A SIMPLE PROOF OF KOSTANT'S THEOREM THAT $U(\mathfrak{g})$ IS FREE OVER ITS CENTER

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0. Introduction

0.1. Let $\mathfrak{g}$ be a reductive Lie algebra over an algebraically closed field k of characteristic 0 and $U(\mathfrak{g})$ its universal enveloping algebra.

In his remarkable paper [K] published in this journal more than thirty years ago B. Kostant showed that $U(\mathfrak{g})$ is a free module over its center $\mathcal{Z}(\mathfrak{g})$. This result plays a fundamental role in the representation theory of $\mathfrak{g}$.

0.2. As usual the freeness result easily follows from the corresponding result for commutative algebras :

($*$) Let $\mathcal{P}(\mathfrak{g})$ be the algebra of polynomial functions on $\mathfrak{g}$ and $I = \mathcal{P}(\mathfrak{g})^G$ be the subalgebra of functions invariant with respect to the adjoint group G. Then $\mathcal{P}(\mathfrak{g})$ is a free I-module.

Kostant deduced the statement ($*$) from some sophisticated results of commutative algebra by careful analysis of the variety $\mathcal{N} \subset \mathfrak{g}$ of nilpotent elements in $\mathfrak{g}$.

Later it was realized that this and other similar results follow from general theorems of commutative algebra and estimates of the dimension of $\mathcal{N}$ (see [S], Proposition 4.3).

0.3. All these proofs rely on rather nontrivial results of commutative algebra. This has several drawbacks.

1) First of all, it creates a misleading impression that the statement ($*$) is a difficult result. For example, Dixmier in his textbook [D] does not present the complete proof of this result; in a crucial place, he just refers the reader to Kostant's original paper.
2) These proofs are by their nature not constructive – they prove existence of a free basis but do not tell anything about its structure.
3) When one attempts to generalize these results to, say, quantum groups, everything breaks down since one cannot apply commutative algebra.

0.4. In this note we show how to deduce the statement ($*$) directly from the standard results by Chevalley on the structure of the algebra I. Moreover, we exhibit an explicit basis of $\mathcal{P}(\mathfrak{g})$ as an I-module.

The idea of the proof can be described as follows. Suppose we are given a regular algebra I and an I-module M and suppose that we suspect that the following property $P(I, M)$ holds:

M is a projective I-module.

If the module M is finitely generated, there are many tools which may help to prove such a statement (e.g. it is enough to check that M is Cohen-Macauley,

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which is usually much easier to do). If M is not finitely generated these tools usually do not work.

So one can try the following general strategy for proving the statement $P(I, M)$: find another pair (R, N), where R is a regular algebra and N an R-module, such that the statement $P(R, N)$ implies $P(I, M)$ and such that N is a finitely generated R-module; after this one should try to prove the statement $P(R, N)$.

The interesting thing is that such a pair often exists and that the proof of a stronger statement $P(R, N)$ is usually much simpler than the proof of a weaker statement $P(I, M)$.

0.5. In section 1 we describe our method and use it to prove a stronger version of Kostant's freeness result. In section 2 we show how to apply this result to simplify the proof of another remarkable Kostant's theorem - his description of the space of functions on the nilpotent cone.

In section 3 we describe some applications of these results to the theory of $\mathfrak{g}$-modules. In particular we introduce the notion of a centrally free Lie subalgebra of $\mathfrak{g}$ and use our method to show that some subalgebras are centrally free. These are main new results of this note.

The notion of centrally free subalgebra appeared in our study of equivariant derived categories on flag varieties (see [BL]). Trying to prove the results of section 3 we came up with the method described in section 1.

1. Generalization of Kostant's theorem

1.1. Our proof is based on the following elementary observation.

Let V be a k-vector space and $\mathcal{P}(V)$ be the algebra of polynomial functions on V, considered as a graded algebra. Given a subspace $H \subset V$ we naturally construct two morphisms of algebras $r : \mathcal{P}(V) \to \mathcal{P}(H)$ (restriction) and $i : \mathcal{P}(V/H) \to \mathcal{P}(V)$.

Let I be a graded subalgebra in $\mathcal{P}(V)$. We would like to find out whether $\mathcal{P}(V)$ is free as an I-module.

Proposition. *Suppose that the algebra I has the following properties with respect to a subspace $H \subset V$*

(i) The restriction morphism $r : I \to \mathcal{P}(H)$ is injective.

(ii) The algebra $\mathcal{P}(H)$ is a free module over the algebra $I' = r(I)$.

Then $\mathcal{P}(V)$ is a free R-module, where $R = \mathcal{P}(V/H) \otimes I$. In particular, $\mathcal{P}(V)$ is a free I-module.

Moreover, suppose we are given a free basis $\{a_\kappa\}$ of I'-module $\mathcal{P}(H)$ consisting of homogeneous elements. Fix any system of homogeneous elements $\{b_\kappa \in \mathcal{P}(V)\}$ such that $r(b_\kappa) = a_\kappa$. Then $\{b_\kappa\}$ is a free basis of the R-module $\mathcal{P}(V)$.

This proposition is proved by a standard play with filtrations. We will present the proof in section 4.

1.2. Let us see how proposition 1.1 implies Kostant's theorem 0.2 $(*)$. Fix a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$. Then we have the following two results by Chevalley:

CH1. The restriction morphism $r : I \to \mathcal{P}(\mathfrak{h})$ defines an isomorphism of the algebra I with the algebra $I' = \mathcal{P}(\mathfrak{h})^W$ of W-invariant polynomials on $\mathfrak{h}$, where W is the Weyl group of $\mathfrak{h}$ in $\mathfrak{g}$ (see [D], Theorem 7.3.7).

CH2. $\mathcal{P}(\mathfrak{h})$ is a free I'-module with some basis $\{a_w \mid w \in W\}$ (see [B], V.5.2).

These results and proposition 1.1 imply

Theorem. *$\mathcal{P}(\mathfrak{g})$ is a free module over the algebra $R = I \otimes \mathcal{P}(\mathfrak{g}/\mathfrak{h})$ with basis $\{b_w\}$. In particular, $\mathcal{P}(\mathfrak{g})$ is a free I-module.*

1.3. Remark. The construction outlined in 1.2 allows us to describe a very explicit basis for the I-module $\mathcal{P}(\mathfrak{g})$. Namely, let us fix a system of positive roots Σ^+ of $\mathfrak{h}$ in $\mathfrak{g}$. Then the construction described in [BGG] provides a very specific basis $\{a_w\}$ of the I'-module $\mathcal{P}(\mathfrak{h})$ connected with Schubert cells. Using the Killing form we can canonically write $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{g}/\mathfrak{h}$ which allows us to extend a_w to elements $b_w \in \mathcal{P}(\mathfrak{g})$.

Now if we fix a Chevalley basis in $\mathfrak{g}$, it would define a very specific basis in $\mathfrak{g}/\mathfrak{h}$ and hence a monomial basis in $\mathcal{P}(\mathfrak{g}/\mathfrak{h})$.

Together this provides a specific basis of the I-module $\mathcal{P}(\mathfrak{g})$.

1.4. The same proof as in 1.2 also works in the Kostant-Rallis situation (see [KR]). In fact, as was pointed to us by N. Wallach, in this situation a very similar argument is presented in [W], section 11.4.

More generally, the same proof works for the polar representations of Dadok and Kac (see [DK]).

Namely, consider a polar action of a group G on a vector space V and denote by I the space $\mathcal{P}(V)^G$ of G-invariant polynomial functions on V. Dadok and Kac have shown in [DK] that there exists a subspace $H \subset V$ and a finite reflection group W of automorphisms of H such that the restriction map $r : I \to \mathcal{P}(H)$ gives an isomorphism of I with the algebra $I' = \mathcal{P}(H)^W$. Since for a finite reflection group W the space $\mathcal{P}(H)$ is a free finitely generated I'-module (see [B], V.5.2), this implies that $\mathcal{P}(V)$ is a free finitely generated module over the algebra $\mathcal{P}(V/H) \otimes I$. In particular, it is a free I-module.

2. Functions on Nilpotent Cone

In the process of proving his theorem, Kostant also established some properties of the nilpotent cone, which are very important for the theory of $\mathfrak{g}$-modules. In this section we would like to show how to streamline Kostant's arguments using theorem 1.2.

2.1. Let $I \subset \mathcal{P}(V)$ be a graded subalgebra as in 1.1. Consider the ideal $I^+ \subset I$ generated by elements of positive degree and denote by J the ideal in $\mathcal{P}(V)$ generated by I^+, $J = \mathcal{P}(V)I^+$.

