Part I. Derived category $D_G(X)$ and functors.

0. Some preliminaries.

0.1. Let G be a topological group and X be a topological space. We say that X is a G-**space** if G acts continously on X. This means that the multiplication map

$$m: \ G \times X \to X, \quad (g, x) \mapsto gx$$

is continuous.

Let X, Y be G-spaces. A continuous map $f: X \to Y$ is called a G-**map** if it commutes with the action of G on X and Y.

More generally, let $\phi: H \to G$ be a homomorphism of topological groups. Let X be an H-space and Y be a G-space and $f: X \to Y$ be a continuous map. We call f a ϕ-**map** if

$$f(hx) = \phi(h)f(x)$$

for all $x \in X, \ h \in H$.

Let X be a G-space. We denote by $\overline{X} := G\backslash X$ the quotient space (the space of G-orbits) of X and by $q: X \to \overline{X}$ the natural projection. By definition q is a continuous and open map.

0.2. Let X be a G-space. Consider the diagram of spaces

$$G \times G \times X \ \begin{array}{c} \xrightarrow{d_0} \\ \xrightarrow{d_1} \\ \xrightarrow{d_2} \end{array} \ G \times X \ \begin{array}{c} \xrightarrow{d_0} \\ \xleftarrow{s_0} \\ \xrightarrow{d_1} \end{array} \ X$$

where

$$d_0(g_1, \dots, g_n, x) = (g_2, \dots, g_n, g_1^{-1}x)$$

$$d_i(g_1, \dots, g_n, x) = (g_1, \dots, g_i g_{i+1}, \dots, g_n, x), \quad 1 \le i \le n-1$$

$$d_n(g_1, \dots, g_n, x) = (g_1, \dots, g_{n-1}, x)$$

$$s_0(x) = (e, x)$$

A G-**equivariant sheaf** on X is a pair (F, θ), where $F \in Sh(X)$ and θ is an isomorphism

$$\theta: d_1^* F \simeq d_0^* F,$$

satisfying the cocycle condition

$$d_0^*\theta \circ d_2^*\theta = d_1^*\theta, \quad s_0^*\theta = id_F.$$

We will always assume that F is an abelian sheaf or, more generally, a sheaf of R - modules for some fixed ring R.

A **morphism** of equivariant sheaves is a morphism of sheaves $F \to F'$ which commutes with θ.

Equivariant sheaves form an abelian category which we denote by $Sh_G(X)$.

Examples.

1. $Sh_G(G) \simeq R - mod.$
2. If G is a connected group, then $Sh_G(pt) \simeq R - mod.$

Remark. In case G is a discrete group, a G-equivariant sheaf is simply a sheaf F together with an action of G which is compatible with its action on X (cf. [Groth]).

0.3. Consider the quotient map $q : X \to \overline{X}$. Let $H \in Sh(\overline{X})$. Then $q^*(H) \in Sh(X)$ is naturally a G-equivariant sheaf. This defines a functor

$$q^* : Sh(\overline{X}) \to Sh_G(X).$$

Let $F \in Sh_G(X)$. Then the direct image $q_*F \in Sh(\overline{X})$ has a natural action of G. Denote by $q_*^G F = (q_*F)^G$ the subsheaf of G-invariants of q_*F. This defines a functor

$$q_*^G : Sh_G(X) \to Sh(\overline{X}).$$

Definition. A G-space X is **free** if

a) the stabilizer $G_x = \{g \in G | gx = x\}$ of every point $x \in X$ is trivial, and
b) the quotient map $q : X \to \overline{X}$ is a locally trivial fibration with fibre G.

A free G-space X is sometimes called a principal G-homogeneous space over $\overline{X}$.

The following lemma is well known.

Lemma. Let X be a free G-space. Then the functor $q^* : Sh(\overline{X}) \to Sh_G(X)$ is an equivalence of categories. The inverse functor is $q_*^G : Sh_G(X) \to Sh(\overline{X})$.

0.4. The last lemma shows that in case of a free G-space we may identify the equivariant category $Sh_G(X)$ with the sheaves on the quotient $Sh(\overline{X})$. Hence in this case one may define the **derived category** $D_G(X)$ of equivariant sheaves on X to be the derived category of the abelian category $Sh_G(X)$, i. e.

$$D_G(X) := D(Sh_G(X)) = D(Sh(\overline{X})).$$

If X is not a free G-space, the category $D(Sh_G(X))$ does not make much sense in general. (However, it is still the right object in case G is a discrete group (see section 8 below)).

It turns out that in order to give a good definition of $D_G(X)$ one has first of all to resolve the G-space X, i.e. replace X by a free G-space, and then to use the above naive construction of D_G for a free space. This allows us to define all usual functors in D_G with all usual properties.

It is possible to give a more abstract definition of D_G using simplicial topological spaces (see Appendix B). However, we do not know how to define functors using this definition and hence never use it.

1. Review of sheaves and functors.

This section is a review of the usual sheaf theory on locally compact spaces and on pseudomanifolds. The subsections on the smooth base change (1.8) and on acyclic maps (1.9) will be especially important to us. We will mostly follow [Bo1].

1.1. Let X be a topological space. We fix a commutative ring R with 1 and denote by C_X the constant sheaf of rings on X with stalk R. We denote by $Sh(X)$ the abelian category of C_X-modules (i.e., sheaves of R-modules) on X.

Let $f : X \to Y$ be a continuous map of topological spaces. We denote by $f^* : Sh(Y) \to Sh(X)$ the inverse image functor and by $f_* : Sh(X) \to Sh(Y)$ the direct image functor. The functor f^* is exact and $f^*(C_Y) = C_X$. The functor f_* is left exact and we denote by $R^i f_*$ its right derived functors.

Our main object of study is the category $D^b(X)$ - the bounded derived category of $Sh(X)$. We also consider the bounded below derived category $D^+(X)$.

A continuous map $f : X \to Y$ defines functors

$$f^* : D^b(Y) \to D^b(X) \quad \text{and} \quad Rf_* : D^+(X) \to D^+(Y).$$

Remark. Since we mostly work with derived categories, we usually omit the sign of the derived functor and write f_* instead of $Rf_*, \otimes$ instead of $\overset{L}{\otimes}$ and so on.

1.2. Truncated derived categories (see [BBD])

For any integer a we denote by $D^{\leq a}(X)$ the full subcategory of objects $A \in D^+(X)$ which satisfy $H^i(A) = 0$ for $i > a$. The natural imbedding $D^{\leq a}(X) \to D^+(X)$ has a right adjoint functor $\tau_{\leq a} : D(X)^+ \to D^{\leq a}(X)$, which is called the truncation functor.

Similarly we define the subcategory $D^{\geq a}(X) \subset D^+(X)$ and the truncation functor $\tau_{\geq a} : D^+(X) \to D^{\geq a}(X)$.

Given a segment $I = [a, b] \subseteq \mathbf{Z}$ we denote by $D^I(X)$ the full subcategory $D^{\geq a}(X) \cap D^{\leq b}(X) \subset D^b(X)$.

Subcategories $D^{\geq a}(X)$, $D^{\leq b}(X)$ and $D^I(X)$ are closed under extensions (i.e. if in an exact triangle $A \to B \to C$ objects A and C lie in a subcategory, then B also lies in the subcategory). All these subcategories are preserved by inverse image functors.

If $J \subset I$, we have a natural fully faithful functor $D^J(X) \to D^I(X)$. The category $D^b(X)$ can be reconstructed from the system of finite categories $D^I(X)$, namely

$$D^b(X) = \lim_{\substack{\to \\ I}} D^I(X).$$

Since all functors $D^J(X) \to D^I(X)$ are fully faithful, there are no difficulties in defining this limit.

In the case when $I = [0,0]$ the subcategory $D^I(X)$ is naturally equivalent to $Sh(X)$. This is the heart of the category $D^b(X)$ with respect to t-structure defined by truncation functors τ (see [BBD]).

1.3. We assume that the coefficient ring R is noetherian of finite homological dimension (in fact we are mostly interested in the case when R is a field, usually of characteristic 0). Then we can define functors of tensor product $\otimes : D^b(X) \times D^b(X) \to D^b(X)$ and $Hom : D^b(X)^0 \times D^+(X) \to D^+(X)$ (see [Bol], V.6.2 and V.7.9).

1.4. For locally compact spaces one has additional functors $f_!, f^!$ and the Verdier duality functor D. In order to define these functors we will work only with a special class of topological spaces. Namely, we say that a topological space X is **nice** if it is Hausdorff and locally homeomorphic to a pseudomanifold of dimension bounded by $d = d(X)$ (see [Bol]).

Every nice topological space is locally compact, locally completely paracompact and has finite cohomological dimension (see [Bol]). In particular every object in $D^b(X)$ can be realized by a bounded complex of injective sheaves. In fact we could consider instead of nice spaces the category of topological spaces satisfying these properties.

Let $f : X \to Y$ be a continuous map of nice topological spaces. Then following [Bol] we define functors $f_*, f_! : D^b(X) \to D^b(Y)$, and $f^*, f^! : D^b(Y) \to D^b(X)$.

Functors described above are connected by some natural morphisms. We will describe some of them; one can find a pretty complete list in [GoMa]. These properties are important for us since we would like them to hold in the equivariant situation as well.

We denote by $\mathcal{T}$ the category of topological spaces.

In the rest of this section 1 (except for 1.9) we assume that all spaces are nice.

1.4.1. We have the following natural functorial isomorphisms.

$$Hom(A \otimes B, C) \simeq Hom(A, Hom(B, C)).$$

$$f^*(A \otimes B) \simeq f^*(A) \otimes f^*(B).$$

1.4.2. Composition. Given continuous maps $f : X \to Y$ and $g : Y \to Z$ there are natural isomorphisms of functors $(fg)^* = g^* \cdot f^*, (fg)^! = g^! \cdot f^!, (fg)_* = f_* \cdot g_*, (fg)_! = f_! \cdot g_!$.

1.4.3. Adjoint functors. The functor f^* is naturally left adjoint to f_* and the functor $f_!$ is naturally left adjoint to $f^!$.

1.4.4. There is a canonical morphism of functors $f_! \to f_*$ which is an isomorphism when f is proper.

1.4.5. Exact triangle of an open subset. Let $U \subset X$ be an open subset, $Y = X \setminus U, i : Y \to X$ and $j : U \to X$ natural inclusions. Then for every $F \in D^b(X)$ adjunction morphisms give exact triangles

$$i_! i^! F \to F \to j_* j^* F$$

and

$$j_! j^! F \to F \to i_* i^* F.$$

In this case $i_! = i_*$ and $j_!$ are extensions by zero, $j^* = j^!$ is the restriction to an open subset, $i_! i^!$ is the derived functor of sections with support in Y.

1.4.6. Base change. In applications we usually fix a topological space S (a base) and consider the category $\mathcal{T}/S$ of topological spaces over the base S. An object of this category is a pair $X \in \mathcal{T}$ and a map $X \to S$.

Every continuous map $\nu : T \to S$ defines a base change $\sim: \mathcal{T}/S \to \mathcal{T}/T$ by $X \mapsto \widetilde{X} = X \times_S T$.

Given a space X/S we will use the projection $\nu : \widetilde{X} \to X$ to define a base change functor $\nu^* : D^b(X) \to D^b(\widetilde{X})$. This functor commutes with functors f^* and $f_!$, i.e. there are natural functorial isomorphisms

$$\nu^* f^* = f^* \nu^* \quad \text{and} \quad \nu^* f_! = f_! \nu^*.$$

Similarly, there are natural isomorphisms

$$\nu^! f^! = f^! \nu^! \quad \text{and} \quad \nu^! f_* = f_* \nu^!.$$

1.4.7. Properties of the functor $f^!$.

The object $D_f := f^!(C_Y) \in D^b(X)$ is called the dualizing object of f.

1. We say that the map f is locally fibered if for every point $x \in X$ there exist neighbourhoods U of x in X and V of $y = f(x)$ in Y such that the map $f : X \to Y$ is homeomorphic to a projection $F \times V \to V$.

Assume that f is locally fibered. Then for every $A \in D^b(Y)$ there is a natural isomorphism

$$f^!(A) \simeq f^*(A) \otimes f^!(C_Y)$$

(see [Ve2]).

2. Let $f : X \hookrightarrow Y$ be a closed embedding. We say that f is relatively smooth if there exists an open neighbourhood U of X in Y, such that $U = X \times \mathbf{R}^d$ and f is the embedding of the zero section $f(x) = (x, 0)$. Let $p : U \to X$ be the projection. An object $F \in D^b(Y)$ is called smooth relative to X if $F_U = p^* F'$ for some $F' \in D^b(X)$.

Assume $f : X \to Y$ is a relatively smooth embedding. Then $D_f \in D^b(X)$ is invertible (see 1.5 below). Let $F \in D^b(Y)$ be smooth relative to X. Then we have a natural isomorphism in $D^b(X)$

$$f^! F = f^* F \otimes D_f.$$

In particular the dualizing object D_Y (see 1.6.1 below) of Y is smooth relative to X and we have

$$D_X = f^! D_Y = f^* D_Y \otimes D_f.$$

3. Let

$$\begin{array}{ccc} Z_p & \xrightarrow{j} & Z \\ \downarrow f & & \downarrow f \\ \{p\} & \overset{i}{\hookrightarrow} & W \end{array}$$

be a pullback square, where $f : Z \to W$ is a locally trivial fibration, and $j : Z_p \to Z$ is the inclusion of the fiber. Then we have a canonical isomorphism of functors

$$j^! \cdot f^* = f^* \cdot i^!.$$

1.5. Twist. An object $L \in D^b(X)$ is called **invertible** if it is locally isomorphic to $C_X[n]$ - the constant sheaf C_X placed in degree $-n$. Then for $L^{-1} := Hom(L, C_X)$ the natural morphism $L \otimes L^{-1} \to C_X$ is an isomorphism. Every invertible object L defines a twist functor $L : D^b(X) \to D^b(X)$ by $A \longmapsto L \otimes A$. If L, M are invertible objects, then $N = L \otimes M$ is also invertible and the twist by N is isomorphic to the product of twists by L and M. In particular, the twist functor by L has an inverse given by the twist by L^{-1}.

The twist is compatible with all basic functors. For example $L \otimes (A \otimes B) \simeq (L \otimes A) \otimes B$ and $L \otimes Hom(A, B) = Hom(A, L \otimes B) = Hom(L^{-1} \otimes A, B)$.

Fix a base S and an invertible object L in $D^b(S)$. It defines a family of twist functors L in categories $D^b(X)$ for all spaces X/S; namely if $p : X \to S$ and $A \in D^b(X)$, then $L(A) = p^*(L) \otimes A$. This twist is compatible with all our functors, i.e., for every continuous map $f : X \to Y$ over the base S there are canonical isomorphisms of functors

$$f^* L = L f^*, \; f^! L = L f^!, \; f_* L = L f_*, \; f_! L = L f_!.$$

These isomorphisms are compatible with isomorphisms in 1.4.

1.6. Verdier duality

1.6.1. Let us fix an invertible object D_{pt} in $D^b(pt)$ and call it a dualizing object over the point. For any nice topological space X we define its **dualizing object** $D_X \in D^b(X)$ to be $p^!(D_{pt})$, where $p : X \to pt$. If X is a smooth manifold of dimension d the dualizing object D_X is invertible (1.5) and locally isomorphic to $C_X[d]$. Using this dualizing object we define the Verdier duality functor $D : D^b(X) \to D^b(X)$ by $D(A) = Hom(A, D_X)$.

For any object $A \in D^b(X)$ we have a canonical functorial biduality morphism

$$A \to D(D(A)).$$

1.6.2. Theorem (Verdier duality). *For any continuous map f there are canonical functorial isomorphisms*

$$Df_! = f_* D \quad and \quad f^! D = Df^*.$$

1.6.3. Different choices of the object D_{pt} give rise to different duality functors, which differ by a twist. We will choose the standard normalization $D_{pt} = C_{pt}$ (see [Bo1]).

Remark. This standard normalization is not always natural. For example, if $R = k(M)$ is an algebra of functions on a nonsingular algebraic variety M, the natural choice for D_{pt} is a dualizing module for M, equal to Ω_M $[dim M]$.

1.7. Smooth maps. Let $f : X \to Y$ be a continuous map of topological spaces. We say that f is smooth of relative dimension d if for every point $x \in X$ there exist neighborhoods U of x in X and V of $f(x)$ in Y such that the restricted map $f : U \to V$ is homeomorphic to the projection $V \times \mathbf{R}^d \to V$.

For a smooth map f the dualizing object $D_f \in D^b(X)$ is invertible and is locally isomorphic to $C_X[d]$.

1.8. Smooth base change. Consider a smooth base change $\nu : T \to S$. If X is a nice topological space (see 1.4.), then the space $\widetilde{X} = X \times_S T$ is also nice. The crucial observation, which makes our approach possible, is that in this situation the base change functor $\nu^* : D^b(X) \to D^b(\widetilde{X})$ essentially commutes with all other functors.

Theorem (Smooth base change).
(i) We have canonical functorial isomorphisms

$$\nu^*(A \otimes B) \;=\; \nu^*(A) \otimes \nu^*(B),$$

$$\nu^*(Hom(A,B)) \;=\; Hom(\nu^*(A),\nu^*(B)).$$

(ii) Let $f : X \to Y$ be any map of spaces over S. Let us denote by the same symbol the corresponding map $\widetilde{X} \to \widetilde{Y}$. Then for $A \in D^b(X)$, $B \in D^b(Y)$ we have canonical isomorphisms

$$\nu^* f_*(A) \simeq f_*\nu^*(A), \quad \nu^* f_!(A) \simeq f_!\nu^*(A)$$

$$\nu^* f^*(B) \simeq f^*\nu^*(B), \quad \nu^* f^!(B) \simeq f^!\nu^*(B).$$

These isomorphisms are compatible with isomorphisms in 1.4.

(iii) The Verdier duality commutes with ν^* up to a twist by the (invertible) dualizing object D_ν of $\nu : T \to S$. Namely

$$D(\nu^*(A)) = D_\nu \otimes \nu^*(D(A)).$$

This isomorphism is compatible with the identities in 1.6. For example, if we identify $\nu^*(DD(A)) \simeq DD(\nu^* A)$ using the last isomorphism then ν^* preserves the biduality morphism (1.6.1).

We will discuss this theorem in Appendix **A**.

1.9. Acyclic maps. Fix $n \geq 0$. In this section we consider general topological spaces. The proofs are given in Appendix A below.

1.9.1. Definition. We say that a continuous map $f : X \to Y$ is **n-acyclic** if it satisfies the following conditions:

a) For any sheaf $B \in Sh(Y)$ the adjunction morphism $B \to R^0 f_* f^*(B)$ is an isomorphism and $R^i f_* f^*(B) = 0$ for $i = 1, 2, \ldots, n$.

b) For any base change $\widetilde{Y} \to Y$ the induced map $f : \widetilde{X} = X \times_Y \widetilde{Y} \to \widetilde{Y}$ satisfies the property a).

We say that f is ∞-acyclic if it is n-acyclic for all n.

It is convenient to rewrite the condition a) in terms of derived categories. Namely, consider the functor $\sigma = \tau_{\leq n} \cdot f_* : D^b(X) \to D^b(Y)$. Then the adjunction morphisms $B \to f_* f^*(B)$ and $f^* f_*(A) \to A$ define functorial morphisms $\tau_{\leq n}(B) \to \sigma f^*(B)$ and $f^*\sigma(A) \to \tau_{\leq n}(A)$.

The condition a) can be now written as

a′) For any sheaf $B \in Sh(Y) \subset D^b(Y)$ the natural morphism $B \to \sigma f^*(B)$ is an isomorphism.

1.9.2. It turns out that for an n-acyclic map $f : X \to Y$ large pieces of the category $D^b(Y)$ can be realized as full subcategories in $D^b(X)$. Namely, let us say that an object $A \in D^+(X)$ **comes from** Y if it is isomorphic to an object of the form $f^*(B)$

for some $B \in D^+(Y)$. We denote by $D^+(X|Y) \subset D^+(X)$ the full subcategory of objects which come from Y.

Let us fix a segment $I = [a, b] \subset \mathbf{Z}$ and consider the truncated subcategory $D^I(X|Y) = D^I(X) \cap D^+(X|Y)$.

Proposition (see Appendix **A**). *Let $f : X \to Y$ be an n-acyclic map, where $n \geq |I| = b - a$ (resp ∞-acyclic). Then*
(i) The functor $f^* : D^I(Y) \to D^I(X|Y)$ (resp. $f^* : D^+(Y) \to D^+(X|Y)$) is an equivalence of categories. The inverse functor is given by $\sigma = \tau_{\leq b} \circ f_* : D^b(X) \to D^b(Y)$ (resp. $f_* : D^+(X) \to D^+(Y)$).
(ii) The functor f^* gives a bijection of the sets of equivalence classes of exact triangles in $D^I(Y)$ and $D^I(X|Y)$ (resp. in $D^+(Y)$ and $D^+(X|Y)$). In other words a diagram (T) in $D^I(Y)$ is an exact triangle iff the diagram $f^*(T)$ in $D^I(X)$ is an exact triangle.
(iii) The subcategory $D^I(X|Y) \subset D^b(X)$ (resp. $D^+(X|Y) \subset D^+(X)$) is closed under extensions and taking direct summands.

1.9.3. The following lemma gives a criterion, when an object $A \in D^I(X)$ comes from Y.

Lemma. *Suppose we have a base change $q : \widetilde{Y} \to Y$ in which q is epimorphic and admits local sections. Set $\widetilde{X} = X \times_Y \widetilde{Y}$ and consider the induced map $\widetilde{f} : \widetilde{X} \to \widetilde{Y}$. Then*
(i) The induced map $\widetilde{f}$ is n-acyclic if and only if f is n-acyclic.
(ii) Suppose $f, \widetilde{f}$ are n-acyclic. Let $A \in D^I(X)$, where $|I| \leq n$. Then A comes from Y if and only if its base change $\widetilde{A} = q^*(A) \in D^I(\widetilde{X})$ comes from $\widetilde{Y}$.
(iii) The above assertions hold if we replace "n-acyclic" by "∞-acyclic" and D^I by D^+.

1.9.4. The following criterion, which is a version of the Vietoris-Begle theorem, gives us a tool for constructing n-acyclic maps.

We say that a topological space M is n-**acyclic**, if it is non-empty, connected, locally connected (i.e. every point has a fundamental system of connected neighborhoods) and for any R-module A we have $H^0(M, A) \simeq A$ and $H^i(M, A) = 0$ for $i = 1, 2, \dots, n$.

Criterion. *Let $f : X \to Y$ be a locally fibered map (1.4.7). Suppose that all fibers of f are n-acyclic. Then f is n-acyclic.*

1.10. Constructible complexes.

Suppose that X is a pseudomanifold with a given stratification $\mathcal{S}$ (see [Bo1] I.1). We denote by $D^b_c(X; \mathcal{S})$ the full subcategory of $\mathcal{S}$-constructible complexes in $D^b(X)$, i.e. complexes whose cohomology sheaves are constructible with respect to $\mathcal{S}$ (see [Bol]). Then $D^b_c(X;\mathcal{S})$ is a triangulated subcategory of $D^b(X)$, closed with respect to extensions and taking direct summands. It is also preserved by functors τ, $\otimes$, Hom and D. For a constructible complex A the biduality morphism $A \to DD(A)$ is an isomorphism. If $f : (X,\mathcal{S}) \to (Y,\mathcal{S}')$ is a stratified map of pseudomanifolds, then functors $f^*, f^!, f_*$ and $f_!$ preserve constructibility (see [Bol]).

Consider the natural partial order on the set of all stratifications of X ($\mathcal{S} \geq \mathcal{T}$ if strata of $\mathcal{S}$ lie inside strata of $\mathcal{T}$). If $\mathcal{S} \geq \mathcal{T}$ we have a natural fully faithful inclusion functor $D^b(X;\mathcal{T}) \to D^b(X;\mathcal{S})$.

We define a **constructible space** to be a topological space X together with a system $\{\mathcal{S}\}$ of stratifications of X (allowable stratifications), which is a directed system with respect to $\geq$. For constructible space X we define

$$D^b_c(X) = \lim_{\mathcal{S}} D^b_c(X,\mathcal{S}).$$

Let $(X,\mathcal{S})$ and $(Y,\mathcal{T})$ be constructible spaces. A continuous map $f : X \to Y$ is called constructible if for any allowable stratifications $\mathcal{S}$ and $\mathcal{T}$ there exist allowable stratifications $\mathcal{S}' \geq \mathcal{S}$ and $\mathcal{T}' \geq \mathcal{T}$ such that $f : (X,\mathcal{S}') \to (Y,\mathcal{T}')$ is a stratified map. For a constructible map f functors $f^*, f^!, f_*$, and $f_!$ preserve constructibility.
Examples. 1. Let X be a complex algebraic variety. Then as a topological space X has a canonical structure of a constructible space. Namely a stratification $\mathcal{S}$ is allowable if all its strata are algebraic. Any algebraic map $f : X \to Y$ is constructible.

2. Similarly every real semialgebraic set has a canonical structure of a constructible space.

Appendix A.

A1. Proof of Theorem 1.8

(i) See [Bol], V.10.1,10.21.
(ii) Consider the pullback diagram

$$\begin{array}{ccc} \widetilde{X} & \xrightarrow{\nu} & X \\ \downarrow f & & \downarrow f \\ \widetilde{Y} & \xrightarrow{\nu} & Y \end{array}$$

The isomorphism for f^* follows from 1.4.2 and for $f_!$ is a base change isomorphism 1.4.6 (here we do not use that ν is smooth).

By 1.4.7 the functor $\nu^!$ is obtained from ν^* by the twist (see 1.5) by the relative dualizing sheaf $D_\nu \in D^b(T)$, i.e., we have canonical isomorphisms $\nu^! \simeq D_\nu \otimes \nu^*$ on both $\widetilde{X}$ and $\widetilde{Y}$. Since all functors commute with the twist (see 1.5) it suffices to find canonical isomorphisms $\nu^! f^! \simeq f^! \nu^!$ and $\nu^! f_* \simeq f_* \nu^!$. Again this follows from 1.4.2 and 1.4.6.

The proof of the fact that smooth base change preserves all functorial identities mentioned in the theorem is quite lengthy and is based on case by case considerations. We omit the details.
(iii) This follows from 1.4.7 and 1.7.

A2. Proof of proposition 1.9.2

(i) Consider the functor $\sigma : D^b(X) \to D^b(Y)$ given by $\sigma(A) = \tau_{\leq b} f_*(A)$. Using adjunction morphisms $Id \to f_* f^*$ and $f^* f_* \to Id$ in combination with the truncation functor $\tau_{\leq b}$ we construct morphisms of functors $\alpha : \tau_{\leq b} \to \sigma f^*$ and $\beta : f^*\sigma \to \tau_{\leq b}$.

Let $C \subset D^b(Y)$ be the full subcategory of objects B for which the morphism α is an isomorpism. This subcategory is closed under extensions and by the acyclicity condition it contains subcategories $Sh(Y)[-i]$ for $i \geq a$. Hence C contains $D^{\geq a}$.

In particular on the category $D^I(Y)$ we have a functorial isomorphism $B \to \sigma f^*(B)$, which shows that the functor σ on this subcategory is left inverse to f^*.

Let $B \in D^I(Y)$. Properties of adjunction morphisms imply that the composition morphism $f^*(B) \to f^*\sigma f^*(B) \to f^*(B)$ is an identity. This implies the following

Criterion. An object $A \in D^I(X)$ lies in $D^I(X|Y)$ if and only if the morphism $\beta : f^*\sigma(A) \to A$ is an isomorphism.

This criterion shows that the functors f and σ are inverse equivalences of categories $D^I(Y)$ and $D^I(X|Y)$.
(ii) This follows from the fact that inverse functors f^* and σ are exact.
(iii) This follows from the criterion in (i).

The assertions about $D^+(Y)$ and $D^+(X|Y)$ in case f is ∞-acyclic are proved similarly.

A3. Proof of Lemma 1.9.3.

(i) Since $\widetilde{f}$ is obtained from f by a base change, the n-acyclicity of f implies that of $\widetilde{f}$ (see 1.9.1). Conversely, suppose $\widetilde{f}$ is n-acyclic. Locally on Y the map $q : \widetilde{Y} \to Y$ has a section s and the map f can be obtained from $\widetilde{f}$ by the base change with s. Thus f is n-acyclic locally on Y and hence is n-acyclic.

(ii) If A comes from Y then clearly $\widetilde{A}$ comes from Y. Let us show that if $\widetilde{A}$ comes from $\widetilde{Y}$ then A comes from Y.

Using the criterion in A2(i) it is enough to check this locally on Y. But locally A is obtained from $\widetilde{A}$ by the base change with the morphism s, and hence it comes from Y.

(iii) Is proved similarly.

A4. Proof of criterion 1.9.4

Our proof is a refined version of the argument in [Bol],pp.80-82.

Step 1. Let $f : X = Y \times F \to Y$ be a projection with a nonempty connected fiber F. Then for any $B \in Sh(Y)$ one has $\Gamma(X, f^*B) \simeq \Gamma(Y, B)$.

Step 2. If $f : X \to Y$ is locally fibered with locally connected fibers, then the functor f^* preserves direct products of sheaves.

Indeed, let $B = \prod B_\alpha$ be a product of sheaves on Y. Consider the natural morpism $\gamma : f^*(B) \to \prod f^*(B_\alpha)$. We claim that γ is an isomorphism. It is enough to check that γ induces isomorphisms on sections over small enough open sets $U \subset X$. By our assumption we can choose open subsets $U \subset X$ and $V \subset Y$ such that the map $f : U \to V$ is homeomorphic to a projection $V \times F \to V$ with a non-empty connected fiber F. Then by Step 1

$$\Gamma(U, f^*(B)) = \Gamma(V, B) = \prod \Gamma(V, B_\alpha)$$

and

$$\Gamma(U, \prod f^*(B_\alpha)) = \prod \Gamma(U, f^*(B_\alpha)) = \prod \Gamma(V, B_\alpha)$$

which implies that γ is an isomorphism.

Step 3. Let $i : y \to Y$ be an imbedding of a point (not necessarily closed), $j : F = f^{-1}(y) \to X$ the corresponding imbedding of the fiber over y. Then functors i_* and j_* are exact and we have the base change $f^*i_*(A) \simeq j_*f^*(A)$ for $A \in Sh(y)$.

Indeed, since the statement is local, we can assume that $X = Y \times S$. Consider a point $x = (y', s) \in X$ and a sheaf $B \in Sh(S)$. Then it is easy to see that the stalk of $j_*(B)$ at the point x equals the stalk of B at the point s if y' lies in the closure of y and equals 0 otherwise, and similarly for the map i. This implies the exactness of functors i_*, j_*. Comparison of stalks proves the base change.

Step 4. As in A2(i) let us consider the functor $\sigma = \tau_{\leq b} f_* : D(X)^b \to D^b(Y)$ and the functorial morphism $\alpha : B \to \sigma f^*(B)$.

Let $C = C(Y) \subset Sh(Y)$ be the subcategory of sheaves B such that α is an isomorphism $\alpha : B \simeq \sigma f^*(B)$.

Let $i : y \to Y$ be an imbedding of a point and $B \in Sh(y)$. Then by the assumption on fibers of the map f, the sheaf B lies in $C(y)$. Using step 3 we deduce that its image $i_*(B)$ lies in $C(Y)$. Since the functor f_* preserves direct products, step 2 implies that $C(Y)$ is closed under direct product. Hence any sheaf $B \in Sh(Y)$ can be imbedded in a sheaf E which lies in $C(Y)$.

Step 5. Using standard devissage one shows by induction on n that all sheaves on Y lie in $C(Y)$.

Step 6. For any base change $\widetilde{Y} \to Y$ the corresponding map $\widetilde{X} \to \widetilde{Y}$ satisfies the same conditions as f. Hence f is n-acyclic.

2. Equivariant derived categories.

This section contains the definition of our main object - the derived category $D_G(X)$. This is not a derived category in the usual sense, i.e. it is not the derived category of an abelian category. However, $D_G(X)$ is a triangulated category with a t-structure, whose heart is equivalent to the abelian category $Sh_G(X)$ of equivariant sheaves on X. We give several equivalent definitions of $D_G(X)$ - each one appears to be useful.

We start with the bounded derived category $D^b_G(X)$ - definition 2.2.4. Other definitions of $D^b_G(X)$ are given in 2.4 and 2.7. The bounded below category $D^+_G(X)$ is defined in 2.9 in a similar way. One notices that there is a quick definition of categories D^b_G, D^+_G using ∞- dimensional spaces (2.7, 2.9.9). However, the most part of this section is devoted to showing that in the case of the bounded category $D^b_G(X)$ we may work only with finite dimensional spaces. This is important for the definition of functors in section 3 below. On the other hand, ∞-dimensional spaces appear to be convenient for D^+_G. In particular the definition (in section 6) of our main functor Q_{f*} - the general direct image - essentially uses ∞-dimensional spaces.

In this section we work with arbitrary topological spaces; in particular we do not assume that they are Hausdorff.

We fix a topological group G and consider the category of G-spaces.

2.1. Categories $D_G(X, P)$.

Let us recall some definitions from section 0.

2.1.1. Definition. a) A G-space is a topological space X together with a continuous (left) action of G on X. A G-map $f : X \to Y$ is a continuous G-equivariant map.

For a G-space X we denote by $\overline{X}$ the quotient space $\overline{X} = G \setminus X$ and by q the quotient map $q : X \to \overline{X}$.

b) A G-space X is called **free** if G acts freely on X and the quotient map $q : X \to \overline{X}$ is a locally trivial fibration with fiber G.

Lemma. *Let $\nu : P \to X$ be a G-map. Suppose the G-space X is free. Then $P = X \times_{\overline{X}} \overline{P}$ and in particular P is free.*

This lemma is proved in 2.3.1.

2.1.2. Definition. Let X be a G-space. A **resolution** of X is a G-map $p : P \to X$ in which the G-space P is free. A morphism of resolutions is a G-map over X.

We denote the category of resolutions of X by $Res(X)$ or $Res(X, G)$.

We will be mostly interested in resolutions which are epimorphic and moreover n-acyclic for large n.

Examples. 1. Let $T = G \times X$ be a G-space with the diagonal action of G. Then the projection $p : T \to X$ is a resolution of X, which we call the **trivial** resolution of X.

2. More generally, let M be any free G-space. Then the projection $p : X \times M \to X$ is a resolution of X.

3. If $P \to X$ and $R \to X$ are two resolutions of X, then their product $S = P \times_X R$ also is a resolution of X, which has natural projections on P and R (this is the product of P and R in the category $Res(X)$).

4. Let $f : X \to Y$ be a G-map. Then every resolution $P \to X$ can be considered as a resolution of Y. This defines a functor $Res(X) \to Res(Y)$.

This functor has a right adjoint functor $f^0 : Res(Y) \to Res(X)$. Namely for any resolution $R \to Y$ we set $f^0(R) = R \times_Y X$ (it is called the **induced** resolution of X).

2.1.3. For any resolution $p : P \to X$ of a G-space X we consider the following diagram of topological spaces

$$Q(p): \qquad X \xleftarrow{p} P \xrightarrow{q} \overline{P} = G \setminus P.$$

Definition. We define the category $D^b_G(X, P)$ as follows:

an object F of $D^b_G(X, P)$ is a triple $(F_X, \overline{F}, \beta)$ where $F_x \in D^b(X)$, $\overline{F} \in D^b(\overline{P})$ and $\beta : p^*(F_X) \simeq q^*(\overline{F})$ is an isomorphism in $D^b(P)$.

a morphism $\alpha : F \to H$ in $D^b_G(X, P)$ is a pair $\alpha = (\alpha_X, \overline{\alpha})$, where $\alpha_X : F_X \to H_X$ and $\overline{\alpha} : \overline{F} \to \overline{H}$ satisfy $\beta \cdot p^*(\alpha_X) = q^*(\overline{\alpha}) \cdot \beta$.

Examples. If $G = \{e\}$, the category $D^b_G(X, P)$ is canonically equivalent to the category $D^b(X)$. If X is free and $P = X$, the category $D^b_G(X, P)$ is canonically equivalent to the category $D(\overline{X})$.

2.1.4. For any G-space X we define the forgetful functor $For : D^b_G(X, P) \to D^b(X)$ by $For(F) = F_X$.

2.1.5. Let $\nu : P \to R$ be a morphism of two resolutions of a G-space X. Then we define the inverse image functor $\nu^* : D^b_G(X, R) \to D^b_G(X, P)$ by $\nu^*(F_X, \overline{F}, \beta) = (F_X, \overline{\nu}^*(\overline{F}), \gamma)$, where $\overline{\nu} : \overline{P} \to \overline{R}$ is the quotient map and $\gamma = \nu^*(\beta) : p^*(F_X) = \nu^* r^*(F_X) \to \nu^* q^*(\overline{F}) = q^* \overline{\nu}^*(\overline{F})$.

2.1.6. More generally, let $f : X \to Y$ be a G-map. Suppose we are given two resolutions $p : P \to X$ and $r : R \to Y$ and a morphism $\nu : P \to R$ compatible with f (i.e., $f \cdot p = r \cdot \nu$). In this situation we define the inverse image functor $f^* : D^b_G(Y, R) \to D^b_G(X, P)$ by $f^*(F_Y, \overline{F}, \beta) = (f^*(F_Y), \overline{\nu}^*(\overline{F}), \gamma)$, where $\overline{\nu} : \overline{P} \to \overline{R}$ is the quotient map and $\gamma = \nu^*(\beta) : p^*(f^*(F_Y)) = \nu^* r^*(F_Y) \to \nu^* q^*(F) = q^* \overline{\nu}^*(\overline{F})$.

We will use this functor mostly in two situations: when $R = P$ and when $R = f^0(P)$ is the induced resolution (see 2.1.2, example 4).

2.1.7. Let $p : P \to X$ be a resolution of a G-space X and $\overline{X}$ be the quotient space of X. We define the quotient functor $q^* : D^b(\overline{X}) \to D^b_G(X,P)$ by $q^*(A) = (q^*(A), \overline{p}^*(A), \gamma)$, where $q : X \to \overline{X}$ is the quotient map, $\overline{p} : \overline{P} \to \overline{X}$ the natural projection and γ the natural isomorphism $p^*q^*_X(A) \simeq q^*\overline{p}^*(A)$.

2.2. Categories $D^I_G(X)$ and $D^b_G(X)$.

2.2.1. We want to define the equivariant derived category $D^b_G(X)$ as a limit of categories $D^b_G(X,P)$ when resolutions $P \to X$ become more and more acyclic.

We fix a segment $I = [a,b] \subset \mathbf{Z}$ and first define the category $D^I_G(X)$.

Definition. For any resolution $p : P \to X$ we define a full subcategory $D^I_G(X,P) \subset D^b_G(X,P)$ using the forgetful functor, i.e., $F \in D^I_G(X,P)$ if $F_X \in D^I(X)$ (see 1.2). For an epimorphic map p this is equivalent to the condition $\overline{F} \in D^I(\overline{P})$.

We say that a resolution $p : P \to X$ is n-acyclic if the continuous map p is n-acyclic (see 1.9). The following proposition, which we prove in 2.3.3. is central for our purposes.

Proposition. *Let $p : P \to X$ be an n-acyclic resolution, where $n \geq |I|$. Suppose that X is a free G-space. Then the quotient functor $q^* : D^I(\overline{X}) \to D^I_G(X,P)$ is an equivalence of categories (2.1.7).*

2.2.2. We will mostly work with the following corollary of the proposition, which describes how the category $D^I_G(X,P)$ depends on the resolution. Let P, R be two resolutions of a G-space X, $S = P \times_X R$ their product and $pr_R : S \to R$ the natural projection.

Corollary. *Suppose that the resolution $P \to X$ is n-acyclic, where $n \geq |I|$. Then the functor $pr^*_R : D^I_G(X,R) \to D^I_G(X,S)$ is an equivalence of categories.*

2.2.3. Fix an n-acyclic resolution $p : P \to X$, where $n \geq |I|$. For any resolution $R \to X$ we define the functor $C_{R,P} : D^I_G(X,P) \to D^I_G(X,R)$ as a composition $C_{R,P} = (pr^*_R)^{-1} \cdot pr^*_P : D^I_G(X,P) \to D^I_G(X,S) \simeq D^I_G(X,R)$, where $S = P \times_X R$ and pr_P, pr_R - projections of S on P and R. This functor is defined up to a canonical isomorphism.

Let us list some properties of this functor, which immediately follow from the definition.

(i) If $\nu : R \to R'$ is a morphism of resolutions, we have the canonical isomorphism of functors $C_{R,P} \simeq \nu^* \cdot C_{R',P}$ (2.1.5).

In particular, for any morphism $\nu : R \to P$ we have the canonical isomorphism $C_{R,P} \simeq \nu^*$.

(ii) Let Q be another n-acyclic resolution. Then we have a canonical isomorphism of functors $C_{R,P} \simeq C_{R,Q} \cdot C_{Q,P}$; since the functor $C_{P,P}$ is the identity, in this case the functor $C_{Q,P}$ is an equivalence of categories (the inverse functor is $C_{P,Q}$).

2.2.4. From now on we assume the following property of a G-space X:

$(*)$ For every $n > 0$ there exists an $n-$acyclic resolution $p : P \to X$.

Definition. For every segment $I \subset \mathbf{Z}$ we define the category $D^I_G(X)$ to be $D^I_G(X, P)$, where P is some n-acyclic resolution of X with $n \geq |I|$. As follows from 2.2.3 this category is defined up to a canonical equivalence. If $J \subset I$, we have a fully faithful functor $i : D^J(X) \to D^I(X)$, defined uniquely up to a canonical isomorphism. We define the equivariant derived category $D^b_G(X)$ to be the limit

$$D^b_G(X) = \lim_{\underset{I}{\to}} D^I_G(X)$$

(compare with 1.2).

Passing to the limit in constructions defined in 2.1.4, 2.1.6 and 2.1.7 we define the following functors:
(i) The forgetful functor $For : D^b_G(X) \to D^b(X)$
(ii) The inverse image functor $f^* : D^b_G(Y) \to D^b_G(X)$ for a G-map $f : X \to Y$.
(iii) The quotient functor $q^* : D^b(\overline{X}) \to D^b_G(X)$.

For example, let us describe the inverse image functor. Let $F \in D^b_G(Y)$. Choose a segment $I \subset \mathbf{Z}$ such that $F \in D^I_G(Y)$. Fix $n \geq |I|$ and find an n-acyclic resolution $R \to Y$. Then by definition F is an object of $D^I_G(Y, R)$. Consider the induced resolution $P = f^0(R) \to X$ with the natural projection $\nu : P \to R$. Using the construction from 2.1.6 we define the inverse image $\nu^*(F) \in D^I_G(X, P)$. Since P is an n-acyclic resolution of X this gives an object $f^*(F) \in D^I_G(X)$.

These constructions are compatible, which means that we have canonical isomorphisms of functors $For \cdot f^* \simeq f^* \cdot For$, $q^* \cdot \overline{f}^* \simeq f^* \cdot q^*$, where $\overline{f} : \overline{X} \to \overline{Y}$ is the quotient map, and $For \cdot q^* \simeq q^*$, where $q : X \to \overline{X}$ is the quotient map.

2.2.5. Proposition. *Let X be a free G-space. Then the quotient functor q^* : $D^b(\overline{X}) \to D^b_G(X)$ is an equivalence of categories.*

Indeed, in this case X is an ∞-acyclic resolution of X, so $D^b_G(X) \simeq D^b_G(X, X) \simeq D^b(\overline{X})$.

2.3. Proofs.

2.3.1. Proof of lemma 2.1.1. Set $Y = \overline{X}$ and consider the natural map $\alpha : P \to S = X \times_Y \overline{P}$. Since G acts freely on X, α is a bijection and G acts freely on

P. In order to prove that α is a homeomorphism we can replace Y by its small open subset. Since $X \to Y$ is a locally trivial fibration with fiber G we can assume $X = G \times Y$.

Let $\pi : X \to G$ be the projection and $\mu = \pi\nu : P \to G$. Consider the continuous map $\tau : P \to P$ given by $\tau(p) = \mu(p)^{-1}(p)$. Since μ is a G-map, τ is constant on G-orbits. Hence τ induces a continuous map $\overline{\tau} : \overline{P} \to P$, such that $\tau = \overline{\tau}q$. The action of G then defines a continuous map $\beta : G \times \overline{P} \to P$. Identifying $G \times \overline{P}$ with $X \times_Y \overline{P}$ we see that β is the inverse to α. This proves lemma 2.1.1.

2.3.2. Lemma. *Let $I \subset \mathbf{Z}$ be a segment and $p : P \to X$ be an n-acyclic resolution of a G-space X, where $n \geq |I|$. Let $D^I(\overline{P}|p)$ be the full subcategory of $D^I(\overline{P})$ defined by*

$$D^I(\overline{P}|p) = \{H \in D^I(\overline{P}) | q^*(H) \in D^I(P|X)\}$$

(i.e. it consists of objects H for which $p^*(H)$ comes from X, see 1.9.2).

Then the restriction functor $D^I_G(X, P) \to D^I(\overline{P}|p)$, $F \longmapsto \overline{F}$, is an equivalence of categories. The subcategory $D^I(P|p) \subset D(\overline{P})$ is closed under extensions and taking direct summands.

Indeed, by definition an object $F \in D^I_G(X, P)$ is a triple $(F_X, \overline{F}, \beta)$ with $F_X \in D^I(X)$, $\overline{F} \in D^I(\overline{P})$ and $\beta : p^*(F_X) \simeq q^*(\overline{F})$. By Proposition 1.9.2 the functor $p^* : D^I(X) \to D^I(P|X)$ is an equivalence of categories. Hence we can describe F by a triple $(H \in D^I(P|X), \overline{F} \in D^I(\overline{P}), \beta : H \simeq q^*(\overline{F}))$.

Such a triple is determined by $\overline{F}$ up to a canonical isomorphism, so F can be described by an object $\overline{F} \in D^I(\overline{P})$ such that $q^*(\overline{F}) \in D^I(P|X)$.

Since the subcategory $D^I(P|X) \subset D^I(P)$ is closed under extensions and taking direct summands (1.9.2), the subcategory $D^I(\overline{P}|p) \subset D(\overline{P})$ also has these properties. This proves the lemma.

2.3.3. Proof of proposition 2.2.1.

By lemma 2.1.1 we have $P = X \times_{\overline{X}} \overline{P}$. Since the resolution map $p : P \to X$ is n-acyclic and $q : X \to \overline{X}$ is epimorphic and admits local sections, we can apply lemma 1.9.3. In particular, we see that the map $\overline{p} : \overline{P} \to \overline{X}$ is n-acyclic and hence $D^I(\overline{X}) \simeq D^I(\overline{P}|\overline{X}) = \{H \in D^I(\overline{P}) | H$ comes from $\overline{X}\}$. By Lemma 1.9.3 this last category equals $\{H \in D^I(\overline{P}) | q^*(H) \in D^I(P|X)\} \simeq D^I(\overline{P}|p)$. It remains to apply lemma 2.3.2. This proves the proposition.

2.3.4. Proof of corollary 2.2.2. Let $F \in D^I_G(X, R)$. By definition F is described by a triple $F_X \in D^I(X)$, $\overline{F} \in D^I(\overline{R})$ and $\beta : q^*(\overline{F}) \simeq r^*(F_X)$. Applying proposition 2.2.1 to the n-acyclic resolution $p : S \to R$ of a free G-space R we can replace the object $\overline{F} \in D^I(\overline{R})$ by an object H of an equivalent category $D^I_G(R, S)$. Thus F can be described by the 5-tuple

$$(F_X, H_R, \overline{H}, \beta : r^*(F_X) \simeq H_R,\ \gamma : p^*(H_R) \simeq q^*(\overline{H})).$$

Clearly the triple (F_X, H_R, β) is determined by the object F_X up to a canonical isomorphism. Replacing in our 5-tuple this triple by F_X, we see that F can be described by a triple $(F_X, \overline{H}, \delta : s^*(F_X) \simeq q^*(\overline{H}))$, where $\delta = \gamma \cdot p^*(\beta)$. But the category of such triples is by definition $D^I_G(X, S)$. This proves the corollary.

2.4. Description of the category $D^b_G(X)$ in terms of fibered categories.

We will mostly work with another description of the equivariant derived category $D^b_G(X)$, which uses the notion of a fibered category over the category $\mathcal{T}$ of topological spaces. Let us recall this notion.

2.4.1. Definition. A **fibered category** $C/\mathcal{T}$ is a correspondence which assigns to every object $X \in \mathcal{T}$ a category $C(X)$, and to every continuous map $f : X \to Y$ a functor $f^* : C(Y) \to C(X)$, and to every pair of composable maps $f : X \to Y$ and $h : Y \to Z$ an isomorphism $(hf)^* \simeq f^*h^*$, which satisfy the natural compatibility conditions.

We will work with the following two examples of fibered categories: $C(X) = Sh(X)$ and $C(X) = D^b(X)$.

Usually the fibered category $C/\mathcal{T}$ is described in a slightly different way. Namely, consider the category C defined as follows:

An object of C is a pair (X, A), $X \in \mathcal{T}, A \in C(X)$.

A morphism $\phi : (Y, B) \to (X, A)$ in C is a pair , consisting of a continuous map $f : X \to Y$ and a morphism $\phi : f^*(B) \to A$ in $C(X)$.

A composition of morphisms is defined in a natural way.

By definition we have the natural contravariant projection functor $\pi : C \to \mathcal{T}$. This functor completely describes the fibered category $C/\mathcal{T}$, since one can reconstruct the category $C(X)$ as the fiber of π over an object X (see [Gi]).

A morphism $\phi : (Y, B) \to (X, A)$ in C is called **cartesian** if the corresponding morphism $\phi : f^*(B) \to A$ is an isomorphism. This notion can be described directly in terms of the functor $\pi : C \to \mathcal{T}$ (see [Gi]).

Let K be any category . We call a functor $\Phi : K \to C$ **cartesian** if for any morphism $\alpha \in Mor(K)$ its image $\Phi(\alpha) \in Mor(C)$ is Cartesian.

2.4.2. Fix a fibered category $\pi : C \to \mathcal{T}$. Let K be any category and $\Phi : K \to \mathcal{T}$ be a covariant functor. We want to define the category $C(\Phi)$ which is the **fiber** of the fibered category C over the functor Φ.

By definition, an object $F \in C(\Phi)$ is a *cartesian* functor $F : K^0 \to C$, such that $\pi \cdot F = \Phi$. A morphism in $C(F)$ is a morphism of functors $\alpha : F \to H$ such that $\pi(\alpha)$ is the identity morphism of the functor Φ.

In other words, an object $F \in C(\Phi)$ is a correspondence which assigns to every object $a \in K$ an object $F(a) \in C(\Phi(a))$ and to every morphism $\alpha : a \to b$ in K an isomorphism $F(\alpha) : \Phi(\alpha)^* F(b) \simeq F(a)$ in $C(\Phi(a))$ such that
a) If $\alpha = id$, then $F(\alpha) = id$.
b) For any pair of morphisms $\alpha : a \to b$, $\beta : b \to c$ in K we have

$$F(\beta\alpha) = F(\alpha) \cdot \Phi(\alpha)^*(F(\beta)).$$

Examples. 1. Let K be the trivial category (one object, one morphism). Then a functor $\Phi : K \to \mathcal{T}$ is nothing else but a topological space X and $C(\Phi) = C(X)$.
2. Let K be the category with 3 objects in which morphisms are described by the following diagram

$$\circ \xleftarrow{p} \circ \xrightarrow{q} \circ\ .$$

For any resolution $p : P \to X$ of a G-space X the diagram $Q(p)$ (2.1.3) represents a functor $\Phi : K \to \mathcal{T}$ and the category $D^b(\Phi)$ is equivalent to $D^b_G(X, P)$.

2.4.3. Let X be a G-space. Consider the category $K = Res(X, G)$ of resolutions of X and the functor $\Phi : K \to \mathcal{T}$, $\Phi(P) = \overline{P} = G \setminus P$.
Proposition. *The fiber $D^b(\Phi)$ of the fibered category $D^b/\mathcal{T}$ is naturally equivalent to the category $D^b_G(X)$.*

In other words, we can think of an object $F \in D^b_G(X) = D^b(\Phi)$ as a collection of objects $F(P) \in D^b(\overline{P})$ for all resolutions $P \to X$ together with a collection of isomorphisms $\nu^*(F(R)) \simeq F(P)$ for morphisms of resolutions $\nu : P \to R$, satisfying natural compatibility conditions.
Proof. (i) Suppose we are given an object $F \in D^b_G(X)$. For any resolution $p : P \to X$ consider the object $p^*(F) \in D^b_G(P)$. Since P is a free G-space, by proposition 2.2.5 we can find an object $F(P) \in D^b(\overline{P})$ and an isomorphism $p^*(F) \simeq q^*(F(P))$. For any morphism of resolutions $\nu : P \to R$ we have a canonical isomorphism $\nu^*(F(R)) \simeq F(P)$ in $D^b(\overline{P})$, which corresponds to a natural isomorphism $\nu^* r^*(F) \simeq p^*(F)$ in $D^b_G(P)$. This collection defines an object $F \in D^b(\Phi)$.
(ii) Conversely, let $H = \{H(P) \in D^b(\overline{P}), H(\nu)\}$ be an object of $D^b(\Phi)$. Denote by T the trivial resolution $T = G \times X \to X$. For any resolution P consider a diagram of resolutions

$$T \longleftarrow P_+ = T \times_X P \longrightarrow P,$$

in which morphisms are projections. Note that the diagram

$$\overline{T} \longleftarrow \overline{P}_+ \longrightarrow \overline{P}$$

in $\mathcal{T}$ coincides with the diagram $Q(p)$ in 2.1.3. Thus H defines objects $H_X = H(T) \in D^b(X)$, $\overline{H} = H(P) \in D^b(\overline{P})$ and an isomorphism $p^*(H_X) \simeq H(P_+) \simeq q^*(\overline{H})$, i.e., an object $H(P) \in D_G(X,P)$. It is clear that the collection of objects $H(P)$ is compatible with morphisms of resolutions.

Chose a segment $I \subset \mathbf{Z}$ such that $H(T) \in D^I(X)$. Choose an n-acyclic resolution $P \to X$ where $n > |I|$. Then the object $H(P) \in D^I_G(X,P)$ by definition can be considered as an object of $D^I_G(X)$.

If R is another n-acyclic resolution and $S = P \times_X R$, then the objects $pr_P^*(H(P))$ and $pr_R^*(H(R))$ in $D^I_G(X,S)$ are canonically isomorphic to $H(S)$, which shows that the objects $H(P)$ and $H(R)$ in $D^I_G(X)$ are canonically isomorphic. This defines the inverse functor $D^b(\Phi) \to D^b_G(X)$.

Remark. Let us describe explicitly the functor $D^b_G(X) \to D^b(\Phi)$.

Given an object $F \in D^I_G(X)$ we can find an n-acyclic resolution $P \to X$ with $n \geq |I|$ and realize F as an object in $D^I(\overline{P})$. For any resolution $R \to X$ consider the product resolution $S = P \times_X R$ and define an object $F(R) \in D^I(\overline{R})$ by $F(R) = (pr_R^*)^{-1} pr_P^*(F)$ (here we use the fact that the functor pr_R^* gives an equivalence of categories since the map $pr_R : \overline{S} \to \overline{R}$ is n-acyclic; see also 2.2.3). This construction gives us a collection of objects $F(R) \in D^b(\overline{R})$, i.e., an object in $D^b(\Phi)$.

2.4.4. Analyzing the proof of proposition 2.4.3 we see that in order to reconstruct the category $D^b_G(X)$ we do not need to consider all resolutions and all morphisms between them. We can work with a smaller family of resolutions and their morphisms, provided it is rich enough. The following proposition, whose proof is just a repetition of the proof in 2.4.3, gives the precise statement of the result.

Proposition. *Let J be a category and $j : J \to K = Res(X,G)$ be a functor. Consider the composed functor $\Psi = \Phi \cdot j : J \to \mathcal{T}$ and denote by j the fiber functor $j : D^b(\Phi) \to D^b(\Psi)$.*

Suppose that the pair J, j has the following properties:

a) The category J has direct products and the functor j preserves direct products.

b) The image $j(J)$ contains the trivial resolution T.

c) For every $n > 0, j(J)$ contains an n-acyclic resolution.

Then $j : D^b_G(X) \simeq D^b(\Phi) \to D^b(\Psi)$ is an equivalence of categories.

Example. Let $f : X \to Y$ be a G-map. Set $J = Res(Y)$, $j = f^0 : Res(Y) \to Res(X)$ (2.1.2 example 4), $\Psi = j \cdot \Phi$. Then $D^b(\Psi) \simeq D^b(\Phi)$. In other words in order to compute $D^b_G(X)$ it is enough to use only those resolutions which come from Y.

2.4.5. Summarizing, in order to describe an object $F \in D^b_G(X)$ it is enough to do the following:

(i) To fix some sufficiently rich category J of resolutions of X closed with respect to direct products.

(ii) For every resolution $P \in J$ to describe an object $F(P) \in D^b(\overline{P})$.

(iii) For every morphism of resolutions $\nu : P \to R$ in J to construct an isomorphism $\alpha_\nu : \nu^*(F(R)) \simeq F(P)$.

(iv) To check that the system of isomorphisms constructed in (iii) is compatible with the composition of morphisms in J. Namely, given a composition of two morphisms $P \xrightarrow{\nu} R \xrightarrow{\mu} S$, we should have an equality $\alpha_\nu \cdot \nu^* \alpha_\mu = \alpha_{\mu\nu}$.

Remark. Similar results hold for the fibered category $Sh/\mathcal{T}$. Namely, the category $Sh(\Phi)$, and more generally $Sh(\Psi)$ as in 2.4.4, is naturally equivalent to the category $Sh_G(X)$ of G-equivariant sheaves on X (0.2,2.5.3). We will prove a slightly stronger statement in Appendix **B**.

We will construct the category $D_G^+(X)$ in 2.9 by the same method.

2.4.6. Let us describe some basic functors for the equivariant derived category using the language of fibered categories.

(i) The forgetful functor $For : D_G^b(X) \in D^b(X)$ (2.1.4). Identifying $D_G^b(X)$ with the category $D^b(\Phi)$ we can describe this functor by $For(F) = F(T) \in D^b(\overline{T}) = D^b(X)$, where $T = G \times X$ is the trivial resolution of X.

(ii) The inverse image functor (2.1.6). Let $f : X \to Y$ be a G-map. It defines a functor $Res(X) \to Res(Y)$, $P \mapsto P$ (see 2.1.2, example 4). Using this functor we define the inverse image functor $f^* : D_G^b(Y) \to D_G^b(X)$ by $f^*(F)(P) = F(P) \in D^b(\overline{P})$.

We also have another description of this functor, which uses induced resolutions. Namely, consider the category $Res(Y)$ of resolutions of Y and the functor $f^0 : Res(Y) \to Res(X)$, $R \mapsto f^0(R)$ (2.1.2, example 4). For every resolution $R \to Y$ we have the canonical map $f : f^0(R) \to R$. We define the inverse image functor $f^* : D_G^b(Y) \to D_G^b(X)$ by $f^*(F)(f^0(R)) = f^*(F(R)) \in D^b(\overline{f^0(R)})$. Note that this formula defines the value of the functor $f^*(F) \in D^b(\Phi)$ not on all resolutions of X, but only on resolutions of the form $f^0(P)$. However the proposition 2.4.4 shows, that this gives a well defined object in $D_G^b(X) = D^b(\Phi)$.

(iii) The quotient functor (2.1.7). The quotient functor $q^* : D^b(\overline{X}) \in D_G^b(X)$ is defined by $q^*(A)(P) = \overline{p}^*(A) \in D^b(\overline{P})$, where $\overline{p} : \overline{P} \to \overline{X}$ is the natural map.

2.5. Truncation and the structure of a triangulated category on $D_G^b(X)$.

2.5.1. Let $F \in D_G^b(X)$. We will interpret F as a functor $P \mapsto F(P)$ on the category $Res(X)$ as in 2.4.

Fix an interval $I \subset \mathbf{Z}$ and suppose that for some resolution P for which the map $P \to X$ is epimorphic we have $F(P) \in D^I$. Then for any other resolution R,

$F(R)$ also lies in D^I, since for $S = P \times R$ the projection of $\overline{S}$ on $\overline{R}$ is epimorphic and inverse images of $F(P)$ and $F(R)$ are isomorphic.

It is clear that the full subcategory of objects $F \in D^b_G(X)$ satisfying this condition is the subcategory $D^I_G(X)$ described in 2.2.1.

Similarly we define subcategories $D^{\leq a}_G(X)$, $D^{\geq a}_G(X)$ and truncation functors $\tau_{\leq a}$ and $\tau_{\geq a}$ (for example, $\tau_{\leq a}(F)(P) = \tau_{\leq a}(F(P))$).

2.5.2. Definition. A diagram $F \to F' \to F'' \to F[1]$ in $D_G(X)$ is called an **exact triangle** if for any resolution $P \to X$ the diagram $F(P) \to F'(P) \to F''(P) \to F(P)[1]$ is an exact triangle in $D(\overline{P})$.

Proposition. *(i) The collection of exact triangles makes $D^b_G(X)$ into a triangulated category.*

(ii) The truncation functors $\tau_{\leq}$ and $\tau_{\geq}$ define a t-structure on the category $D^b_G(X)$ (see [BBD]).

Proof. (i) We have to check the axioms of triangulated categories. Each axiom deals with a finite number of objects $A_i \in D^b_G(X)$. Let us choose a segment $I \subset \mathbf{Z}$ such that all objects $A_i[k]$ for $k = -1, 0, 1$ lie in $D^I_G(X)$. Then choose $n > |I|$, fix an n-acyclic resolution $P \to X$ and identify $D^I_G(X)$ with $D^I_G(X, P)$.

As was proved in 2.3.2, the natural restriction functor $D^I_G(X, P) \to D^I(\overline{P})$ gives an equivalence of the category $D^I_G(X, P)$ with the full subcategory $D^I(\overline{P}|p) \subset D^b(\overline{P})$ which is closed under extensions.

Each axiom of triangulated categories asserts the existence of some objects and morphisms, such that certain diagrams are commutative and certain diagrams are exact triangles. We find the corresponding objects and morphisms in the category $D^I(\overline{P})$ and extend these diagrams to other resolutions of X as in remark in 2.4.3. Using proposition 1.9.2 we see that this extension preserves commutative diagrams and exact triangles.

(ii) Since all the functors in the definition of the category $D^b_G(X)$ commute with truncation functors, the assertion is obvious.

2.5.3. Proposition. *The heart $C = D^{\leq 0}_G(X) \cap D^{\geq 0}_G(X)$ of the t-category $D^b_G(X)$ is naturally equivalent to the category $Sh_G(X)$.*

We will prove this result in Appendix B.

By definition, an object $F \in D^b_G(X)$ lies in C iff $F(P)$ is a sheaf for every resolution P. In other words, we can identify the category $Sh_G(X)$ with the fiber $Sh(\Phi)$ of the fibered category $Sh/\mathcal{T}$ over the functor $\Phi : Res(X) \to \mathcal{T}$ (or with the fiber $Sh(\Psi)$ as in 2.4.4).

2.5.4. There is a natural functor $i : D^b(Sh_G(X)) \to D^b_G(X)$ from the bounded derived category of equivariant sheaves to the equivariant derived category. Namely, if F is a complex of equivariant sheaves on X, then for any resolution $p : P \to X$ it defines a complex $p^*(F)$ of equivariant sheaves on P . We denote by $i(F)(P)$ the corresponding object in $D^b(\overline{P})$ (use lemma 0.3). This functor i induces an equivalence of abelian categories $i : Sh_G(X) \simeq D_G^{\leq 0}(X) \cap D_G^{\geq 0}(X)$ (2.5.3), but in general is not an equivalence of the triangulated categories. However, it is an equivalence if the group G is discrete (8.3.1).

2.6. Change of groups. The quotient and the induction equivalences.

2.6.1. Let $H \subset G$ be a subgroup and X be a G-space. Then it is intuitively clear that restricting the action of G to H one should get a restriction functor $Res_{H,G} : D_G^b(X) \to D_H^b(X)$.

Here is an explicit description of this restriction functor.

For any H-resolution $p : P \to X$ we consider the induced G-resolution $ind(P) = G \times_H P$, with the morphism $\nu : ind(P) \to X$ given by $\nu(g,l) = g(p(l))$, where $l \in P$. Note that we have a canonical isomorphism $\overline{ind(P)} = \overline{P}$.

Let $F \in D_G^b(X)$ be an object, which we interpret as a functor $P \to F(P)$ like in 2.4.3. Then we define the object $Res_{H,G}(F) \in D_H^b(X)$ by $Res_{H,G}(F)(P) = F(ind(P)) \in D^b(\overline{P})$.

In case when H is a trivial group this functor is naturally isomorphic to the forgetful functor $For : D_G^b(X) \to D_H^b(X) = D^b(X)$.

More generally, let X be an H-space and Y be a G-space. Let $f : X \to Y$ be a ϕ-map, where $\phi : H \to G$ is the embedding (0.1). In this situation we define the inverse image functor

$$f^* : D_G^b(Y) \to D_H^b(X) \quad \text{by} \quad f^*(F)(P) = (F(ind(P)).$$

(Here $ind(P) = G \times_H P$ and the G-map $\nu : ind(P) \to Y$ is given by $\nu(g,l) = gfp(l)$ for $g \in G$ and $l \in P$).

By definition the functor f^* preserves the t-structure.

2.6.2. Quotient equivalence. We saw in proposition 2.2.5. that for a free G-space X we have the equivalence $D_G^b(X) \simeq D^b(\overline{X})$. This is also clear from the description of the category $D_G^b(X)$ in terms of fibered categories, since in this case the category $Res(X,G)$ has a final object X.

Similar arguments prove the following more general result.

Let $H \subset G$ be a normal subgroup, $B = G/H$ the quotient group. Then for any G-space X the space $Z = H \setminus X$ is a B-space and the projection $q : X \to Z$ is a ϕ-map, where $\phi : G \to B$ is the quotient homomorphism.

Theorem. *If X is free as an H-space then the t-categories $D_B^b(H \setminus X)$ and $D_G^b(X)$ are naturally equivalent.*

Proof. Let $P \to X$ be a resolution of a G-space X. Then $P_0 = H \backslash P$ is a resolution of the B-space Z. This defines a functor $Res(X,G) \to Res(Z,B)$. Since X is free as an H-space, lemma 2.1.1 implies that this functor is an equivalence of categories (the inverse functor is $P_0 \mapsto P = X \times_Z P_0$). Since $\overline{P}_0 = \overline{P}$ we see that this equivalence is compatible with quotient functors $\Phi_X : Res(X) \to \mathcal{T}$ and $\Phi_Z : Res(Z) \to \mathcal{T}$. This implies that the fiber categories $D^b(\Phi_Z)$ and $D^b(\Phi_X)$ are equivalent.

2.6.3. Induction equivalence. Let $\phi : H \to G$ be a embedding of a subgroup. For an H-space X consider the induced G-space $ind(X) = G \times_H X$. We have a canonical ϕ-map $\nu : X \to ind(X), x \mapsto (e,x)$. It induces an inverse image functor $\nu^* : D^b_G(ind(X)) \to D^b_H(X)$ (2.6.1).

Theorem. *The functor ν^* is an equivalence of t-categories.*

Proof. The functor $P \to ind(P)$ is an equivalence of categories $ind : Res(X,H) \to Res(ind(X),G)$ such that $\overline{P} = \overline{ind(P)}$. Hence by definition of the functor ν^* in 2.6.1 it is an equivalence of categories.

Remark. Let X be a G-space and $H \subset G$ be a subgroup. Then X can be considered as an H-space and we have a canonical G-map $\pi : ind(X) \to X$. Using the inverse image functor and the induction equivalence we get a functor

$$\pi^* : D^b_G(X) \to D^b_G(ind(X)) \simeq D^b_H(X).$$

It is easy to see that this functor is canonically isomorphic to the restriction functor $Res_{G,H}$ (2.6.1).

2.7. Other descriptions of the category $D^b_G(X)$.

We are going to give two more alternative descriptions of the category $D^b_G(X)$.

2.7.1. Fix a sequence of resolutions $P_0 \to P_1 \to \cdots \to P_n \to \cdots$ of X where P_n is an n-acyclic resolution (if G is a Lie group, we can take $P_i = X \times M_i$, where $\{M_i\}$ is the sequence of free G-manifolds constructed using the Stiefel manifolds as in section 3.1. below). Then we can define $D^b_G(X)$ as the 2-limit of the categories $D^b_G(X, P_i)$. In other words, an object $F \in D^b_G(X)$ is a sequence $F = \{F_X \in D^b(X), \overline{F}_n \in D^b(\overline{P_n})\}$, together with a system of isomorphisms $p_n^*(F_X) \simeq q_n^*(\overline{F}_n)$, $\nu_{in}^*(\overline{F}_n) \simeq \overline{F}_i$, where $p_n : P_n \to X$, $q_n : P_n \to \overline{P}_n$ and $\nu_{i,n} : P_i \to P_n$, satisfying obvious compatibility conditions. Corollary 2.2.2 implies that this category is equivalent to the category $D^b_G(X)$.

2.7.2. The following description of the category $D^b_G(X)$ probably provides the most satisfactory intuitive picture.

Let us fix an ∞-acyclic locally connected free G-space M (for example, take

$$M = \lim_{\rightarrow} M_n$$

as in section 3.1. below). Then $P = X \times M$ is an ∞-acyclic resolution of X and by definition $D^b_G(X) = D^b_G(X, P)$.

Note that if M is in addition contractible, then the fibration $\overline{P} \to \overline{M}$ is nothing else but the standard fibration $X_G \to BG$ over the classifying space of G with the fiber X. By definition an object $F \in D^b_G(X, P)$ is essentially an object in $D^b(\overline{P})$ whose restrictions to all fibers are isomorphic.

For example, let us describe the category $D^b_G(pt)$.

Proposition. *Let M be a contractible locally connected free G-space, $BG := G \setminus M$ - the classifying space for G. Then the category $D^b_G(pt)$ is equivalent to the full subcategory of $D^b(BG)$ which consists of complexes with locally constant cohomology sheaves. If G is a connected Lie group, then this subcategory consists of complexes with constant cohomology sheaves.*

Proof. By the criterion 1.9.4 the map $p : M \to pt$ is ∞-acyclic. Hence by 2.3.2 $D^b_G(pt)$ is equivalent to the full subcategory $D^b(BG|p) \subset D^b(BG)$. We claim that an object $F \in D^b(BG)$ lies in this subcategory iff its cohomology sheaves are locally constant. Indeed, consider the object $H = q^*(F) \in D^b(M)$, where $q : M \to BG$ is the quotient map. If $F \in D^b(BG|p)$, then H comes from pt and hence its cohomology sheaves are constant. Therefore the cohomology sheaves of F are locally constant.

Conversely, suppose that F has locally constant cohomology sheaves. Let us show that $F \in D^b(BG|p)$. Since by 2.3.2 this subcategory is closed under extensions we can assume that F is a sheaf. Then H is a locally constant sheaf on a contractible space M. Hence it is constant, i.e. comes from pt.

If G is a connected Lie group, then BG is simply connected and hence every locally constant sheaf on BG is constant. This proves the proposition.

This picture will be used for example to describe the behavior of the equivariant derived category when we change the group. Unfortunately the space M in this case is usually infinite-dimensional, so it is difficult to define functors like $f^!$ or the Verdier duality using this picture. However we will use ∞-acyclic resolutions in section 2.9 below in the discussion of the category $D^+_G(X)$.

2.8. Constructible objects.

Suppose that a G-space X is a stratified pseudomanifold with a given stratification $\mathcal{S}$.

Definition. We say that an object $F \in D^b_G(X)$ is $\mathcal{S}$-constructible if the object $F_X \in D^b(X)$ is $\mathcal{S}$-constructible. We denote the full subcategory of $\mathcal{S}$-constructible objects in $D^b_G(X)$ by $D^b_{G,c}(X)$.

Remark. We do not assume that the stratification $\mathcal{S}$ is G-invariant.

2.9. Category $D_G^+(X)$.

Let G be a topological group with the following property:

$$(**) \qquad \text{There exists a } \infty - \text{acyclic free } G - \text{space } M.$$

For example, any Lie group satisfies this condition. It follows that every G-space X has an ∞-acyclic resolution $X \times M \to X$ (1.9.4).

We now proceed exactly as in section 2.4 replacing everywhere D^b (or D^I) by D^+ and n-acyclic resolutions by ∞-acyclic ones. Namely, fix a G-space X. Consider the category $K = Res(X, G)$ of resolutions of X and the functor $\Phi : K \to \mathcal{T}$, $\Phi(P) = \overline{P} = G\backslash P$.

2.9.1. Definition. Define $D_G^+(X) := D^+(\Phi)$ - the fiber of the fibered category $D^+/\mathcal{T}$ over the functor Φ (2.4.1, 2.4.2).

In other words to define an object $F \in D_G^+(X)$ means for every resolution $P \to X$ to give an element $F(P) \in D^+(\overline{P})$ and for every morphism of resolutions $\nu : P \to R$ to give an isomorphism $\nu^* F(R) \simeq F(P)$ satisfying natural compatibility conditions (2.4.2).

2.9.2. Given a resolution $P \to X$ we define the category $D_G^+(X, P)$ replacing D^b by D^+ in the definition 2.1.3.

Lemma. (A D^+-version of lemma 2.3.2.) *Let $p : P \to X$ be an ∞-acyclic resolution. Let $D^+(\overline{P}|p) \subset D^+(\overline{P})$ be the full subcategory consisting of objects H such that q^*H comes from X. Then the restriction functor $D_G^+(X, P) \to D^+(\overline{P}|p)$ is an equivalence of categories.*

2.9.3. Let $\overline{X}$ be the quotient space. We have the obvious quotient functor $q^* : D^+(\overline{X}) \to D_G^+(X, P)$ as in 2.1.7.

The following D^+-analogues of proposition 2.2.1 and corollary 2.2.2 are proved similarly using proposition1.9.2, lemma 1.9.3 and lemma 2.9.2 above.

Proposition. *Let $P \to X$ be an ∞-acyclic resolution. Suppose that X is a free G-space. Then the quotient functor $q^* : D^+(\overline{X}) \to D_G^+(X, P)$ is an equivalence of categories.*

Corollary. *Let $P \to X$ and $R \to X$ be two resolutions and $S = P \times_X R$ be their product. Assume that P is ∞-acyclic. Then the functor $pr_R^* : D_G^+(X, R) \to D_G^+(X, S)$ is an equivalence of categories.*

2.9.4. The following proposition provides a "realization" of the categoty $D_G^+(X)$ as in 2.7.2.

Proposition. *Let $P \to X$ be an ∞-acyclic resolution. Then the categories $D_G^+(X)$ and $D_G^+(X, P)$ are naturally equivalent.*

The proof is similar to the proof of the proposition 2.4.3 above.

2.9.5. Combining the above proposition with lemma 2.9.2 we get the following geometric realization of the category $DG^+(pt)$.

Proposition. *Let M be a contractible locally connected free G-space, $BG := G\backslash M$ - the classifying space for G. Then the category $D_G^+(pt)$ is naturally equivalent to the full subcategory of $D^+(BG)$ which consists of complexes with locally constant cohomology sheaves. If G is a connected Lie group, then this subcategory consists of complexes with constant cohomology sheaves.*

The proof is the same as in proposition 2.7.2.

2.9.6. The proposition 2.4.4 remains true if we replace D^b by D^+ and require that $j(J)$ contains an ∞-acyclic resolution. For example, if $P \to X$ is ∞-acyclic and $T = G \times X \to X$ is the trivial resolution, then the following subcategory of $Res(X, G)$ is rich enough to define $D_G^+(X)$:

$$T \longleftarrow T \times_X P \longrightarrow P$$

(cf. example 2.4.2 and the proof of proposition 2.4.3).

Also the discussion in 2.4.5 is valid for D^+.

2.9.7. We define exact triangles and truncation functors $\tau_{\leq}$, $\tau_{\geq}$ in $D_G^+(X)$ exactly as in 2.5.1, 2.5.2. The same proof (using ∞-acyclic resolutions) shows that D_G^+ is a triangulated category with a t-structure given by the functors τ.

2.9.8. All constructions and results of section 2.6 are valid for the category D_G^+. Namely, just replace the symbol D_G^b by D_G^+ everywhere.

2.9.9. Remark. We see that using ∞-acyclic resolutions $P \to X$ one gets a quick and "geometric" definition of the category $D_G^+(X) \simeq D_G^+(X, P) \subset D^+(\overline{P})$ (2.9.4, 2.9.2). We could do the same with the bounded category $D_G^b(X)$. However, as was already mentioned, the space P is usually ∞-dimensional, which makes it difficult to apply this construction in the algebraic situation or in the case of functors like $f^!$ or the Verdier duality.

The category D_G^+ is needed in order to define the general direct image functor Q_* (when the group changes) (section 6). Although we did not use ∞-acyclic resolutions in the definition of D_G^+ (2.9.1), they become essential in the definition of the direct image Q_*. This is unpleasant in the algebraic setting. We manage to avoid this problem in some special cases (see section 9 below).

Appendix B. A Simplicial description of the category $D_G(X)$.

B1. Let us describe the category $D^b_G(X)$ in the simplicial language. For a discussion of simplicial spaces and simplicial sheaves we refer to [D1].

Let X be a G-space. Following [D1], we denote by $[G\backslash X].$ the usual simplicial space

$$[G\backslash X]. \; = \; ...G \times G \times X \;\substack{\longrightarrow\\ \longleftarrow\\ \longrightarrow\\ \longleftarrow\\ \longrightarrow}\; G \times X \;\substack{\longrightarrow\\ \longleftarrow\\ \longrightarrow}\; X.$$

Recall that a **simplicial sheaf** $F^\cdot$ on $[G\backslash X].$ is a collection of sheaves $F^\cdot = \{F^n \in Sh(G^n \times X)\}_{n \geq 0}$ with the following additional structure. Let $h : G^n \times X \to G^m \times X$ be a composition of arrows in $[G\backslash X]..$ Then h defines a structure morphism $\alpha_h : h^* F^m \to F^n$, such that

$$\alpha_{h'h} = \alpha_h \cdot h^* \alpha_{h'},$$

whenever the composition $h'h$ makes sense.

The abelian category of simplicial sheaves on $[G\backslash X].$ is denoted by $Sh([G\backslash X].)$.

B2. Definition. Denote by $Sh_{eq}([G\backslash X].)$ the full subcategory of $Sh([G\backslash X].)$ consisting of simplicial sheaves $F^\cdot$ for which all structure morphisms are isomorphisms.

B3. Fact. The category $Sh_{eq}([G\backslash X].)$ is naturally equivalent to the category $Sh_G(X)$ of G-equivariant sheaves on X (see [D1](6.1.2,b)).

In proposition **B4** below we extend this equivalence to derived categories.

Let $D^b([G\backslash X].)$ be the bounded derived category of simplicial sheaves on $[G\backslash X]..$ Denote by $D^b_{eq}([G\backslash X].)$ the full subcategory of $D^b([G\backslash X].)$ consisting of complexes A, such that $H^i(A) \in Sh_{eq}([G\backslash X].)$.

B4. Proposition. *Triangulated categories $D^b_{eq}([G\backslash X].)$ and $D^b_G(X)$ are naturally equivalent.*

Proof. Let us first prove a special case.

B5. Lemma. *The proposition holds if X is a free G-space.*

Proof of the lemma. Consider the quotient map $q : X \to \overline{X}$. Then q defines an augmentation of the simplicial space $[G\backslash X].$ and hence defines two functors

$$(1) \qquad q^* : Sh(\overline{X}) \to Sh_{eq}([G\backslash X].),$$

$$(2) \qquad q^* : D^b(\overline{X}) \to D^b_{eq}([G\backslash X].).$$

We know that $D^b_G(X) \simeq D^b(\overline{X})$ (2.2.5). Hence it suffices to prove that the second functor is an equivalence.

It is known (see [D1]) that the first functor is an equivalence. So it suffices to show that for any two sheaves $A, B \in Sh(\overline{X})$ we have

$$\mathrm{Ext}^i_{D(\overline{X})}(A, B) = \mathrm{Ext}^i_{D([G\backslash X].)}(q^* A, q^* B).$$

Using the standard method (see [H],7.1) we reduce the proof to the case of an elementary sheaf $A = C_U$ for an open set $U \subset S$. Then we may assume that $\overline{X} = U$ and the above equality becomes

$$H^i(\overline{X}, B) = H^i([G\backslash X]., q^*B),$$

which is well known (see[D1](6.1.2,c)). This proves the lemma.

Let X be any G-space. Let $P \to X$ be an ∞-acyclic resolution. It defines a map of simplicial spaces $p : [G\backslash X]. \to [G\backslash P].$ and hence induces the functor

$$p^* : D^b_{eq}([G\backslash X].) \to D^b_{eq}([G\backslash P].).$$

By the lemma this last category is equivalent to $D^b(\overline{P})$. So we get the functor

$$q^{*-1} \cdot p^* : D^b_{eq}([G\backslash X].) \to D^b(\overline{P}).$$

It is clear that for $F \in D^b_{eq}([G\backslash X].)$ its image $q^{*-1}p^*F$ lies in the subcategory $D^b(\overline{P}|p) \simeq D^b_G(P|p) \simeq D^b_G(X)$ (lemmas 2.3.2, 2.9.2, proposition 2.9.4). So we actually have the functor

$$(*) \qquad q^{*-1} \cdot p^* : D^b_{eq}([G\backslash X].) \to D^b_G(X).$$

We claim that it is an equivalence.

Indeed, by a simplicial version of section 1.9 we conclude that the map $p : [G\backslash P]. \to [G\backslash X].$ is ∞-acyclic, that is the functor $p^* : D^b([G\backslash X].) \to D^b([G\backslash P].)$ is fully faithful and its right inverse is p_*. This implies that the functor (*) is fully faithful. On the other hand, it is clear that if $F \in Sh(X)$ is such that $p^*F \in Sh_G(P)$, then also canonically $F \in Sh_G(X)$ (use again the acyclicity of the map p). Therefore the functor (*) induces the equivalence of abelian categories $Sh_G(X)$ and $D^{\leq 0}_G(X) \cap D^{\geq}_G(X)$ and hence is an equivalence. This proves the proposition.

B6. Remark. Note that the last argument also proves the proposition 2.5.3

3. Functors.

In this section we consider a G-map $f : X \to Y$ and describe functors $\otimes$,Hom, f^*,$f^!$,f_*,$f_!$ and D in the categories $D^b_G(X)$ and $D^b_G(Y)$. We also study relations between these functors and the ones introduced earlier in 2.6. In section 3.7 we define the integration functors $D^b_H(X) \to D^b_G(X)$ which are (left and right) adjoint to the restriction functor $D^b_G(X) \to D^b_H(X)$ for a closed subgroup $H \subset G$.

3.1. In this section 3 we assume that G is a Lie group, satisfying the following condition

(*+) For every n there exists an n-acyclic free G-space M which is a manifold.

It follows that every G-space X has an n-acyclic *smooth* resolution $M \times X \to X$ (see 1.7, 1.9, 2.1.2).

Let us show that this property holds in most interesting cases.

Lemma. *Let G be a Lie group with one of the following properties:*

a) G is a linear group, i.e., a closed subgroup of $GL(k, \mathbf{R})$ for some k,

b) G has a finite number of connected components.

Then the property (+) holds for G.*

Proof. a) The Stiefel manifold M_n of k-frames in $\mathbf{R}^{n+k}$ is an n-acyclic free G-manifold.

b) By a result of G.Mostow there exists a compact Lie subgroup $K \subset G$ such that the manifold G/K is contractible. By Peter-Weyl theorem K is a linear group, so by a) it has an n-acyclic free K-manifold M'. Then $M = G \times_K M'$ is an n-acyclic free G-manifold. This proves the lemma.

For a G-space X we denote by $SRes(X)$ the category of smooth resolutions of X and smooth morphisms between them. It follows from the property (*+) that this category is sufficiently rich, so we can define $D^b_G(X)$ to be the fiber of $\pi : D^b \to \mathcal{T}$ over the functor $\Phi_X : SRes(X) \to \mathcal{T}, P \to \overline{P}$ (see 2.4.5). This is the definition of the category $D^b_G(X)$ which we use in order to define all functors. The main reason for sticking to smooth resolutions is the fact that all functors commute with a smooth base change (see 1.8).

3.2. Functors $\otimes$ and Hom.

Let $F, H \in D^b_G(X)$. We will consider F and H as functors on the category $SRes(X)$ of smooth resolutions of X (see 2.4.5).

Now define objects $F \otimes H$ and $Hom(F, H)$ in $D^b_G(X)$ by

$$F \otimes H(P) := F(P) \otimes H(P),$$

$$Hom(F,H)(P) := Hom(F(P), H(P))$$

for every smooth resolution P. For any smooth morphism of smooth resolutions $\nu : S \to P$ we define isomorphisms

$$F(S) \otimes H(S) \simeq \nu^*(F(P) \otimes H(P)),$$

$$Hom(F(S), H(S)) \simeq \nu^*(Hom(F(P), H(P)))$$

using the smooth base change (see 1.8).

3.3. Functors $f^*, f^!, f_*$ and $f_!$.

Let $f : X \to Y$ be a G-map of nice topological spaces. Then $P \longmapsto f^0(P)$ defines the functor $f^0 : SRes(Y) \to SRes(X)$ (see 2.1.2, example 4), which takes n-acyclic resolutions into n-acyclic ones. Hence the category $f^0(SRes(Y))$ is rich enough to define objects in $D^b_G(X)$.

We will define the above functors using the smooth base change. For example, consider the functor f_*.

Given an object $F \in D^b_G(X)$ we define $f_*(F) \in D^b_G(Y)$ by

$$f_*(F)(P) := f_*(F(f^0(P))),$$

for $P \in SRes(Y)$, where $f : f^0(P) \to P$ is the natural projection.

Given a smooth morphism $\nu : S \to P$ in $SRes(Y)$ we define the isomorphism $f_*(F)(\nu) : \nu^*(f_*(F)(P)) \simeq f_*(F)(S)$ as $f_*(F)(\nu) := f_*(F(\nu))$ using the smooth base change applied to the pullback diagram

$$\begin{array}{ccc} \overline{f^0(S)} & \xrightarrow{\nu} & \overline{f^0(P)} \\ \downarrow f & & \downarrow f \\ \overline{S} & \xrightarrow{\nu} & \overline{P} \end{array}$$

The collection of objects $f_*(F)(P)$ and isomorphisms $f_*(F)(\nu)$ defines an object $f_*(F) \in D^b_G(Y)$.

Similarly we define functors $f^*, f^!, f_!$. Note that functors f^* and $f_!$ can be defined using arbitrary (not necessarily smooth) resolutions.

3.4. Properties of functors $\otimes, Hom, f^*, f^!, f_*, f_!$.

3.4.1. Theorem. *(i) Let $H \subset G$ be a subgroup of G. Then the above functors commute with the restriction functor $Res_{H,G}$ (2.6.1). This means that there exist canonical isomorphisms of functors $Res_{H,G} \cdot f_* \simeq f_* \cdot Res_{H,G}$, etc. In particular, these functors commute with the forgetful functor $For : D^b_G(X) \to D^b(X)$.*
(ii) Let $H \subset G$ be a normal subgroup, $B = G/H$. Let $f : X \to Y$ be a G-map of G-spaces which are free as H-spaces. Then the above functors commute with the

quotient equivalences $q^*: D^b_B(H\setminus X) \to D^b_G(X)$ *and* $q^*: D^b_B(H\setminus Y) \to D^b_G(Y)$ *in theorem 2.6.2. In particular, when* $H = G$ *these functors commute with the quotient equivalences* $q^*: D^b(\overline{X}) \to D^b_G(X)$ *and* $q^*: D^b(\overline{Y}) \to D^b_G(Y)$.
(iii) Let $H \subset G$ *be a subgroup. Then the above functors commute with the induction functor* $D^b_H(Y) \simeq D^b_G(G\times_H Y)$ *for an* H*-space* Y *as in theorem 2.6.3.*
Proof. Immediately follows from definitions.

3.4.2. Examples.
1. We want to show that the constant sheaf C_X and the dualizing object D_X for a G-space X always have a natural structure of a G-equivariant object. Let $p: X \to pt$ be a map to a point.

We define the equivariant constant sheaf $C_X = C_{X,G} \in D^b_G(X)$ by $C_X(P) := C_{\overline{P}}$. Clearly $C_{X,G} = p^*(C_{pt,G})$ and $For(C_{X,G}) = C_X$.

Define the equivariant dualizing object on X to be $D_{X,G} = p^!(C_{pt,G}) \in D^b_G(X)$. By theorem 3.4.1 we have $For(D_{X,G}) = D_X$.

More generally, for a G-map $f: X \to Y$ we define the equivariant relative dualizing object $D_{f,G} := f^!(C_Y) \in D^b_G(X)$. Again by theorem 3.4.1 it corresponds to the usual relative dualizing object under the forgetful functor.
2. Let $Y \subset X$ be a closed G-subspace, $U = X\setminus Y$. Consider the natural imbeddings $i: Y \to X$ and $j: U \to X$. Then for $F \in D^b_G(X)$ we have the exact triangles

$$i_!i^!(F) \to F \to j_*j^*(F) \quad \text{and} \quad j_!j^!(F) \to F \to i_*i^*(F).$$

These triangles are functorial in F. They are compatible with the forgetful functor, the restriction functor $Res_{H,G}$ (2.6.1) and with the quotient and the induction equivalences of theorems 2.6.2 and 2.6.3.

3.4.3. Theorem. *All properties of the functors* $\otimes$, Hom, f^*, f_*, $f^!$, $f_!$ *listed in 1.4 hold in the equivariant case.*
Proof. Since the functorial identities listed in 1.4 commute with the smooth base change (theorem 1.8), they automatically lift to the equivariant category (see the argument in 3.3).

3.5. The equivariant Verdier duality.

Assume that X is a nice G-space (1.4).
3.5.1. Definition. Consider the equivariant dualizing object $D_{X,G} = p^!C_{pt,G} \in D^b_G(X)$ (3.4.2 example 1). For $F \in D^b_G(X)$ define its Verdier dual as

$$D(F) = Hom(F, D_{X,G}).$$

3.5.2. Theorem (The equivariant Verdier duality).
(i) There exists a canonical biduality morphism

$$F \to D(D(F))$$

in $D^b_G(X)$.
(ii) For a G-map $f : X \to Y$ *we have canonical isomorphisms of functors between the categories* $D^b_G(X)$ *and* $D^b_G(Y)$

$$Df_! \simeq f_* D \quad \text{and} \quad f^! D \simeq Df^*.$$

(iii) The Verdier duality commutes with the forgetul functor $For : D^b_G(X) \to D^b(X)$.
Proof. As in section 3.3 we work with the category $SRes(X, G)$ of smooth resolutions of X.

(i). Let us first of all analize the equivariant dualizing object $D_{X,G}$.

Let $P \to pt$ be a smooth resolution of the point pt, i. e. a smooth free G-space. Consider the induced resolution of X:

$$p^0(P) = P \times X \to X$$

and the corresponding map of quotient spaces

$$\overline{p} : \overline{p^0(P)} \to \overline{P}.$$

By definition $D_{X,G}(p^0(P)) = \overline{p}^! C_{\overline{P}} \in D^b(\overline{p^0(P)})$. Since P is a smooth free G-space, the quotient space $\overline{P}$ is also smooth. Hence the dualizing object $D_{\overline{P}}$ is invertible (1.6.1). Since $\overline{p}$ is a locally trivial fibration, by 1.4.7 we have

$$D_{X,G}(p^0(P)) := p^! C_{\overline{P}} = p^! D_{\overline{P}} \otimes p^* D_{\overline{P}}^{*-1} = D_{\overline{p^0(P)}} \otimes p^* D_{\overline{P}}^{-1}.$$

Therefore, for $F \in D^b_G(X)$, the object $D(F)(P)$ is canonically isomorphic to the usual Verdier dual $D(F(P))$ of $F(P) \in D^b(\overline{P})$ twisted by $D_{\overline{P}}^{-1}$ (1.5). In particular, $D(D(F))(P)$ is canonically isomorphic to $D(D(F(P)))$. So the usual biduality morphism

$$F(P) \to D(D(F(P)))$$

(1.6.1) induces the desired biduality morphism

$$F(P) \to D(D(F))(P).$$

It remains to check that this morphism is compatible with the smooth base change by smooth maps $R \to P$ of smooth resolutions of pt. This follows from theorem 1.8(iii). This proves (i).

(ii) is proved similarly.

(iii) follows immediately from theorem 3.4.1(i).

The theorem is proved.

Some further properties of the Verdier duality (in particular its behavior under the quotient and the induction equivalences) will be studied in section 7 below.

3.6. Equivariant constructible sheaves.

Let X be a G-space, which is a pseudomanifold with a stratification $\mathcal{S}$. Let $D^b_{G,c}(X) \subset D^b_G(X)$ be the full subcategory of G-equivariant $\mathcal{S}$-constructible objects (2.8). Then it is preserved by functors $\otimes, Hom$ and D and the biduality morphism $F \to D(D(F))$ is an isomorphism for $F \in D^b_{G,c}(X)$.

If $f : X \to Y$ is a stratified G-map of pseudomanifolds, functors $f^*, f^!, f_*$ and $f_!$ preserve constructibility. This follows from the corresponding properties of the category $D^b_c(X)$ (see 1.10), the definition of $D^b_{G,c}$ in terms of D^b_c (see 2.8) and the fact that all functors commute with the forgetful functor (theorem 3.4.1(i)).

Similarly for a constructible G-space X, as described in 1.10, we define the full subcategory $D^b_{G,c}(X) \subset D^b_G(X)$ of G-equivariant constructible objects. This category is preserved by all functors for constructible G-maps.

3.7. Integration functors.

Let X be a nice G-space (1.4) and $H \subset G$ be a closed subgroup.

3.7.1. Theorem. *The restriction functor* $Res_{H,G} : D^b_G(X) \to D^b_H(X)$ *has a right adjoint functor* Ind_* *and a left adjoint functor* $Ind_!$.

In particular, in case of a trivial subgroup H *we have a right and a left adjoint functors to the forgetful functor* $For : D^b_G(X) \to D^b(X)$.

Proof. Consider X as an H-space, and denote by Z the induced G-space $Z = ind(X) = G \times_H X$ and by $\pi : Z \to X$ the natural G-map. Denote by $\nu : X \to Z$, $x \mapsto (e, x)$ the natural embedding. By theorem 2.6.3 we have an equivalence of categories $\nu^* : D^b_B(Z) \simeq D^b_H(X)$. As was remarked in 2.6.3 the restriction functor $Res_{H,G} : D^b_G(X) \to D^b_H(X)$ is naturally isomorphic to $\nu^* \cdot \pi^*$.

Let us put $Ind_* = \pi_* \cdot \nu^{*-1} : D^b_H(X) \to D^b_G(X)$. By theorem 3.4.3 this functor is the right adjoint to $Res_{H,G}$.

Consider the equivariant dualizing object $D_\pi = D_{\pi,G} \in D^b_G(Z)$ of the smooth map π (1.4.7, 1.7, 3.4.2). Then we have the canonical isomorphism of functors $\pi^* \simeq D_\pi^{-1} \cdot \pi^!$, where D_π^{-1} stands for the twist functor by D_π^{-1} (1.4.7, 1.5, 3.4.3).

Let us put $Ind_! = \pi_! \cdot D_\pi \cdot \nu^{*-1} : D^b_H(X) \to D^b_G(X)$. Then by theorem 3.4.3 this functor is left adjoint to $Res_{H,G}$.

3.7.2. Proposition. *Let $f : X \to Y$ be a G-map of nice topological. Then the functor Ind_* commutes with functors f_* and $f^!$, and the functor $Ind_!$ commutes with functors $f_!$ and f^*.*

Proof. This follows from 3.4.1, 3.4.3, 1.4.6.

3.7.3. Theorem. *Suppose that the space G/H is ∞-acyclic (1.9), for example contractible. Then the restriction functor $Res_{H,G}$ is fully faithful and its left inverse is the functor Ind_*.*

In particular, if G is an ∞-acyclic group and H is trivial, then the forgetful functor $For : D^b_G(X) \to D^b(X)$ is fully faithful.

Proof. Since the functor Ind_* is the right adjoint to Res it is enough to check that $Ind_* \cdot Res$ is isomorphic to the identity functor. Using the explicit description of the functor Ind_* in the proof of 3.7.1 we see that it amounts to an isomorphism $\pi_*\pi^* \simeq Id$. Since the map $\pi : Z \to X$ is a fibration with the ∞-acyclic fiber G/H it is ∞-acyclic (see 1.9.4). Then the statement follows from 1.9.2 and 3.4.3.

4. Variants.

In sections 2, 3 we have shown that the standard theory of constructible sheaves on topological spaces has a natural extension to the eqivariant situation. The reason for this was the existence of the smooth base change (see 1.8).

There are several theories which are parallel to the theory of constructible sheaves - étale sheaves on algebraic varietes, mixed sheaves on varieties over a finite field, D-modules on complex varieties. In all these situations there exists a smooth base change, so they have natural equivariant extensions. In this section we briefly discuss some of them.

4.1. Let X be a complex algebraic variety, $D^b_c(X)$ be the category of complexes on the topological space X which are constructible with respect to some algebraic stratification. If X is acted upon by a linear algebraic group G we can define the equivariant category $D^b_{G,c}(X)$ in the same way as in section 2. All properties discussed in sections 2 and 3 hold in this case.

4.2. Let X be a complex algebraic variety. Consider the derived category $D^b(D_X)$ of D-modules on X and full subcategories $D^b_h(D_X)$ and $D^b_{rh}(D_X)$ of holonomic and regular holonomic complexes (see [Bo2]).

These categories have functors and properties similar to category $D^b(X)$ in section 1. In particular the smooth base change holds for D-modules (this easily follows from the definition of the functor $f^!$ in [Bo2]).

Let G be a linear algebraic group acting algebraically on X. Using smooth (complex) resolutions as in section 3 we define the equivariant derived categories $D^b_G(D_X), D^b_{G,h}(D_X)$ and $D^b_{G,rh}(D_X)$ and functors between them.

Let $D^b_{G,c}(X)$ be the category described in 4.1. Then the de Rham functor DR, described in [Bo2], establishes the equivariant Riemann-Hilbert correspondence

$$DR : D^b_{G,rh}(D_X) \simeq D^b_{G,c}(X).$$

Remark. A. Beilinson has shown that the category $D^b(D_X)$ can be described directly in terms of D-modules on X in a language analogous to [DV]. We will discuss this interpretation elsewhere.

4.3. Let k be an algebraically closed field. Fix a prime number l prime to $char(k)$. For an algebraic variety X over k we denote by $D^b_c(X)$ the bounded derived category of constructible $\mathbf{Q}_l$-sheaves on X (see [D2]). This category has all functorial properties listed in section 1.

Let G be a linear algebraic group defined over k which acts on X. Then using smooth resolutions as in section 3 we can define the eqivariant derived category $D^b_{G,c}(X)$ and corresponding functors.

5. Equivariant perverse sheaves.

In this section we assume that G is a complex linear algebraic group acting algebraically on a complex variety X. We are interested in the category $D^b_{G,c}(X)$ of equivariant constructible objects on X, defined in 4.1. For a complex variety M we denote by d_M its complex dimension. In the algebraic setting we always assume that the basic ring R is a field of characteristic 0.

5.1 Equivariant perverse sheaves.

We want to define the subcategory of equivariant perverse sheaves $Perv_G(X) \subset D^b_{G,c}(X)$.

Definition. An object $F \in D^b_{G,c}(X)$ lies in the subcategory $Perv_G(X)$ if F_X lies in $Perv(X)$.

It is clear from this definition that all the elementary results about perverse sheaves hold in the equivariant situation. For example, this category is the heart of the "perverse" t-structure on the category $D^b_{G,c}(X)$; in particular it is an abelian category. Every object in $Perv_G(X)$ has finite length and we can describe simple objects in $Perv_G(X)$ in the usual way (see 5.2 below).

Proposition. *(i)* $D(Perv_G(X)) = Perv_G(X)$.

(ii) Let $H \subset G$ *be a closed complex normal subgroup, acting freely on* X, $B = H\backslash G$. *Then the quotient equivalence* $q^* : D^b_B(H\backslash X) \to D^b_G(X)$ *induces the equivalence* $q* : Perv_B(H\backslash X) \to Perv_G(X)[-d_H]$ *(2.6.2).*

(iii) Let $H \subset G$ *be a closed complex subgroup,* X *- a complex* H*-variety,* $Y = G \times_H X$ *and* $\nu : X \to Y$ *the obvious inclusion. Then the induction equivalence* $\nu^* : D^b_G(Y) \to D^b_H(X)$ *induces the equivalence* $Perv_G(Y) \to Perv_H(X)[d_G - d_H]$ *(2.6.3).*

Proof. (i) and (ii) are obvious, since all functors commute with the forgetful functor and the category $Perv_G(X)$ is defined in terms of $Perv(X)$.

(iii) Note that it suffices to work only with complex smooth resolutions $P \to X$ (for example using complex Stiefel manifolds as in 3.1). If P is such a resolution, then an object $F \in D^b_{H,c}(X)$ lies in $Perv_H(X)$ iff the object $F(P) \in D^b(\overline{P})$ lies in $Perv(\overline{P})[d_X + d_H - d_P]$. This implies (iii).

5.2. The equivariant intersection cohomology sheaf.

Let $j : V \hookrightarrow X$ be the inclusion of a smooth locally closed irreducible G-invariant subset, and $\mathcal{L} \in Sh_G(V)$ be a G-equivariant local system on V. Consider the intermediate extension $j_{!*}\mathcal{L}[d_V] \in Perv_G(X)$, where d_V is the complex dimension of V ([BBD]). We call this extension the equivariant intersection cohomology sheaf $IC_G(\overline{V},\mathcal{L})$. In case $\overline{V}=X$ and the local system is trivial $\mathcal{L}=C_V\in Sh_G(V)$ we denote it by $IC_G(X)$. As in the nonequivariant case one can show that simple objects in $Perv_G(X)$ are exactly the intersection cohomology sheaves $IC_G(\overline{V},\mathcal{L})$, for an irreducible local system $\mathcal{L}\in Sh_G(V)$.

Remark. Note that the quotient and the induction equivalences of proposition 5.1 (ii),(iii) preserve the equivariant intersection cohomology sheaves (up to a shift).

5.3. Decomposition theorem.

An object $F\in D^b_G(X)$ is called semisimple if it is isomorphic to a direct sum of objects $L_i[n_i]$, for some irreducible perverse sheaves $L_i\in D^b_G(X)$.

Theorem. *Let $f: X\to Y$ be a proper G-map of complex algebraic varieties. Let $F\in D^b_G(X)$ be a semisimple object. Then its direct image $H=f_*(F)\in D^b_G(Y)$ is semisimple.*

Proof. Choose a large enough segment $I\subset \mathbf{Z}$ such that $H\in D^I_G(Y)$ (see 2.2). Choose a smooth complex n-acyclic resolution $p: P\to Y$ with $n>|I|$. Then the category $D^I_G(Y)$ is by definition equivalent to the category $D^I_G(X,P)$. By lemma 2.3.2 this category is equivalent to the full subcategory $D^I(\overline{P}|p)\subset D^I(\overline{P})$.

Using the usual decomposition theorem (see [BDD]), we deduce that the object $H\in D^I(\overline{P})$ is semisimple, i.e. is of the form $\oplus H_i$, $H_i\simeq L_i[n_i]$, where L_i are simple perverse sheaves in $D^b(\overline{P})$. Since the subcategory $D^I(\overline{P}|p)\subset D^I(\overline{P})$ is closed with respect to direct summands, all objects H_i lie in this subcategory, which gives the decomposition $H\simeq\oplus H_i$ in $D^I_G(X,P)\simeq D^I_G(X)$. It is clear that every H_i in $D^b_G(X)$ has the form $H_i\simeq L_i[n_i']$, where L_i are irreducible perverse sheaves. This proves the theorem.

6. General inverse and direct image functors Q^*, Q_*.

6.0. Let $\phi : H \to G$ be a homomorphism of topological groups and $f : X \to Y$ be a ϕ-map of of topological spaces (0.1). In this situation we will define functors $Q^* : D^b_G(Y) \to D^b_H(X)$ $(D^+_G(Y) \to D^+_H(X))$ and $Q_* : D^+_H(X) \to D^+_G(Y)$. Many of the functors defined earlier are special cases of these general functors (see 6.6, 6.12 below).

6.1. Assume that the groups H, G satisfy the condition (*) in 2.2.4 (for example, they may be Lie groups). As usual, denote by $Res(X) = Res(X, H)$ and $Res(Y) = Res(Y, G)$ the categories of resolutions of X and Y (2.1.2). We interpret the categories $D^b_H(X)$ and $D^b_G(Y)$ as fibers of the fibered category $D^b/\mathcal{T}$ over the functor Φ (2.4.3).

6.2. Definition. Let $P \to X$, $R \to Y$ be resolutions, and $f : P \to R$ be a ϕ-map, such that the diagram

$$\begin{array}{ccc} X & \longleftarrow & P \\ \downarrow f & & \downarrow f \\ Y & \longleftarrow & R \end{array}$$

is commutative. Then we call resolutions P and R **compatible.**

The following construction produces many compatible resolutions.

Consider the bifunctor

$$\times_f : Res(X) \times Res(Y) \to Res(X), \quad (S, R) \mapsto S \times_X f^0(R).$$

Indeed, $f^0(R) = X \times_Y R$ is naturally an H-space and the projection $f^0(R) \to X$ is an H-map. Note that if $R \to Y$ is n-acyclic then $f^0(R) \to X$ is also n-acyclic. However, $f^0(R)$ is not a free H-space in general, hence not a resolution of X.

We have the obvious map of quotients $\overline{f} : \overline{S \times_f R} \to \overline{R}$, induced by the projection $f : S \times_f R \to R$.

Remarks. 1. If S, R are n-acyclic, then $S \times_f R$ is also such.

2. The trivial resolutions $H \times X \to X$ and $G \times Y \to Y$ are naturally compatible.

3. If $P_1 \to R_1$ and $P_2 \to R_2$ are compatible resolutions then $P_1 \times_X P_2 \to R_1 \times_Y R_2$ are also compatible.

6.3. Definition. A resolution $P \in Res(X)$ is *compatible* (with the map $f : X \to Y$) if it fits into a compatible pair $P \to R$ (6.2). A morphism $P_1 \to P_2$ between compatible resolutions is *compatible* if it fits into a commutative square

$$\begin{array}{ccc} P_1 & \longrightarrow & P_2 \\ \downarrow & & \downarrow \\ R_1 & \longrightarrow & R_2 \end{array}$$

where columns are compatible pairs. Denote by $CRes(X) \subset Res(X)$ the subcategory of compatible resolutions and compatible morphisms.

6.4. Remark. It follows from remarks in 6.2 and proposition 2.4.4 that the category $CRes(X)$ is rich enough to define an object in $D^b_H(X)$.

6.5. Definition of Q^*_f. Let $F \in D^b_G(Y)$ and $P \in CRes(X)$. Let $P \to R$ be a compatible pair of resolutions and $\overline{f} : \overline{P} \to \overline{R}$ be the induced map of quotients. Define

$$Q^*_f F(P) := \overline{f}^* F(R).$$

We must check that the value of $Q^*_f F$ on P is well defined (i.e. is independent of the choice of R), and that $Q^*_f F$ is an object on $D^b_H(X)$.

Let $P \to R'$ be a different compatible pair with the induced map $\overline{f}' : \overline{P} \to \overline{R}'$. Then $P \to R \times_Y R'$ is also a compatible pair, which shows that objects $\overline{f}^* F(R)$ and $\overline{f}^* F(R')$ in $D^b(\overline{P})$ are canonically isomorphic.

Let $\nu : P_1 \to P_2$ be a morphism in $CRes(X)$. We can complete it to a diagram

$$\begin{array}{ccc} P_1 & \longrightarrow & P_2 \\ \downarrow & & \downarrow \\ R_1 & \longrightarrow & R_2 \end{array}$$

as in definition 6.3 above. This shows that objects $\nu^*(Q^*_f F(P_2))$ and $Q^*_f F(P_1)$ are canonically isomorphic. Hence $Q^*_f F$ is a well defined object in $D^b_H(X)$.

6.6. Properties of Q^*.

The properties listed below follow immediately from the definitions.

1. The functor $Q^* : D^b_G(Y) \to D^b_H(X)$ is exact and preserves the t-structure, i.e.

$$Q^* : Sh_G(Y) \to Sh_H(X).$$

2. Let B be another topological group satisfying the condition (*) in 2.2.4. Let $\psi : G \to B$ be a homomorphism and $g : Y \to Z$ be a ψ-map. Then we have a canonical isomorphism of functors

$$Q^*_f \cdot Q^*_g = Q^*_{gf}.$$

3. Suppose that $H = G$ and $\phi = id$. Then Q^* is canonically isomorphic to the inverse image functor f^* in 2.2.4, 3.3.

4. Suppose that $\phi : H \hookrightarrow G$ is an embedding of a subgroup. Then Q^* is canonically isomorphic to the inverse image f^* in 2.6.1. In particular if $X = Y$, we have $Q^* = Res_{H,G}$, and if moreover $H = \{e\}$, then $Q^* = For$.

5. Suppose that $\phi : H \to G$ is surjective with the kernel K, such that X is a free K-space and $q : X \to Y$ is the quotient map by the action of K. Then Q_q^* is naturally isomorphic to the quotient equivalence $q^* : D_G^b(Y) \simeq D_H^b(X)$ in 2.6.2.
6. Suppose that $\phi : H \to G$ is injective, $Y = G \times_H X$ and $\nu : X \to Y, \quad x \mapsto (e, x)$. Then Q_ν^* is naturally isomorphic to the induction equivalence $\nu^* : D_G^b(Y) \simeq D_H^b(X)$.

6.7. Let us assume that the groups H, G satisfy the condition (**) in (2.9). Then we can use the definition of D_H^+, D_G^+ in 2.9.1 to define the inverse image functor

$$Q_f^* : D_G^+(Y) \to D_H^+(X),$$

by replacing everywhere in 6.5 the category D^b by D^+.

Alternatively, let $P \to R$ be compatible ∞-acyclic resolutions of X and Y respectively, and $\overline{f} : \overline{P} \to \overline{R}$ be the induced map of quotients. By lemma 2.9.2 the categories $D_H^+(X)$ and $D_G^+(Y)$ are naturally identified as certain full subcategories in $D^+(\overline{P})$ and $D^+(\overline{R})$ respectively. Under this identification we have

$$Q_f^* = \overline{f}^* : D^+(\overline{R}) \to D^+(\overline{P}).$$

All the remarks in 6.6 apply also to D^+.

6.8. Let us define the direct image $Q_{f*} : D_H^+(X) \to D_G^+(Y)$. For simplicity we assume that H, G are Lie groups (and hence satisfy the condition (**) in 2.9).

First we need some local terminology.

Let $p : W \to Z$ be a continuous map of topological spaces. We call p a **good** map if p is locally fibered (1.4.7) with a locally acyclic fiber. This means that for every point $w \in W$ there exist neighbourhoods U of w in W and V of $z = p(w)$ in Z such that $U \simeq F \times V$ where F is acyclic and p is the projection.

Let $GRes(Y) \subset Res(Y, G)$ be the subcategory consisting of good resolutions $r : R \to Y$ (i.e. the map r is good) and good morphisms between them. It follows from our assumptions on the group G that the category $GRes(Y)$ is rich enough to define the category $D_G^+(Y)$ (2.4.4, 2.9.6). Indeed, let N be a locally acyclic free G-space. Then the following diagram of resolutions lies in $GRes(Y)$:

$$G \times Y \leftarrow G \times Y \times N \to Y \times N,$$

(all maps are projections).

Note that if $R_1 \to R_2$ is a morphism in $GRes(Y)$, then the induced map $\overline{R_1} \to \overline{R_2}$ is also good.

6.9. Definition of Q_{f*}. Let $F \in D_H^+(X)$ and $R \in GRes(Y)$. Choose an ∞-acyclic resolution $P \to X$. Consider the compatible pair of resolutions (6.2)

$$P \times_f R \to R$$

and the induced map of quotients

$$\overline{f} : \overline{P \times_f R} \to \overline{R}.$$

We define

$$Q_{f*}F(R) := \overline{f}_* F(P \times_f R).$$

We must check that the value of $Q_{f*}F$ on R is well defined (i.e. does not depend on the choice of P), and that $Q_{f*}F$ is an object in $D^+_G(Y)$.

Let $P' \to X$ be a different ∞-acyclic resolution. Consider the commutative diagram

$$\begin{array}{ccc} \overline{(P \times_X P') \times_f R} & \xrightarrow{t} & \overline{P' \times_f R} \\ \downarrow s & & \downarrow \overline{f'} \\ \overline{P \times_f R} & \xrightarrow{\overline{f}} & \overline{R} \end{array}$$

where all maps are induced by the obvious projections. Put $\overline{F}_P = F(P \times_f R)$, $\overline{F}_{P'} = F(P' \times_f R)$. By definition, we have a canonical isomorphism

$$s^* \overline{F}_P = t^* \overline{F}_{P'}.$$

Note that morphisms s, t are ∞-acyclic, and hence $t_* \cdot t^* = id$, $s_* \cdot s^* = id$ (Proposition 1.9.2(i)). So we have canonical isomorphisms

$$t_* s^* \overline{F}_P = \overline{F}_{P'},$$

$$s_* s^* \overline{F}_P = \overline{F}_P,$$

and therefore a canonical isomorphism

$$\overline{f}_* \overline{F}_P = \overline{f'}_* \overline{F}_{P'},$$

which shows that $Q_{f*}F(R)$ is independent of the choice of P.

Let $g : S \to R$ be a good morphism of resolutions. Consider the pullback diagram

$$\begin{array}{ccc} \overline{P \times_f S} & \xrightarrow{\overline{g}} & \overline{P \times_f R} \\ \downarrow \overline{f} & & \downarrow \overline{f} \\ \overline{S} & \xrightarrow{\overline{g}} & \overline{R} \end{array}$$

In order for $Q_{f*}F$ to be a well defined object in $D^+_G(Y)$ it suffices to show that the base change morphism

$$\overline{g}^* \overline{f}_* \overline{F} \to \overline{f}_* \overline{g}^* \overline{F}$$

is an isomorphism, where $\overline{F} = F(P \times_f R) \in D^+(\overline{P \times_f R})$. Since $\overline{g}$ is a good map this follows from the good base change lemma C.1 proved in Appendix C below. So $Q_{f*}F$ is an object in $D^+_G(Y)$.

6.10. Let us give an alternative description of the functor Q_{f*}. Choose a compatible pair $S \to R$ of ∞-acyclic resolutions (6.2) such that $R \in GRes(Y)$. Let $\overline{f} : \overline{S} \to \overline{R}$ be the induced map of quotients. Recall that categories $D^+_H(X)$ and $D^+_G(Y)$ are canonically identified as certain full subcategories in $D^+(\overline{S})$ and $D^+(\overline{R})$ (2.9.2). After this identification we have

$$Q_{f*} = \overline{f}_* : D^+(\overline{S}) \to D^+(\overline{R})$$

(cf. 6.7 above).

6.11. Example. Let $X = Y = pt$. Let

$$\overline{\phi} : BH \to BG$$

be the map of classifying spaces induced by the homomorphism $\phi : H \to G$. By proposition 2.9.5 the categories $D^+_H(pt)$ and $D^+_G(pt)$ are naturally realized as full subcategories of $D^+(BH)$ and $D^+(BG)$ consisting of complexes with locally constant cohomology. After this identification we have

$$Q^* = \overline{\phi}^* : D^+(BG) \to D^+(BH),$$

$$Q_* = \overline{\phi}_* : D^+(BH) \to D^+(BG).$$

6.12. Properties of Q_*.

1. The functor Q_* is the right adjoint to $Q^* : D^+_G(Y) \to D^+_H(X)$.

Indeed, this is clear from 6.7 and 6.10.

2. Let B be another Lie group and $\psi : G \to B$ be a homomorphism. Let $g : Y \to Z$ be a ψ-map. Then

$$Q_{g*}Q_{f*} = Q_{gf*}.$$

This is clear.

3. Let $K \subset H$ be the kernel of the homomorphism ϕ. Assume that X is a free K-space. For example, ϕ may be injective. The Q_* commutes with the forgetful functor. In particular, if X and Y are nice topological spaces, then Q_* preserves the bounded equivariant category $Q_* : D^b_H(X) \to D^b_G(X)$.

Indeed, let $R \to Y$ be a good ∞-acyclic resolution. Then the H-space $f^0(R) = X \times_Y R$ is free, and hence is an ∞-acyclic resolution of X. So we can use the map $\overline{f} : \overline{f^0(R)} \to \overline{R}$ to define the direct image $Q_* = \overline{f}_*$. Consider the commutative diagram, where both squares are cartesian

$$\begin{array}{ccccc} X & \longleftarrow & f^0(R) & \longrightarrow & \overline{f^0(R)} \\ \downarrow f & & \downarrow f & & \downarrow \overline{f} \\ Y & \longleftarrow & R & \longrightarrow & \overline{R} \end{array} \quad .$$

Since horizontal arrows are good morphisms (6.8) we conclude by the good base change lemma C1 that Q_* commutes with the forgetful functor.

4. Suppose that $H = G$, $\phi = id$ and $f : X \to Y$ be a G-map of nice topological spaces. Then Q_* preserves the bounded category D^b_G and is naturally isomorphic to f_* in 3.3.

This follows immediately from the property 3 above, since both f_* and Q_* are right adjoint to the same functor $f^* = Q^* : D^b_G(Y) \to D^b_H(X)$.

5. Suppose that $\phi : H \to G$ is injective. Let $X = Y$ and $f = id$. Then Q_* preserves the bounded category D^b and is canonically isomorphic to the integration functor Ind_* (3.7).

This follows immediately from the properties 1,3 above and from 6.6(4).

6. In the situation of the quotient or the induction equivalence (6.6 (5,6)) the functor Q_* preserves the bounded category D^b and is equal to Q^{*-1}.

Indeed, in both cases the functor Q_{f*} preserves the bounded category D^b (see 3 above) and hence is equal to Q_f^{*-1} by 1 above and 6.6(5, 6).

7. Some relations between functors.

We establish some relations between the earlier defined functors that we found useful. Roughly speaking, subsections 7.1 - 7.3 contain some commutativity statements, and in 7.4 -7.6 we discuss the behavior of the quotient and the induction equivalences with respect to the Verdier duality.

In this section we assume for simplicity that all groups are Lie groups satisfying the condition (*+) in 3.1 and all spaces are nice (1.4).

7.1. Theorem. (Smooth base change) *Let $\phi : H \to G$ be a homomorphism of groups. Consider the pullback diagram*

$$\begin{array}{ccc} \tilde{X} & \xrightarrow{g} & X \\ \downarrow f & & \downarrow f \\ \tilde{Y} & \xrightarrow{g} & Y \end{array}$$

where $g : \tilde{X} \to X$ (resp. $g : \tilde{Y} \to Y$) is a smooth map of H-spaces (resp. G-spaces) and f is a ϕ-map. Then

(i) All functors $\otimes$, Hom, f^*, f_*, $f^!$, $f_!$, Q_f^*, Q_{f*} between categories D_H^b, D_G^b, (or D_H^+, D_H^+) when defined commute with the smooth base change g^*.

(ii) The functor g^* commutes with the Verdier duality $D : D_H^b(X) \to D_H^b(X)$ up to the twist by the dualizing object $D_{g,H} \in D_H^b(\tilde{X})$ (3.4.2(1)), i.e.

$$D \cdot g^* = D_{g,H} \otimes g^* \cdot D.$$

Proof. This is nothing but the smooth base change (1.8). When ∞-dimensional spaces are involved (functor Q_{f*}) one may use the good base change lemma C1.

7.2. Proposition. *Let $\phi : H \to G$ be a homomorphism of groups and $f : X \to Y$ be a G-map of G-spaces. Consider X as an H-space and $id : X \to X$ as a ϕ-map. Let $Q^* = Q_{id}^* : D_G(X) \to D_H(X)$ be the corresponding inverse image. We have a similar inverse image for the space Y.*

Then all functors $\otimes$, Hom, f^*, f_*, $f^!$, $f_!$, D, Q_f^*, Q_{f*} between categories D_G, D_H when defined commute with the inverse image Q^*. In particular Q^* preserves the dualizing objects.

For example, if H is a subgroup in G then all the above functors commute with the restriction functor $Res_{H,G} = Q^*$.

Proof. In this case $Q_f^* = f^*$, $Q_{f*} = f_*$, so it suffices to consider the first seven functors defined in section 3.

Let N be a free smooth G-space, M' be a free smooth H-space. Later on we can assume that N, M' are sufficiently acyclic. Put $M = M' \times N$ - free smooth

H-space. Denote by $P_G = N \times X \to X$, $P_H = M \times X \to X$, $R_G = N \times Y \to Y$, $R_H = M \times Y \to Y$ the corresponding smooth G- and H-resolutions of X and Y. They form the pullback diagram

$$\begin{array}{ccc} P_H & \longrightarrow & P_G \\ \downarrow & & \downarrow \\ R_H & \longrightarrow & R_G \end{array}$$

where horizontal arrows are projections and vertical ones are induced by f.

This induces the pullback diagram of quotients

$$\begin{array}{ccc} \overline{P}_H & \xrightarrow{g} & \overline{P}_G \\ \downarrow \overline{f} & & \downarrow \overline{f} \\ \overline{R}_H & \xrightarrow{g} & \overline{R}_G \end{array}$$

where the map g is smooth. By the smooth base change (1.8) the functor g^* commutes with all functors $\otimes$, Hom, $\overline{f}^*$, $\overline{f}_*$, $\overline{f}^!$, $\overline{f}_!$. But g^* represents the inverse image Q^* on the given resolutions. Hence Q^* commutes with the corresponding functors in the equivariant category.

Let us prove that Q^* commutes with the Verdier duality. It suffices to show that Q^* preserves the dualizing objects.

Consider the pullback diagram

$$\begin{array}{ccc} \overline{P}_H & \xrightarrow{g} & \overline{P}_G \\ \downarrow p & & \downarrow p \\ \overline{M} & \xrightarrow{g} & \overline{N} \end{array}$$

Then by definition $D_{X,G}(P_G) = p^! C_{\overline{N}}$, $D_{X,H}(P_H) = p^! C_{\overline{M}}$. Hence again by the smooth base change

$$g^* D_{X,G}(P_G) = D_{X,H}(P_H)$$

and so $Q^* D_{X,G} = D_{X,H}$. This proves the proposition.

