Part II. DG-modules and equivariant cohomology.

The main purpose of the three sections 10,11,12 is to prove theorem 12.7.2 - the detailed algebraic description of the categories $D^b_G(pt)$ and $D^+_G(pt)$ for a connected Lie group G. So we suggest that the reader goes directly to this theorem and, if he understands the statement, he may proceed to next sections which use very little from sections 10-12 besides the mentioned theorem.

In section 10 we review the important language of DG-modules over a DG-algebra. In section 11 we study a very special DG-algebra in terms of which we eventually describe the categories $D^b_G(pt)$ and $D^+_G(pt)$. In section 12 the main theorem 12.7.2 is proved. Unfortunately, the proof is quite technical, mainly for the bounded below category $D^+_G(pt)$.

10. DG - modules.

Our goal is to introduce the homotopy category and the derived category of DG-modules, and to define the derived functors of *Hom* and $\otimes$.

Most of the general material is contained in [Iℓ], but we review the basic definitions for the sake of completeness.

10.1. Definition. A DG-**algebra** $\mathcal{A} = (A, d)$ is a graded associative algebra $A = \oplus_{i=-\infty}^{\infty} A^i$ with a unit $1_A \in A^0$ and an additive endomorphism d of degree 1 s.t.

$$d^2 = 0$$

$$d(a \cdot b) = da \cdot b + (-1)^{\deg(a)} a \cdot db,$$

and

$$d(1_A) = 0.$$

10.2. Definition. A left DG-**module** (M, d_M) over a DG–algebra $\mathcal{A} = (A, d)$ (or simply an $\mathcal{A}$–module) is a graded unitary left A-module $M = \oplus_{i=-\infty}^{\infty} M^i$ with an additive endomorphism $d_M : M \to M$ of degree 1 s.t. $d_M^2 = 0$ and

$$d_M(am) = da \cdot m + (-1)^{\deg(a)} a \cdot d_M m$$

for $a \in A, m \in M$. A morphism of DG–modules is a morphism of A–modules of degree zero, which commutes with d.

We will write for short M for (M, d_M) if this causes no confusion.

10.2.1. Denote by $\mathcal{M}_\mathcal{A}$ the abelian category of left $\mathcal{A}$–modules.

Note that if $A = A^0$, then an $\mathcal{A}$–module is just a complex of A–modules. In particular $\mathcal{M}_\mathbb{Z}$ is the category of complexes of abelian groups.

10.2.2. The **cohomology** $H(M)$ of an $\mathcal{A}$–module M is $H(M) := \mathrm{Ker} d_M / Im d_M$. Note that $H(M)$ is naturally a graded left module over the graded ring $H(\mathcal{A})$.

10.3. The **translation functor** $[1] : \mathcal{M}_\mathcal{A} \to \mathcal{M}_\mathcal{A}$ is an automorphism of $\mathcal{M}_\mathcal{A}$ s.t.

$$(M[1])^i = M^{i+1}, \quad d_{M[1]} = -d_M$$

and the A–module structure on $M[1]$ is twisted, that is

$$a \circ m = (-1)^{\deg(a)} am,$$

where $a \circ m$ is the multiplication in $M[1]$ and am is the multiplication in M.

10.3.1. Two morphisms $f, g : M \to N$ in $\mathcal{M}_\mathcal{A}$ are **homotopic** if there exists a morphism of A–modules (possibly not of $\mathcal{A}$–modules) $M \xrightarrow{s} N[-1]$ s.t.

$$f - g = s d_M + d_N s$$

Null homotopic morphisms $\mathrm{Hot}(M, N)$ form a 2–sided ideal in $\mathrm{Hom}_{\mathcal{M}_\mathcal{A}}(M, N)$ and we define the **homotopy category** $\mathcal{K}_\mathcal{A}$ to have the same objects as $\mathcal{M}_\mathcal{A}$ and morphisms

$$\mathrm{Hom}_{\mathcal{K}_\mathcal{A}}(M, N) := \mathrm{Hom}_{\mathcal{M}_\mathcal{A}}(M, N)/\mathrm{Hot}(M, N).$$

We now proceed to define the cone of a morphism and the standard triangle in exactly the same way as for complexes of $\mathbb{Z}$–modules.

