Lectures on Lie Algebras

Joseph Bernstein

Abstract This is a lecture course for beginners on representation theory of semisimple finite dimensional Lie algebras. It is shown how to use infinite dimensional representations (Verma modules) to derive the Weyl character formula. We also provide a proof for Harish–Chandra's theorem on the center of the universal enveloping algebra and for Kostant's multiplicity formula.

Keywords Lie algebra • Verma module • Weyl character formula • Kostant multiplicity formula • Harish–Chandra center

1991 Mathematics Subject Classification. 17A70 (Primary) 17B35 (Secondary)

Introduction

These notes originally were a draft of the transcript of my lectures in the Summer school in Budapest in 1971. For the lectures addressed to the advanced part of the audience, see [Ge]. The beginners' part was released a bit later see [Ki]. It contains a review by Feigin and Zelevinsky, which *expands* my lectures. Therefore, the demand in a short and informal guide for the beginners still remains, I was repeatedly told. So here it is.

We will consider finite dimensional representations of semisimple finite dimensional complex Lie algebras. The facts presented here are well known ([Bu], [Di], [Se]) and in a more rigorous setting. But our presentation of these facts is comparatively new (at least, it was so in 1971) and is based on the systematic usage of the Verma modules M_χ.

J. Bernstein (✉)
Department of Mathematics, Tel-Aviv University, Tel-Aviv, Israel
e-mail: bernstei@math.tau.ac.il

B. Krötz et al. (eds.), *Representation Theory, Complex Analysis, and Integral Geometry*,
DOI 10.1007/978-0-8176-4817-6_6,

The reader is supposed to be acquainted with the main notions of Linear Algebra ([Pr] will be just fine). The knowledge of the first facts and notions from the theory of Lie algebra will not hurt but is not required.

The presentation is arranged as follows:

In Sect. 1, we discuss general facts regarding Lie algebras, their universal enveloping algebras, and their representations.

In Sect. 2, we discuss in detail the case of the simplest simple Lie algebra $\mathfrak{g} = \mathfrak{sl}(2)$. The results of this section provide essential tools for treating the general case.

In Sect. 3, we provide without proofs a list of results on the structure of semisimple Lie algebras and their root systems.

In Sect. 4, we introduce some special category of $\mathfrak{g}$-modules, so-called category $\mathcal{O}$. We construct basic objects of this category – Verma modules M_χ – and describe some of their properties.

In Sect. 5, we construct, for every semisimple Lie algebra $\mathfrak{g}$, a family of irreducible finite dimensional representations A_λ.

In Sect. 6, we formulate one of the central results – Harish–Chandra's description of the algebra $\mathfrak{Z}(\mathfrak{g})$ – center of the enveloping algebra of $\mathfrak{g}$. For the proof see Sect. 9.

In Sect. 7, we describe various properties of the category $\mathcal{O}$ that follow from the Harish–Chandra theorem.

In Sect. 8, we prove Weyl's character formula for irreducible $\mathfrak{g}$-modules A_λ and derive Kostant's formula for the multiplicities of weights for these representations. We also prove that every finite dimensional $\mathfrak{g}$-module is decomposable into a direct sum of irreducible modules isomorphic to A_λ.

In Sect. 9, we present a proof of the Harish–Chandra theorem.

1 General Facts About Lie Algebras

All vector spaces considered in what follows are defined over a ground field $\mathbb{K}$. We assume that $\mathbb{K}$ is algebraically closed of characteristic 0. The reader can assume $\mathbb{K} = \mathbb{C}$.

1.1 Lie Algebras

Definition. A *Lie algebra* is a $\mathbb{K}$-vector space $\mathfrak{g}$ equipped with a bilinear multiplication $[\ ,\ ] : \mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g}$ (it is called *bracket*) that satisfies the following identities:

$$[X, Y] + [Y, X] = 0 \quad \text{for any } X, Y \in \mathfrak{g} \qquad \text{(S - S)}$$

$$[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0 \text{ for any } \quad X, Y, Z \in \mathfrak{g}. \qquad \text{(J.I.)}$$

The identity (S-S) signifies skew-symmetry of the bracket, (J.I.) is called the *Jacobi identity*.

Example. Let A be an associative algebra. By means of the subscript L we will denote the Lie algebra $\mathfrak{g} = A_L$ whose underlying vector space is a copy of A and the bracket is given by the formula $[X, Y] = XY - YX$. Clearly, A_L is a Lie algebra: (S-S) and (J.I.) are subject to a direct verification.

If V is a vector space, we denote $\mathfrak{gl}(V)$ its *general linear Lie algebra* that is defined as $\mathfrak{gl}(V) = (\mathrm{End}_{\mathbb{K}}(V))_L$.

We abbreviate $\mathfrak{gl}(\mathbb{K}^n)$ to $\mathfrak{gl}(n)$. Note that this is just the algebra $Mat(n)$ of $n \times n$-matrices with the operation $[X, Y] = XY - YX$.

1.2 Representations of Lie Algebra

A *representation* γ of a Lie algebra $\mathfrak{g}$ in a vector space V is a morphism of Lie algebras $\gamma : \mathfrak{g} \to \mathfrak{gl}(V)$. We will denote by the same symbol γ the corresponding morphism of vector spaces $\mathfrak{g} \otimes V \to V$.

We will also use the following equivalent terms for representations: "γ is an action of Lie algebra $\mathfrak{g}$ on V"; "V is $\mathfrak{g}$-module".

Morphisms of $\mathfrak{g}$-modules are defined as usual. The category of $\mathfrak{g}$-modules will be denoted by $\mathcal{M}(\mathfrak{g})$.

An important example of a representation is the adjoint representation ad of a Lie algebra $\mathfrak{g}$ on the vector space $V = \mathfrak{g}$. It is defined by formula $ad(X)(Y) := [X, Y]$. The fact that this is a representation follows from Jacobi identity.

1.3 Tensor Product Representation

Given representations γ, δ of a Lie algebra $\mathfrak{g}$ in spaces V and E we construct the tensor product representation $\eta = \gamma \otimes \delta$ in the space $V \otimes E$ via Leibnitz rule $\eta(X) = \gamma(X) \otimes Id + Id \otimes \delta(X)$.

Lemma. *Let $\gamma : \mathfrak{g} \otimes V \to V$ be any representation of a Lie algebra $\mathfrak{g}$. Consider on the space $\mathfrak{g} \otimes V$ the structure of $\mathfrak{g}$-module given by representation $Ad \otimes \gamma$. Then $\gamma : \mathfrak{g} \otimes V \to V$ is a morphism of $\mathfrak{g}$-modules.*

The verification is left to the reader.

1.4 Some Examples of Lie Algebras

Example 1. Let $\mathfrak{n}^-$, $\mathfrak{n}_-$, and $\mathfrak{h}$ be the subspaces of $\mathfrak{g} = \mathfrak{gl}(n)$ consisting of all strictly upper triangular, strictly lower triangular and diagonal matrices, respectively.

Clearly, $\mathfrak{n}_+$, $\mathfrak{n}_-$, and $\mathfrak{h}$ are Lie subalgebras of $\mathfrak{gl}(n)$. Important role in representation theory plays a triangular decomposition $\mathfrak{gl}(n) = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$ (this is a direct sum decomposition of vector spaces, but not of Lie algebras).

Example 2. The space of $n \times n$ matrices with trace zero is a Lie algebra; it is called the *special linear algebra* and denoted by $\mathfrak{sl}(n)$.

Example 3. Let B be a bilinear form on a vector space V. Consider the space $Der(B)$ of all operators $X \in \mathfrak{gl}(V)$ that preserve B, i.e., $B(Xu, v) + B(u, Xv) = 0$ for any $u, v \in V$.

It is easy to see that this subspace is closed under the bracket and so is a Lie subalgebra of $\mathfrak{gl}(V)$.

If B is nondegenerate, we distinguish two important subcases:

- B is symmetric, then $Der(B)$ is called the *orthogonal Lie algebra* and denoted by $\mathfrak{o}(V, B)$.
- B is skew-symmetric, then $Der(B)$ is called the *symplectic Lie algebra* and denoted by $\mathfrak{sp}(V, B)$.

It is well known that over $\mathbb{C}$ all nondegenerate symmetric forms on V are equivalent to each other and the same applies to skew-symmetric forms. So Lie algebras $\mathfrak{o}(V, B)$ and $\mathfrak{sp}(V, B)$ actually depend only on the dimension of V, and we will sometimes denote them by $\mathfrak{o}(n)$ and $\mathfrak{sp}(2m)$.

The Lie algebras $\mathfrak{gl}(n)$, $\mathfrak{o}(n)$, and $\mathfrak{sp}(2m)$ are called *classical Lie algebras*.

For the proof of the statements of this section, see ([Bu], [Di], [OV], [Se]).

1.5 Universal Enveloping Algebra

Let $\mathfrak{g}$ be a Lie algebra over $\mathbb{K}$. To $\mathfrak{g}$ we assign an *associative* $\mathbb{K}$-algebra with unit, $U(\mathfrak{g})$, called the *universal enveloping* algebra of the Lie algebra $\mathfrak{g}$. Namely, consider the tensor algebra $T(\mathfrak{g})$ of the *space* $\mathfrak{g}$, i.e.,

$$T^{\bullet}(\mathfrak{g}) = \bigoplus_{n \geq 0} T^n(\mathfrak{g}),$$

where $T^0(\mathfrak{g}) = \mathbb{K}$, $T^n(\mathfrak{g}) = \mathfrak{g} \otimes \cdots \otimes \mathfrak{g}$ (n factors). Consider also the two-sided ideal $I \subset T(\mathfrak{g})$ generated by the elements $X \otimes Y - Y \otimes X - [X, Y]$ for any $X, Y \in \mathfrak{g}$. Set $U(\mathfrak{g}) = T(\mathfrak{g})/I$.

We will identify the elements of $\mathfrak{g}$ with their images in $U(\mathfrak{g})$. Under this identification, any $\mathfrak{g}$-module may be considered as a (left, unital) $U(\mathfrak{g})$-module and, conversely, any $U(\mathfrak{g})$-module may be considered as a $\mathfrak{g}$-module. We will not distinguish the $\mathfrak{g}$-modules from the corresponding $U(\mathfrak{g})$-modules.

The algebra $U(\mathfrak{g})$ has a natural increasing filtration $U(\mathfrak{g})_n = \sum_{i \leq n} T^i(\mathfrak{g})$. We denote by $\mathrm{gr}U(\mathfrak{g})$ the associated graded algebra $\mathrm{gr}U(\mathfrak{g}) = \oplus_{n \geq 0} \mathrm{gr}_n U(\mathfrak{g})$, where $\mathrm{gr}_n U(\mathfrak{g}) := U(\mathfrak{g})_n / U(\mathfrak{g})_{n-1}$. This algebra is clearly commutative and hence the natural morphism $i : \mathfrak{g} \to \mathrm{gr}_1 U(\mathfrak{g})$ extends to a morphism of graded commutative algebras $i : S^\bullet(\mathfrak{g}) \to \mathrm{gr}U(\mathfrak{g})$, where $S^\bullet(\mathfrak{g})$ is the symmetric algebra of the linear space $\mathfrak{g}$. The following result will be used repeatedly in the lectures.

Theorem (Poincaré–Birkhoff–Witt). The morphism $i : S^\bullet(\mathfrak{g}) \longrightarrow \mathrm{gr}U(\mathfrak{g})$ is an isomorphism of graded commutative algebras.

Corollary. *(1) $U(\mathfrak{g})$ is a Noetherian ring without zero divisors.*
(2) Let symm$'$: $S^\bullet(\mathfrak{g}) \longrightarrow T^\bullet(\mathfrak{g})$ be the map determined by the formula

$$X_1 \otimes \cdots \otimes X_k \mapsto \frac{1}{k!} \sum_{\sigma \in \mathfrak{S}_k} X_{\sigma(1)} \otimes \cdots \otimes X_{\sigma(k)}.$$

Denote by symm: $S^\bullet(\mathfrak{g}) \longrightarrow T^\bullet(\mathfrak{g}) \longrightarrow U(\mathfrak{g})$ the composition of symm$'$ and the projection onto $U(\mathfrak{g})$. The map symm is an isomorphism of linear spaces (not algebras).
(3) If $X_1, \ldots, X_k$ is a basis of $\mathfrak{g}$, then the set of monomials $X_1^{n_1} X_2^{n_2} \ldots X_k^{n_k}$, where the n_i run over the set $\mathbb{Z}_{\geq 0}$ of nonnegative integers, is a basis of $U(\mathfrak{g})$.

Remark. The version of PBW as stated above is in [Di] 2.3.6. A direct proof can be found in [BG]. For point one of the corollary, see 2.3.8 and 2.3.9 of loc. cit. For the second point, see 2.4 in [Di] and for the last point see 2.1.8 in [Di].

1.6 Some Finiteness Results

In order to analyze finite dimensional representations of a Lie algebra, we will often use infinite dimensional representations that satisfy some finiteness assumptions.

1.6.1 Locally Finite Representations

Definition. Let A be an associative algebra. An A-module V is called *locally finite* if it is a union of finite dimensional A-submodules.

Notice that the subcategory $\mathcal{M}(A)^{lf} \subset \mathcal{M}(A)$ of locally finite A-modules is closed with respect to subquotients. It is easy to check that if algebra A is finitely generated then $\mathcal{M}(A)^{lf}$ is also closed under extensions.