7.3. Let us prove another base change theorem. Let $\phi : H \to G$ be a homomorphism of groups. Let $G' \subset G$ be a closed subgroup, and $H' = \phi^{-1}(G') \subset H$ be its preimage in H. So we get a commutative diagram of group homomorphisms

$$\begin{array}{ccc} H' & \longrightarrow & H \\ \downarrow & & \downarrow \\ G' & \longrightarrow & G \end{array}$$

Let $f : X \to Y$ be a ϕ-map. We get a diagram of functors

$$\begin{array}{ccc} D^+_{H'}(X) & \xleftarrow{Res_{H',H}} & D^+_H(X) \\ \downarrow Q_{f*} & & \downarrow Q_{f*} \\ D^+_{G'}(Y) & \xleftarrow{Res_{G',G}} & D^+_G(Y) \end{array}$$

Theorem. *Assume that ϕ induces an isomorphism $H/H' \simeq G/G'$. Then there is a natural isomorphism of functors*

$$Q_{f*} \cdot Res_{H',H} = Res_{G',G} \cdot Q_{f*}$$

from $D^+_H(X)$ to $D^+_{G'}(Y)$.

Proof. Consider the H-space $H \times_{H'} X$ with the H-map

$$g : H \times_{H'} X \to X, \quad (h,x) \mapsto hx,$$

and the H'-map

$$\nu : X \to H \times_{H'} X, \quad x \mapsto (e,x).$$

Similarly for Y with G and G'.

The restriction functor $Res_{H',H} : D^+_H(X) \to D^+_{H'}(X)$ is isomorphic to the composition of the inverse image

$$g^* : D^+_H(X) \to D^+_H(H \times_{H'} X)$$

with the induction equivalence

$$\nu^* = Q^*_\nu : D^+_H(H \times_{H'} X) \to D^+_{H'}(X).$$

The induction equivalence Q^*_ν commutes with the direct image Q_{f*} because $(Q^*_\nu)^{-1} = Q_{\nu*}$. So it remains to prove that the inverse image g^* commutes with Q_{f*}.

We have a commutative diagram

$$\begin{array}{ccc} H \times_{H'} X & \xrightarrow{g} & X \\ \downarrow f & & \downarrow f \\ G \times_{G'} Y & \xrightarrow{g} & Y \end{array}$$

where f is the ϕ-map and g is an H-map (resp. a G-map). Because of our assumption $H/H' = G/G'$ it follows that this is a pullback diagram. Since g is smooth, the assertion follows from 7.1.

7.4. Definition. Let $\phi : H \to G$ be a homomorphism of groups and $f : X \to Y$ be a ϕ-map. Consider Y as an H-space via the homomorphism ϕ and define the dualizing object $D_f = D_{f,H} \in D^b_H(X)$ of the map f as

$$D_f := f^! C_{Y,H}.$$

7.4.1. Lemma. *In the above notations assume that the map f is smooth. Then the inverse image*

$$Q^*_f : D^b_G(Y) \to D^b_H(X)$$

commutes with the Verdier duality up to a twist by the invertible object D_f, i.e.

$$D \cdot Q_f^* = D_f \otimes Q_f^* \cdot D.$$

Proof. Follows immediately from 7.1, 7.2.

7.5. We know that the quotient equivalence q^* commutes with the functors $\otimes$, Hom, f^*, f_*, $f^!$, $f_!$ (3.4.1). Since q^* is the inverse image (and q^{*-1} is the direct image) it also commutes with the functors Q^* and Q_* in the appropriate setting. We claim that q^* commutes with the Verdier duality up to a twist by the invertible dualizing object D_q (7.4).

Namely, let $0 \to H \to G \xrightarrow{\phi} B \to 0$ be an exact sequence of groups. Let X be a G-space which is free as an H-space. Consider $\overline{X} = H\backslash X$ as a B-space and the quotient morphism $q : X \to \overline{X}$ as a ϕ-map. The following proposition is a special case of lemma 7.4.1 above.

7.5.1. Proposition. *The quotient equivalence*

$$q^* : D_B^b(\overline{X}) \simeq D_G^b(X)$$

commutes with the Verdier duality D up to a twist by D_q (7.4). Namely

$$D \cdot q^* = D_q \otimes q^* \cdot D.$$

For a manifold M denote by d_M its dimension.

7.5.2. Proposition. *Assume that in the previous proposition the group G is connected. Then the dualizing object D_q is the constant sheaf shifted by the dimension of H*

$$D_q = C_{X,G}[d_H]$$

and hence

$$D \cdot q^* = (q^* \cdot D)[d_H].$$

That is the quotient equivalence commutes with the Verdier duality up to the shift by d_H.

Proof. Put $\overline{X} = Y$. Consider both X, Y as G-spaces and $f : X \to Y$ as a G-map. Let $P \to Y$ be a smooth resolution of Y and $f^0(P) \to X$ the induced resolution of X. Consider the obvious pullback diagram

$$\begin{array}{ccc} f^0(P) & \xrightarrow{q} & \overline{f^0(P)} \\ \downarrow f & & \downarrow \overline{f} \\ P & \xrightarrow{q} & \overline{P} \end{array}$$

We must show that

$$\overline{f}^! C_{\overline{P}} = C_{\overline{f^o(P)}}[d_H].$$

Since $\overline{f}$ is a smooth map we know that $\overline{f}^! C_{\overline{P}} = D_{\overline{f}}$ is invertible (1.4.7, 1.6.1) and locally isomorphic to $C_{\overline{f^o(P)}}[d_H]$. The map q is a smooth fibration with connected fibers (=G). Hence it suffices to show that the local system $q^*\overline{f}^! C_{\overline{P}}$ is trivial. By the smooth base change

$$q^*\overline{f}^! C_{\overline{P}} = f^! q^* C_{\overline{P}} = f^! C_P.$$

But the map f is a principal H-bundle, so $f^! C_P = C_{f^o(P)}[d_H]$ by the following lemma.

7.5.3. Lemma. *Let T be a Lie group and $g : Z \to W$ be a principal T-bundle. Then*

$$g^! C_W = C_Z[d_T].$$

Proof. The fibers of g are orientable (=T) and we claim that one can orient the fibers in a compatible way. Indeed, the transition functions in the principal T-bundle are given by the (right) multiplication by elements of T, which preserves the orientation of the fibers.

7.5.4. Corollary. *Under the assumptions of the previous proposition 7.5.2 we have a canonical isomorphism of functors*

$$Q_{q*} \cdot D = (D \cdot Q_{q*})[d_H].$$

Proof. Indeed, $Q_{q*} = q^{*-1}$ (6.12(6)). So apply 7.5.2.

7.6. We know that the induction equivalence (2.6.3) commutes with all functors in section 3 except for the Verdier duality (3.4.1). Here we prove that it commutes with the duality up to a twist by the invertible object D_ν (7.4).

Namely, let $\phi : H \hookrightarrow G$ be an embedding of a closed subgroup and X be an H-space. Consider the induced G-space $Y = G \times_H X$ and the natural ϕ-map $\nu : X \hookrightarrow Y$.

7.6.1. Proposition. *The induction equivalence*

$$Q_\nu^* : D_G^b(Y) \simeq D_H^b(X)$$

commutes with the Verdier duality D up to a twist by the invertible dualizing object $D_\nu \in D^b_H(X)$:

$$D \cdot Q^*_\nu = D_\nu \otimes Q^*_\nu \cdot D.$$

Proof. Since the embedding $\nu : X \hookrightarrow Y$ is relatively smooth (1.4.7(2)), the dualizing object D_ν is indeed invertible.

Since Q^*_ν commutes with Hom it suffices to prove that

$$D_{X,H} = D_\nu \otimes Q^*_\nu D_{Y,G}.$$

By proposition 7.2 $Q^* D_{Y,G} = D_{Y,H}$. So the desired identity is

$$D_{X,H} = D_\nu \otimes \nu^* D_{Y,H},$$

which is the equivariant analogue of 1.4.7(2) (see theorem 3.4.3). This proves the proposition.

7.6.2. Proposition. *Assume that in the previous proposition the group H is connected. Then the dualizing object D_ν is the constant sheaf shifted by the difference of dimensions of H and G*

$$D_\nu = C_{X,H}[d_H - d_G]$$

and hence

$$D \cdot Q^*_\nu = (Q^*_\nu \cdot D)[d_H - d_G].$$

That is the induction equivalence commutes with the Verdier duality up to the shift by $d_H - d_G$.

Proof. Consider the obvious pullback diagram

$$\begin{array}{ccc} X & \xrightarrow{\nu} & Y \\ \downarrow p & & \downarrow p \\ pt & \xrightarrow{i} & G/H \end{array}$$

where $p : Y \to G/H$ is a locally trivial fibration with fiber X. By the equivariant analogue of 1.4.7(3) we have

$$\nu^! \cdot p^* = p^* \cdot i^!.$$

Hence it suffices to show that

$$i^! C_{G/H,H} = C_{pt,H}[d_H - d_G].$$

Note that G/H is a manifold of dimension $d_G - d_H$. Hence $i^! C_{G/H,H} \in D^b_H(pt)$ is an invertible equivariant sheaf concentrated in degree $d_G - d_H$. But H is connected, hence it is actually constant.

7.6.3. Corollary. *Under the assumptions of the previous proposition 7.6.2 we have a canonical isomorphism of functors*

$$D \cdot Q_{\nu *} = (Q_{\nu *} \cdot D)[d_G - d_H].$$

Proof. Indeed, $Q_{\nu *} = (Q_\nu^*)^{-1}$ (6.12(6)). So apply 7.6.2.

Appendix C.

In this appendix we prove the good base change lemma.

C1. Lemma. *Consider a pullback diagram of continuous maps of topological spaces*

$$\begin{array}{ccc} \tilde{X} & \xrightarrow{g} & X \\ \downarrow f & & \downarrow f \\ \tilde{Y} & \xrightarrow{g} & Y, \end{array}$$

where g is a good map (6.8). Then the base change morphism of functors

$$g^* \cdot f_* \to f_* \cdot g^*$$

is an isomorphism.

Proof. Let $S \in D^+(X)$ and $z \in \tilde{Y}$. We must prove that the induced map on stalks

$$(g^* f_* S)_z \to (f^* g_* S)_z$$

is an isomorphism.

Since g is a good map, there exists a fundamental system of neighbourhoods U of z such that $U = F \times V$, where F is acyclic and $g : U \to V$ is the projection. It suffices to show that over such U we have a quasiisomorphism of complexes

$$(*) \qquad g^* f_* S(U) \simeq f_* g^* S(U).$$

Let us compute both sides in (*). We may assume that the complex S consists of injective sheaves and that $Y = V$, $\tilde{Y} = U$. Since F is connected the cohomology on the left hand side of (*) is

$$H^i(\Gamma(g^* f_* S)) = H^i(\Gamma(f_* S)) = H^i(X, S).$$

On the other hand, since tha map $g : \tilde{X} \to X$ is ∞-acyclic (1.9.4, 1.9.2) the complex $g^* S$ consists of Γ-acyclic sheaves and we have

$$H^i(\Gamma(f_* g^* S)) = H^i(\Gamma(g^* S)) = H^i(X, S).$$

This proves the lemma.

8. Discrete groups and functors.

In this section all groups are assumed to be **discrete**, except in subsection 8.7.

8.0. Let G be a group and X be a G-space. As we mentioned in section 0, a G-equivariant sheaf F on X in this case is simply a sheaf with an action of G which is compatible with its action on X. As usual $Sh_G(X)$ denotes the abelian category of G-equivariant sheaves (of R-modules) on X. This category was studied by A. Grothendieck in [Groth] who showed in particular that $Sh_G(X)$ has enough injectives. He also considered some functorial properties of $Sh_G(X)$ and our discussion here is a variation on the theme of [Groth]. The main point is that we have a natural equivalence of categories

$$D(Sh_G(X)) \simeq D_G(X),$$

where D stands for D^b or D^+ (8.3.1). In other words in the case of a discrete group G the "naive" category $D(Sh_G(X))$ is good enough. Our main objective here is to give a different description of the direct image Q_* in this case (8.4.2) and to study its properties when the action is almost free. In section 9 we apply these results to some actions of algebraic groups.

Notice that given a discrete group G and a G-map $X \to Y$ (of locally compact G-spaces), it is not absolutely clear how to extend the usual functors like $f^!$ to the derived categories $D^b(Sh_G(X))$, $D^b(Sh_G(Y))$. So it may be still useful to work with the equivalent category D^b_G and to use the definition of functors in section 3 above.

8.1. Let $\phi : H \to G$ be a homomorphism of groups and $f : X \to Y$ be a ϕ-map. It induces a natural inverse image functor

$$f^* : Sh_G(Y) \to Sh_H(X).$$

Namely, if $F \in Sh_G(Y)$, then $f^*F \in Sh(X)$ is a sheaf associated to the presheaf

$$f^0F : U \mapsto F(f(U)), \quad U \text{ open in } X.$$

The group G acts naturally on the presheaf f^0F (and hence on the sheaf f^*F) by the formula

$$g : F(f(U)) \xrightarrow{\phi(g)} F(f(V)),$$

where $g(U) = V$.

Assume that the homomorphism ϕ is surjective with the kernel K. Let $S \in Sh_H(X)$. Then the direct image $f_*S \in Sh(Y)$ is naturally an H-equivariant sheaf on Y, considered as an H-space. Hence its subsheaf of K-invariants $(f_*S)^K = f_*^K S$ is naturally a G-equivariant sheaf on Y.

8.1.1. Proposition. *Let $\phi : H \to G$ be a surjective homomorphism of groups with the kernel K. Let X be an H-space which is free as a K-space. Let $Y = K\backslash X$ and*

$f : X \to Y$ be the quotient map. Clearly, Y is a G-space and f is a ϕ-map. Then $f^* : Sh_G(Y) \to Sh_H(X)$ is an equivalence of categories, and the inverse functor is f_*^K.

Proof. One checks immediately that the natural morphisms of functors

$$Id_{Sh_G(Y)} \to f_*^K f^*,$$

$$f^* f_*^K \to Id_{Sh_H(X)}$$

are isomorphisms.

8.2. Let $H \subset G$ be a subgroup and X be a G-space. Consider the natural restriction functor

$$Res_G^H : Sh_G(X) \to Sh_H(X).$$

8.2.1 Definition. Let $S \in Sh_H(X)$. We say that $F \in Sh_G(X)$ is **induced** from S, $F = Ind_H^G(S)$ if

(a) S is a subsheaf of F,

(b) $F = \oplus_{s \in G/H} sS$.

Clearly,

$$\mathrm{Hom}_{Sh_G}(Ind_H^G(S), F') = \mathrm{Hom}_{Sh_H}(S, Res_G^H F')$$

for every $F' \in Sh_G(X)$.

8.2.2. Lemma. *For every $S \in Sh_H(X)$ there exists a unique induced sheaf $Ind_H^G(S) \in Sh_G(X)$. The functor*

$$Ind_H^G : Sh_H(X) \to Sh_G(X)$$

is exact.

Proof. Consider the $G \times H$-space $G \times X$ with the action $(g,h)(g',x) = (gg'h^{-1}, hx)$ and the projection $p : G \times X \to X$. Then p^*S is naturally a $G \times H$-equivariant sheaf on $G \times X$. Since $G \times X$ is a free H-space, by proposition 8.1.1 there exists a unique G-equivariant sheaf S' on the G-space $G \times_H X$ such that $p^*S = q^*S'$, where $q : G \times X \to G \times_H X$ is the quotient map by H. Let $m : G \times_H X \to X$ be the G-map $(g,x) \mapsto gx$. Note, that $G \times_H X = \coprod_{s \in G/H} (s, X)$, where (s, X) is homeomorphic to X. Consider the exact functor $m_! : Sh_G(G \times_H X) \to Sh_G(X)$. Then $m_! S' = \oplus_{s \in G/H} sS = Ind_H^G(S)$ is the desired induced sheaf. The uniqueness of the induced sheaf is obvious. This proves the lemma.

8.2.3. Corollary. *The restriction functor Res_G^H (8.2) has a left adjoint exact induction functor Ind_H^G.*

8.2.4. Corollary. *The restriction functor Res^H_G maps injective sheaves to injective. In particular, the forgetful functor $Res^{\{e\}}_G = For : Sh_G(X) \to Sh(X)$ preserves injectives.*

Proof. Indeed, the functor is the right adjoint to the exact functor Ind^G_H.

8.2.5. Corollary. *The functor $Ind^G_H : D(Sh_H(X) \to D(Sh_G(X))$ is the left adjoint to $Res^H_G : D(Sh_G(X)) \to D(Sh_H(X))$, where D denotes D^b or D^+, and the functors are the trivial extensions of the corresponding exact functors between the abelian categories.*

Proof. This follows immediately from 8.2.3 and 8.2.4.

8.2.6. Proposition. *The induction functor Ind commutes with the inverse image. Namely, let $\phi : G_1 \to G_2$ be a homomorphism and $f : X \to Y$ be a ϕ-map. Let $H_2 \subset G_2$ be a subgroup and $H_1 = \phi^{-1}(H_2) \subset G_1$. Assume that ϕ induces a bijection $G_1/H_1 = G_2/H_2$. Then there is a natural isomorphism o f functors*

$$Ind^{G_1}_{H_1} \cdot f^* \simeq f^* \cdot Ind^{G_2}_{H_2} : Sh_{H_2}(Y) \to Sh_{G_1}(X).$$

Proof. Indeed, let $S \in Sh_{H_2}(Y)$. Then the inverse image $f^*(Ind^{G_2}_{H_2}(S)) \in Sh_{G_1}(X)$ is the induced sheaf $Ind^{G_1}_{H_1}(f^*F)$ (the functor f^* preserves direct sums).

8.3. Let us recall the definition of the functor

$$i : D(Sh_G(X)) \to D_G(X),$$

where D denotes D^b or D^+ (2.5.4).

Let M be a contractible free G-space. Consider the ∞-acyclic resolution $p : P = M \times X \to X$ of X and let $q : P \to \overline{P}$ be the quotient map. Then $D_G(X) \simeq D_G(X, P)$, where the last category is the full subcategory of $D(\overline{P})$ consisting of objects $\overline{F}$ such that $q^*\overline{F}$ comes from X.

Let $S \in D(Sh_G(X))$ be a complex of equivariant sheaves on X. Then $p^*S \in D(Sh_G(P))$, and since P is a free G-space there exists a unique $T \in D(\overline{P})$ such that $q^*T = p^*S$ (lemma 0.3). We put $i(S) = T$.

8.3.1. Theorem. *The above functor $i : D^+(Sh_G(X)) \to D^+_G(X)$ is an equivalence of categories.*

Proof. Using 2.5.3 it suffices to prove that for $F, H \in D^+(Sh_G(X))$

$$\mathrm{Hom}_{D(Sh_G(X))}(F, H) = \mathrm{Hom}_{D_G(X)}(i(F), i(H)).$$

Note that

$$\mathrm{Hom}_{D_G(X)}(i(F), i(H)) = \mathrm{Hom}_{D(\overline{P})}(i(F), i(H)) = \mathrm{Hom}_{D(Sh_G(P))}(p^*F, p^*H).$$

Hence we should prove that

$$(*) \qquad \mathrm{Hom}_{D(Sh_G(X))}(F, H) = \mathrm{Hom}_{D(Sh_G(P))}(p^*F, p^*H).$$

We will reduce (*) to its nonequivariant version, which holds because the map p is ∞-acyclic.

Put $Ind(.) = Ind_{\{e\}}^G(.)$. Assume first that $F = Ind(C_U) \in Sh_G(X)$. Then $p^*F = Ind(C_{p^{-1}(U)})$ (8.2.6). By corollary 8.2.5 the left and right terms in (*) are equal to $\mathrm{Hom}_{D(X)}(C_U, For(H))$ and $\mathrm{Hom}_{D(P)}(C_{p^{-1}(U)}, p^*For(H))$ respectively. But these two groups are equal, since the map p is ∞-acyclic.

Let $F \in Sh_G(X)$. We can find a left resolution of F of the form

$$\ldots \to \oplus_V Ind(C_V) \to \oplus_U Ind(C_U) \to F \to 0.$$

By standard arguments (see [H],7.1) we deduce that (*) holds for $F \in D^b(Sh_G(X))$.