10.3.2. The **cone** $C(u)$ of a morphism $M \xrightarrow{u} N$ in $\mathcal{M}_\mathcal{A}$ is defined in the usual way. Namely, $C(u) = N \oplus M[1]$ with the differential $d_{N \oplus M[1]} = (d_N + u, -d_M)$. We have the obvious diagram

$$M \xrightarrow{u} N \to C(u) \to N[1]$$

in $\mathcal{M}_\mathcal{A}$ which is called a **standard triangle**.

10.3.3. An **exact triangle** in $\mathcal{K}_\mathcal{A}$ is a diagram isomorphic (in $\mathcal{K}_\mathcal{A}$) to a standard triangle above.

10.3.4. Definition. A short exact sequence

$$K \to M \to N$$

of $\mathcal{A}$-modules is called A-split if it splits as a sequence of A-modules.

One can show that an A-split sequence as above can be complemented to an exact triangle

$$K \to M \to N \to K[1]$$

in $\mathcal{K}_{\mathcal{A}}$.

10.3.5. Proposition. *The homotopy category $\mathcal{K}_{\mathcal{A}}$ with the translation functor [1] and the exact triangles defined as above forms a triangulated category (see [Ve1]).*

Proof. The proof for complexes of $\mathbb{Z}$-modules applies here without any changes.

10.4. A morphism $M \xrightarrow{u} N$ in $\mathcal{M}_{\mathcal{A}}$ is a **quasiisomorphism** if it induces an isomorphism on the cohomology $H(M) \xrightarrow{\sim} H(N)$.

10.4.1. The **derived category** $D_{\mathcal{A}}$ is the localization of $\mathcal{K}_{\mathcal{A}}$ with respect to quasi-isomorphisms (see [Ve1]).

10.4.2. Lemma. *The collection of quasi-isomorphisms in $\mathcal{K}_{\mathcal{A}}$ forms a localizing system (see [Ve1]).*
Proof. Same as for complexes of $\mathbb{Z}$-modules.

10.4.3. Corollary. *The derived category $D_{\mathcal{A}}$ inherits a natural triangulation from $\mathcal{K}_{\mathcal{A}}$.*

Proof. Same as for complexes.

Later we will develop the formalizm of derived functors between the derived categories $D_{\mathcal{A}}$ (see 10.12 below).

10.4.4. Remark. One can check that a short exact sequence

$$0 \to M \to N \to K \to 0$$

in $\mathcal{M}_{\mathcal{A}}$ defines an exact triangle in $D_{\mathcal{A}}$.

10.5. As in the case of complexes, the functors $\mathrm{Hom}_{\mathcal{K}_{\mathcal{A}}}(M,\cdot)$, $\mathrm{Hom}_{\mathcal{K}_{\mathcal{A}}}(\cdot,N)$, $\mathrm{Hom}_{D_{\mathcal{A}}}(M,\cdot)$, $\mathrm{Hom}_{D_{\mathcal{A}}}(\cdot,N)$, $H(\cdot)$ from the category $\mathcal{K}_{\mathcal{A}}$ or $D_{\mathcal{A}}$ to the category of graded abelian groups are **cohomological**. That is, they take exact triangles into long exact sequences.

10.6. Right DG–modules. One can develop a similar theory for right DG–modules.

10.6.1. Definition. A right DG–module (M, d_M) over $\mathcal{A} = (A, d)$ is a right graded A–module $M = \oplus_{i=-\infty}^{\infty} M^i$ with an additive endomorphism $d_M : M \to M$ of degree 1, s.t. $d_M^2 = 0$ and

$$d_M(ma) = d_M m \cdot a + (-1)^{\deg(m)} m \cdot da.$$

Denote the category of right DG–modules over $\mathcal{A}$ by $\mathcal{M}^r_{\mathcal{A}}$.

One can either proceed to define the homotopy category $\mathcal{K}^r_{\mathcal{A}}$ and the derived category $D^r_{\mathcal{A}}$ in a way similar to left DG–modules, or simply reduce the study of right modules to that of left modules using the following remark 10.6.3 (the two approaches yield the same result).

10.6.2. For a DG–algebra $\mathcal{A} = (A, d)$ we define its **opposite** $\mathcal{A}^{\mathrm{op}} = (A^{\mathrm{op}}, d)$ to have the same elements and the same differential d, but a new multipliciation $a \circ b$ defined by

$$a \circ b := (-1)^{\deg(a)\cdot\deg(b)} ba,$$

where ba denotes the multiplication in A.

10.6.3. Remark. Let $\mathcal{A}$ be a DG–algebra, $\mathcal{A}^{\mathrm{op}}$ its opposite. