Introduction.

Let $f : X \to Y$ be a continuous map of locally compact spaces. Let $Sh(X)$, $Sh(Y)$ denote the abelian categories of sheaves on X and Y, and $D(X)$, $D(Y)$ denote the corresponding derived categories (maybe bounded $D = D^b$ or bounded below $D = D^+$ if necessary). It is well known that there exist functors

$$f^*,\ f_*,\ f^!,\ f_!,\ D,\ Hom,\ \otimes$$

between the categories $D(X)$ and $D(Y)$, which satisfy certain identities.

Now assume that X, Y are in addition G-spaces for a topological group G, and that f is a G-map. Instead of sheaves let us consider the equivariant sheaves $Sh_G(X)$, $Sh_G(Y)$. One wants to have triangulated categories $D_G(X)$, $D_G(Y)$ – "derived categories of equivariant sheaves" – together with all the above functors. More precisely, there should exist the forgetful functor

$$For : D_G \to D,$$

so that the functors in categories D_G are compatible with the usual ones in categories D under this forgetful functor. Simple examples show that the derived category $D(Sh_G)$ of the abelian category Sh_G cannot be taken for D_G (unless the group G is discrete). The main purpose of this work is to introduce the suitable category D_G and to define the corresponding functors.

Actually, we get more structure. Namely, let $\phi : H \to G$ be a homomorphism of groups, X be an H-space, Y be a G-space, and $f : X \to Y$ be a map compatible with the homomorphism ϕ. In this situation we have functors of inverse and direct image

$$Q_f^* : D_G(Y) \to D_H(X),$$

$$Q_{f*} : D_H(X) \to D_G(Y).$$

The direct image functor Q_{f*} is probably the most interesting one. It does not in general commute with the forgetful functor.

For a connected Lie group G we give an algebraic description of the triangulated category $D_G(pt)$ in terms of DG-modules over a natural DG-algebra $\mathcal{A}_G$. This description is our main tool in applications of the theory. As an example of an application we "compute" the equivariant intersection cohomology (with compact supports) of toric varieties.

Let us explain briefly the structure of the text. Part I is devoted mainly to the definition of the category $D_G(X)$ and of various functors. In part II we use DG-modules to study the category $D_G(pt)$ and discuss equivariant cohomology. Finally, in the last part III the general theory is applied to toric varieties, which yields some applications to combinatorics.

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