A general $F \in D^+(Sh_G(X))$ can be represented as an inductive limit of bounded complexes

$$F = \varinjlim \tau_{\leq n} F.$$

Assume that H consists of injective sheaves. Then the complex $Hom^{\cdot}(F, H)$ is the surjective inverse limit

$$Hom^{\cdot}(F, H) = \varprojlim Hom^{\cdot}(\tau_{\leq n}F, H).$$

Hence the isomorphism (*) for $F \in D^b$ implies the isomorphism for $F \in D^+$. This proves the theorem.

8.4. By the above theorem we may identify the categories $D^+(Sh_G(X))$ and $D_G^+(X)$.

Let $\phi : H \to G$ be a homomorphism of groups anf $f : X \to Y$ be a ϕ-map. Clearly, the inverse image $f^* : D(Sh_G(Y)) \to D(Sh_H(X))$ defined in 8.1 corresponds to the inverse image $Q^* : D_G(Y) \to D_H(X)$ under the identification $D(Sh_G) = D_G$. We want to identify the direct image Q_* (6.9) explicitly when the homomorphism ϕ is surjective.

Assume that $\phi : H \to G$ is surjective, $K = ker(\phi)$. Consider the functor $f_*^K : Sh_H(X) \to Sh_G(Y)$ defined in 8.1. This is a left exact functor (as a composition of two left exact functors), and we denote by Rf_*^K its right derived functor

$$Rf_*^K : D^+(Sh_H(X)) \to D^+(Sh_G(Y)).$$

8.4.1. Proposition. *Let $\phi : H \to G$ be a surjective homomorphism, $K = ker(\phi)$ and $f : X \to Y$ be a ϕ-map. Then the following hold.*

(i) The functor $f_*^K : Sh_H(X) \to Sh_G(Y)$ is the right adjoint to $f^* : Sh_G(Y) \to Sh_H(X)$.

(ii) The functor $Rf_*^K : D^+(Sh_H(X)) \to D^+(Sh_G(Y))$ is the right adjoint to $f^* : D^+(Sh_G(Y)) \to D_H^+(X))$.

8.4.2. Corollary. *The functor Rf_*^K corresponds to the direct image Q_* under the identification $D^+(Sh_G) = D_G^+$.*

Indeed, both functors are right adjoint to the same functor $f^* = Q^*$.

Proof of 8.4.1. (i) We must show that

$$\mathrm{Hom}_{Sh_H(X)}(f^*F, S) = \mathrm{Hom}_{Sh_G(Y)}(F, f_*^K S),$$

where $F \in Sh_G(Y)$, $S \in Sh_H(X)$.

As in the proof of theorem 8.3.1 we may assume that $F = Ind(C_U)$ for some open subset $U \subset Y$. Then $f^*F = Ind_K^H(C_{f^{-1}(U)})$ (8.2.6). We have

$$\mathrm{Hom}_{Sh_G(Y)}(F, f_*^K S) = \mathrm{Hom}_{Sh(Y)}(C_U, For f_*^K S) = \Gamma(U, f_*^K S) = \Gamma(f^{-1}(U), S)^K,$$

$$\mathrm{Hom}_{Sh_H(X)}(f^*F, S) = \mathrm{Hom}_{Sh_K(X)}(C_{f^{-1}(U)}, Res_H^K S) = \Gamma(f^{-1}(U), S)^K.$$

This proves (i). Now, (ii) follows from (i) and the fact that f_*^K maps injectives to injectives (being the right adjoint to the exact functor f^*).

8.5. In this section we work in the following setup. Let $\phi : H \to G$ be a surjective homomorphism, $K = ker(\phi)$. Let $f : X \to Y$ be a ϕ-map which is the quotient map by the action of K.

8.5.1. Lemma. *(i) Let $F' \in Sh_G(Y)$. The adjunction map*

$$F' \to f_*^K f^* F'$$

is an isomorphism.

(ii) Let $F \in Sh_H(X)$. The adjunction map

$$\alpha : f^* f_*^K F \to F$$

is a monomorphism. Moreover α is an isomorphism if and only if F comes from Y, i.e. $F = f^*F'$ for some $F' \in Sh_G(Y)$.

Proof. (see [Groth] in case $G = \{e\}$.)

8.5.2. Corollary. *(i) The functor $f^* : Sh_G(Y) \to Sh_H(X)$ is fully faithful.*

(ii) For each $F \in Sh_H(X)$ the subsheaf $im(\alpha) \subset F$ is the maximal subsheaf of F that comes from Y.

8.5.3. Consider the following full subcategories of $Sh_H(X)$:

$$I_K = \{F \in Sh_H(X) | \text{ for each } x \in X, \text{ its stabilizer } K_x \subset K \text{ acts trivially on } F_x\},$$

$$S_K = \{F \in Sh_H(X) | \text{ for each } x \in X, \text{ the stalk } F_x \text{ has no nonzero } K_x\text{-invariants}\}.$$

Remark. The subcategoty I_K is closed under subquotients, but not under extensions in general. The subcategory S_K is closed under extensions, but not under subquotients in general. However, if for each point $x \in X$ the stabilizer K_x is finite and if the basic ring R is a field of characteristic 0, then both subcategories are closed under extensions and subquotients.

8.5.4. Recall the following

Definition. Let B be a group and Z be a B-space. We say that B acts on Z properly discontinuously, if

(i) the stabilizer B_z of each point $z \in Z$ is finite,

(ii) each point $z \in Z$ has a neighbourhood V_z such that $bV_z \cap V_z = \emptyset$ if $b \in B,\ b \notin B_z$.

8.5.5. Lemma. *Assume that the group K acts properly discontinuously on X. Let $F \in Sh_H(X)$ and consider the adjunction map*

$$\alpha : f^* f_*^K F \to F.$$

Then for every point $x \in X$ the stalk $im(\alpha)_x \subset F_x$ is equal to K_x-invariants of F_x.

Proof. Fix a point $x \in X$. Since K acts properly discontinuously on X, there exists a fundamental system of neighbourhoods V_x of x with the following properties.

1. $K_x V_x = V_x$,
2. $kV_x \cap V_x = \emptyset$ if $k \in K,\ k \notin K_x$.

The stalk $(f^* f_*^K F)_x$ is equal to the limit

$$(f^* f_*^K F)_x = \lim_{\substack{\to \\ V_x}} F(f^{-1}(f(V_x)))^K.$$

But $f^{-1}(f(V_x)) = \coprod_{s \in K/K_x} sV_x$ and hence

$$F(f^{-1}(f(V_x)))^K = F(V_x)^{K_x}.$$

Therefore,

$$(f^* f_*^K F)_x = \lim_{\substack{\to \\ V_x}} F(V_x)^{K_x} = F_x^{K_x}.$$

This proves the lemma.

8.5.6. Proposition. *Assume that the subgroup $K \subset H$ acts properly discontinuously on X. Then the following hold.*

(i) A sheaf $F \in Sh_H(X)$ belongs to I_K (8.5.3) if and onty if it comes from Y.

(ii) Assume that the basic ring of coefficients is a field of characteristic 0. Let $F \in Sh_H(X)$. Consider the canonical exact sequence

$$0 \to F_I \to F \to F_S \to 0,$$

where the map $F_I \to F$ is the adjunction inclusion $\alpha : f^* f_*^K F \to F$ (8.5.1(ii)). Then $F_I \in I_K$, $F_S \in S_K$.

Proof. Let $F \in Sh_H(X)$. Consider the adjunction monomorphism

$$\alpha : f^* f_*^K F \to F.$$

By lemma 8.5.1(ii) we know that it is an isomorphism if and only if F comes from Y. On the other hand, by the previous lemma, it is an isomorphism if and only if $F \in I_K$. This proves (i).

Fix a point $x \in X$. Consider the exact sequence of stalks

$$0 \to F_{I,x} \to F_x \to F_{S,x} \to 0.$$

By the previous lemma, the image of $F_{I,x}$ in F_x coincides with the K_x-invariants. Since K_x is finite and the basic ring R is a field of characteristic 0, the stalk $F_{S,x}$ has no K_x-invariants. This proves (ii).

8.6. Let the basic ring R be a field of characteristic 0.

Let $\phi : H \to G$ be a surjective homomorphism of groups, $K = ker(\phi)$. Let $f : X \to Y$ be a ϕ-map, which is the quotient map by the action of K on X. Assume that K acts on X properly discontinuously.

8.6.1. Theorem. *Under the above assumptions the following hold.*

(i) The functor $f_*^K : Sh_H(X) \to Sh_G(Y)$ is exact.

(ii) $Rf_*^K \cdot f^* \simeq Id_{D^+(Sh_G(Y))}$.

(iii) $Q_* Q^* \simeq Id_{D^+_G(Y)}$.

(iv) Let $F \in Sh_H(X)$. If $F \in S_K$, then $Q_* F = Rf_*^K F = 0$.

(v) The functor f^* induces an equivalence of categories $f^* : Sh_G(Y) \simeq I_K$.

Proof. (ii) and (iii) are equivalent using the identification $D^+(Sh_G) = D^+_G$, $Rf_*^K = Q_*$, $f^* = Q^*$ in 8.4.2. In view of lemma 8.5.1(i) the assertion (i) implies (ii). So it remains to prove (i), (iv), (v).

(i). Let $F \to F'$ be a surjective morphism in $Sh_H(X)$. It suffices to show that $f_*^K F \to f_*^K F'$ is surjective. Fix a point $y \in Y$ and let $x \in X$ be one of its preimages, $f(x) = y$. There exists a fundamental system of neighbourhoods U of y such that $f^{-1}(U) = \coprod_{s \in K/K_x} sV_x$, where V_x is a neighbourhood of x with the following properties:

1. $K_x V_x = V_x$.
2. $kV_x \cap V_x = \emptyset$ if $k \in K,\ k \notin K_x$.

We have

$$f_*^K F(U) = (\prod_{s \in K/K_x} F(sV_x))^K = F(V_x)^{K_x},$$

and similarly

$$f_*^K F'(U) = F'(V_x)^{K_x}.$$

Hence $f_*^K F_y = F_x^{K_x}$, $f_*^K F'_y = F'^{K_x}_x$ and it suffices to show that the map $F_x^{K_x} \to F'^{K_x}_x$ is surjective. This follows from the surjectivity of $F_x \to F'_x$, since taking K_x-invariants is an exact functor (K_x is a finite group and we work in characteristic 0). This proves (i) and hence also (ii) and (iii).

(iv). It suffices to show that $f_*^K F = 0$ if $F \in S_K$ and then to use (i). So let $F \in S_K$. Using the above argument we find that $f_*^K F_y = F_x^{K_x} = 0$. This proves (iv).

(v). We have $f^* Sh_G(Y) \subset I_K$ (8.5.6(i)), and $f_*^K \cdot f^* = Id_{Sh_G(Y)}$ (8.5.1(i)). Hence it suffices to show that $f_*^K : I_K \to Sh_G(Y)$ is injective on morphisms. But this again follows from the proof of (i) above. This proves (v) and the theorem.

8.7. Let us consider the algebraic situation.

As usual, the basic ring R is assumed to be a field of characteristic 0. Let $\phi : H \to G$ be a surjective homomorphism of groups with a *finite* kernel $K = ker(\phi)$. Let $f : X \to Y$ be a ϕ-map. Assume that X, Y are complex algebraic varieties, f is an algebraic morphism, which is also the quotient map by the action of K.

8.7.1. Theorem. *Under the above assumptions the following hold.*

(i) The functor $Q_* : D_H^+(X) \to D_G^+(Y)$ preserves the t-structure, i.e. $Q_* : Sh_H(X) \to Sh_G(Y)$.

(ii) $Q_* Q^* \simeq Id_{D_G^+(Y)}$.

(iii) The functor Q_* is exact in the perverse t-structure, i.e. $Q_* : Perv_H(X) \to Perv_G(Y)$.

(iv) $Q_* IC_H(X) = IC_G(Y)$.

Proof. Since X is a Hausdorff topological space (in the classical topology) and the group K is finite, it acts on X properly discontinuously. Hence (i), (ii) follow from theorem 8.6.1(i),(iii).

(iii). Notice that $f : X \to Y$ is a finite morphism of algebraic varieties and hence $f_* Perv(X) \subset Perv(Y)$.

Let $P \in Perv_H(X)$ be a complex of equivariant sheaves (using the equivalence $D_H^b(X) \simeq D^b(Sh_H(X))$). We have to check that the complex $f_*^K P \in D^b(Sh_G(Y))$ satisfies the support and the cosupport conditions. Namely, let $i : Z \hookrightarrow Y$ be an inclusion of a locally closed subvariety of codimension k. We need to show that

$$(*) \qquad H^s(i^* f_*^K P)_z = 0, \quad s > k$$

$$(**) \qquad H^s(i^! f_*^K P)_z = 0, \quad s < k$$

at the generic point $z \in Z$.

We know (*) and (**) for $f_* P$ instead of $f_*^K P$. The functor i^* commutes with taking K-invariants, which is an exact functor. Hence $H^s(i^* f_*^K P) = (H^s(i^* f_* P))^K$ and (*) holds.

In order to prove (**) we may assume that the complex P consists of injective H-equivariant sheaves. Since functors f_*, f_*^K, For preserve injectives, the complexes $f_* P \in D^b(Sh(Y))$ and $f_*^K P \in D^b(Sh_G(Y))$ also consist of injective sheaves. The functor $i^!$ - taking sections with support in Z - is left exact, hence applicable to the complexes $f_* P$ and $f_*^K P$. It commutes with taking K-invariants and $H^s(i^! f_*^K P) = H^s(i^! f_* P)^K$, so (**) holds. This proves (iii).

(iv) Since $Q_* = f_*^K$ we may assume that $H = K$, $G = \{e\}$. Recall that the intersection cohomology sheaf $IC(W)$ on a stratified pseudomanifold W can be constructed from the constant sheaf C_U on the open stratum U by pushing it forward to the union with smaller and smaller strata and by truncating (see [Bo1]).

Let $\mathcal{S}$ be a stratification of Y and $\mathcal{T} = f^{-1}(\mathcal{S})$ be the induced stratification of X such that $IC(Y)$ and $IC(X)$ are constructible with respect to $\mathcal{S}$ and $\mathcal{T}$. Let $V \subset Y$ be the open stratum on Y and $f^{-1}(V) = U$ be the one on X. Let $C_{U,K} \in Sh_K(U)$ be the constant K-equivariant sheaf on U. Its direct image $f_*^K C_{U,K} = C_V$ is the constant sheaf on V. We claim that constructions of $IC(X)$ and $IC(Y)$ from C_U and C_V commute with the direct image f_*^K. Indeed, the operations of pushing forward (direct image) and truncation obviously commute with the exact functor f_*. The functor $(\ )^K$ of taking K-invariants commutes with the pushforward, and since $(\ )^K$ is exact it also commutes with the truncation. This proves (iv) and the theorem.

9. Almost free algebraic actions.

9.0. In this section we consider almost free actions (only finite stabilizers) of reductive algebraic groups and extend theorem 8.7.1 to this situation. As always in the algebraic setting we assume that the basic ring R is a field of characteristic 0. We denote by d_M the complex dimension of a complex algebraic variety M.

Let $\phi : H \to G$ be a surjective (algebraic) homomorphism of affine reductive complex algebraic groups with the kernel $K = ker(\phi)$. Let X and Y be complex algebraic varieties with algebraic actions of H and G respectively. Let $f : X \to Y$ be an algebraic morphism which is a ϕ-map. Assume that the following hold.

a) The group K acts on X with only finite stabilizers.

b) The morphism f is affine and is the geometric quotient map by the action of K (all K-orbits in X are closed).

9.1. Theorem. *Under the above assumptions the following hold.*

(i) The functor $Q_* : D^+_H(X) \to D^+_G(Y)$ *preserves the t-structure, i.e.* $Q_* : Sh_H(X) \to Sh_G(Y)$. *In particular* Q_* *preserves the bounded category* D^b.

(ii) $Q_* Q^* = Id_{D_G(Y)}$.

(iii) The functor Q_* *preserves the perverse t-structure, i.e.* $Q_* : Perv_H(X) \to Perv_G(Y)[d_K]$.

(iv) $Q_* IC_H(X) = IC_G(Y)[d_K]$.

Proof. Let us first prove the theorem in the absolute case $H = K$, $G = \{e\}$.

All assertions of the theorem are local on Y (at least for the Zariski topology). Hence we may assume that X is affine. By our assumptions all K-orbits are closed. Hence by a fundamental theorem of D. Luna (see [Lu]) at each point $x \in X$ there exists an etale slice. This means the following.

There exists an affine K_x-invariant subvariety $S \subset X$ containing x with the following properties. Consider the following natural diagram

$$\begin{array}{ccc} K \times_{K_x} S & \xrightarrow{\tilde{\alpha}} & X \\ \downarrow \tilde{f} & & \downarrow f \\ U & \xrightarrow{\alpha} & Y \end{array}$$

where f and $\tilde{f}$ are quotient maps by K, $\tilde{\alpha}$ is the obvious K-map and α is the induced map of the quotients. Then

1. This is a pullback diagram.
2. The morphism α (and hence also $\tilde{\alpha}$) is etale.

Claim. It suffices to prove the theorem for the map $\tilde{f}$ instead of f.

Indeed, by the smooth base change theorem (7.1) the assertions (i),(ii),(iii) for $Q_{\tilde{f}*}$ imply those for Q_{f*}. Suppose we proved (iv) for $Q_{\tilde{f}*}$. Then $\alpha^* \cdot Q_{f*} IC_H(X) = IC(U)$ and hence $Q_{f*} IC_H(X)$ is a simple perverse sheaf on Y. It remains to show that for some smooth open dence subset $V \subset Y$ we have $Q_{f*} IC_H(X)|_V = C_V$. We can choose a smooth V in such a way that $f^{-1}(V) \subset X$ is also smooth and hence $IC_H(X)|_{f^{-1}(V)} = C_{f^{-1}(V)} = Q^*C_V$. But the adjunction map $C_V \to Q_*Q^*C_V$ is an isomorphism by (ii), which is what we need. This proves the claim.

So we may (and will) assume that $U = Y$, $K \times_{K_x} S = X$.

Denote by $\psi : K_x \hookrightarrow K$ the natural embedding.

Consider S as a K_x-space and the embedding $\nu : S \to X$ as a ψ-map. Put $g = f|_S : S \to Y$. We have the following commutative diagram

$$\begin{array}{ccc} S & \xrightarrow{\nu} & X \\ \downarrow g & & \downarrow f \\ Y & = & Y \end{array}$$

Hence $Q_{g*} : D^+_{K_x}(S) \to D^+(Y)$ is equal to the composition of $Q_{\nu*} : D^+_{K_x}(S) \to D^+_K(X)$ with $Q_{f*} : D^+_K(X) \to D^+(Y)$. But $Q_{\nu*}$ is the inverse functor to the induction equivalence $Q^*_\nu = \nu^* : D^+_K(X) \simeq D^+_{K_x}(S)$ (6.12(6)) and hence it preserves the t-structure and the intersection cohomology sheaves (up to a shift) (5.2). Hence it suffices to prove the theorem for Q_{g*} instead of Q_{f*}. It remains to apply theorem 8.7.1 above. This proves the theorem in the absolute case $H = K$ and $G = \{e\}$.

Let us treat the general case.

Let $P \to Y$ be a smooth complex resolution of the G-space Y. Consider the pullback diagram

$$\begin{array}{ccc} P \times_Y X & \longrightarrow & X \\ \downarrow & & \downarrow \\ P & \longrightarrow & Y \end{array}$$

where horizontal arrows are smooth H- and G-maps respectively and vertical arrows are ϕ-maps. The statements (i),(ii),(iii) of the theorem are invariant under the smooth base change (7.1). If the fibers of $P \to Y$ are connected (which we can assume) then (iv) is also invariant. Hence it suffices to prove the theorem for the map $P \times_Y X \to P$ instead of $X \to Y$. In other words we may (and will) assume that Y is a free G-space.

Let $q : Y \to \overline{Y}$ be the quotient map by the action of G. Consider the composed morphism $q \cdot f : X \to \overline{Y}$ as a ψ-map, where $\psi : H \to \{e\}$. Then this map qf satisfies the assumptions of the theorem, i.e. it is affine and is the quotient map of X by the action of H, which has only finite stabilizers. So by the absolute case of the theorem which was proved above we know (i)-(iv) for the map qf.

Notice that $Q_{qf*} = Q_{q*}Q_{f*}$ and Q_{q*} is the inverse functor to the quotient equivalence $Q^*_q = q^* : D^+(\overline{Y}) \simeq D^+_G(Y)$ (6.12(6) , which preserves the t-structure and intersection cohomology sheaves (5.2). Hence we deduce that (i)-(iv) hold for Q_{f*} as well. This proves the theorem.