$, where $M_\Sigma^{(n)} = M^n/M^{n-1}$. We define the mapping $\sigma\colon M \to M_\Sigma$ in analogy with Definition 1.1. The space $\sigma(L)$ is a Σ-module. Since the ring Σ is Noetherian, it follows that $\sigma(L)$ contains a finite number of generators v_i, which can be assumed to be homogeneous elements. Let $v_i = \sigma(u_i)$, $u_i \in L$.

We will show that any element $e \in M^n \cap L$ belongs to $D^n(u_i)$. Suppose that this has been proved for all $e \in M^{n-1} \cap L$. We write $\sigma(e)$ in the form $\sigma(e) = \sum c_i v_i$, where the c_i are homogeneous elements of Σ of degrees $n - \deg v_i \leq n$. Let $\mathcal{D}_i \in D$ be such that $\sigma(\mathcal{D}_i) = c_i$. Then $\mathcal{D}_i \in D^n$, $e - \sum \mathcal{D}_i u_i \in M^{n-1} \cap L$, and by hypothesis $e - \sum \mathcal{D}_i u_i \in D^{n-1}(u_i)$; hence $e \in D^n(u_i)$. The assertion of the lemma has now been proved for a free module M, since $D^n(u_i) \subset M^{n+k} \cap L$, where k is the maximal degree of the elements u_i.

We now consider an arbitrary D-module M and a system of generators $e_1, \ldots, e_s$. We denote by $\widehat{M}$ the free D-module with generators $f_1, \ldots, f_s$ and by τ the mapping $\tau\colon \widehat{M} \to M$, given by the formula $\tau(f_i) = e_i$. Let $\widehat{L} = \tau^{-1}(L) \subset M$, let $\widehat{u}_i$ be the generators of $\widehat{L}$, chosen in the manner indicated above, and let $u_i = \tau(\widehat{u}_i)$. Then u_i are the generators of L, and $L \cap M^n = \tau(\widehat{L} \cap \widehat{M}^n) \subset \tau(\widehat{L}^n) = L^n$. Since $u_i \in M^k$ for some k, it follows that $L^n \subset L \cap M^{n+k}$. This completes the proof of the proposition.

Definition 1.5. Given a set of elements $e_1, \ldots, e_s$ in the D-module M, we denote by $\mathrm{Ann}(e_1, \ldots, e_s)$ the left ideal in D consisting of those operators $\mathcal{D}$, such that $\mathcal{D} e_i = 0$ for all i.

PROPOSITION 1.4. Let the D-module M be generated by the elements $e_1, \ldots, e_s$, and let c be a homogeneous element in Σ. Then the following conditions are equivalent.

1. $c \in \operatorname{rad} \sigma(\mathrm{Ann}(e_1, \ldots, e_s))$.‡

2. If $\mathcal{D} \in D$ is an operator such that $\sigma(\mathcal{D}) = c$ and $\{M^n\}$ is the standard filtration of the module M, then for all n $\mathcal{D}^k M^n \subset M^{n+k\deg\mathcal{D} - q(k)}$, where q(k) grows unboundedly together with k.

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*By a D-module we mean a left unitary D-module.

†This concept is an equivalence relation in the set of D-filtrations of the module M.

‡If J is an ideal in Σ, then $\operatorname{rad} J = \{c \in \Sigma \mid \text{for some } n\ c^n \in J\}$.

Proof. Since condition 2 does not depend on the choice of standard filtration, we will assume that $M^n = D^n(e_1, \ldots, e_s)$.

Condition 2 is equivalent to the statement that for some k $\mathscr{D}^k M^n \subset M^{n+k\deg\mathscr{D}-1}$ for all n. Therefore, condition 2 depends only on $\sigma(\mathscr{D})$.

$1 \Rightarrow 2$. Let $c \in \operatorname{rad} \sigma(\operatorname{Ann}(e_1, \ldots, e_s))$.

It is sufficient to verify condition 2 for the element c^p for some p. We choose p such that $c^p \in \sigma(\operatorname{Ann}(e_1, \ldots, e_s))$, i.e., $c^p = \sigma(\mathscr{D})$, where $\mathscr{D} \in \operatorname{Ann}(e_1, \ldots, e_s)$.

Let $e = \sum \mathscr{D}_i e_i$, where $\mathscr{D}_i \in D^n$. Then

$$\mathscr{D}e = \sum \mathscr{D}\mathscr{D}_i e_i = \sum [\mathscr{D}, \mathscr{D}_i] e_i \in M^{n+\deg\mathscr{D}-1}.$$

Thus, $\mathscr{D}M^n \subset M^{n+\deg\mathscr{D}-1}$; i.e., c^p satisfies condition 2.

$2 \Rightarrow 1$. Let c satisfy condition 2. We will show that for each element $f \in M$ it is possible to construct an operator $\mathscr{D}_f \in D$, such that a) $\mathscr{D}_f \cdot f = 0$; b) $\sigma(\mathscr{D}_f)$ is the degree of the element c.

Indeed, let $\mathscr{D} \in D$ be an element such that $\sigma(\mathscr{D}) = c$, and let $L_f = D(f)$. Since the filtration $\{\tilde{L}_f^n\} = \{L_f \cap M^n\}$ is equivalent to the standard one, it follows that $\mathscr{D}^k f \in D^{k\deg\mathscr{D}-1}(f)$ for some k; i.e., $\mathscr{D}^k f = \mathscr{D}' f$, where $\deg \mathscr{D}' < k \cdot \deg \mathscr{D}$. It is clear that the operator $\mathscr{D}^k - \mathscr{D}'$ satisfies conditions a) and b).

We now consider the operators

$$\mathscr{D}_1 = \mathscr{D}_{e_1}, \mathscr{D}_2 = \mathscr{D}_{\mathscr{D}_1 e_2} \cdot \mathscr{D}_1, \ \ldots \ , \mathscr{D}_s = \mathscr{D}_{(\mathscr{D}_{s-1} e_s)} \cdot \mathscr{D}_{s-1}.$$

Then $\mathscr{D}_s \in \operatorname{Ann}(e_1, \ldots, e_s)$, and $\sigma(\mathscr{D}_s)$ is the degree of the element c. Therefore, $c \in \operatorname{rad} \sigma(\operatorname{Ann}(e_1, \ldots, e_s))$.

Definition 1.6. 1. For any ideal J in the ring Σ we denote by $Z(J) \subset W$ the set of zeros of the ideal J (i.e., the set of all maximal ideals in Σ, which contain J).

2. Let M be a finitely generated D-module. We put $J(M) = \operatorname{rad} \sigma(\operatorname{Ann}(e_1, \ldots, e_s))$ and $\Delta(M) = Z(\sigma(\operatorname{Ann}(e_1, \ldots, e_s))) = Z(J)$, where $e_1, \ldots, e_s$ is any system of generators of the module M.

It follows from Proposition 1.4 that J(M) and hence $\Delta(M)$ are independent of the choice of system of generators. Indeed, J(M) can be defined as the ideal in Σ, generated by homogeneous elements satisfying condition 2 of Proposition 1.4.

LEMMA 1.5. If $0 \to M_1 \to M \to M_2 \to 0$ is an exact sequence of finitely generated D-modules, then $\Delta(M) = \Delta(M_1) \cup \Delta(M_2)$ and $J(M) = J(M_1) \cap J(M_2)$.

Proof. It is sufficient to show that $J(M) = J(M_1) \cap J(M_2)$. Let c be a homogeneous element of Σ. We choose a system of generators $e_1, \ldots, e_s$ in M and let $f_1, \ldots, f_s$ be their images in M_2. We introduce the standard filtrations $\{M^n\}$, $\{M_1^n\}$, and $\{M_2^n\}$. It follows immediately from Propositions 1.3 and 1.4 that $J(M) \subset J(M_1) \cap J(M_2)$.

We now prove the reverse inclusion. Let $c \in J(M_1) \cap J(M_2)$. There exist a number p and an operator $\mathscr{D} \in \operatorname{Ann}(f_1, \ldots, f_s)$, such that $\sigma(\mathscr{D}) = c^p$. Then $\mathscr{D}(e_i) \in M_1$, and since $\sigma(\mathscr{D}) \in J(M_1)$, it follows that $\mathscr{D}^{k+1}(e_i) \subset M_1^{k\deg\mathscr{D}-q(k)} \subset M^{k\deg\mathscr{D}-q'(k)}$, where q'(k) grows unboundedly, together with k. Therefore, $\sigma(\mathscr{D}) = c^p \in J(M)$ This means that $c \in J(M)$.

§2. Beginning in this section, the ring D will be a ring of differential operators with polynomial coefficients on an N-dimensional complex linear space V (see the example of § 1).

We fix a system of coordinates $x_1, \ldots, x_N$ in V and a dual system of coordinates $y_1, \ldots, y_N$ in V*.

We denote by R the ring of polynomial functions on V and by Σ the ring of polynomial functions on $V \times V^*$ ($R = \mathbf{C}[x_1, \ldots, x_N]$, $\Sigma = \mathbf{C}[x_1, \ldots, x_N, y_1, \ldots, y_N]$).

Each operator $\mathscr{D} \in D$ can be written in a unique way in the form $\mathscr{D} = \sum_{\alpha,\beta} c_{\alpha\beta}(\mathscr{D}) x^\alpha \left(\frac{\partial}{\partial x}\right)^\beta$, where $\alpha = (i_1, \ldots, i_N)$, $\beta = (j_1, \ldots, j_N)$, $c_{\alpha\beta} \in \mathbf{C}$, $x^\alpha = x_1^{i_1} \cdot \ldots \cdot x_N^{i_N}$, $(\partial/\partial x)^\beta = (\partial/\partial x_1)^{j_1} \cdot \ldots \cdot (\partial/\partial x_N)^{j_N}$, and each element $c \in \Sigma$ can be written uniquely in the form $c = \sum_{\alpha,\beta} c_{\alpha\beta}(c) x^\alpha y^\beta$, where $y^\beta = y_1^{j_1} \cdot \ldots \cdot y_N^{j_N}$.

In the ring D we introduce the filtration $\{D^n\}$, by putting $D^n = \{\mathscr{D} \in D \mid c_{\alpha\beta}(\mathscr{D}) = 0 \text{ for } |\beta| > n\}$.* The accompanying graded ring relative to this filtration is isomorphic to Σ. The mapping σ: $D \to \Sigma$ is given by the formula $c_{\alpha\beta}(\sigma(\mathscr{D})) = c_{\alpha\beta}(\mathscr{D}) \cdot \delta_{|\beta|}^{\deg \mathscr{D}}$, where δ is the Kronecker delta.

The affine manifold W, corresponding to the ring Σ, is isomorphic to $V \times V^*$. We denote by π the natural projection π: $W \to V$.

We will consider V as the complexification of the real linear space V_R. The coordinates x_i will be assumed to be real.

We denote by S the space of rapidly decreasing, infinitely differentiable differential forms of degree N on V_R and by S' the dual space of slowly increasing generalized functions (see [1]). We consider S' as a D-module (see [1]).

Definition 2.1. $S_i' = \{\mathscr{E} \in S' \mid \dim \Delta(D(\mathscr{E})) \leqslant N + i\}$.

PROPOSITION 2.1. S_i' is a D-submodule of S'.

Proof. Let $\mathscr{E}_1, \mathscr{E}_2 \in S_i'$, $\mathscr{D}_1, \mathscr{D}_2 \in D$ and $\mathscr{E} = \mathscr{D}_1\mathscr{E}_1 + \mathscr{D}_2\mathscr{E}_2$.

It follows from Lemma 1.5 that

$$\Delta(D(\mathscr{E})) \subset \Delta(D(\mathscr{E}_1, \mathscr{E}_2)) \subset \Delta(D(\mathscr{E}_1) \oplus D(\mathscr{E}_2)) = \Delta(D(\mathscr{E}_1)) \cup \Delta(D(\mathscr{E}_2)).$$

Therefore, $\dim \Delta(D(\mathscr{E})) \leqslant N + i$, i.e., $\mathscr{E} \in S_i'$.

We proceed to the study of functions $\mathscr{E} \in S_0'$. Each such function $\mathscr{E}$ satisfies a system of equations $I(\mathscr{E}) = 0$, where I is some ideal in D such that $\dim Z(\sigma(I)) \leq N$. We fix such an ideal I and introduce the notation

$$\Delta = Z(\sigma(I)) = \Delta(D/I), \quad \Delta_R = \Delta \cap (V_R \times V_R^*), \quad \tilde{\Delta} = \pi(\Delta \setminus (V \times 0)), \quad \tilde{\Delta}_R = \pi(\Delta_R \setminus (V \times 0)).$$

We note that $\dim \tilde{\Delta} \leq N-1$, since the manifold Δ is given in W by equations homogeneous in $\{y_i\}$.

THEOREM 2.2. Any solution $\mathscr{E} \in S'$ of the system of equations $I(\mathscr{E}) = 0$ is analytic in the region $V_R \setminus \tilde{\Delta}_R$.

Proof. Let $c_1, \ldots, c_k \in \sigma(I)$ be a choice of homogeneous elements which are simultaneously zero nowhere except on Δ. By raising them to the appropriate power it can be assumed that $c_1, \ldots, c_k$ have the same degree of homogeneity n. Let $\mathscr{D}_1, \ldots, \mathscr{D}_k \in I$ be elements such that $\sigma(\mathscr{D}_i) = c_i$.

We consider the operator $\mathscr{D} = \sum \bar{\mathscr{D}}_i \mathscr{D}_i$ (here $\bar{\mathscr{D}}_i$ is obtained from $\mathscr{D}_i$ by replacing all the coefficients by their complex conjugates). $\mathscr{D}$ has order 2n, and $\sigma(\mathscr{D}) = \sum \sigma(\bar{\mathscr{D}}_i)\sigma(\mathscr{D}_i) = \sum \bar{c}_i c_i$. On the set $V_R \times V_R^* \setminus \Delta_R$ this symbol is nonzero. Therefore, the operator $\mathscr{D}$ is elliptic off the set $\tilde{\Delta}_R$ (see [7]). Since $\mathscr{D}(\mathscr{E}) = 0$, it follows from Theorem 7.5.1 of [7] that $\mathscr{E}$ is an analytic function off $\tilde{\Delta}_R$. This completes the proof of the theorem.

We now study the analytic continuation of the function $\mathscr{E}$. We denote by Ω a connected component of the set $V_R \setminus \tilde{\Delta}_R$.

THEOREM 2.3. 1. Any solution $\mathscr{E}$ of the system $I(\mathscr{E}) = 0$ can be continued analytically from the set Ω to the set $V \setminus \tilde{\Delta}$ as a multivalued analytic function.

2. In a neighborhood of any point $x \in V \setminus \tilde{\Delta}$ all the analytic solutions of the system $I(\mathscr{E}) = 0$ from a finite-dimensional space.

COROLLARY. The multivalued function obtained as the analytic continuation of $\mathscr{E}$ is finitely dependent, i.e., all its branches are linear combinations of a finite number of branches.

Proof of the Theorem. We consider the module M = D/I as an R-module. Let U be an arbitrary open affine subset of $V \setminus \tilde{\Delta}$, (see [6]), and let R_U be the ring of regular functions on U. We study the behavior of the function $\mathscr{E}$ in the region U.

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*$|\beta| = j_1 + \ldots + j_N$.

LEMMA 2.4. For each such set U the module $M_U = R_U \otimes_R M$ has a finite number of generators as an R_U-module.

Proof. We consider elements $e_1, e_2, \ldots, \in M$, which are images of operators $(\partial/\partial x)^\beta$ under the mapping $D \to M$. They generate M as an R-module. We consider the set $\pi^{-1}(U) \subset V \times V^*$ and the ideal $\sigma_U(I)$ in the ring of regular functions on it generated by $\sigma(I)$. Since $Z(\sigma_U(I)) \subset U \times 0$, it follows from Hilbert's Nullstellensatz* that there exists a number m such that $y^\beta \in \sigma_U(I)$ for $|\beta| > m$. This implies that the R_U-module M_U is generated by the images of the operators $(\partial/\partial x)^\beta$ with $|\beta| \leq m$.

We choose a system of generators $e_1 = 1, e_2, \ldots, e_k$ of the R_U-module M_U. Then for any i and j

$$\frac{\partial}{\partial x_i}(e_j) = \sum_{l=1}^{k} A_{ij}^l e_l, \text{ where } A_{ij}^l \in R_U.$$

Now let $\mathscr{E}$ be an analytic solution of the system $I((\varepsilon)) = 0$ in (an analytic) neighborhood of the point $x \in U$. We consider the vector-valued function $\{\mathscr{E}_j\} = \{e_j(\mathscr{E})\}$, $j = 1, 2, \ldots, k$. It satisfies the system of equations

$$\frac{\partial}{\partial x_i}(\mathscr{E}_j) = \sum_{l=1}^{k} A_{ij}^l \mathscr{E}_l \ (1 \leqslant j,\ l \leqslant k,\ 1 \leqslant i \leqslant N). \tag{*}$$

The next lemma is proved by standard methods from the theory of ordinary differential equations.

LEMMA 2.5. In a simply connected region V_0 of the complex linear space V with coordinates $x_1, \ldots, x_N$ let there be given the system of equations (*), where the A_{ij}^l are analytic functions in V_0. Then:

1. If the system (*) has an analytic solution $\hat{\mathscr{E}} = (\mathscr{E}_1, \ldots, \mathscr{E}_k)$ in a neighborhood of the point $x \in V_0$, then this solution can be continued analytically to the entire region V_0.

2. Each analytic solution $\hat{\mathscr{E}}$ of the system (*) in the region V_0 is defined by its value at the point x (in particular, the dimension of the space of such solutions does not exceed k).

Each point $x \in V \setminus \tilde{\Delta}$ is contained in an affine neighborhood $U_x \subset V \setminus \tilde{\Delta}$. Therefore, Theorem 2.3 is easily deduced from Lemmas 2.4 and 2.5.

§3. In this section we shall give a certain method of computing $\dim \Delta(M)$ for the D-module M.

In the ring D we introduce the filtration $\{D_1^n\}$, by putting

$$D_1^n = \{\mathscr{D} \in D \mid c_{\alpha\beta}(\mathscr{D}) = 0 \text{ for } |\alpha| + |\beta| > n\}.$$

If $\mathscr{D} \in D$, then $\sigma_1(\mathscr{D}) \in \Sigma$, and

$$c_{\alpha\beta}(\sigma_1(\mathscr{D})) = c_{\alpha\beta}(\mathscr{D}) \cdot \delta_{|\alpha|+|\beta|}^{\deg_1 \mathscr{D}}.$$

Similarly, we define in Σ the filtration Σ_1^n and the symbol by putting $\tilde{\sigma}_1 \colon \Sigma \to \Sigma$:

$$\Sigma_1^n = \{c \in \Sigma \mid c_{\alpha\beta}(c) = 0 \text{ for } |\alpha| + |\beta| > n\}, \quad c_{\alpha\beta}(\tilde{\sigma}_1(c)) = c_{\alpha\beta}(c) \cdot \delta_{|\alpha|+|\beta|}^{\deg c}.$$

The filtrations $\{D_1^n\}$ and $\{\Sigma_1^n\}$ are convenient, since all the spaces D_1^n and Σ_1^n are finite dimensional.

Definition 3.1. 1. Let M be a D-module (or a Σ-module), and let $e_1, \ldots, e_s$ be its system of generators. We put $d_1^n(M) = \dim D_1^n(e_1, \ldots, e_s)$ (respectively $\dim \Sigma_1^n(e_1, \ldots, e_s)$).

2. If L is a subspace of D (or of Σ), then we set $L_1^n = L \cap D_1^n$ (respectively $L \cap \Sigma_1^n$).

It is easy to verify that if I is an ideal in D and $M = D/I$, then $d_1^n(M) = \dim D_1^n - \dim I_1^n$.†

THEOREM 3.1. Let M be a finitely generated D-module. Then the following conditions are equivalent:

*We present the formulation of Hilbert's theorem. Let A be a finitely generated algebra over **C**, and let Π be the corresponding affine variety. Let J_1, J_2 be ideals in A with $Z(J_1 \subset Z(J_2)$. Then for some n $J_2^n \subset J_1$.
†We assume that in the module $M = D/I$ a generator is fixed which is the image of the identity under the mapping $D \to M$.

1. $\dim \Delta(M) \leq m$ ($\Delta(M)$ is constructed with the filtration D^n).

2. $d_1^n(M) = O(n^m)$.*

It follows from Lemma 1.5 that it is sufficient to consider the case $M = D/I$.

For the proof of Theorem 3.1 we need an analogous result for the ring Σ.

PROPOSITION 3.2 (see Theorems 41 and 42 of [2]). If J is an ideal in Σ, then $d_1^n(\Sigma/J)$ is a polynomial in n for large n. The degree of this polynomial is equal to $\dim Z(J)$.

We proceed to the proof of Theorem 3.1.

$2 \Rightarrow 1$. It is easy to verify that $\sigma(I_1^n) \subset \sigma(I)_1^n$. Therefore, $\dim \sigma(I)_1^n \geq \dim \sigma(I_1^n) = \dim I_1^n$, whence $d_1^n(\Sigma/\sigma(I)) \leq d_1^n(D/I) = O(n^m)$. It follows from Proposition 3.2 that $\dim \Delta(M) \leq m$.

$1 \Rightarrow 2$. In order to prove this implication, we introduce in the ring D a countable number of filtrations $\{D_k^n\}$ such that $\dim D_k^n < \infty$ and for large k the filtrations $\{D_k^n\}$ approximate the filtration $\{D^n\}$. Namely, we set

$$D_k^n = \{\mathscr{D} \in D \mid c_{\alpha\beta}(\mathscr{D}) = 0 \text{ for } |\alpha| + k|\beta| > n\}$$

and define the mapping σ_k: $D \to \Sigma$ by the formula $c_{\alpha\beta}(\sigma_k(\mathscr{D})) = c_{\alpha\beta}(\mathscr{D}) \cdot \delta_{|\alpha|+k|\beta|}^{\deg_k \mathscr{D}}$. Similarly, we define filtrations Σ_k^n of the ring Σ. If I(J) are ideals in D (in Σ) then we define $d_k^n(D/I)$ [respectively $d_k^n(\Sigma/J)$] in analogy with Definition 3.1.

We note that the assertions $\{d_k^n(D/I) = O(n^m)\}$ and $\{d_1^n(D/I) = O(n^m)\}$ are equivalent for any k. Indeed, $D_k^n \subset D_1^n \subset D_k^{kn}$, whence $d_k^n(D/I) \leq d_1^n(D/I) \leq d_k^{kn}(D/I)$.

A similar argument shows that $d_k^n(\Sigma/\sigma_k(I)) \leq d_1^n(\Sigma/\sigma_k(I))$. Since $d_k^n(\Sigma/\sigma_k(I)) = d_k^n(D/I)$, for the proof of the implication $1 \Rightarrow 2$ it is sufficient to verify that $d_1^n(\Sigma/\sigma_k(I)) = O(n^m)$ for some k.

It follows from Proposition 3.2 that $d_1^n(\Sigma/\sigma(I)) = O(n^m)$. This means that $d_1^n(\Sigma/\tilde{\sigma}_1\sigma(I)) = d_1^n(\Sigma/\sigma(I)) = O(n^m)$.

Thus, for the proof of the implication $1 \Rightarrow 2$ it is sufficient to show that $\sigma_k(I) \supset \tilde{\sigma}_1\sigma(I)$ for some k.

Let $\mathscr{D}_i \in I$ be operators such that the elements $\tilde{\sigma}_1\sigma(\mathscr{D}_i)$ generate the ideal $\tilde{\sigma}_1\sigma(I)$. We take $k = \max \deg_1(\mathscr{D}_i)$. It is then easy to verify that $\tilde{\sigma}_1\sigma(\mathscr{D}_i) = \sigma_k(\mathscr{D}_i)$, i.e., $\sigma_k(I) \supset \tilde{\sigma}_1\sigma(I)$. Therefore, $d_1^n(\Sigma/\sigma_k I) \leq d_1^n(\Sigma/\tilde{\sigma}_1\sigma(I)) = O(n^m)$. This completes the proof of Theorem 3.1.

COROLLARY 3.3. Let there be given an automorphism ω of the algebra D, D-modules M_1 and M_2, and a linear (over C) isomorphism F: $M_1 \to M_2$ such that $F\mathscr{D} = \omega(\mathscr{D})F$ for any operator $\mathscr{D} \in D$. Then for any element $e \in M$ $\dim \Delta(D(e)) = \dim \Delta(D(Fe))$.

Proof. There exists a k such that $D_1^1 \subset \omega(D_1^k)$ and $\omega(D_1^1) \subset D_1^k$. Then $D_1^n \subset \omega(D_1^{kn})$ and $\omega(D_1^n) \subset D_1^{kn}$ for all n. Therefore, $F(D_1^n(e)) \subset D_1^{kn}(Fe) \subset F(D_1^{k^2n}(e))$, i.e., $d_n^1(D(e)) \leq d_1^{kn}(D(Fe)) \leq d_1^{k^2n}(D(e))$. It follows from Theorem 3.1 that $\dim \Delta(D(e)) = \dim \Delta(D(Fe))$.

Example 1. Let $\tilde{\omega}$ be an invertible polynomial mapping $V \to V$, with $\omega(V_R) = V_R$. We put $M_1 = M_2 = S'$ and denote by ω and F the automorphisms of D and S' induced by the mapping $\tilde{\omega}$. Corollary 3.3 shows that $F(S_i') = S_i'$.

Example 2. We put $M_1 = M_2 = S'$. Let F be Fourier transformation with respect to the variables $x_1, \ldots, x_k$ $(k \leq N)$ and let ω be given by the formulas

$$\left\{\omega(x_j) = -i\frac{\partial}{\partial x_j},\ \omega\left(\frac{\partial}{\partial x_j}\right) = -ix_j \text{ for } j \leq k;\ \omega(x_j) = x_j,\ \omega\left(\frac{\partial}{\partial x_j}\right) = \frac{\partial}{\partial x_j} \text{ for } j > k\right\}.$$

It is easy to verify (see [1]) that the conditions of Corollary 3.3 are satisfied. Thus, we obtain

COROLLARY 3.4. If F : S' $\to$ S' is Fourier transformation with respect to part of the variables, then $F(S_i') = S_i'$ for all i.

*It can be shown that $d_1^n(M)$ is a polynomial in n for large n.

CHAPTER 2

The Functions $f(P)$

§4. We fix a polynomial P with real coefficients on V_R. Let f be a generalized function on the line. We wish to study a system of equations which the function $f(P)$ satisfies on V_R.

We first define what we mean by the function $f(P)$. We introduce the notation: T is the line, t is the coordinate on T; $R_T = C[t]$; D_T is the ring of differential operators with polynomial coefficients on T with the filtration $\{D_T^n\}$ with respect to the degree of the operators. The derivatives $(\partial P/\partial x_i)$ we denote by P_i.

We set sing P = $\{x \in V \mid P_i(x) = 0 \text{ for all } i\}$. As is known, $P(\text{sing } P) \subset T$ is a finite set.

We fix an open region $\Theta \subset V_R$ such that $P(\partial\Theta)$ is a finite set ($\partial\Theta$ is the boundary of the region Θ). We set $\Omega = P(\Theta)\backslash P(\partial\Theta)\backslash P(\text{sing } P \cap \Theta)$. $\bar{\Omega}$ is the closure of Ω, $\partial\Omega = \bar{\Omega}\backslash\Omega$.

Definition 4.1. We denote by C the space of continuous functions on Ω and consider the mapping η: $S \to C$ defined as follows: if $\varphi \in S$, $t_0 \in \Omega$, then we set

$$\eta(\varphi)(t_0) = \int_{P^{-1}(t_0)\cap\Theta} \omega,$$

where ω is the differential form of degree $N-1$ on the manifold $P^{-1}(t_0) \cap \Theta$, such that $\omega \wedge dP = \varphi$ (see [1], Ch. 3, §4).

PROPOSITION 4.1. If $\varphi \in S$, then $\eta(\varphi)$ is an infinitely differentiable function on Ω, which is rapidly decreasing at infinity. In a neighborhood of each point $h \in \partial\Omega$ $\eta(\varphi)$ admits a representation of the form $\eta(\varphi) = \sum \varphi_i(z) \cdot f_i(z)$, where $\varphi_i \in C^\infty(T)$, and $\{f_i\}$ is a fixed (independent of φ) finite set of functions of the form $f_i = (\ln |z|)^{k_i}|z|^{r_i}$, $0 \le k_i \le N-1$, $r_i \in \mathbf{Q}$. (Such an expansion is possible separately in a right and left neighborhood of the point h; z is a local parameter in this neighborhood.)

The proof does not differ essentially from arguments given in [3] and [4]; we therefore only sketch the proof.

The definition of $\eta(\varphi)$ is first generalized to the case in which X is a nonsingular algebraic manifold, P is a rational function on X, and φ is an infinitely smooth complex-valued measure on X. Since V is imbedded in a projective space X of dimension N, the original problem reduces to the analogous problem for a compact manifold X. Using the theorem of Hironaka on the resolution of singularities (as is done in [3] and [4]) and then localizing the problem by means of a partition of unity, the problem can be reduced to the following case: $X = \mathbf{R}^N$, $P = x_1^{i_1} \cdot \ldots \cdot x_N^{i_N}$, $\Theta = \{x \in \mathbf{R}^N \mid x_i > 0 \text{ for all } i\}$, φ is an infinitely differentiable form with compact support (the numbers $i_k \in \mathbf{Z}$ are not necessarily positive). In this case $\eta(\varphi)$ can be described directly (see [1], Ch. 3, §4).

Definition 4.2. 1. We denote by C^k the space of k times continuously differentiable functions f on $\bar{\Omega}$ such that $f^{(i)}(h) = 0$ for $h \in \partial\Omega$, $i = 0, 1, \ldots, k$, and $\nu_k(f) = \sup_{t\in\Omega}\left((1+|t|^k)\cdot\left(\sum_{i=0}^{k}|f^{(i)}(t)|\right)\right) < \infty$ (here $f^{(i)} = (\partial/\partial t)^i f$).

The norm ν_k defines a Banach space structure in C^k.

2. We set $\mathcal{L} = \eta(S)$ and $\mathcal{L}^k = \mathcal{L} \cap C^k$ $(k = 0, 1, \ldots)$.

Corollary of Proposition 4.1. $\dim(\mathcal{L}/\mathcal{L}^k) < \infty$.

Definition 4.3. 1. We put $C_k' = (C^k)^*$ and $\mathcal{L}_k' = (\mathcal{L} + C^k)^*$ $(k = 0, 1, \ldots)$. ($\mathcal{L} + C^k$ we provide with the norm induced by the norm ν_k on C^k. Since $(\mathcal{L} + C^k)/C^k = \mathcal{L}/\mathcal{L}^k$ is finite dimensional this can be done uniquely up to equivalence.)

2. We put $C_\infty' = \bigcup C_k'$ and $\mathcal{L}' = \bigcup \mathcal{L}_k'$, $k = 0, 1, \ldots$. The spaces C_∞' and $\mathcal{L}'$ we provide with the inductive limit topologies.

3. We denote by $\varkappa$ the natural projection $\varkappa: \mathcal{L}_k' \to C_k'$ and put $\tilde{\mathcal{L}}_k = \operatorname{Ker}\varkappa \; (= (\mathcal{L}/\mathcal{L}^k)^*)$ and $\tilde{\mathcal{L}} = \bigcup \tilde{\mathcal{L}}_k$, $k = 0, 1, \ldots$.

Remark 1. The spaces $\tilde{\mathcal{L}}_k$ are finite dimensional.

Remark 2. The space C'_∞ is a space of generalized functions on Ω which do not grow rapidly near the boundary.

LEMMA 4.2. 1. The sequence $0 \to \tilde{\mathcal{L}} \to \mathcal{L}' \xrightarrow{\varkappa} C'_\infty \to 0$ is exact.

2. Each element $u \in \mathcal{L}'$ is defined by its values on $\mathcal{L}$.

Proof. Assertion 1 follows from the definitions. We prove assertion 2. From the definition of η it follows easily that $\mathcal{L} \supset C_c^\infty(\Omega)$; therefore, $\mathcal{L}$ is dense in any of the spaces $\mathcal{L} + C^k$. Therefore, any element $u \in \mathcal{L}'_k$ is defined by its values on $\mathcal{L}$.

Definition 4.4. 1. We denote by η^*: $\mathcal{L}' \to S'$ the mapping which is the transpose of the mapping η: $S \to \mathcal{L}$. (It is easy to verify that it is defined and continuous.)

2. If $f \in C'_\infty$, then we put $f(P) = \eta^*(\varkappa^{-1}(f)) \in S'$, where $\varkappa^{-1}(f)$ is any pre-image of f.

We note that the function $f(P) \in S'$ is not defined uniquely, but rather only up to elements of $\eta^*(\tilde{\mathcal{L}})$. If $f \in C'_k$, then this nonuniqueness can be reduced by choosing $\varkappa^{-1}(f) \in \mathcal{L}'_k$. Thus, $f(P)$ is defined up to elements of the finite dimensional space $\eta^*(\tilde{\mathcal{L}}_k)$.

If the function f is continuous on $\bar{\Omega}$, then it is possible to choose for $f(P) \in S'$ the following regular function on $V_{\mathbf{R}}$: $f[P] = (0$ for $x \notin \Theta$, $f(P(x))$ for $x \in \Theta)$.

We now proceed to study the differential equations which the functions $f(P)$ satisfy.

THEOREM 4.3. If $f \in C'_\infty$ and satisfies a nontrivial equation $\mathcal{D}(f) = 0$, where $\mathcal{D} \in D_T$, then the function $f(P)$ lies in S'_0.

Example. The function $e^{iP} \in S'_0$. In particular, its Fourier transform is analytic everywhere except for a certain semialgebraic set.

Theorem 4.3 follows easily from the following two theorems, the proofs of which form the content of §§5, 6.

THEOREM 4.3'. Let $f \in C'_\infty$ and $\mathcal{D}f = 0$, where $\mathcal{D} \in D_T, \mathcal{D} \neq 0$. We denote by $\overline{f(P)}$ the image of the function $f(P)$ in the D-module $S'/D(\eta^*(\tilde{\mathcal{L}}))$. Then $\dim \Delta(D(\overline{f(P)})) \leq N$.

THEOREM 4.3". If $u \in \tilde{\mathcal{L}}$, then $\eta^*(u) \in S'_0$).

§ 5. Proof of Theorem 4.3'.

PROPOSITION 5.1. Let $f \in C'_\infty$. Then

$$\left[\frac{\partial}{\partial x_i}(f(P)) - P_i\left(\frac{\partial f}{\partial t}(P)\right)\right] \in D(\eta^*(\tilde{\mathcal{L}})). \tag{*}$$

Proof. We note that the assertion (*) does not depend on the choice of the functions $f(P)$ and $(\partial f/\partial t)(P)$. Let $f \in C'_k$. It is clear that the assertion is local in t, and it may therefore be assumed that f is concentrated in a neighborhood of the point $t_0 = 0$.

Let $\lambda \in \mathbf{C}$ and $\operatorname{Re}\lambda \geq 0$. We set $f_\lambda = f \cdot |t|^\lambda \in C'_k$. This is an analytic function of λ. When $\operatorname{Re}\lambda$ is large, we define analytic functions of λ, u_f, and v_f with values in $\mathcal{L}'_{k+1}$ by the formulas $u_f(\psi) = f(|t|^\lambda \psi)$ and $v_f(\psi) = -f(|t|^\lambda (\partial/\partial t)\psi)$, where $\psi \in \mathcal{L}$. It is clear that $\varkappa(u_f(\epsilon)) = f_\lambda$ and $\varkappa(v_f(\lambda)) = (\partial/\partial t) f_\lambda$. We will prove that $(\partial/\partial x_i)\eta^*(u_f) = P_i \cdot \eta^*(v_f)$.