If V is an arbitrary A-module, then the sum of all locally finite submodules is the maximal locally finite submodule of V. We denote it $V^{\text{A-finite}}$.

We use the same definitions for a module V over a Lie algebra $\mathfrak{a}$. In particular, we denote by $V^{\mathfrak{a}\text{-finite}}$ the maximal locally finite $\mathfrak{a}$-submodule of V.

Lemma. *(i) Let $\mathfrak{a}$ be a Lie algebra. Then the tensor product of locally finite representations is locally finite.*

(ii) Let $\mathfrak{g}$ *be a finite-dimensional Lie algebra and* $\mathfrak{a} \subset \mathfrak{g}$ *its Lie subalgebra. Given a* $\mathfrak{g}$*-module* V *consider its maximal* $\mathfrak{a}$*-locally finite submodule* $L = V^{\mathfrak{a}\text{-}finite}$*. Then* L *is a* $\mathfrak{g}$*-submodule of* V*.*

Proof. The proof of (i) is straightforward. Then (i) implies that the morphism of $\gamma : \mathfrak{g} \otimes V \to V$ maps $\mathfrak{g} \otimes L$ into L, i.e., L is a $\mathfrak{g}$-submodule. □

Exercise. Show that the same result is true under weaker assumptions. Namely, it is enough to assume that the adjoint action of the Lie algebra $\mathfrak{a}$ on the space $\mathfrak{g}/\mathfrak{a}$ is locally finite.

1.6.2 Locally Nilpotent Representations

Definition. Let $\mathfrak{a}$ be a Lie algebra. An $\mathfrak{a}$-module V is called *nilpotent* if for some natural number k we have $\mathfrak{a}^k(V) = 0$. An $\mathfrak{a}$-module V is called *locally nilpotent* if it is a sum of nilpotent submodules.

As before we denote by $V^{\mathfrak{a}\text{-nilp}}$ the maximal locally nilpotent submodule of V.

Lemma. *(i) Tensor product of locally nilpotent representations is locally nilpotent.*

(ii) Let $\mathfrak{g}$ *be a Lie algebra and* $\mathfrak{a} \subset \mathfrak{g}$ *its Lie subalgebra such that the adjoint action of* $\mathfrak{a}$ *on* $\mathfrak{g}$ *is locally nilpotent. Given a* $\mathfrak{g}$*-module* V *consider its maximal* $\mathfrak{a}$*-locally nilpotent submodule* $L = V^{\mathfrak{a}-nilp}$*. Then* L *is a* $\mathfrak{g}$*-submodule of* V*.*

The proof is the same as in Lemma 1.6.1.

1.7 Representations of Abelian Lie Algebras

Let $\mathfrak{a}$ be an abelian Lie algebra (i.e., the bracket on $\mathfrak{a}$ is identically 0). Let V be a locally finite $\mathfrak{a}$-module.

For every character $\chi \in \mathfrak{a}^*$, we denote by $V(\chi)$ the space of generalized eigenvectors of $\mathfrak{a}$ with eigencharacter χ.

Proposition. V *is a direct sum of the subspaces* $V(\chi)$*.*

This is a standard result of linear algebra, see Proposition A.1 in the appendix.

Definition. A module V over an abelian Lie algebra $\mathfrak{a}$ is called *semisimple* if it is spanned by eigenvectors of $\mathfrak{a}$.

For any $\mathfrak{a}$-module V, we denote by $V^{\mathfrak{a}\text{-ss}}$ the maximal semisimple $\mathfrak{a}$-submodule of V.

Lemma. *(i) Tensor product of semisimple representations is semisimple.*

(ii) Let $\mathfrak{g}$ *be a Lie algebra and* $\mathfrak{a} \subset \mathfrak{g}$ *its abelian Lie subalgebra such that the adjoint action of* $\mathfrak{a}$ *on* $\mathfrak{g}$ *is semisimple. Given a* $\mathfrak{g}$*-module* V *consider its maximal* $\mathfrak{a}$*-semisimple submodule* $L = V^{\mathfrak{a}\text{-}ss}$*. Then* L *is a* $\mathfrak{g}$*-submodule of* V*.*

Again, the proof is the same as in Lemma 1.6.1.

2 The Representations of $\mathfrak{sl}(2)$

In this section, we will describe representations of the simplest simple Lie algebra $\mathfrak{g} = \mathfrak{sl}(2)$.

2.1 The Lie Algebra $\mathfrak{sl}(2)$

The Lie algebra $\mathfrak{sl}(2)$ consists of matrices $\mathbf{x} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ over field $\mathbb{K}$ such that $\operatorname{tr} \mathbf{x} = a + d = 0$. In $\mathfrak{sl}(2)$, select the following basis

$$E_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad E_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

The commutation relations between the elements of the basis are:

$$[H, E_+] = 2E_+ \,; [H, E_-] = -2E_- \,; [E_+, E_-] = H.$$

Remark. We will see that in any semisimple Lie algebra $\mathfrak{g}$ we can find many triples of elements (E_+, H, E_-) of $\mathfrak{g}$ that satisfy above relation. We call such a triple an $\mathfrak{sl}(2)$-*triple*. In this way, the study of the representations of the Lie algebra $\mathfrak{sl}(2)$ provides us with lots of information on the representations of any semisimple Lie algebra $\mathfrak{g}$.

The above relations between E_-, H, and E_+ and a simple inductive argument yield the following relations in $U(\mathfrak{sl}(2))$:

$$[H, E_+^k] = 2kE_+^k, \; [H, E_-^k] = -2kE_-^k, \; [E_+, E_-^k] = kE_-^{k-1}(H - (k-1)).$$

Besides, it is easy to verify that the element

$$C = 4E_-E_+ + H^2 + 2H$$

belongs to the center of $U(\mathfrak{sl}(2))$. The element C is called the *Casimir operator*.

Let V be an $\mathfrak{sl}(2)$-module. A vector $v \in V$ is called a *weight vector* if it is an eigenvector of the operator H, i.e. $Hv = \chi v$; the number $\chi \in \mathbb{K}$ is called the *weight* of v.

We denote by $V^{ss}(\chi)$ the subspace of all such vectors. Similarly, we define $V(\chi)$ to be the space of generalized weight vectors for H (see appendix for definitions).

Lemma.

$$E_+(V^{ss}(\chi)) \subset V^{ss}(\chi + 2), \; E_+(V(\chi)) \subset V(\chi + 2)$$
$$E_-(V^{ss}(\chi)) \subset V^{ss}(\chi - 2), \; E_-(V(\chi)) \subset V(\chi - 2).$$

Proof. Let $v \in V^{ss}(\chi)$. Then $(H - \chi - 2)E_+v = E_+(H - \chi)v = 0$, i.e. $E_+v \in V^{ss}(\chi + 2)$. Similarly if $v \in V(\chi)$ then $(H - \chi - 2)^n E_+v = E_+(H - \chi)^n v = 0$ for large n, i.e. $E_+v \in V(\chi + 2)$.

The proof for E_- is similar. □

A nonzero vector v is called *a highest weight vector* if it is a weight vector with some weight χ and $E_+v = 0$.

2.2 A Key Lemma

Lemma 1. *Let V be a representation of $\mathfrak{sl}(2)$ and $v \in V$ a highest weight vector of weight χ. Consider the sequence of vectors $v_k = E_-^k v$, $k = 0, 1 \ldots$. Then*

1) $Hv_k = (\chi - 2k)v_k$, $E_+v_{k+1} = (k + 1)(\chi - k)v_k$
2) *The subspace $L \subset V$ spanned by vectors v_k is an $\mathfrak{sl}(2)$-submodule and all non-zero vectors v_k are linearly independent.*
3) *Suppose that $v_k = 0$ for large k. Then $\chi = l \in \mathbb{Z}_{\geq 0}$, $v_k \neq 0$ for $0 \leq k \leq l$ and $v_k = 0$ for $k > l$.*

Proof. (1) is proved by induction in k.

(2) follows from 1) since v_k are eigenvectors of H with distinct eigenvalues.

(3) Let l be the first index such that $v_{l+1} = 0$. Then $0 = E_+v_{l+1} = (l+1)(\chi - l)v_l$ and hence $\chi = l$. □

2.3 Construction of Representations A_l

Let us now describe a family of irreducible finite dimensional representations of $\mathfrak{sl}(2)$. For every $l \in \mathbb{Z}_{\geq 0}$, we construct a representation A_l of dimension $l + 1$. This representation is generated by a highest weight vector v_l of weight l.

First we describe this representations geometrically. Consider the natural action of the group $G = SL(2, \mathbb{K})$ on the plane $\mathbb{K}^2$ with coordinates (x, y). It induces the action of G on the space V of polynomial functions on $\mathbb{K}^2$.

The action of the group G on V induces a representation of its Lie algebra $\mathfrak{g} = \mathfrak{sl}(2)$. It can be described via explicit formulas using differential operators

$$E_+ = x\partial_y, H = x\partial_x - y\partial_y, E_- = y\partial_x.$$

The representation V is a direct sum of invariant subspaces $A_l, l \in \mathbb{Z}_{\geq 0}$, where A_l is the space of homogeneous polynomials of degree l.

In particular, the representations A_l extend to representations of the group $G = SL(2, \mathbb{K})$.

Let us describe these representations explicitly. The space A_l has a basis consisting of monomials $\{a_{-l}, a_{-l+2}, \dots, a_{l-2}, a_l\}$, where $a_i = x^{(l+i)/2}y^{(l-i)/2}$. The action of the algebra $\mathfrak{sl}(2)$ is as follows:

$$Ha_i = ia_i, E_-a_i = \frac{l+i}{2}a_{i-2}, E_+a_i = \frac{l-i}{2}a_{i+2}.$$

Exercise. (i) Show that the module A_ℓ is irreducible.

(ii) Consider the $\mathfrak{sl}(2)$-module M generated by a vector m subject to the relations $H(m) = \ell m$ (i.e. m has weight ℓ), $E_+(m) = 0$ and $E_-^{\ell+1}(m) = 0$. Prove that M is isomorphic to the module A_ℓ described above.

2.4 Classification of Irreducible Finite Dimensional Modules of the Lie Algebra $\mathfrak{sl}(2)$

Proposition. *(1) In any finite dimensional nonzero $\mathfrak{sl}(2)$-module V, there is a submodule isomorphic to one of A_l.*

(2) The Casimir operator C acts on A_l as the scalar $l(l+2)$.

(3) The modules A_l are irreducible, distinct, and exhaust all (isomorphism classes of) finite dimensional irreducible $\mathfrak{sl}(2)$-modules.

Proof. (1) Consider all eigenvalues of H in V and choose an eigenvalue χ such that $\chi + 2$ is not an eigenvalue. Let v_0 be a corresponding eigenvector. Then $Hv_0 = \chi v_0$, $E_+v_0 = 0$. Since V is finite dimensional, Key Lemma implies that $\chi = \ell \in \mathbb{Z}_{\geq 0}$, $E_-^{\ell+1}v_0 = 0$ and the space spanned by $E_-^r v_0$, where $r = 0, 1, \dots, \ell$, forms an $\mathfrak{sl}(2)$-submodule $L \subset V$. The Exercise above implies that L is isomorphic to A_ℓ.

(2) It is quite straightforward that $Ca_l = l(l+2)a_l$. If $a \in A_l$, then $a = Xa_l$ for a certain $X \in U(\mathfrak{sl}(2))$. Hence, $Ca = CXa_l = XCa_l = l(l+2)a$.

(3) If A_l contains a nontrivial submodule V, then it contains A_k, where $k < l$, contradicting the fact that $C = l(l+2)$ on A_l and $C = k(k+2)$ on A_k.

Heading (1) implies that A_l, where $l \in \mathbb{Z}_{\geq 0}$, exhaust all irreducible $\mathfrak{sl}(2)$-modules. □

2.5 Complete Reducibility of $\mathfrak{sl}(2)$-Modules

Proposition. *Any finite dimensional $\mathfrak{sl}(2)$-module V is isomorphic to a direct sum of modules of type A_l. In other words, finite dimensional representations of $\mathfrak{sl}(2)$ are completely reducible.*

Proof. We will use the following general lemma that we prove below.

Lemma. *Let $\mathcal{C}$ be an abelian category. Suppose that any object $V \in \mathcal{C}$ of length 2 is completely reducible. Then any object $V \in \mathcal{C}$ of finite length is completely reducible.*

This implies that it is enough to prove the proposition for a module V of length two. Let $S \simeq A_k$ be an irreducible submodule of V and $Q = V/S \simeq A_l$ a quotient module.

If $k \neq l$, then the Casimir operator has two distinct eigenvalues on V and hence V splits as a direct sum of generalized eigenvectors of C and this decomposition is $\mathfrak{sl}(2)$-invariant. Thus, we can assume that $k = l$.

Consider now the decomposition of $V = \oplus V(i)$ with respect to generalized eigenspaces of the operator H. Since V is glued from two copies of representation A_l, it is clear that $\dim V(i) = 2$ if $i = -l, -l+2, ..., l$ and there are no other summands. Also, it is clear that $E_-^l : V(l) \to V(-l)$ is an isomorphism.

Let us show that the action of H on the space $V(l)$ is given by a scalar operator. Indeed consider the identity $E_+ E_-^{l+1} - E_-^{l+1} E_+ = E_-^l (H - l)$. The left-hand side is 0 on the space $V(l)$ so the right-hand side is 0. Since the operator E_-^l does not have kernel on $V(l)$, we conclude that $H = l$ on $V(l)$.

Now let us choose a vector $v \in V(l)$ that does not lie in the submodule S. Then it is a highest weight vector and by the Key Lemma it generates a submodule $Q' \subset V$ isomorphic to A_l. It is clear that this submodule isomorphically maps to the quotient module $Q = V/S$, i.e. $V \simeq S \oplus Q'$. □

Proof of lemma. We proceed by induction on the length of the object V. Find a simple submodule $S \subset V$ and consider the quotient module $Q = V/S$. By the induction assumption, we can write the quotient module $Q = V/S$ as a direct sum of simple objects $Q = \oplus W_i$. It is enough to show that the natural projection $p : V \to Q$ has a section $\nu : Q \to V$. We construct this section ν separately on every summand W_i. Namely, consider the module $V_i = p^{-1}(W_i)$. This module has length two and by assumption is completely reducible. Hence, the projection $p_i : V_i \to W_i$ has a section $\nu_i : W_i \to V_i \subset V$.

Corollary. *Let V be a finite dimensional $\mathfrak{sl}(2)$-module. Then*

(1) H is diagonalizable and each of the operators E_-^i and E_+^i gives an isomorphism between $V(i)$ and $V(-i)$.
(2) The action of $\mathfrak{sl}(2)$ uniquely extends to the action ρ of the group $SL(2, \mathbb{K})$ on V that satisfies the following condition: Let X equal E_+ or E_-, $t \in \mathbb{K}$ and $g = exp(tX) \in SL(2, \mathbb{K})$. Then the operator $\rho(g)$ in V equals $exp(tX)$.

Remark. The same conclusion holds under the weaker assumption that the module V is $\mathfrak{sl}(2)$-finite. This is left as an exercise to the reader.

3 A Crash Course on Semi-Simple Lie Algebras

3.1 Killing Form

Any Lie algebra $\mathfrak{g}$ admits a unique maximal solvable ideal called the radical $Rad(\mathfrak{g})$. The Lie algebra $\mathfrak{g}$ is called *semisimple* iff its *radical* is zero.

For finite dimensional Lie algebras over a field $\mathbb{K}$ of characteristic 0, there is an equivalent definition, often more convenient. It is given in terms of the *Killing* form, which is the symmetric bilinear form on $\mathfrak{g}$ defined by

$$(X, Y) = \mathrm{tr}(\mathrm{ad}\ X \cdot \mathrm{ad}\ Y).$$

Theorem (Cartan–Killing). $\mathfrak{g}$ is semisimple iff its Killing form is nondegenerate.

3.2 Cartan Subalgebra

There exists a maximal commutative subalgebra $\mathfrak{h} \subset \mathfrak{g}$ such that the adjoint action of $\mathfrak{h}$ on $\mathfrak{g}$ is semisimple.

Such subalgebra is called a *Cartan* subalgebra of $\mathfrak{g}$. In what follows we will fix a Cartan subalgebra $\mathfrak{h}$. One can show that any two Cartan subalgebras are conjugate, so we do not lose information fixing one of them. The number $r = \dim \mathfrak{h}$ is called the *rank* of $\mathfrak{g}$.

3.3 Root System

Consider the adjoint action of the Cartan subalgebra $\mathfrak{h}$ on $\mathfrak{g}$. We obtain a decomposition $\mathfrak{g} = \oplus \mathfrak{g}_\chi$, where for $\chi \in \mathfrak{h}^*$ we have

$$\mathfrak{g}_\chi = \{X \in \mathfrak{g} : [H, X] = \chi(H)X\}.$$

This is called the weight decomposition of $\mathfrak{g}$. Since Killing form is $\mathfrak{h}$-invariant, we see that $(\mathfrak{g}_\chi, \mathfrak{g}_\nu) = 0$ unless $\chi + \nu = 0$. Since this form is nondegenerate, it gives a nondegenerate pairing between $\mathfrak{g}_\chi$ and $\mathfrak{g}_{-\chi}$. In particular, the restriction of the Killing form to $\mathfrak{g}_0$ is nondegenerate.

Proposition. *(1)* $\mathfrak{g}_0 = \mathfrak{h}$
(2) For $\chi \neq 0$, *we have* $\dim_{\mathbb{K}}(\mathfrak{g}_\chi) \leq 1$.

Let

$$R = \{\chi \in \mathfrak{h}^* - \{0\} : \mathfrak{g}_\chi \neq \{0\}\}$$

Then $R \subset \mathfrak{h}^*$ is a finite subset of nonzero elements of the dual space $\mathfrak{h}^*$.

Elements of R are called *roots*.

For every $\gamma \in R$, we fix a nonzero element $E_\gamma \in \mathfrak{g}$. It is called a *root vector*. We will see later that if $\gamma \in R$, then $-\gamma \in R$ and $\lambda\gamma \notin R$ for $\lambda \neq \pm 1$.

3.4 $\mathfrak{sl}(2)$-Triples

Proposition. *We can choose root vectors E_γ for all roots $\gamma \in R$ in such a way that for every root γ the triple of elements $E_\gamma \in \mathfrak{g}_\gamma$, $H_\gamma := [E_\gamma, E_{-\gamma}] \in \mathfrak{h}$ and $E_{-\gamma} \in \mathfrak{g}_{-\gamma}$ form an $\mathfrak{sl}(2)$-triple.*

Essentially, this means that we can find an element $H_\gamma \in [\mathfrak{g}_\gamma, \mathfrak{g}_{-\gamma}] \subset \mathfrak{h}$ such that $\gamma(H_\gamma) = 2$.

The vector $H_\gamma \in \mathfrak{h}$ is called a *coroot* corresponding to the root $\gamma \in \mathfrak{h}^*$.

Corollary. *Let γ, δ be roots. If $\delta + \gamma \notin R$, then $[E_\gamma, E_\delta] = 0$. If $\delta + \gamma \in R$, then $[E_\gamma, E_\delta] = C E_{\gamma+\delta}$, where $C \neq 0$.*

3.5 Integral Structure: Weight Lattice and Root Lattice

From properties of $\mathfrak{sl}(2)$-representations, we see that all eigenvalues of the operator H_γ are integers. In particular for any root δ we have $\delta(H_\gamma) \in \mathbb{Z}$.

Let $\check{Q}$ denote the subgroup of $\mathfrak{h}$ generated by all coroots H_γ (it is called a *coroot lattice*).

For elements $H \in \check{Q}$, we have $(H, H) = \sum \delta(H)^2 \geq 0$, i.e. the Killing form is positive on $\check{Q}$. In fact, it is strictly positive since for any vector H in its kernel we have $\delta(H) = 0$ for all $\delta \in R$ and hence H acts trivially in the adjoint representation. The same reason shows that $\check{Q}$ is a lattice in $\mathbb{K}$-vector space $\mathfrak{h}$.

Let us denote by P the lattice in $\mathfrak{h}^*$ dual to the lattice $\check{Q}$ (it is usually called the *weight lattice*; the elements of P are called *integral weights*). It contains a sublattice Q generated by all roots (it is called *root lattice*).

Since the restriction of the Killing form to $\mathfrak{h}$ is nondegenerate, it induces a bilinear form $\langle \cdot, \cdot \rangle$ on $\mathfrak{h}^*$.

One can describe the coroot $H_\gamma \in \mathfrak{h}$, with $\gamma \in R$ by the property

$$\chi(H_\gamma) = \frac{2\langle \chi, \gamma \rangle}{\langle \gamma, \gamma \rangle} \quad \text{for any } \chi \in \mathfrak{h}^*.$$

3.6 The Weyl Group of the Lie Algebra $\mathfrak{g}$

We will consider the $\mathbb{R}$ vector space $\mathfrak{a} = \mathbb{R} \otimes \check{Q}$ equipped with Euclidean structure defined by positive definite Killing form on it. It is convenient to use convex geometry of this space to state and prove many results about roots and weights.

For any root $\gamma \in R$, consider the linear transformation in the space $\mathfrak{h}^*$ defined by the formula

$$\sigma_\gamma(\chi) = \chi - \chi(H_\gamma)\gamma.$$

The transformation σ_γ is the reflection in the hyperplane defined by the equation $\langle \chi, \gamma \rangle = 0$. In particular $\sigma_\gamma^2 = \mathrm{Id}$ and $\det(\sigma_\gamma) = -1$. The corresponding reflection on the space $\mathfrak{h}$ is given by the formula $\sigma_\gamma(H) = H - \gamma(H)H_\gamma$.

The group of linear transformations of $\mathfrak{h}^*$ generated by operators σ_γ, where $\gamma \in R$, is called the *Weyl group* of $\mathfrak{g}$ and will be denoted by W.

The group W is a group of orthogonal transformations of the space $\mathfrak{h}^*$. It naturally acts on the space $\mathfrak{h}$. The action of W preserves the Killing form, the set of roots R, the set of coroots, lattices P, Q, and $\check{Q}$. Since the Killing form on the lattice P is positive definite, the Weyl group W is finite.

If $\chi_1, \chi_2 \in \mathfrak{h}^*$, then we write $\chi_1 \sim \chi_2$ whenever χ_1 and χ_2 belong to the same orbit of the Weyl group, i.e., when $\chi_1 = w\chi_2$ for a certain $w \in W$.

We also consider the induced actions of W on the Euclidean space $\mathfrak{a}$ and on its dual. In this realization, the Weyl group is a finite group generated by reflections and we can use many geometric facts about actions of such groups.

3.7 Weyl Chamber

For every root $\gamma \in R$ consider the hyperplane Π_γ in the space $\mathfrak{a}^*$ orthogonal to γ, i.e. the set of weights that vanish on H_γ. Consider in $\mathfrak{a}^*$ an open subset $\mathfrak{a}^* \setminus \bigcup_{\gamma \in R} \Pi_\gamma$ obtained by removing all root hyperplanes and fix a connected component C of this set. We denote by $\overline{C}$ the closure of C in $\mathfrak{a}$. The set $\overline{C}$ is called the *Weyl chamber*.

The choice of this set plays central role in the theory. We will see that all Weyl chambers are conjugate under the action of W.

We have the following

Proposition. *$\overline{C}$ is a fundamental domain for the W-*action *on $\mathfrak{a}$. More precisely:*

(1) If $\chi \in \mathfrak{a}$, then $w\chi \in \overline{C}$ for a certain $w \in W$.
(2) If $\chi, w\chi \in \overline{C}$, then $\chi = w\chi$. If, moreover, $\chi \in C$, then $w = e$.

3.8 Positive Roots and Simple Roots

In what follows we fix a Weyl chamber C. A root γ is called *positive* if the coroot H_γ is positive on C, i.e. if $(\chi, \gamma) > 0$ for all $\chi \in C$.

We denote by R^+ the subset of positive roots. It is clear that R is a disjoint union of sets R^+ and $R^- = -R^+$. Also R^+ is closed under addition, i.e. if γ, δ are positive roots and their sum is a root then this root is positive.

A positive root α is called a *simple root* if it cannot be written as a sum of two positive roots. We denote by $B \subset R^+$ the subset of simple roots .

Proposition. *(1) B is a base of the root lattice Q. Every positive root γ is a sum of simple roots with nonnegative integer coefficients.*

(2) Simple roots correspond to hyperplanes in $\mathfrak{a}^*$ that are walls of the Weyl chamber $\bar{C}$.

(3) The Weyl group W is generated by reflections σ_α corresponding to simple roots (they are called simple reflections).

(4) Let α be a simple root. Then for any positive root γ different from α the root $\sigma_\alpha(\gamma)$ is positive. In particular, if β is a simple root different from α, then $(\alpha, \beta) \leq 0$.

(5) Let $\rho \in \mathfrak{h}^*$ be half of the sum of all positive roots. Then for any simple root α we have $\rho(H_\alpha) = 1$ and $\sigma_\alpha(\rho) = \rho - \alpha$. In particular, ρ lies in the lattice P.

Let us denote by Q^+ the subsemigroup of the root lattice Q generated by positive roots. In other words, Q^+ is a free semigroup generated by the set B.

Using this semigroup, we introduce a partial order $\prec$ on the space $\mathfrak{h}^*$ by $\chi \prec \psi$ if $\psi = \chi + q$ with $q \in Q^+$.

Note that a weight χ lies in P iff $\chi(H_\alpha) \in \mathbb{Z}$ for every simple root α. A weight χ is called *dominant* if $\chi(H_\alpha) \in \mathbb{Z}_{\geq 0}$ for every simple root α. Equivalent condition: $\sigma_\alpha(\chi) \prec \chi$.

We denote the semigroup of dominant weights by P^+. Note that the cone generated by P^+ in $\mathfrak{a}^*$ is usually much smaller than the cone generated by Q^+.

3.9 The Triangular Decomposition of a Lie Algebra $\mathfrak{g}$

From this description of the root system R, we derive the following decomposition :

$$\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+,$$

where $\mathfrak{n}_-$ and $\mathfrak{n}_+$ are subspaces generated by E_γ for $\gamma \in R^-$ and $\gamma \in R^+$, respectively. This is a decomposition of linear spaces (not of Lie algebras). We have

Lemma. *(i) $\mathfrak{n}_+$ (resp. $\mathfrak{n}_-$) is the Lie subalgebra of $\mathfrak{g}$ generated by E_α (resp. by $E_{-\alpha}$), where $\alpha \in B$.*

(ii) $[\mathfrak{h}, \mathfrak{n}_+] = \mathfrak{n}_+$ and $[\mathfrak{h}, \mathfrak{n}_-] = \mathfrak{n}_-$.

(iii) *The Lie algebras* $\mathfrak{n}_+$ *and* $\mathfrak{n}_-$ *are nilpotent. Moreover, if* $X \in \mathfrak{n}_+$ *or* $\mathfrak{n}_-$, *then* $\mathrm{ad}X$ *is a nilpotent operator on* $\mathfrak{g}$.
(iv) $U(\mathfrak{g}) \simeq U(\mathfrak{n}_-) \otimes U(\mathfrak{h}) \otimes U(\mathfrak{n}_+) \simeq U(\mathfrak{n}_-) \otimes U(\mathfrak{n}_+) \otimes U(\mathfrak{h})$.

4 Category $\mathcal{O}$ and Verma Modules M_χ

The aim of these lectures is the description of finite dimensional $\mathfrak{g}$-modules. In the sixties, it was noted that it is more natural to describe the finite dimensional modules in the framework of a wider class of $\mathfrak{g}$-modules. First, let us give several preparatory definitions.

4.1 Weight Spaces

Let V be a $\mathfrak{g}$-module. For any $\chi \in \mathfrak{h}^*$ denote by $V^{ss}(\chi)$ the space of vectors $v \in V$ such that $Hv = \chi(H)v$ for any $H \in \mathfrak{h}$ and call it the *weight space* of weight χ. If $V^{ss}(\chi) \neq 0$, then χ is called a *weight* of the $\mathfrak{g}$-module V and any $v \in V^{ss}(\chi)$ is called a *weight vector*. A module V is called $\mathfrak{h}$-*diagonalizable* if $V = \sum_{\chi \in \mathfrak{h}^*} V^{ss}(\chi)$.

Similarly, we introduce a generalized weight space $V(\chi)$ as the space of vectors $v \in V$ such that for any $H \in \mathfrak{h}$ one has $(H - \chi(H))^n = 0$ for large n. If V is $\mathfrak{h}$-finite, it has decomposition $V = \oplus V(\chi)$ (see appendix A). We denote by $P(V)$ the set of weights $\chi \in \mathfrak{h}^*$ such that $V(\chi) \neq 0$ (the weight support of V).

Lemma. *Let* V *be a* $\mathfrak{g}$*-module. For any* $\gamma \in R$, $\chi \in \mathfrak{h}^*$ *we have* $E_\gamma V^{ss}(\chi) \subset V^{ss}(\chi + \gamma)$ *and* $E_\gamma V(\chi) \subset V(\chi + \gamma)$

The proof is the same as in $\mathfrak{sl}(2)$ case.

4.2 The Category $\mathcal{O}$

Let us now introduce a class of $\mathfrak{g}$-modules that we will consider. The objects of *category* $\mathcal{O}$ are $\mathfrak{g}$-modules M satisfying the following conditions.

(1) M is a finitely generated $U(\mathfrak{g})$-module.
(2) M is $\mathfrak{h}$-diagonalizable.
(3) M is $\mathfrak{n}_+$-finite.

Clearly, if a $\mathfrak{g}$-module M belongs to $\mathcal{O}$, then so does any submodule of M and any quotient module of M, and if $M_1, M_2 \in \mathcal{O}$, then $M_1 \oplus M_2 \in \mathcal{O}$.

Lemma. *Let* $\mathfrak{g}$ *be a semisimple Lie algebra. Then any finite dimensional* $\mathfrak{g}$*-module* V *lies in* $\mathcal{O}$.

Proof. It suffices to verify that V is $\mathfrak{h}$-diagonalizable. Since the operators H_γ, where $\gamma \in R$, generate $\mathfrak{h}$ and commute, it suffices to verify that V is H_γ-diagonalizable.

Now V is a finite dimensional $\mathfrak{s}_\gamma$-module, with $\mathfrak{s}_\gamma \subset \mathfrak{g}$ generated by E_γ, H_γ and $E_{-\gamma}$. Since $\mathfrak{s}_\gamma$ is isomorphic to $\mathfrak{sl}(2)$, the result follows from Corollary 2.5. □

4.3 Highest Weight

A nonzero weight vector $m \in M$ is called *a highest weight vector* if $\mathfrak{n}_+ m = 0$.

Since $\mathfrak{n}_+$ is generated by E_α for $\alpha \in B$, we have

Lemma. *A weight vector m is a highest weight vector if and only if $E_\alpha m = 0$ for every $\alpha \in B$.*

Proposition. *Let $M \in \mathcal{O}$ be nonzero. Then M contains a nonzero highest weight vector.*

Proof. The proof is the same as in the case of $\mathfrak{sl}(2)$. We choose an $\mathfrak{h}$-invariant finite dimensional vector subspace $V \subset M$ that generates M. Replacing it by $U(\mathfrak{n}_+)V$ we can assume that it is also $\mathfrak{n}_+$-invariant. Consider all weights χ of $\mathfrak{h}$ in V. Since this is a finite set, there exists a weight χ in V such that for every positive root γ the weight $\chi + \gamma$ is not a weight in V. Any nonzero vector $v \in V(\chi)$ is a highest weight vector. □

4.4 Verma Modules

We now introduce a family of central objects in the category $\mathcal{O}$. These are the *Verma modules* M_χ.

Lemma. *Let $\chi \in \mathfrak{h}^*$. There exists a pair (M_χ, m_χ) of a $\mathfrak{g}$-module and a highest weight vector $m_\chi \in M_\chi(\chi - \rho)$ that satisfies the following universality condition.*

For any $\mathfrak{g}$-module M and highest weight vector $v \in M$ of weight $\chi - \rho$, there exists a unique morphism of $\mathfrak{g}$-modules $i_v : M_\chi \to M$ with $i(m_\chi) = v$.

Remark. By abstract nonsense, if such a module exists it is unique up to a canonical isomorphism.

Proof. Let $\chi \in \mathfrak{h}^*$. In $U(\mathfrak{g})$, consider the left ideal I_χ generated by E_γ, where $\gamma \in R^+$, and by $H + \rho(H) - \chi(H)$, where $H \in \mathfrak{h}$. Define the $\mathfrak{g}$-module M_χ setting $M_\chi = U(\mathfrak{g})/I_\chi$. Let m_χ stand for the natural generator of M_χ (over $\mathfrak{g}$), i.e., the image of $1 \in U(\mathfrak{g})$ under the mapping $U(\mathfrak{g}) \longrightarrow M_\chi$. The module M_χ and the vector m_χ clearly satisfy the universal property. □

Since Verma module is generated by a highest weight vector, the results of Sect. 1.6 imply that it lies in category $\mathcal{O}$.

Lemma. *Let $\chi \in \mathfrak{h}^*$. Then M_χ is a free $U(\mathfrak{n}_-)$ – module with one generator m_χ.*

Proof. The statement follows from the decomposition $U(\mathfrak{g}) = U(\mathfrak{n}_-) \otimes U(\mathfrak{h}) \otimes U(\mathfrak{n}_+)$. □

Corollary. *(1) The set of weights $P(M_\chi)$ of the module M_χ equals to $(\chi-\rho)-Q^+$, i.e. weights of M_χ are of the form $\chi - \rho - q$ for $q \in Q^+$.*
(2) Let M be an arbitrary $\mathfrak{g}$-module, $m \in M$ a highest weight vector of weight $\chi - \rho$ and $i_m : M_\chi \longrightarrow M$ be the corresponding unique morphism. Then i_m is an embedding if and only if $Xm \neq 0$ for any nonzero $X \in U(\mathfrak{n}_-)$.

4.5 The Irreducible Objects L_χ

The next lemma provides a precise parametrization of isomorphism classes of irreducible objects in the category $\mathcal{O}$ in terms of characters of $\mathfrak{h}$.

Lemma. *(1) Let $\chi \in \mathfrak{h}^*$. Then Verma module M_χ has a unique irreducible quotient L_χ and* $\mathrm{Hom}(M_\chi, L_\chi) \approx \mathbb{K}$.
(2) Any irreducible module $L \in \mathcal{O}$ is isomorphic to a module L_χ for a unique weight $\chi \in \mathfrak{h}^*$.

In other words, up to isomorphism L_χ is the unique irreducible $\mathfrak{g}$-module that has highest weight vector of weight $\chi - \rho$. Modules L_χ for different χ are not isomorphic and every irreducible object L in category $\mathcal{O}$ is isomorphic to one of the modules L_χ.

Proof. (1) Consider the weight decomposition $M = M^{\mathrm{top}} \oplus M'$ where M^{top} is the one-dimensional space $M(\chi-\rho)$ and $M' = \oplus M(\mu)$ with sum over $\mu \lneq \chi-\rho$. Any $\mathfrak{g}$-submodule $N \subset M_\chi$ splits with respect to this decomposition, i.e $N = N \bigcap M^{\mathrm{top}} \oplus N \bigcap M'$. Since any non-zero vector of the space M^{top} generates the module M_χ, we see that any proper $\mathfrak{g}$-submodule of M_χ is contained in M'. Thus, the sum of all proper submodules is contained in M'. This shows that M_χ has a unique maximal proper submodule and hence it has unique simple quotient.
(2) Lemma 4.3. implies that every simple module L in $\mathcal{O}$ has a highest weight vector. Using 4.4 we construct a non-zero morphism $M_\chi \to L$ and this implies that L is isomorphic to the module L_χ.

Note that the set of weights $P(L_\chi)+\rho$ has χ as the unique maximal element. This shows how to reconstruct the weight χ from the simple module L. □

Remark. An alternative argument that yields the uniqueness of an irreducible module with highest weight $\chi - \rho$ is as follows. Let M_1, M_2 be two irreducible modules of highest weight $\chi - \rho$ and m_1, m_2 be their highest weight vectors. Then $N = U(\mathfrak{n}\_)(m_1 \oplus m_2) \subset M_1 \oplus M_2$ is a $U(\mathfrak{g})$-submodule. Since both projections $N \to M_1$ and $N \to M_2$ are non zero we see that $M_1 \approx M_2$.

4.6 Characters

In the study of modules from the category $\mathcal{O}$ we will use the notion of the character of a $\mathfrak{g}$-module M.

More generally, let M be a $\mathfrak{g}$-module such that it is $\mathfrak{h}$-finite and in the weight decomposition all the weight spaces $M(\chi)$ are finite-dimensional. In this case, we define the *character* π_M to be the function on $\mathfrak{h}^*$ defined by the equation

$$\pi_M(\chi) = \dim M(\chi).$$

On $\mathfrak{h}^*$, define the *Kostant function* K by the equality

$$K(\chi) = \text{ the number of presentations of the weight } \chi \text{ in the form}$$
$$\chi = -\sum_{\gamma \in R^+} n_\gamma \gamma, \text{ where } n_\gamma \in \mathbb{Z}_{\geq 0}.$$

For any function u on $\mathfrak{h}^*$ set $\operatorname{supp} u = \{\chi \in \mathfrak{h}^* \mid u(\chi) \neq 0\}$. Denote by $\mathcal{E}$ the space of $\mathbb{Z}$-valued functions u on $\mathfrak{h}^*$ such that $\operatorname{supp} u$ is contained in the union of a finite number of sets of the form $\psi - Q^+$, where $\psi \in \mathfrak{h}^*$. For example, $\operatorname{supp} K = -Q^+$, hence, $K \in \mathcal{E}$.

Lemma. *(i)* $\pi_{M_\chi}(\psi) = K(\psi - \chi + \rho)$.
(ii) If $M \in \mathcal{O}$, *then* π_M *is defined and* $\pi_M \in \mathcal{E}$.

Proof. (1) Let us enumerate the elements of R^+, e.g., $\gamma_1, \dots, \gamma_s$. Then the elements $E_{-\gamma_1}^{n_1} \dots E_{-\gamma_s}^{n_s} m_\chi$, where $n_1, \dots, n_s \in \mathbb{Z}_{\geq 0}$, form a basis in M_χ. Hence, $\pi_{M_\chi}(\psi) = K(\psi - \chi + \rho)$.

(2) Choose a finite-dimensional $\mathfrak{h}$-invariant subspace $V \subset M$ that generates M. Replacing V by $U(\mathfrak{n}_+)V$ we can assume that V is also $\mathfrak{n}_+$-invariant. This implies that $M = U(\mathfrak{n}_-)V$. Thus we can write $M = \sum U(\mathfrak{n}_-)(v_i)$, where v_i is a basis of V consisting of weight vectors.

As in heading (i) we have $\dim M(\psi) \leq \sum_{1 \leq i \leq k} K(\psi - \chi_i + \rho)$ implying lemma. □

Exercise. Prove the converse statement: Let M is a finitely generated $U(\mathfrak{g})$-module such that it is $\mathfrak{h}$-diagonalizable, its character π_M is defined and lies in $\mathcal{E}$. Then $M \in \mathcal{O}$.

5 The Weyl Modules A_λ, $\lambda \in P^+$

In this section we construct for every $\lambda \in P^+$ a finite dimensional $\mathfrak{g}$-module A_λ of highest weight λ. Later we will show that $A_\lambda \cong L_{\lambda+\rho}$ and that these modules exhaust all irreducible finite dimensional $\mathfrak{g}$-modules.

5.1 Injections Between Verma Modules

We begin with the following key Proposition:

Proposition. *Let M be a $\mathfrak{g}$-module and $m \in M$ a highest weight vector of weight $\chi - \rho$. Suppose that $k = \chi(H_\alpha) \in \mathbb{Z}_{\geq 0}$. Then the vector $m' = E_{-\alpha}^k m$ is either zero or a highest weight vector of weight $\sigma_\alpha(\chi) - \rho$.*

Proof. Clearly, the weight of the vector $m' = E_{-\alpha}^k m_\chi$ is equal to $\chi - \rho - k\alpha = \sigma_\alpha(\chi) - \rho$.

By Lemma 4.3. it suffices to show that $E_\beta m' = 0$ for $\beta \in B$. If $\beta \neq \alpha$, then

$$E_\beta m' = E_\beta E_{-\alpha}^k m_\chi = E_{-\alpha}^k E_\beta m_\chi = 0,$$

because $[E_\beta, E_{-\alpha}] = 0$. Further,

$$E_\alpha m' = E_\alpha E_{-\alpha}^k m_\chi = E_{-\alpha}^k E_\alpha m_\chi + k E_{-\alpha}^{k-1}(H_\alpha - (k-1)) m_\chi = 0,$$

since $H_\alpha m_\chi = (\chi - \rho)(H_\alpha) m_\chi = (k-1) m_\chi$. □

Remark. This last point is just a repetition of $\mathfrak{sl}(2)$ computation in 2.2.

Corollary. *Suppose $\chi \in \mathfrak{h}^*$ and $\alpha \in B$ are such that $\sigma_\alpha(\chi) < \chi$.*

Then there is a canonical embedding $M_{\sigma_\alpha(\chi)} \longrightarrow M_\chi$ that maps $m_{\sigma_\alpha(\chi)}$ to $E_{-\alpha}^k m_\chi$, where $k = \chi(H_\alpha)$

5.2 π_M is σ_α-Invariant

Lemma. *Let $\alpha \in B$ be a simple root and let $\mathfrak{s}_\alpha \subset \mathfrak{g}$ be the corresponding $\mathfrak{sl}(2)$-subalgebra. Let $M \in \mathcal{O}$ be a $\mathfrak{s}_\alpha$-finite module. Then the character π_M is σ_α invariant.*

Proof. Consider the decomposition $M = \oplus M(k)$ with respect to the action of $H_\alpha \in \mathfrak{g}_\alpha$. By $\mathfrak{sl}(2)$ theory, we have $E_{-\alpha}^k : M(k) \longrightarrow M(-k)$ is an isomorphism for any $k \geq 0$. Decomposing $M(k) = \oplus M(\chi)$, where $\chi \in \mathfrak{h}^*$ with $\chi(H_\alpha) = k$ it is clear that $E_{-\alpha}^k$ induces an isomorphism between $M(\chi)$ and $M(\sigma_\alpha(\chi))$. □

5.3 Construction of the Weyl Modules

For any $\lambda \in P^+ = P \cap \overline{C}$ we have $\sigma_\alpha(\lambda + \rho) \lneqq \lambda + \rho$ and hence by Corollary 5.1. we have the containment $M_{\sigma_\alpha(\lambda+\rho)} \subsetneqq M_{\lambda+\rho}$

We now set

$$A_\lambda = M_{\lambda+\rho} / \sum_{\alpha \in B} M_{\sigma_\alpha(\lambda+\rho)}.$$

Theorem. *(1)* $\pi_{A_\lambda}(\lambda) = 1$.

(2) $P(A_\lambda) \subset \lambda - Q^+ \subset P$ *and* $\pi_{A_\lambda}(w\nu) = \pi_{A_\lambda}(\nu)$ *for any* $w \in W$ *and* $\nu \in P$.

(3) If ν *is a weight of* A_λ, *then either* $\nu \sim \lambda$ *or* $|\nu| < |\lambda|$, *where* $|\nu|$ *is the length of the weight* ν.

(4) $\dim A_\lambda < \infty$.

Proof. (1) The modules $M_{\sigma_\alpha(\lambda+\rho)}$ do not contain vectors of weight λ; hence, these modules are contained in $\sum_{\psi \in \mathfrak{h}^* \setminus \{\lambda\}} M_{\lambda+\rho}(\psi)$. Therefore, $\dim A_\lambda(\lambda) = \dim M_{\lambda+\rho}(\lambda) = 1$.

(2) Since W is generated by σ_α, where $\alpha \in B$, it is enough to verify heading (2) for these elements. Fix $\alpha \in B$. Since A_λ is generated by $\mathfrak{s}_\alpha$-finite vector, Lemma 1.6.1. implies that A_λ is $\mathfrak{s}_\alpha$-finite. The result now follows from Sect. 5.2.

(3) It is clear that

$$\operatorname{supp} \pi_{A_\lambda} \subset \operatorname{supp} \pi_{M_{\lambda+\rho}} = \lambda - Q^+.$$

Let $\pi_{A_\lambda}(\nu) \neq 0$. By replacing ν with a W-equivalent element we can assume that $\nu \in \bar{C}$. Hence, $\lambda = \nu + q$, where $q \in Q^+$. Further on

$$|\lambda|^2 = |\nu|^2 + |q|^2 + 2(\nu, q) \geq |\nu|^2 + |q|^2.$$

Hence, either $|\lambda| > |\nu|$ or $q = 0$ and then $\lambda = \nu$.

(4) $\operatorname{supp} \pi_{A_\lambda}$ is contained in the intersection of the lattice P with the ball $|\nu| \leq |\lambda|$, and, therefore, is finite. Hence, $\dim A_\lambda < \infty$. □

We can now deduce a few results concerning the modules L_χ that are finite dimensional.

Corollary. *An irreducible module* L_χ *is finite dimensional if and only if* $\chi - \rho \in P^+$.

What is missing is the irreducibility of the modules A_λ as this identifies them with $L_{\lambda+\rho}$. This will be proven in Sect. 8.

6 Statement of Harish–Chandra's Theorem on $\mathfrak{Z}(\mathfrak{g})$

The center of the associative algebra $U(\mathfrak{g})$ plays an important role in the study of representations of $\mathfrak{g}$. It is common to denote this commutative algebra by $\mathfrak{Z}(\mathfrak{g})$. In this section I formulate the Harish–Chandra theorem that describes the algebra $\mathfrak{Z}(\mathfrak{g})$. The description of the Harish–Chandra homomorphism is very simple when we consider the action of $\mathfrak{Z}(\mathfrak{g})$ on Verma modules. Indeed, it is easy to see that any element $z \in \mathfrak{Z}(\mathfrak{g})$ acts by a scalar on each of the modules M_χ. Thus we obtain, for each $z \in \mathfrak{Z}(\mathfrak{g})$ a complex valued function on $\mathfrak{h}^*$. We show below that this is a polynomial function on $\mathfrak{h}^*$ that is invariant with respect to the Weyl group. The complete proof of Harish–Chandra's theorem is carried out in Sect. 9.

6.1 The Harish–Chandra Projection

In what follows we will identify the algebra $U(\mathfrak{h}) = S(\mathfrak{h})$ with the algebra $Pol(\mathfrak{h}^*)$ of polynomial functions on the space $\mathfrak{h}^*$.

For any element $X \in U(\mathfrak{g})$ we will construct a function $j(X)$ on the space $\mathfrak{h}^*$ as follows. Given a weight $\chi \in \mathfrak{h}^*$ consider the Verma module M_χ, its highest weight vector $m = m_\chi$ of weight $\chi-\rho$ and a functional $f = f_\chi$ on M_χ such that $f(m) = 1$ and f vanishes on the complementary subspace $M' = \oplus_{\psi\neq\chi-\rho} M_\chi(\psi)$.

It is clear that such functional f exists and is uniquely defined. Now we define $j(X)(\chi) := f_\chi(Xm_\chi)$.

Lemma. *For any $X \in U(\mathfrak{g})$ the function $j(X)$ is a polynomial function in χ.*

Proof. Using triangular decomposition we can write $X = X_0 + X_+ + X_-$, where $X_0 \in U(\mathfrak{h})$, $X_+ \in U(\mathfrak{g})\mathfrak{n}_+$ and $X_- \in \mathfrak{n}_- U(\mathfrak{g})$. This implies that $j(X)(\chi) = j(X_0)(\chi) = X_0(\chi - \rho)$ and this is a polynomial function in χ. □

This proof shows that up to a ρ shift the function $j(X)$ coincides with the "central" part X_0 of the element $X \in U(\mathfrak{g})$; this part is often called *the Harish–Chandra projection*.

6.2 The Harish–Chandra's Homomorphism

Lemma. *(1) For any $z \in \mathfrak{Z}(\mathfrak{g})$ the operator z on the Verma module M_χ is a scalar operator $j(z)(\chi) \cdot Id_{M_\chi}$.*
(2) $j : \mathfrak{Z}(\mathfrak{g}) \to Pol(\mathfrak{h}^*)$ is a morphism of algebras (it is called Harish–Chandra homomorphism).
(3) For any $z \in \mathfrak{Z}(\mathfrak{g})$ the function $j(z) \in Pol(\mathfrak{h}^*)$ is W-invariant.

Proof. (1) Since z commutes with action of $\mathfrak{h}$ we see that $zm_\chi \in M_\chi(\chi - \rho)$ and hence $zm_\chi = cm_\chi$. Since vector m_χ generates M_χ we see that $z = c \cdot \mathrm{Id}$. It is clear that $c = j(z)(\chi)$.
(2) immediately follows from 1.
(3) We would like to show that for any $w \in W$ we have $j(z)(w\chi) = j(z)(\chi)$. It suffices to consider the case when $w = \sigma_\alpha$ for $\alpha \in B$.

Since $j(z)(\chi)$ and $j(z)(\sigma_\alpha(\chi))$ are polynomial functions in χ, it suffices to prove the equality for $\chi \in P^+$. But in this case $M_{\sigma_\alpha(\chi)} \subset M_\chi$, and that implies that the action of z on the Verma modules M_χ and $M_{\sigma_\alpha(\chi)}$ is given by the same scalar. □

6.3 The Harish–Chandra Theorem

By the previous lemmas, the correspondence $z \mapsto j_z$ defines a ring homomorphism $j : \mathfrak{Z}(\mathfrak{g}) \longrightarrow Pol(\mathfrak{h}^*)^W$. We can now state the following important result of Harish–Chandra.

Theorem. *The Harish–Chandra morphism* $j : \mathfrak{Z}(\mathfrak{g}) \longrightarrow Pol(\mathfrak{h}^*)^W$ *is an isomorphism of algebras.*

Remark. In [Di], the map j is described as a composition of the so-called Harish–Chandra projection with a shift. It is easy to trace both in our construction.

Remark. Our construction of the Harish–Chandra map appears to depend on a choice of ordering on the root system.

A different choice of ordering yields the same map, although this statement requires a proof.

7 Corollaries of the Harish–Chandra Theorem

7.1 Description of Infinitesimal Characters

Denote by $\Theta = Spec(\mathfrak{Z}(\mathfrak{g}))$ the set of all homomorphisms $\theta : \mathfrak{Z}(\mathfrak{g}) \to \mathbb{K}$ – such morphisms are usually called *infinitesimal characters.* The Harish–Chandra morphism $j : \mathfrak{Z}(\mathfrak{g}) \to Pol(\mathfrak{h}^*)$ defines a map of sets $\sigma : \mathfrak{h}^* \longrightarrow \Theta$. We usually denote the image $\sigma(\chi)$ by θ_χ.

One of the important corollaries of the Harish–Chandra theorem is the following.

Proposition. *The map* σ *gives a bijection* $\sigma : \mathfrak{h}^*/W \simeq \Theta$.

We have seen that $\sigma(w\chi) = \sigma(\chi)$ so σ defines a map of sets $\sigma : \mathfrak{h}^*/W \to \Theta$.

First let us show that this map is an imbedding.

Lemma. $\theta_{\chi_1} = \theta_{\chi_2}$ *only if* $\chi_1 \sim \chi_2$.

Proof. Let $\chi_1 \not\sim \chi_2$. Let us construct a polynomial $T \in Pol(\mathfrak{h}^*)^W$ such that $T(\chi_1) = 0$, while $T(\chi_2) \neq 0$. For this, take a polynomial $T' \in Pol(\mathfrak{h}^*)$ such that $T'(w\chi_1) = 0$ and $T'(w\chi_2) = 1$ for any $w \in W$ and set $T(\chi) = \sum_{w\in W} T'(w\chi)$.

As follows from the Harish–Chandra theorem, there is an element $z \in \mathfrak{Z}(\mathfrak{g})$ such that $j_z = T$. But then

$$j_z(\chi_1) = \theta_{\chi_1}(z) \neq \theta_{\chi_2}(z) = j_z(\chi_2). \qquad \square$$

The proof of the surjectivity of the map $\sigma : \mathfrak{h}^*/W \rightarrow \Theta$ requires some knowledge of commutative algebra. In fact we will not need this statement so we leave it as an exercise for the reader.

Exercise. Show that any homomorphism of algebras $\theta : Pol(\mathfrak{h}^*)^W \longrightarrow \mathbb{K}$ is of the form θ_χ for a certain $\chi \in \mathfrak{h}^*$.

Hint. First show that $Pol(\mathfrak{h}^*)$ is finitely generated $Pol(\mathfrak{h}^*)^W$-module. Then using Nakayama lemma prove the following general fact from commutative algebra:

Let A be a commutative $\mathbb{K}$-algebra and $B \subset A$ is a $\mathbb{K}$-subalgebra such that A is finitely generated as B-module. Then any morphism of algebras $\theta : B \rightarrow \mathbb{K}$ can be extended to a morphism of algebras $A \rightarrow \mathbb{K}$ (see e.g. lemma 1.4.2 in [Ke]).

7.2 Decomposition of the Category $\mathcal{O}$

Lemma. *Let $M \in \mathcal{O}$. Then there exist an ideal $J \subset \mathfrak{Z}(\mathfrak{g})$ of finite codimension such that $JM = 0$.*

Proof. We can find finite family of weights $\chi_1, ..., \chi_r$ such that $V = \oplus M(\chi_i)$ generates M. The space V is $\mathfrak{Z}(\mathfrak{g})$-invariant. The ideal $J = ker(\mathfrak{Z}(\mathfrak{g}) \rightarrow End(V))$ has the desired property. $\square$

Corollary. *Any $M \in \mathcal{O}$ is $\mathfrak{Z}(\mathfrak{g})$-finite and hence has a direct sum decomposition $M = \oplus_\theta M(\theta)$. Moreover, the set of characters $\theta \in \Theta$ such that $M(\theta) \neq 0$ is finite.*

This follows from the Lemma and Proposition A.2 of the Appendix.

Remark. In our case, the submodule $M(\theta) \subset M$ can be described explicitly as

$$M(\theta) = Ker(I_\theta^n)$$

for sufficiently large n, where $I_\theta = Ker(\theta : \mathfrak{Z}(\mathfrak{g}) \rightarrow \mathbb{K})$.

Exercise. Show that the category $\mathcal{O}$ admits the following decomposition $\mathcal{O} = \oplus \mathcal{O}_\theta$, where the sum runs over $\theta \in \Theta = Spec(\mathfrak{Z}(\mathfrak{g}))$.

Deduce that if N is a subquotient of M then $\Theta(N) \subset \Theta(M)$.

7.3 Finite Length

Proposition. *Any module $M \in \mathcal{O}$ has a finite length.*

Proof. We will prove a more precise statement. Fix $S \subset \Theta$ and consider the full subcategory $\mathcal{O}_S$ of $\mathcal{O}$ consisting of all objects M such that $\Theta(M) \subset S$. Consider the set $\Xi := \Xi(S) \subset \mathfrak{h}^*$ consisting of weights $\chi \in \mathfrak{h}^*$ such that $\theta_{\chi+\rho} \in S$. Consider the exact functor $Res_\Xi : \mathcal{O} \to Vect$, defined by

$$Res_\Xi(M) = \oplus_{\chi \in \Xi} M(\chi).$$

Lemma. *Res_Ξ is faithful on the subcategory $\mathcal{O}_S$*

The lemma follows from the fact that for any irreducible object L in $\mathcal{O}_S$ we have $Res_\Xi(L) \neq \{0\}$.

The lemma implies that for any $M \in \mathcal{O}_S$ we have that the length of M is bounded by $\dim Res_\Xi(M)$. □

Exercise. Show that if $L_{\chi'}$ is a subquotient of M_χ, then $\chi' \sim \chi$. Furthermore, if $L_{\chi'}$ lies in the kernel of $M_\chi \to L_\chi$, then $\chi' \lneqq \chi$

7.4 The Grothedieck Group of the Category $\mathcal{O}$

We will use the standard construction that assigns to every (small) abelain category $\mathcal{C}$ an abelian group $K(\mathcal{C})$ that is called **Grothendieck group** of $\mathcal{C}$.

Namely, denote by A the free abelian group generated by symbols $[M]$, where M runs through the isomorphism classes of objects of $\mathcal{C}$. Let B be the subgroup of A generated by expressions $[M_1] + [M_2] - [M]$ for all exact sequences

$$0 \longrightarrow M_1 \longrightarrow M \longrightarrow M_2 \longrightarrow 0.$$

By definition, the Grothendieck group $K(\mathcal{C})$ of the category $\mathcal{C}$ is the quotient A/B.

Exercise. Suppose we know that every object of an abelian category $\mathcal{C}$ is of finite length. Show that

(i) The map $\mathbb{Z}[Irr C] \longrightarrow K(\mathcal{C})$ is an epimorphism. In other words, the classes of simple objects of $\mathcal{C}$ generate $K(\mathcal{C})$.
(ii) Prove that the map above is an isomorphism. In particular, $K(\mathcal{C})$ is a free abelian group. Hint: Jordan-Hoelder.

In what follows we will use the fact that the collection $\{[L_\chi]\}_{\chi \in \mathfrak{h}^*}$ forms a basis for $K(\mathcal{O})$.

Proposition. *The collection $\{[M_\chi]\}_{\chi \in \mathfrak{h}^*}$ forms a basis of $K(\mathcal{O})$.*

Proof. We can write $K(\mathcal{O}) = \oplus K(\mathcal{O}_\theta)$. We will show that for a given infinitesimal character θ the collection $\{[M_\chi] : \chi \in \mathfrak{h}^* \text{ is such that } \theta_\chi = \theta\}$ forms a basis for $K(\mathcal{O}_\theta)$. We note that the collection $\{[L_\chi] : \chi \in \mathfrak{h}^* \text{ is such that } \theta_\chi = \theta\}$ forms a basis $K(\mathcal{O}_\theta)$. Recall that for any $\psi \in \mathfrak{h}^*$

$$[M_\psi] = [L_\psi] + \sum_{\varphi \gneqq \psi,\ \varphi \sim \psi} n_\varphi [L_\varphi],$$

where $n_\varphi \in \mathbb{Z}$. Inverting this unipotent matrix yields the result. □

7.5 Realization of the Grothendieck group $K(\mathcal{O})$

It will be convenient to have a realization of the group $K(\mathcal{O})$ by embedding it into the group $\mathcal{E}$, the group of $\mathbb{Z}$-valued functions on $\mathfrak{h}^*$ (see Sect. 4.6). Namely, we introduce the convolution product on $\mathcal{E}$ by setting

$$(u * v)(\chi) = \sum_{\varphi \subset \mathfrak{h}^*} u(\varphi) v(\chi - \varphi) \quad \text{for} \quad u, v \in \mathcal{E}.$$

Note that only a finite number of the summands are non-zero. Since $u * v \in \mathcal{E}$, the convolution endows $\mathcal{E}$ with a commutative algebra structure.

For any $\chi \in \mathfrak{h}^*$ define $\delta_\chi \in \mathcal{E}$ by setting $\delta_\chi(\varphi) = 0$ for $\varphi \neq \chi$ and $\delta_\chi(\chi) = 1$.

Clearly, δ_0 is the unit of $\mathcal{E}$.

Set

$$L = \prod_{\gamma \in R+} (\delta_{\gamma/2} - \delta_{-\gamma/2}) = \delta_\rho \prod_{\gamma \in R+} (\delta_0 - \delta_{-\gamma})$$

Here Π is the convolution product in $\mathcal{E}$.

We can now define a homomorphism $\tau : K(\mathcal{O}) \longrightarrow \mathcal{E}$ by the formula

$$\tau([M]) = L * \pi_M,$$

where $M \in \mathcal{O}$,

Theorem. *(1)* $\tau(M_\chi) = \delta_\chi$.

(2) The mapping $\tau : K(\mathcal{O}) \longrightarrow \mathcal{E}$ gives an isomorphism of $K(\mathcal{O})$ with the subgroup $\mathcal{E}_c \subset \mathcal{E}$ consisting of functions with compact support.

Proof. The second point is an immidiate consequence of the first in lieu of the fact that the family $\{[M_\chi]\}$ generates $K(\mathcal{O})$. The proof of the first point is based on Lemma 4.6 and the following Lemma.

Lemma. *Let K be the Kostant function, see Sect. 4.6. Then*

$$K * \delta_{-\rho} * L = \delta_0.$$

Proof. For any $\gamma \in R^+$ set $a_\gamma = \delta_0 + \delta_{-\gamma} + \ldots + \delta_{-n\gamma} + \ldots$. The definition of K implies that

$$K = \prod_{\gamma \in R^+} a_\gamma.$$

Further, $(\delta_0 - \delta_{-\gamma})a_\gamma = \delta_0$. Since L can be represented as $\prod_{\gamma \in R^+}(\delta_0 - \delta_{-\gamma})\delta_\rho$, we are done. □

Remark. The theorem implies that finding the exact transition matrix between the basis $\{[M_\chi]\}$ and the basis $\{[L_\chi]\}$ is equivalent to the determination of $\tau(L_\chi)$. This is the subject of the Kazhdan–Lusztig conjecture.

8 Description of Finite Dimensional Representations

8.1 Complete Reducibility of Finite Dimensional Modules

In this section, we will describe all finite dimensional representations of a semisimple Lie algebra $\mathfrak{g}$. As was shown in Sect. 4.2, all such representations belong to $\mathcal{O}$. Recall that in Sect. 5 we constructed a collection of finite dimensional $\mathfrak{g}$-modules A_λ parameterized by weights $\lambda \in P^+$. We will now show that any finite dimensional module is isomorphic to a direct sum of such modules, and that these are irreducible. This yields complete reducibility.

Theorem. *(1) Let M be a finite dimensional $\mathfrak{g}$-module. Then M is isomorphic to a direct sum of modules of the form A_λ for $\lambda \in P^+$.*
(2) All the modules A_λ, where $\lambda \in P^+$, are irreducible.

Proof. (1) We may assume that $M = M(\theta)$, where $\theta \in \Theta$. Let m be any highest weight vector of M and λ its weight. Then $\theta = \theta_{\lambda+\rho}$. Besides, for any simple root α we have $E^k_{-\alpha}m = 0$ for large k, and hence by Lemma 2.2 $\lambda(H_\alpha) \in \mathbb{Z}_{\geq 0}$. Therefore, $\lambda \in P^+$.

Since $\lambda \in P^+$ the element $\lambda + \rho$ lies inside the interior of the Weyl chamber and thus is uniquely recovered from the infinitesimal character of the module A_λ.

Let $m_1, \ldots, m_l$ be a basis of $M(\lambda)$. Let us construct the morphism $p : \bigoplus_{1 \leq i \leq l} M(\lambda + \rho) \longrightarrow M$ so that each generator $(m_{\lambda+\rho})_i$ for $i = 1, 2, \ldots, l$ goes to m_i. As follows from Lemma 2.2, for any simple root α we have $E^{k_\alpha}_{-\alpha}m_i = 0$, where $k_\alpha = (\lambda + \rho)(H_\alpha)$

Hence p may be considered as the morphism $p : \oplus_{1 \leq i \leq l}(A_\lambda)_i \longrightarrow M$.

Let L_1 and L_2 be the kernel and cokernel of the morphism p. Then $\Theta(L_i) = \{\theta\}$ and $L_i(\lambda) = 0$, where $i = 1, 2$. As was shown above, $L_1 = L_2 = 0$, i.e., $M \cong \bigoplus_{1 \leq i \leq l}(A_\lambda)_i$.

(2) Let M be a nontrivial submodule of A_λ. Then $\Theta(M) = \theta_{\lambda+\rho}$, hence, $M(\lambda) \neq 0$, i.e., M contains a vector of weight λ. But then $M = A_\lambda$. Thus, the module A_λ is irreducible and the proof of the Theorem is complete. □

Corollary. *$A_\lambda \cong L_{\lambda+\rho}$, where $\lambda \in P^+$.*

Remark. The module A_λ is an irreducible module of highest weight λ. The strange shift in its numbering as an irreducible module corresponds to the Harish–Chandra shift.

8.2 Characters of Highest Weight Modules A_λ

Consider the natural action of the group W on the space of functions on $\mathfrak{h}^*$ defined by

$$(wu)(\chi) = u(w^{-1}\chi) \text{ for } w \in W, \chi \in \mathfrak{h}^*.$$

Lemma. *$wL = \det w \cdot L$ for any $w \in W$.*

Proof. It suffices to verify that $\sigma_\alpha L = -L$ for $\alpha \in B$. Since σ_α permutes the elements of the set $R^+ \setminus \{\alpha\}$ and transforms α into $-\alpha$, then

$$\sigma_\alpha L = (\delta_{-\alpha/2} - \delta_{\alpha/2}) \prod_{\gamma \in R^+ \setminus \{\alpha\}} (\delta_{\gamma/2} - \delta_{-\gamma/2}) = -L.$$ □

The next theorem provides a formula for the formal character of the finite dimensional irreducible module L_λ. This will give us Kostant multiplicity formulas, Weyl character formula and Weyl dimension formula.

Theorem. *Suppose L_λ is finite dimensional. Then*

$$L * \pi_{L_\lambda} = \sum_{w \in W} \det w \cdot \delta_{w(\lambda)}.$$

Proof. We have

$$[L_\lambda] = \sum_{\mu \sim \lambda} a_\mu [M_\mu]$$

with $a_\lambda = 1$.

Applying τ to this equation, we obtain

$$\tau([L_\lambda]) = \sum_{\mu \sim \lambda} a_\mu \delta_\mu.$$

Since π_{L_λ} is W-invariant and L is W-skew invariant, we see that $\tau([L_\lambda]) = L * \pi_{L_\lambda}$ is W-skew-invariant as well.