Consider V as an algebraic variety over k and denote by $(X, \mathcal{O}_X)$ a subscheme of V defined by the ideal J. This is an affine scheme with the algebra of global functions $\mathcal{O}(X) = \Gamma(X, \mathcal{O}_X)$ isomorphic to $\mathcal{P}(V)/J$.

Suppose that there exists a subspace $H \subset V$ satisfying the conditions of proposition 1.1 with some finite set $\{a_\kappa\}$. Consider $Y = V/H$ as an algebraic variety and denote by p the natural projection $p : X \to Y$. Then proposition 1.1. immediately implies that $\mathcal{O}(X)$ is a free finitely generated module over a regular algebra $\mathcal{P}(Y)$. A scheme X with such a property is called Cohen-Macaulay.

2.2. Lemma. *Let X be a Cohen-Macaulay scheme and $U \subset X$ an open subscheme. Consider algebras $\mathcal{O}(X)$ and $\mathcal{O}(U)$ of global functions on X and U and denote by r the restriction morphism $r : \mathcal{O}(X) \to \mathcal{O}(U)$*

a) Suppose that $\dim(X \setminus U) < \dim X$. *Then r is injective.*

b) Suppose that $\dim(X \setminus U) \leq \dim X - 2$. *Then r is an isomorphism.*

Proof. Since $\mathcal{O}(X)$ is a finitely generated $\mathcal{O}(Y)$-module, p is a finite morphism. In particular it is closed.

Without loss of generality we can replace U by a smaller open subset $p^{-1}(W)$ where $W = Y \setminus p(X \setminus U)$ is an open subset of Y.

Let us consider the free $\mathcal{O}_Y$-module $F = p_*(\mathcal{O}_X)$. Then the restriction map $r : \mathcal{O}(X) \to \mathcal{O}(U)$ is the same as the restriction map $r : \Gamma(Y, F) \to \Gamma(W, F)$.

Suppose that $\dim(X \setminus U) < \dim X$. Then $\dim(Y \setminus W) < \dim Y$, which implies that r is injective.

Suppose that $\dim(X \setminus U) \leq \dim X - 2$. Then $\dim(Y \setminus W) \leq \dim Y - 2$. This implies that every regular function on W extends to a regular function on Y. Since the sheaf F is a free sheaf of $\mathcal{O}_Y$-modules this implies that r is bijective. $\square$

2.3. Corollary. *Let X be a Cohen-Macaulay scheme and $U \subset X$ an open subscheme.*

a) *Suppose that* $\dim(X \setminus U) < \dim X$ *and that the scheme U is reduced (i.e. $\mathcal{O}(U)$ does not have nilpotent elements). Then the scheme X is reduced.*

b) *Suppose that* $\dim(X \setminus U) \leq \dim X - 2$ *and that the scheme U is reduced, irreducible and normal. Then the scheme X is reduced, irreducible and normal.*

This immediately follows from 2.2.

2.4. Consider now Kostant's case $V = \mathfrak{g}$, $I = \mathcal{P}(\mathfrak{g})^G$.

Set $J = \mathcal{P}(\mathfrak{g}) \cdot I^+$ and denote by $\mathcal{N}$ the subscheme of $\mathfrak{g}$ defined by the ideal J.

Kostant has proved the following results:

a) As a set, $\mathcal{N}$ coincides with the set of nilpotent elements in $\mathfrak{g}$.
b) Let $U \subset \mathcal{N}$ be the subset of regular nilpotent elements. Then U is an open subset of $\mathcal{N}$ and $\dim(\mathcal{N} \setminus U) \leq \dim \mathcal{N} - 2$.
c) $(U, \mathcal{O}_U)$ is a nonsingular subscheme in $\mathfrak{g}$. In particular it is reduced and normal (more precisely, Kostant proved that if $\{z_i\}$ is a system of generators of the algebra I, then their differentials are linearly independent at points of U; this implies that U is nonsingular).

The proof in 1.2 shows that the scheme $\mathcal{N}$ is Cohen-Macaulay. Thus lemma 2.2 and corollary 2.3 imply the following results about the nilpotent cone $\mathcal{N}$:

a) $(N, \mathcal{O}_{\mathcal{N}})$ is a reduced scheme corresponding to the subset of nilpotent elements in $\mathfrak{g}$. In other words, the ideal J consists of all polynomial functions which vanish on nilpotent elements.
b) $\mathcal{N}$ is a normal variety.
c) Consider the restriction map $r : \mathcal{O}(\mathcal{N}) \to \mathcal{O}(U)$, where U is the variety of regular nilpotent elements in $\mathfrak{g}$. Then r is an isomorphism.

3. Some Applications to $\mathfrak{g}$-Modules

3.1. For applications to enveloping algebras it is better to reformulate proposition 1.1 in terms of dual spaces. Namely, let M be a vector space and $I \subset S(M)$ be a graded subalgebra. Choose a subspace $K \subset M$ and set $C = M/K$. Consider the corresponding morphisms of symmetric algebras $S(K) \xrightarrow{i} S(M) \xrightarrow{p} S(C)$.

Proposition. *Let us assume that the graded subalgebra $I \subset S(M)$ satisfies the following conditions with respect to the subspace K:*

(i) $p : I \to S(C)$ is an embedding.

(ii) $S(C)$ is a free $I' = p(I)$ – module with basis $\{a_\kappa\}$ consisting of homogeneous elements.

Lift each element a_κ to a homogeneous element $b_\kappa \in S(M)$. Then $S(M)$ is a free module over the algebra $R = S(K) \otimes I$ with basis $\{b_\kappa\}$.

This is just a dual version of proposition 1.1.

3.2. Theorem. *Let $\mathfrak{g}$ be a reductive Lie algebra. Consider its triangular decomposition $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$. Choose a basis $\{a_w\} \subset U(\mathfrak{h})$ of $U(\mathfrak{h})^W$-module $U(\mathfrak{h})$ (as in 1.2).*

Consider the algebra $R = U(\mathfrak{n}^-) \otimes U(\mathfrak{n}^+)^0 \otimes \mathcal{Z}(\mathfrak{g})$, where $U(\mathfrak{n}^+)^0$ is the opposite algebra, and define its action on the space $U(\mathfrak{g})$ by $(x \otimes y \otimes z)(u) = xuzy$, where $x \in U(\mathfrak{n}^-), y \in U(\mathfrak{n}^+), z \in \mathcal{Z}(\mathfrak{g})$. Then $U(\mathfrak{g})$ is a free R-module with basis $\{a_w\}$.

In particular, $U(\mathfrak{g})$ is a free $\mathcal{Z}(\mathfrak{g})$-module.

Proof. It is enough to check the corresponding statement for associated graded modules and algebras (see 4.2). Thus we have to show that $S(\mathfrak{g})$ is a free module over the algebra $R' = S(\mathfrak{n}^-) \otimes S(\mathfrak{n}^+) \otimes I$ with basis $\{a_w\}$. This follows from Chevalley theorems and proposition 3.1 with $M = \mathfrak{g}$, $K = \mathfrak{n}^- \oplus \mathfrak{n}^+$, $C = \mathfrak{h}$, since $S(K) = S(\mathfrak{n}^-) \otimes S(\mathfrak{n}^+)$. □

3.3. Centrally free subalgebras.

We say that a subalgebra $\mathfrak{l} \subset \mathfrak{g}$ is **centrally free** if $U(\mathfrak{g})$ is a free $U(\mathfrak{l}) \otimes \mathcal{Z}(\mathfrak{g})$-module. If we fix a central character $\eta : \mathcal{Z}(\mathfrak{g}) \to \mathbb{C}$ and denote by U_η the corresponding algebra $U_\eta = U(\mathfrak{g})/U(\mathfrak{g}) \cdot \ker \eta$, then for a centrally free subalgebra $\mathfrak{l} \subset \mathfrak{g}$, the algebra U_η will be a free $U(\mathfrak{l})$-module.

Here are examples of such subalgebras.

Example 1. $\mathfrak{l} = \mathfrak{n}$. The fact that $\mathfrak{l}$ is centrally free follows from theorem 3.2.

Example 2. Let θ be a Cartan involution (i.e. $\theta\big|_{\mathfrak{h}} = -1$) and $\mathfrak{l} = \mathfrak{g}^\theta$. Then $\mathfrak{l}$ is a centrally free subalgebra.

Indeed, consider the algebra $R = U(\mathfrak{l}) \otimes U(\mathfrak{n}^-)^0 \otimes \mathcal{Z}(\mathfrak{g})$ and define its action on $U(\mathfrak{g})$ by $(x, y, z)(u) = xuyz$. Then the same proof as in 2.2 shows that $U(\mathfrak{g})$ is a free R-module (since $\mathfrak{l} \oplus \mathfrak{n}^- = K$, i.e. $S(\mathfrak{l}) \otimes S(\mathfrak{n}^-) = S(K)$). In particular, $U(\mathfrak{g})$ is a free $U(\mathfrak{l}) \otimes \mathcal{Z}(\mathfrak{g})$-module.