Indeed, let f_n be a sequence of functions in C^1 which converges to f in C'_k. Then for each λ $u_{f_n} \to u_f$ and $v_{f_n} \to v_f$ in the space $\mathcal{L}'_{k+1}$. It is therefore sufficient to show that $(\partial/\partial x_i)\eta^*(u_{f_n}) = P_i \cdot \eta^*(v_{f_n})$. But in this case

$$\eta^*(u_{f_n})(\lambda) = f_{n\lambda}[P] \qquad \eta^*(v_{f_n})(\lambda) = \left(\frac{\partial}{\partial t} f_{n\lambda}\right)[P],$$

where $f_{n\lambda} = f_n \cdot |t|^\lambda \in C^1$. Therefore, the equation $(\partial/\partial x_i)\eta^*(u_{f_n}) = P_i \eta^*(v_{f_n})$ is simply the formula for the derivative of a composite function.

We have thus shown that $(\partial/\partial x_i)\eta^*(u_f) = P_i \eta^*(v_f)$.

Since the kernel $\tilde{\mathcal{L}}_{k+1}$ of the mapping $\varkappa: \mathcal{L}'_{k+1} \to C'_{k+1}$ is finite-dimensional, it is possible to find functions u(λ and v(λ) with values in $\mathcal{L}_{k+1}$, defined for Re $\lambda \geq 0$, such that $\varkappa(u(\lambda)) = f_\lambda$ and $\varkappa(v)(\lambda)) = (\partial f_\lambda/\partial t)$. When Re λ is large, it follows that $u(\lambda) - u_f(\lambda) \in \tilde{\mathcal{L}}_{k+1}$ and $v(\lambda) - v_f(\lambda) \in \tilde{\mathcal{L}}_{k+1}$; therefore,

$$\left[\frac{\partial}{\partial x_i}\eta^*(u(\lambda)) - P_i\eta^*(v(\lambda))\right] \in \frac{\partial}{\partial x_i}(\eta^*(\tilde{\mathcal{L}}_{k+1})) + P_i(\eta^*(\tilde{\mathcal{L}}_{k+1})).$$

Since the space on the right is finite-dimensional, and the left side depends analytically on λ, this inclusion is true also for λ = 0, which completes the proof of Proposition 5.1.

Proposition 5.1 enables us to formulate Theorem 4.3' in purely algebraic terms.

Definition 5.1. 1. Let M be a D_T-module. We construct a D-module M_P as follows. As an R-module, M_P is equal to $R \underset{R_T}{\otimes} M$ (R is considered as an R_T algebra relative to the imbedding ρ: $R_T \to R$, $\rho(t) = P$). The action of the operators $(\partial/\partial x_i)$ is given by the formulas $\frac{\partial}{\partial x_i}(r \otimes e) = \frac{\partial r}{\partial x_i} \otimes e + P_i r \otimes \frac{\partial}{\partial t} e$, where $r \in R$, $e \in M$.

2. If $f \in M$, then we put $f_P = 1 \otimes f \in M_P$, $M_P(f) = D(f_P)$, $I_P(f) = \operatorname{Ann}(f_P) \subset D$.

It is easy to verify that Definition 5.1 is good. The mapping $M \to M_P$ gives a functor from the category of D_T-modules to the category of D-modules.

We note that even if M is finitely generated, the module M_P may not be finitely generated.

It follows from Proposition 5.1 that it is possible to construct a mapping of D-modules η^0: $(C'_\infty)_P \to S'/D(\eta^*(\tilde{\mathcal{L}}))$ such that $\eta^0(1 \otimes f) = f(P) \bmod D(\eta^*(\tilde{\mathcal{L}}))$ for any $f \in C'_\infty$. Therefore, Theorem 4.3' follows from the following purely algebraic theorem.

THEOREM 5.2. Let M be a D_T-module, $f \in M$. If $\dim \Delta(M) \leq 1$, then $\dim \Delta(M_P(f)) \leq N$.

Proof. 1. We first consider the case in which the principal part of the polynomial P, which we denote by $\tilde{P}$, is nondegenerate, i.e., sing $\tilde{P} = \{0\}$. In this case the set sing P is compact and hence finite.

Our aim is to construct sufficiently many elements in the ideal $I_P(f)$. We note first of all that for any i and j the operators $\mathcal{H}_{ij} = P_i \frac{\partial}{\partial x_j} - P_j \frac{\partial}{\partial x_i} \in D$ belong to $I_P(f)$.

Let $\mathcal{D}_0(f) = 0$, where $\mathcal{D}_0 \in D_T$. We write $\mathcal{D}_0$ in the form $\mathcal{D}_0 = Q(t)\left(\frac{\partial}{\partial t}\right)^k + \mathcal{D}'$, where $\mathcal{D}' \in D_T^{k-1}$.

LEMMA 5.3. Let $s = k(k-1)/2$. Then for any indices i, j$(1 \leq i, j \leq N)$ there exists an operator $\mathcal{D}_{ij} \in I_P(f)$, such that $\sigma(\mathcal{D}_{ij}) = Q(P)P_i^s y_j^k$.

Proof. If $e \in M$, $r \in R$, $\beta = (j_1, \ldots, j_N)$, then

$$\left(\frac{\partial}{\partial x}\right)^\beta (r \otimes e) = r \cdot P^\beta \otimes \left(\frac{\partial}{\partial t}\right)^{|\beta|} e + e', \text{ where } P^\beta = P_1^{j_1} \cdot \ldots \cdot P_N^{j_N} \text{ и } e' \in R \otimes D_T^{|\beta|-1}(e).$$

From this it follows easily that $P_i^s \otimes D_T^{k-1}(f) \subset D^{k-1}(1 \otimes f)$.

Since $\mathcal{D}_0(f) = \left(Q(t)\left(\frac{\partial}{\partial t}\right)^k + \mathcal{D}'\right) f = 0$, it follows that $Q(t)\left(\frac{\partial}{\partial t}\right)^k f \in D_T^{k-1}(f)$. Therefore, $Q(P)(\partial/\partial x_j)^k (1 \otimes f) \in R \otimes D_T^{k-1}(f)$, and hence $Q(P) \cdot P_i^s (\partial/\partial x_j)^k (1 \otimes f) = \tilde{\mathcal{D}}(1 \otimes f)$, where $\deg \tilde{\mathcal{D}} \leqslant k-1$. It is thus possible to set $\mathcal{D}_{ij} = Q(P) \times P_i^s \left(\frac{\partial}{\partial x_j}\right)^k - \tilde{\mathcal{D}}$.

Thus, in $\sigma(I_P(f))$ there are elements $\sigma(\mathcal{H}_{ij}) = P_i y_j - P_j y_i$ and $\sigma(D_{ij}) = P_i^s \cdot Q(P) \cdot y_j^k$.

Therefore, $\Delta(M_P(f))$ is contained in the union of the following sets: sing $P \times V^*$ and

$$A_i = \{(x, y) \in V \times V^* \mid P_i(x) \neq 0,\ Q(P)(x) = 0,\ y_j = y_i \cdot P_j(x)/P_i(x)\}.$$

All these sets have dimension no greater than N; therefore $\dim \Delta(M_P(f)) \leq N$.

2. We will now prove Theorem 5.2 for an arbitrary polynomial $P \in R$. We wish to show that $d_1^n(\mathcal{D}/I_P(f)) = O(n^N)$ (see Theorem 3.1). For the case in which $\tilde{P}$ is nondegenerate, we have already proved this.

We denote the degree of the polynomial by $q(q > 0)$ and consider a family of polynomials of degree q depending on the parameter $\tau \in \mathbf{C}$,

$$P_\tau = P\cdot(1-\tau) + (x_1^q + \ldots + x_N^q)\tau.$$

Since the set of polynomials of degree q with nondegenerate principal part is open in the Zariski topology and P_1 is contained in this set, it follows for all $\tau \in C$, except for a finite number, that the polynomial $\widetilde{P}_\tau$ is nondegenerate.

LEMMA 5.4. For any natural number n the inequality $\dim I_{P_\tau}(f)_1^n \le \dim I_P(f)_1^n$ is satisfied for all $\tau \in \mathbf{C}$, except a countable number.

Proof. This follows from the fact that the space $I_{P_\tau}(f)_1^n$ is singled out in D_1^n by linear equations whose coefficients are rational functions of τ. We described this in more detail. We set $B = R \otimes_{\mathbf{C}} M$ and define mappings μ_τ^n: $D_1^n \to B$ and ν_τ: $B \to B$:

$$\mu_\tau^n\left(x^\alpha\frac{\partial}{\partial x_{i_1}}\cdot\ldots\cdot\frac{\partial}{\partial x_{i_k}}\right) = x^\alpha\frac{\partial}{\partial x_{i_1}}\left(\ldots\left(\frac{\partial}{\partial x_{i_k}}(1\otimes f)\right)\right),$$

where $(\partial/\partial x_i)(r\otimes e) = (\partial r/\partial x_i)\otimes e + r(P_\tau)_i \otimes (\partial/\partial t)e$ (we assume that each element $\mathcal{D} \in D$ is written in the form $\mathcal{D} = \sum c_{\alpha\beta}(\mathcal{D})x^\alpha\left(\frac{\partial}{\partial x}\right)^\beta$);

$$\nu_\tau(r\otimes e) = Pr\otimes e - r\otimes te.$$

It is easy to verify that ν_0 is an imbedding.

We deduce Lemma 5.4 from the fact that $I_{P_\tau}(f)_1^n = (\mu_\tau^n)^{-1}\nu_\tau(B)$. We set $B_k = D_1^k(1)\otimes D_{T_1}^k(f)$. Then $\nu_\tau(B_k) \subset B_{2k}$ and $\mu_\tau^n(D_1^n) \subset B_k$ for large k, and $\dim I_{P_\tau}(f)_1^n = \lim_{k\to\infty}\dim(\mu_\tau^n)^{-1}\cdot\nu_\tau(B_k)$.

Therefore, Lemma 5.4 follows from the following assertion, the proof of which we omit: if μ_τ: $\widetilde{C} \to \widetilde{B}$ and ν_τ: $\widetilde{A} \to \widetilde{B}$ are linear mappings of finite-dimensional spaces involving τ as a polynomial and ν_0 is an imbedding, then $\dim \mu_\tau^{-1}\nu_\tau(\widetilde{A}) \le \dim \mu_0^{-1}\nu_0(\widetilde{A})$ for all τ except a finite number.

Lemma 5.4 implies the existence of a point $\tau \in \mathbf{C}$, such that $\dim I_{P_\tau}(f)_1^n \le \dim I_P(f)_1^n$ for all n and the polynomial $\widetilde{P}_\tau$ is nondegenerate. For this point the following inequalities are satisfied:

$$d_1^n(D/I_P(f)) \leqslant d_1^n(D/I_{P_\tau}(f)) = O(n^N).$$

Therefore, $\dim \Delta(M_P(f)) \le N$.

This terminates the proof of Theorem 5.2, and hence also Theorem 4.3'.

§6. Proof of Theorem 4.3". We must show that if $u \in \mathcal{L}$, then $\dim \Delta(D(\eta * u)) \le N$. It may be assumed without loss of generality that $0 \in \partial\Omega$ and $u \in \widetilde{\mathcal{L}}_+$, where $\widetilde{\mathcal{L}}_+$ consists of functionals $v \in \mathcal{L}$, for which $v(\varphi)$ is defined by the behavior of the function φ in a small right neighborhood of zero $\mathcal{L}$). Moreover, we will assume that $P(\partial\Theta) = \{0\}$.

Definition 6.1. Let $\lambda \in C$, $\operatorname{Re}\lambda > 0$. We denote by t_+^λ the continuous function on T which is equal to 0 for $t < 0$ and equal to t^λ for $t > 0$. We consider the function t_+^λ as an element of $\mathcal{L}'$.

LEMMA 6.1. 1. The function t_+^λ depends analytically on λ for $\operatorname{Re}\lambda > 0$ and can be continued as a meromorphic function with values in $\mathcal{L}'$ to the entire plane of the variable λ.

2. We write the expansion of t_+^λ in a Laurent series in a neighborhood of the point λ_0:

$$t_+^\lambda = a_{-k}(\lambda_0)(\lambda-\lambda_0)^{-k} + \ldots + a_0(\lambda_0) + \ldots$$

Then the coefficients $a_{-k}(\lambda_0), \ldots, a_{-1}(\lambda_0) \in \widetilde{\mathcal{L}}_+$.

3. The coefficients $a_{-i}(\lambda_0)$ for all possible $i > 0$ and $\lambda_0 \in C$ form an algebraic basis in $\widetilde{\mathcal{L}}_+$.

The proof follows immediately from the asymptotic expansion for functions $\varphi \in \mathcal{L}$ obtained in Proposition 4.1 (see [1], Ch. 1, 4).

We will study the equations satisfied by the functions t_+^λ. For this we will have to consider equations depending on λ.

Let $D[\lambda]$ be the ring of polynomials in the variable λ with coefficients in D. In it we introduce the filtrations $\{D^n[\lambda]\}$ and $\{D^n_I[\lambda]\}$. The associated ring with respect to the filtration $\{D^n[\lambda]\}$ is isomorphic to $\Sigma[\lambda]$, and the corresponding affine variety is equal to $W \times \Lambda$ (Λ is the complex line).

If $\lambda_0 \in C$, then by s_{λ_0} we denote the evaluation mapping $D[\lambda] \to D$ and $\Sigma[\lambda] \to \Sigma$, obtained by replacing $\lambda \to \lambda_0$.

LEMMA 6.2. We set $e_i = t^i \in D_T (i = 0, 1, \ldots)$, $e_i = (\partial/\partial t)^{-i} \in D_T (i = -1, -2, \ldots)$. Then each element $\mathscr{D} \in D_T$ can be uniquely described in the form $\mathscr{D} = \sum_{-\infty}^{\infty} e_i \cdot Q_i \left(t \frac{\partial}{\partial t}\right)$, where the Q_i are polynomials of a single variable.

The proof of the lemma follows immediately by induction on the degree of $\mathscr{D}$.

We consider the $D_T[\lambda]$-module $M = D_T[\lambda]/D_T[\lambda]\,[t(\partial/\partial t) - \lambda]$ and denote its generator by f. Lemma 6.2 implies that the elements e_i form a base for the $C[\lambda]$-module M. In analogy with Definition 5.1, we construct the $D[\lambda]$-module $M_P(f)$ and the ideal $I_P(f)$ in $D[\lambda]$.

THEOREM 6.3. The set $\Delta(M_P(f)) \subset W \times \Lambda$ consists entirely of lines of the form $w \times \Lambda$, $w \in W$.

Proof. We set $M' = D_T[\lambda]/D_T[\lambda]\,(t(\partial/\partial t) - \lambda - 1)$, and let f' be the generator of M'. Since $t(\partial/\partial t)\,(tf) = (\lambda + 1)tf$, it is possible to define a mapping of $D_T[\lambda]$-modules μ: $M' \to M$, by putting $\mu(f') = tf$. It follows from Lemma 6.2 that μ is an imbedding. The mapping which it induces μ_P: $M'_P(f') \to M_P(f)$ (here $\mu_P(1 \otimes f') = P \otimes f$) is also an imbedding. It follows from Lemma 1.5 that $\Delta(M'_P(f')) \subseteq \Delta(M_P(f))$.

The ideal $I_P(f')$ is obtained from the ideal $I_P(f)$ by replacing $\lambda \to \lambda + 1$, and therefore $\Delta(M'_P(f')) = Z(\sigma(I_P(f')))$ is a translation of $\Delta(M_P(f))$ along the line Λ by -1. If the point $(w, \lambda_0) \in \Delta(M_P(f))$, then the points $(w, \lambda_0 - 1), \ldots, (w, \lambda_0 - n) \ldots \in \Delta(M_P(f))$. Since $\Delta(M_P(f))$ is a closed algebraic variety in $W \times \Lambda$, it follows that $w \times \Lambda \subseteq \Delta(M_P(f))$, which completes the proof of Theorem 6.3.

Definition 6.2. 1. We set $\Delta_P = \{w \in W \,|\, w \times \Lambda \subset \Delta(M_P(f))\}$. 2. For any $\lambda \in C$ we denote by M_λ the D_T-module $D_T/D_T[t(\partial/\partial t) - \lambda]$ with generator f_λ and put $I_\lambda = I_P(f_\lambda) \subset D$. The ideal $I_P(f) \subset D[\lambda]$ we denote by I.

It follows from Theorem 6.3 that $\Delta_P = Z(s_{\lambda_0}(\sigma(I)))$ for any $\lambda_0 \in C$.

PROPOSITION 6.4. $\dim \Delta_P \leq N$.

Proof. From Lemma 6.2 it is easy to derive the following Lemma 6.5.

LEMMA 6.5. The $C[\lambda]$-submodule I^n_I of $D^n_I[\lambda]$ is given by linear equations with coefficients in $C[\lambda]$. If in these coefficients we make the replacement $\lambda \to \lambda_0$, where $\lambda_0 \in C$, then they go over into equations for the subspaces $(I_{\lambda_0})^n_I$ in D^n_I.

We choose a point $\lambda_0 \in C$ which is algebraically independent of all the numerical coefficients which enter in the equations defining the ideal I.

It then follows from Lemma 6.5 that $s_{\lambda_0}(I^n_I) = I^n_{\lambda_0 I}$ for all n. Therefore, $s_{\lambda_0}(\sigma(I)) = \sigma(s_{\lambda_0}(I))$, i.e., $\Delta_P = \Delta((M_{\lambda_0})_P(f_{\lambda_0}))$, and by Theorem 5.2 $\dim \Delta_P \leq N$.

LEMMA 6.6. If $\mathscr{D} \in I \subset D[\lambda]$, then $\mathscr{D}(\eta^*(t^\lambda_+)) = 0$ identically in λ.

Proof. We set $m = \deg \mathscr{D}$. It is easy to verify that for $\operatorname{Re} \lambda > m$ the relation $(\partial/\partial x_i)\,\eta^*(u) = P_i \eta^*(\partial u/\partial t)$ is satisfied for all $u \in D^{m-1}_T(t^\lambda_+)$. This implies that $\mathscr{D}(\eta^*(t^\lambda_+)) = 0$ for $\operatorname{Re} \lambda > m$. The proof of the lemma is complete.

PROPOSITION 6.7. Let $t^\lambda_+ = a_{-k}(\lambda - \lambda_0)^{-k} + \ldots + a_0 + a_1(\lambda - \lambda_0) + \ldots +$ be the expansion of the function t^λ_+ in a Laurent series, $a_i \in \mathscr{L}'$. Then $\Delta(D(\eta^*(a_i))) \subset \Delta_{P_n}$ for any $i (i = -k, \ldots, 0, 1, \ldots)$.

Proof. Let $\mathscr{D} = \mathscr{D}_0 + (\lambda - \lambda_0)\mathscr{D}_1 - \ldots + (\lambda - \lambda_0)\mathscr{D}_n \in I$. We put $\mathscr{E}_j = \eta^*(a_j)$. Applying Lemma 6.6, we obtain the system of equations

$$\mathscr{D}_0 \mathscr{E}_{-k} = \mathscr{D}_0 \mathscr{E}_{-k+1} + \mathscr{D}_1 \mathscr{E}_{-k} = \ldots = \mathscr{D}_0 \mathscr{E}_i + \ldots + \mathscr{D}_{k+i} \mathscr{E}_{-k} = 0.$$

We set $M_j = D(\mathscr{E}_{-k}, \ldots, \mathscr{E}_j)$. It is evident from these equations that the generator $\bar{\mathscr{E}}_j$ in the module M_j/M_{j-1} satisfies the equation $s_{\lambda_0}(D)\,(\bar{\mathscr{E}}_j) = 0$ for any $\mathscr{D} \in I$. This means that $\Delta(M_j/M_{j-1}) \subset Z(\sigma(s_{\lambda_0}(I))) \subseteq \Delta_P$.

Proposition 6.7 follows from Lemma 1.5.

Theorem 4.3″ is a direct consequence of Lemma 6.1 and Propositions 6.4 and 6.7.

§7. Fundamental Solutions of Equations with Constant Coefficients

PROPOSITION 7.1. Let P be a polynomial with real coefficients on $V_{\mathbf{R}}$. Then the function $|P|^\lambda$, defined for Re $\lambda > 0$, can be continued analytically as a meromorphic function with values in S' to the entire complex plane of the variable λ. If $\widetilde{P}'$ is any coefficient of the Laurent series for the function $|P|^\lambda$ at any point λ, then $\Delta(D(\widetilde{P}')) \subset \Delta_P$ (in particular, dim $\Delta(D(\widetilde{P}')) \leq N$, i.e., $\widetilde{P}' \in S_0'$).

This proposition follows immediately from Proposition 6.7 and Lemma 6.1.

We will now prove Theorem C of the introduction. We seek a solution of the equation $L(-i(\partial/\partial x_k))(\mathscr{E}_L) = \delta$ in the form $\mathscr{E}_L = F(\widetilde{\mathscr{E}})$ (F is Fourier transformation).

The function $\widetilde{\mathscr{E}}$ must satisfy the equation $L(x_k) \cdot \widetilde{\mathscr{E}} = 1$. We put $P = L \cdot \bar{L}$ and take as $\widetilde{\mathscr{E}}$ the zeroth term of the Laurent series of the function $\bar{L} \cdot P^\lambda$ at the point $\lambda = -1$. Then $L \cdot \widetilde{\mathscr{E}} = 1$, and, as follows from Proposition 7.1, $\widetilde{\mathscr{E}} \in S_0'$. Using Corollary 3.4, we deduce that the fundamental solution $\mathscr{E}_L$ lies in S_0'. This completes the proof of Theorem C.

Hypothesis. $\Delta_P \subset \{(x, y) \in V \times V^* \,|\, (x, y) \in \bar{\square}_P, P(x)\} = 0$, where $\square_P = \{(x, y) \in V \times V^* \,|\, x \notin \text{sing } P, P_i y_j = P_j y_i \text{ for all } i, j\}$.

In the case in which L is a homogeneous polynomial this hypothesis enables us to find a cone containing the singularities of the fundamental solution of the operator $L[-i(\partial/\partial x_k)]$. It evidently contains the cone constructed by Hörmander in [5], but it does not always coincide with it.

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