Thus,

$$L * \pi_{L_\lambda} = \sum_{w \in W} \det w \cdot \delta_{w(\lambda)} \tag{*}$$

Theorem 8.2 is proved. □

Corollary. *(1) for any* $\lambda \in P^+$ *we have* $[A_\lambda] = \sum_{w \in W} \det w \cdot [M_{w(\lambda+\rho)}]$
(2) the Kostant formula for the multiplicity of the weight $\pi_{A_\lambda}(\psi) = \sum_{w \in W} \det w \cdot K(\psi + \rho - w(\lambda + \rho))$ *for any* $\psi \in \mathfrak{h}^*$.

Proof. Since τ is an isomorphism, to verify the first item, we may apply τ to both sides. The second item is a reformulation of the first in view of the Lemma 4.6. □

8.3 Weyl Character Formula

Denote by $F(\mathfrak{h})$ the ring of formal power series in $\mathfrak{h}$, i.e. the completion of the algebra of polynomial functions $Pol(\mathfrak{h})$ at the point zero. For any $\chi \in \mathfrak{h}^*$ set $e^\chi = \sum_{i \geq 0} \frac{\chi^i}{i!}$.

Clearly, $e^\chi \in F(\mathfrak{h})$ and $e^{\chi+\psi} = e^\chi e^\psi$ for $\chi, \psi \in \mathfrak{h}^*$. Let M be a finite dimensional $\mathfrak{g}$-module. Define the *character* $\mathrm{ch}_M \in F(\mathfrak{h})$ of M by the formula

$$\mathrm{ch}_M = \sum_{\chi \in P} \pi_M(\chi) e^\chi.$$

Theorem. *Set*

$$L' = \sum_{w \in W} (\det w) e^{w\rho}.$$

Then for A_λ*, where* $\lambda \in P^+$*, we have*

$$L' \mathrm{ch}_{A_\lambda} = \sum_{w \in W} (\det w) e^{w(\lambda+\rho)}.$$

Proof. The mapping $j : \mathcal{E}_c \longrightarrow F(\mathfrak{h})$ defined by the formula $j(u) = \sum_{\chi \in \mathfrak{h}^*} u(\chi) e^\chi$ is a ring homomorphism. Inserting $\lambda = \rho$ in formula (*) of 8.2, we obtain

$$\sum_{w \in W} \det w \cdot \delta_{w\rho} = L * \pi_{A_0} = L * \delta_0 = L.$$

Hence, $j(L) = L'$. The result now follows by applying j to formula (*) of 8.2 with $A_\lambda = L_{\lambda+\rho}$. □

Remark. (1) When $\mathbb{K} = \mathbb{C}$, all the power series involved in Theorem 8.3 converge and define analytic functions on $\mathfrak{h}$. Theorem 8.3 claims the equality of two such functions.

(2) Let $\mathcal{G}$ be a complex semisimple Lie group with Lie algebra $\mathfrak{g}$ and $\mathcal{H} \subset \mathcal{G}$ the Cartan subgroup corresponding to the Lie subalgebra $\mathfrak{h}$. Consider the finite dimensional representation T of $\mathcal{G}$ corresponding to the $\mathfrak{g}$-module A_λ. Let $h \in \mathcal{H}$. Then $h = \exp(H)$, where $H \in \mathfrak{h}$. It is easy to derive from Theorem 8.3 that

$$\operatorname{Tr} T(h) = \frac{\sum_{w\in W} \det w \cdot \mathrm{e}^{(w(\lambda+\rho))}(H)}{\sum_{w\in W} \det w \cdot e^{(w\rho)}(H)}.$$

This is the well-known H. Weyl's formula for characters of irreducible representations of complex semisimple Lie groups.

8.4 Weyl's Dimension Formula

Theorem. *Let $\lambda \in P^+$. Then*

$$\dim A_\lambda = \prod_{\gamma\in R^+} \frac{\langle \lambda + \rho, \gamma\rangle}{\langle \rho, \gamma\rangle}.$$

Proof. Set

$$F_\chi = \sum_{w\in W} \det w \cdot \mathrm{e}^{w\chi} \text{ for any } \chi \in \mathfrak{h}^*.$$

Clearly, $F_\rho = L' = \prod_{\gamma\in R^+}(\mathrm{e}^{\gamma/2} - \mathrm{e}^{-\gamma/2})$. For any $\chi \in \mathfrak{h}^*$ and $H \in \mathfrak{h}$, we may consider $F_\chi(tH)$ as a formal power series in one variable t.

Let ρ' and λ' be elements of $\mathfrak{h}$ corresponding to ρ and λ, respectively, after the identification of $\mathfrak{h}$ with $\mathfrak{h}^*$ by means of the Killing form. Then

$$\dim A_\lambda = \mathrm{ch} A_\lambda(0) = \frac{F_{\lambda+\rho}(t\rho')}{F_\rho(t\rho')}\Big|_{t=0}.$$

Observe that

$$F_{\lambda+\rho}(t\rho') = \sum_{w\in W} \det w \cdot \mathrm{e}^{t\langle\lambda+\rho, w^{-1}\rho\rangle} = F_\rho(t(\lambda' + \rho')).$$

Hence by the product formula we have

$$\dim A_\lambda = \frac{F_\rho(t(\lambda' + \rho'))}{F_\rho(t\rho')}\Big|_{t=0} = \prod_{\gamma\in R^+}\left(\frac{e^{t/2(\gamma(\lambda'+\rho'))} - e^{-t/2(\gamma(\lambda'+\rho'))}}{e^{t/2(\gamma(\rho'))} - e^{-t/2(\gamma(\rho'))}}\Big|_{t=0}\right)$$

The quantity on the right hand side is evaluated easily to be

$$\prod_{\gamma\in R^+}\frac{\langle\gamma,\lambda+\rho\rangle}{\langle\gamma,\rho\rangle}.$$

□

8.5 Summary of Results

We collect here the results we have proven for finite dimensional representations of $\mathfrak{g}$.

1. For any weight $\lambda \in P^+$ we have constructed a finite dimensional irreducible $\mathfrak{g}$-module A_λ. All such modules are nonisomorphic. Any finite dimensional irreducible $\mathfrak{g}$-module is isomorphic to one of A_λ, where $\lambda \in P^+$.
2. **Complete reducibility**
   Any finite dimensional $\mathfrak{g}$-module M is isomorphic to a direct sum of A_λ.
3. The module A_λ is $\mathfrak{h}$-diagonalizable and has the unique (up to a factor) highest weight vector a_λ. The weight of a_λ is equal to λ. The module A_λ is called a **highest weight module** of highest weight λ.
4. **Harish–Chandra theorem on ideal.**
   The module A_λ is generated by the vector a_λ as $U(\mathfrak{n}_-)$-module (in particular, all the weights of A_λ are less than or equal to λ). The ideal of relations $I = \{X \in U(\mathfrak{n}_-) \mid Xa_\lambda = 0\}$ is generated by the elements $E_{-\alpha}^{m_\alpha+1}$, where $\alpha \in B$ and $m_\alpha = \lambda(H_\alpha)$.
5. The function π_{A_λ} is W-invariant.
6. If ψ is a weight of A_λ, then either $\lambda \sim \psi$ or $|\psi| < |\lambda|$.
7. A_λ has infinitesimal character $\theta_{\lambda+\rho}$. Explicitly, for any $a \in A_\lambda$ and $z \in \mathfrak{Z}(\mathfrak{g})$ we have $za = \theta_{\lambda+\rho}(z)a$.
   If $\lambda_1, \lambda_2 \in P^+$ and $\lambda_1 \neq \lambda_2$, then homomorphisms $\theta_{\lambda_1+\rho}$ and $\theta_{\lambda_2+\rho}$ are distinct.
8. **Weyl character formula**

$$L\cdot \mathrm{ch}_{A_\lambda} = \sum_{w\in W}(\det w)e^{w(\lambda+\rho)}, \quad \text{where } L = \sum_{w\in W}(\det w)e^{w(\rho)}$$

9. **Kostant multiplicity formula.**

$$\pi_{A_\lambda}(\mu) = \sum_{w \in W} (\det w) K(\mu + \rho - w(\lambda + \rho)).$$

10. **Weyl dimension formula**

$$\dim A_\lambda = \prod_{\gamma \in R^+} \frac{\langle \gamma, \lambda + \rho \rangle}{\langle \gamma, \rho \rangle}.$$

11. For any finite dimensional $\mathfrak{g}$-module V, the module V is $\mathfrak{h}$-diagonalizable and its character π_V is W-invariant.

9 Proof of the Harish–Chandra Theorem

The proof we describe here will be obtained by first reducing Harish–Chandra's theorem to Chevalley's restriction theorem. The proof of Chevalley's theorem is obtained using characters of finite dimensional representations A_λ of $\mathfrak{g}$.

The proof we present uses implicitly a group action without defining the group that acts. The existence of the action should not be surprising in view of Corollary 2.5 that finite representations of the Lie algebra $\mathfrak{sl}(2)$ admits an action of the group $SL(2)$. A similar idea applies in general. Instead of providing a formal statement let us briefly explain how to obtain such a group.

Let G be the adjoint group of automorphisms of $\mathfrak{g}$. This is the group generated by groups $SL(2)_\gamma$ corresponding to all the roots γ. This group acts on $\mathfrak{g}$, on $U(\mathfrak{g})$, $S(\mathfrak{g})$ and preserves natural structures on all these spaces. On each of these spaces V the actions of $\mathfrak{g}$ and G are related as follows.

(*) Let $X \in \mathfrak{g}_\alpha$ and $g = \exp\ ad(X) \in G$. Then for any vector $v \in V$ we have

$$gv = \exp(X)v := \sum_k \frac{1}{k!} X^k v.$$

This expression makes sense since $X^k v = 0$ for large k.

In particular the invariants with respect to G and $\mathfrak{g}$ in each of these spaces are the same.

9.1 Reduction to Chevalley's Theorem

We constructed a morphism $j : \mathfrak{Z}(\mathfrak{g}) \to Pol(\mathfrak{h}^*)^W = U(\mathfrak{h})^W$ and would like to show that it is an isomorphism. By construction, j is the restriction to $\mathfrak{Z}(\mathfrak{g})$ of a linear map $j : U(\mathfrak{g}) \to U(\mathfrak{h})$ defined by Harish–Chandra projection (see 6.1).

Morphism j is compatible with natural filtrations on $\mathfrak{Z}(\mathfrak{g})$ and $U(\mathfrak{h})^W$ obtained by restrictions of standard filtrations on $U(\mathfrak{g})$ and $U(\mathfrak{h})$. So in order to show that j is an isomorphism it is enough to check that the associated graded morphism $\alpha := grj : gr\mathfrak{Z}(\mathfrak{g}) \to grU(\mathfrak{h})^W$ is an isomorphism. Let us identify these two spaces.

First of all notice that $\mathfrak{Z}(\mathfrak{g}) = U(\mathfrak{g})^{\mathfrak{g}}$ where we consider the adjoint action of $\mathfrak{g}$ on $U(\mathfrak{g})$, $ad(X)(u) = [X, u]$. Let us also consider the adjoint action of $\mathfrak{g}$ on the algebra $S(\mathfrak{g})$ such that $ad(X)$ is the derivation of the algebra $S(\mathfrak{g})$ satisfying $ad(X)(Y) = [X, Y]$ for $Y \in \mathfrak{g} \subset S(\mathfrak{g})$. Using the morphism *symm* discussed in Corollary 1.5. We see that the space gr $\mathfrak{Z}(\mathfrak{g})$ coincides with the space $S(\mathfrak{g})^{\mathfrak{g}}$ (this follows from the fact that *symm* is a morphism of $\mathfrak{g}$-modules). Similarly, $\mathrm{gr}((U(\mathfrak{h})^W))$ coincides with the space $S(\mathfrak{h})^W$.

Consider the morphism $\beta : S(\mathfrak{g}) = S(\mathfrak{n}_-) \otimes S(\mathfrak{h}) \otimes S(\mathfrak{n}_-) \to S(\mathfrak{h})$ obtained by mapping $\mathfrak{n}_-$ and $\mathfrak{n}_+$ to 0. Analyzing the explicit description of the morphism α described above it is easy to see that it coincides with the restriction of β to $\mathfrak{g}$-invariant elements.

Using Killing form we will identify $\mathfrak{g}$ with $\mathfrak{g}^*$ and $\mathfrak{h}$ with $\mathfrak{h}^*$. In this way we interpret $S(\mathfrak{g})$ as the algebra $Pol(\mathfrak{g})$ of polynomial functions on $\mathfrak{g}$ and $S(\mathfrak{h})$ as the algebra $Pol(\mathfrak{h})$ of polynomial functions on $\mathfrak{h}$. Morphism β after this identification is just the restriction of polynomial functions on $\mathfrak{g}$ to $\mathfrak{h}$.

This shows that Harish–Chandra theorem follow from the following result

Theorem (The Chevalley's restriction theorem). Let $Pol(\mathfrak{g})$ and $Pol(\mathfrak{h})$ be algebras of polynomial functions on $\mathfrak{g}$ and $\mathfrak{h}$, respectively, and $\eta : Pol(\mathfrak{g}) \longrightarrow Pol(\mathfrak{h})$ the restriction homomorphism. Then $Pol(\mathfrak{g})^{\mathfrak{g}} \longrightarrow Pol(\mathfrak{h})^W$ is an isomorphism.