3.4. Using theorem 3.2 we can prove the following general result:

Theorem. *Let $\mathfrak{l}$ be a subalgebra of $\mathfrak{g}$ which acts freely on some open subset of the flag variety X of the Lie algebra $\mathfrak{g}$. Then $\mathfrak{l}$ is centrally free. In particular, for any central character η the algebra U_η described above is free as an $U(\mathfrak{l})$-module.*

In fact we think that the condition in the theorem is necessary and sufficient for a subalgebra $\mathfrak{l} \subset \mathfrak{g}$ to be centrally free.

Proof. Consider the symmetric algebra $S(\mathfrak{g})$ and denote by I the subalgebra of invariants of the adjoint group G. We will use the following simple

Lemma. *There exists a subspace $K \subset \mathfrak{g}$ such that K contains the subspace $\mathfrak{l}$ and the subalgebra I satisfies the conditions of Proposition 3.1 with respect to K.*

Fix a subspace $D \subset K$ complement to $\mathfrak{l}$ and identify $S(D)$ with a subspace in $S(\mathfrak{g})$. Also choose a basis $\{a_\kappa\}$ of this subspace consisting of homogeneous elements.

The proposition 3.1 shows that this provides a free basis for $S(\mathfrak{g})$ as a module over an algebra $Q = I \otimes S(\mathfrak{l})$.

Using the isomorphism $S(\mathfrak{g}) = \mathrm{gr}U(\mathfrak{g})$ we can lift elements a_κ to elements $b_\kappa \in U(\mathfrak{g})$. Since $\mathrm{gr}\mathcal{Z}(\mathfrak{g}) = I$ and $\mathrm{gr}U(\mathfrak{l}) = S(\mathfrak{l})$, lemma 4.2 implies that $\{b_\kappa\}$ is a free basis of $U(\mathfrak{g})$ considered as a module over the algebra $R = \mathcal{Z}(\mathfrak{g}) \otimes U(\mathfrak{l})$.

Now let us see how to prove the lemma. Fix a generic point x of X and denote by $\mathfrak{b}$ the corresponding Borel subalgebra. Choose a triangular decomposition $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$ compatible with $\mathfrak{b}$. By condition of the theorem the subalgebras $\mathfrak{l}$ and $\mathfrak{b}$ do not intersect. We claim that the lemma above holds if we replace $\mathfrak{l}$ by any subspace $L \subset \mathfrak{g}$ which does not intersect $\mathfrak{b}$.

Indeed, without loss of generality we can assume that the subspace L is complementary to $\mathfrak{b}$. Let $n = \dim \mathfrak{n}$. Denote by C the subset of all $2n$ dimensional subspaces $K \subset \mathfrak{g}$ which satisfy conditions of proposition 3.1. Clearly C is a Zariski open subset of the corresponding Grassmannian which is invariant with respect to conjugation.

Let C' be the subset of all those n dimensional subspaces $L \subset \mathfrak{g}$ which can be imbedded into some subspace $K \in C$. Then C' is an open subset of the corresponding Grassmannian which is invariant with respect to conjugation.

Let A be the space of linear morphisms $a : \mathfrak{n}^- \to \mathfrak{b}$. For any such morphism a its graph will be a subspace of $\mathfrak{g}$ complementary to $\mathfrak{b}$, and any subspace complementary to $\mathfrak{b}$ can be obtained in this way.

Denote by A' the subset of all morphisms $a \in A$ whose graphs belong to C'. Clearly A' is a Zariski open subset of A invariant with respect to the adjoint action of the Cartan subgroup H corresponding to the Lie algebra $\mathfrak{h}$. Also A' contains a zero morphism (its graph coincides with $\mathfrak{n}^-$).

Since all the weights of the action of H on the space A are strictly positive this implies that A' equals A. In particular, the lemma above holds for all subspaces L complementary to $\mathfrak{b}$.

This proves the lemma and the theorem. $\square$

4. Proof of Proposition 1.1

4.1. By a graded vector space, we mean a $\mathbb{Z}$-graded space $M = \oplus M_i$, $i \in \mathbb{Z}$. We always assume that the grading is bounded below, i.e. $M_i = 0$ for $i \ll 0$. Similarly a filtered space (M, F) is a vector space M with a family of subspaces F^iM, $i \in \mathbb{Z}$, such that $F^iM \subset F^{i+1}M$ and $\bigcup F^iM = M$. We always assume that $F^iM = 0$ for $i \ll 0$.

A graded vector space $M = \oplus M_i$ is naturally filtered by subspaces $F^iM = \sum M^j$, with $j \leq i$. Conversely, given a filtered vector space (M, F) we denote by $\mathrm{gr}M$ the associated graded space $\mathrm{gr}M = \oplus \mathrm{gr}_i M$, where $\mathrm{gr}_i M = F^iM/F^{i-1}M$. For every $m \in M$ we denote by $\sigma(m)$ its symbol in $\mathrm{gr}M$.

Lemma. *Let $\varphi : M \to N$ be a morphism of filtered vector spaces and let $gr(\varphi)$: $grM \to grN$ be the associated morphism of associated graded spaces. Suppose that $gr\varphi$ is an isomorphism. Then φ is an isomorphism.* $\square$

4.2. Similarly one defines notions of graded and filtered algebras and graded and filtered modules over such algebras.

Lemma. *Let M be a filtered module over a filtered algebra A and $\{m_\kappa\}$ a family of elements of M. Suppose that their symbols $\{\sigma(m_\kappa)\}$ form a free basis of the grA-module grM. Then $\{m_\kappa\}$ is a free basis of the A-module M.*

Proof. Let $d_\kappa = \deg m_\kappa = \{\min i | m_\kappa \in F^i M\}$. Consider a free A-module N with basis $\{n_\kappa\}$ and introduce a filtration on N by $F^i N = \sum_{j,\kappa} F^j A \cdot n_\kappa$ with $j + d_\kappa \leq i$. Consider the morphism $\varphi : N \to M$ of A-modules defined by $\varphi(n_\kappa) = m_\kappa$. It is easy to see that φ is a morphism of filtered A-modules and that grφ is an isomorphism. Now lemma 4.1 implies that φ is an isomorphism, i.e. that $\{m_\kappa\}$ is a free A-basis of M. □

4.3. Now we can prove proposition 1.1.

Let V be a vector space, $\mathcal{P}(V)$ the algebra of polynomial functions on V and $H \subset V$ a linear subspace as in 1.1.

We consider the natural grading on $\mathcal{P}(V)$ and denote by $F^i\mathcal{P}(V)$ the corresponding filtration. We also consider another filtration Φ on $\mathcal{P}(V)$ defined by $\Phi^i\mathcal{P}(V) = \mathcal{P}(V/H) \cdot F^i\mathcal{P}(V)$.

Lemma.

a) gr_Φ $(\mathcal{P}(V)) \approx \mathcal{P}(H) \otimes P(V/H)$, with the standard grading on $\mathcal{P}(H)$ and zero grading on $\mathcal{P}(V/H)$.

b) Let $P \in \mathcal{P}(V)$ be a homogeneous element of degree i. Suppose that $r(P) \in \mathcal{P}(H)$ is nonzero. Then its symbol $\sigma_\Phi(P)$ coincides with $r(P) \in \mathcal{P}(H) \subset \mathcal{P}(H) \otimes \mathcal{P}(V/H)$.

The proof of this lemma is straightforward.

4.4. Consider the filtration on I induced by filtration Φ. Lemma 3.3 implies that for any homogeneous element $P \in I$ its symbol $\sigma_\Phi(P)$ coincides with $r(P) \in \mathcal{P}(H) \subset$ gr$_\Phi(\mathcal{P}(V))$. This implies that the subalgebra $\sigma_\Phi(I) \subset$ gr$_\Phi\mathcal{P}(V)$ coincides with $I' = r(I) \subset \mathcal{P}(H)$.

Similarly, the lemma implies that $\sigma_\Phi(b_\kappa) = a_\kappa$. Since $\{a_\kappa\}$ form a free basis of the I'-module $\mathcal{P}(H)$, they form a free basis of the gr$_\Phi R$ -module gr$_\Phi\mathcal{P}(V) = \mathcal{P}(H) \otimes \mathcal{P}(V/H)$. Lemma 4.2 implies that $\{b_\kappa\}$ form a free basis of the R-module $\mathcal{P}(V)$. This concludes the proof of proposition 1.1.

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