9.2 Proof of Injectivity in Chevalley's Theorem

Let us choose an ordering $\gamma_1, ..., \gamma_r$ of roots of the algebra $\mathfrak{g}$ and consider the algebraic variety $Y = \prod \mathfrak{g}_{\gamma_i} \times \mathfrak{h}$; in fact this is just an affine space isomorphic to $\mathfrak{g}$. Let us define a morphism of algebraic varieties $a : Y \to \mathfrak{g}$ by

$$a(X_1, ..., X_r, H) = \exp \mathrm{ad}(X_1) \exp \mathrm{ad}(X_2) ... \exp \mathrm{ad}(X_r)(H).$$

Clearly, any function $f \in Pol(\mathfrak{g})^{\mathfrak{g}}$ in the kernel of the morphism η will also lie in the kernel of morphism of algebras $a^* : Pol(\mathfrak{g}) \to Pol(Y)$ corresponding to the morphism a.

However, if we choose a regular element $H \in \mathfrak{h}$ (i.e., an element such that $\gamma(H) \neq 0$ for every root γ) and consider the point $y = (0, ..., 0, H) \in Y$, then easy computation shows that the differential da at this point is an isomorphism of linear spaces. This implies that the kernel of the homomorphism a^* is 0.

9.3 Proof of Surjectivity in Chevalley's Theorem

Fix a non-negative integer k. To every finite dimensional representation (ρ, V) of the Lie algebra $\mathfrak{g}$, we assign a polynomial function $P_{k,V}$ on the Lie algebra $\mathfrak{g}$ as follows $P_{k,V}(X) = \mathrm{tr}(\rho(X)^k)$. Clearly, this is a $\mathfrak{g}$-invariant polynomial function on $\mathfrak{g}$. The surjectivity of the morphism η follows from

Proposition. *The collection of functions $P_{k,V}$ on $\mathfrak{h}$ spans $Pol(\mathfrak{h})^W$.*

Proof. Let us denote by $F(\mathfrak{h})$ the completion of the algebra $Pol(\mathfrak{h})$ at maximal ideal $\mathfrak{m}$ corresponding to the point $0 \in \mathfrak{h}$. In other words, if (y_i) is a coordinate system on the linear space $\mathfrak{h}$, then $F(\mathfrak{h}) = \mathbb{K}[[y_1, ..., y_r]]$. Since polynomials $P_{k,V}$ are homogeneous in order to prove the proposition, it is enough to prove that the $\mathbb{K}$-linear span of the collection of polynomials $P_{k,V}$ is dense in the algebra $F(\mathfrak{h})^W$.

To see this we will consider a different model for the algebra $F(\mathfrak{h})$. Namely consider the category $\mathcal{R}(\mathfrak{h})$ of finite dimensional $\mathfrak{h}$-modules. We say that an object V of $\mathcal{R}(\mathfrak{h})$ is **integrable** if the action of $\mathfrak{h}$ is completely reducible and all coroots H_γ have integral spectrum. We denote by $\mathcal{R}$ the full subcategory of $\mathcal{R}(\mathfrak{h})$ of integrable objects. The Grothendieck group $K(\mathcal{R})$ of this category is naturally isomorphic to the group algebra $\mathbb{Z}(P)$ of the lattice P. Namely, a weight $\lambda \in P$ corresponds to a one-dimensional representation T_λ of the Lie algebra $\mathfrak{h}$ of weight λ.

Consider a homomorphism of algebras $\sigma : K(\mathcal{R}) \to F(\mathfrak{h})$ defined by

$$\sigma((\rho, V))(x) = tr_V(\exp(\rho(x)) = \sum_k \frac{1}{k!} tr_V(\rho(x)^k)$$

In particular, $\sigma(T_\lambda) = \exp(\lambda)$.

It is easy to see that the $\mathbb{K}$-span of the image of morphism σ is dense in $F(\mathfrak{h})$ (in fact $F(\mathfrak{h})$ can be realized as the completion of the algebra $\mathbb{K}(P) := \mathbb{Z}(P) \otimes_{\mathbb{Z}} \mathbb{K}$ at the maximal ideal corresponding to the homomorphism $\mathbb{K}(P) = \mathbb{K}(\mathfrak{R}) \otimes \mathbb{K} \to \mathbb{K}$ given by $V \mapsto \dim(V)$.

Now consider the category $\mathcal{R}(\mathfrak{g})$ of finite dimensional $\mathfrak{g}$-modules and the restriction functor $r : \mathcal{R}(\mathfrak{g}) \to \mathcal{R}(\mathfrak{h})$. Based on the $\mathfrak{sl}(2)$ theory we may view r as a functor $r : \mathcal{R}(\mathfrak{g}) \to \mathcal{R}$. Denote by π the corresponding morphism of Grothendieck groups $\pi : K(\mathcal{R}(\mathfrak{g})) \to K(\mathcal{R})$.

For every $V \in \mathcal{R}(\mathfrak{g})$ the element $\pi(V)$ considered as a function on P is just the character π_V of V, which was defined in Sect. 4.6.

Now, the image $\sigma(\pi(V)) \in F(\mathfrak{h})$ equals $\sum_k P_{k,V}/k!$. Thus, in order to show that polynomials $P_{k,V}$ span a dense subset of $F(\mathfrak{h})^W$, it is enough to prove the following.

Lemma. *The image of morphism $\pi : K(\mathcal{R}(\mathfrak{g})) \to K(\mathcal{R})$ equals to the subgroup $K(\mathcal{R})^W \subset K(\mathcal{R})$ of W-invariant elements.*

The lemma easily follows from Theorem 5.3. Namely, if an element $u \in K(\mathcal{R}) \simeq \mathbb{Z}(P)$ is W-invariant, then induction on the maximal length of weights in the support of u implies that u can be written as a $\mathbb{Z}$ -linear combination of $\pi(A_\lambda)$, where $\lambda \in P^+$. □

A. Appendix: Eigenspaces Decomposition

In this section, we present the standard Eigen-space decomposition of linear algebra with few variations that are needed in the text.

A.1 Standard Eigenspace Decomposition

Let $\mathbb{K}$ be an algebraically closed field. Let T be an operator on a finite dimensional $\mathbb{K}$-vector space V.

We denote by $Spec(T, V)$ the set of $\lambda \in \mathbb{K}$ such that the operator $T - \lambda\mathbf{1}$ is not invertible. Since V is finite dimensional, the operator T satisfies some equation $P(T) = 0$ for some monic polynomial P that could be written as $\prod(T - \lambda_i\mathbf{1}) = 0$. This shows that if $V \neq \{0\}$ then the set $Spec(T, V)$ is not empty.

For any $\lambda \in \mathbb{K}$, we denote by $V(\lambda)$ the space of vectors $v \in V$ annihilated by some power of the operator $T - \lambda\mathbf{1}$. Vectors of the these spaces are called generalized eigenvectors. It is clear that $V(\lambda) \neq 0$ iff $\lambda \in Spec(T, V)$.

Denote by $V^{ss}(\lambda) = Ker(T - \lambda\mathbf{1}) \subset V(\lambda)$. Vectors of these spaces are called *eigenvectors*. We say that T is semisimple if V is spanned by eigenvectors of T.

Note that if S is an operator commuting with T then S preserves all the spaces $V^{ss}(\lambda)$, $V(\lambda)$.

Proposition. $V = \oplus V(\lambda)$ *where the sum is taken over all* $\lambda \in \mathbb{K}$.

Proof. (a) We first prove linear independence. Otherwise, take the shortest dependence of the form $v_1 + ... + v_k = 0$, where each v_i is a generalized eigenvector with eigenvalues λ_i, and all eigenvalues are distinct. Clearly, $k \geq 2$. Applying $T - \lambda_1\mathbf{1}$ several times to the above identity, we get a shorter dependency.

(b) For every $\lambda \in \mathbb{K}$ consider the quotient space $Q_\lambda = V/V(\lambda)$. We claim that $Spec(T, Q_\lambda)$ does not contain λ. Indeed, let $V'(\lambda) \subset V$ be the preimage of the space $Q_\lambda(\lambda)$. Then some power of the operator $T - \lambda\mathbf{1}$ maps $V'(\lambda)$ to $V(\lambda)$ and hence some larger power maps it to 0. This implies that $V'(\lambda) = V(\lambda)$ and hence $Q_\lambda(\lambda) = 0$.

Consider now the space $Q = V/\sum_\lambda V(\lambda)$. Since this space is a quotient of all the spaces Q_λ, the set $Spec(T, Q) \subset \cap_\lambda Spec(T, Q_\lambda)$ is empty and hence $Q = 0$. □

Corollary. *If* T *is semisimple then* $V = \oplus V^{ss}(\lambda)$.

A.2 Eigenspace Decomposition for Commuting Families

Let now A be a commutative $\mathbb{K}$-algebra acting on a $\mathbb{K}$-vector space V. For each character $\chi : A \to \mathbb{K}$ we denote by $\mathfrak{m}_\chi = \ker(\chi)$ the corresponding maximal ideal of A. We denote by $V(\chi)$ the subspace of vectors in V that are annihilated by some power of $\mathfrak{m}$. They are called generalized eigenvectors corresponding to the character χ. We denote by $V^{ss}(\chi)$ the space of vectors annihilated by $\mathfrak{m}$. They are called eigenvectors.

We say that the action is *locally finite* if V is a union of finite dimensional A-submodules.

Proposition. *Let A be a commutative algebra and V be a locally finite A-module. We have:*

(1) $V = \oplus V(\chi)$ where the sum is taken over all characters χ of A.
(2) If each $a \in A$ acts semisimply on V, then $V = \oplus V^{ss}(\chi)$.

Proof. We first consider the case $\dim(V) < \infty$.

For (1) note that the linear independence of the spaces $V(\chi)$ follows from the previous proposition. To show that V is a direct sum we argue by induction on dimension of V. If each $a \in A$ has only one eigenvalue $\alpha(a)$, then α is a character and we are done. Otherwise, we can split V, using the previous proposition, as a sum of generalized eigenspaces for some $a \in A$. Since each of these spaces is invariant with respect to the algebra A, we can apply induction. The same proof gives the decomposition in the semi-simple case.

Now the locally finite case is an obvious formal consequence of the finite dimensional case. □

Corollary. *Let A be a finite dimensional commutative algebra over $\mathbb{K}$ with unit. Then*

(1) In A, there is a finite number of maximal ideals $\mathfrak{m}_i$, where $i = 1, \ldots, k$.
(2) There are elements $e_i \in A$, where $i = 1, \ldots, k$, such that

$$e_i e_j = 0 \textit{ for } i \neq j \textit{ and } e_i^2 = e_i;$$

$$e_1 + e_2 + \ldots + e_k = 1;$$

$$e_i \in \mathfrak{m}_j \textit{ for } i \neq j;$$

$$e_i \mathfrak{m}_i^n = 0 \textit{ for } n > \dim A.$$

Proof. Let A act on itself by multiplication. By the previous proposition, we have a projection $P_\chi : A \to A(\chi)$ for each character χ of A. Write the identity operator as a sum $\mathbf{1} = \sum P_i$ where all $P_i = P_{\chi_i}$ are non zero.

If P is one of these projectors, then it is given by multiplication by an element $e = P(1) \in A$ (Indeed, $P(b) = P((b \cdot 1)) = b \cdot P(1) = b \cdot e$).

These elements $e_i = P_i(1)$ and the maximal ideals $\mathfrak{m}_i = \ker(\chi_i)$ satisfy the statement of the corollary. □

Acknowledgements I am thankful to D. Leites for saving these lectures, translating them into English and preparing them for publication, and the Department of Mathematics of Stockholm University for financing a preprint of these lectures as part of Leites' *Seminar on Supermanifolds* proceedings.

I am thankful to E. Sayag for the great help in preparing the final draft of these lectures.

References

[BG] A. Braverman, D. Gaitsgory, *Poincare-Birkhoff-Witt theorem for quadratic algebras of Koszul type*. J. Algebra 181 (1996), 315-328.

[Bu] Bourbaki N., *Groupes et algébres de Lie*. Hermann, Paris, Ch. II–III: 1972; Ch. IV–VI: 1968; Ch. VII–VIII: 1974

[Di] Dixmier J., *Enveloping algebras*, Graduate study in Mathenatics, N. 11, AMS, 1996

[Dy] Dynkin E., The review of principal concepts and results of the theory of linear representations of semi-simple Lie algebras (Complement to the paper "Maximal subgroups of the classical groups") (Russian) Trudy Moskov. Mat. Obšč. 1, (1952). 39–166

[Ge] Gelfand, I. (ed.) *Representations of Lie groups and Lie algebras*. Acad. Kiado, Budapest, 1975

[Ke] Kempf, G.R. *Algebraic varieties*. London Mathematical Society Lecture Note Series, vol. 172, Cambridge University Press, Cambridge, 1993, pp. 163

[Ki] Kirillov, A. (ed.) *Representations of Lie groups and Lie algebras*. Acad. Kiado, Budapest, 1985

[OV] Onishchik A., Vinberg E., *Lie groups and Algebraic Groups*, Springer, 1987

[Pr] Prasolov V., *Problems and theorems in linear algebra*, AMS, 1991

[Se] Serre J.-P. *Lie algebras and Lie groups*. N.Y, W. A. Benjamin, 1965; *Algébres de Lie semi-simples complexes*. N.Y, W. A. Benjamin, 1966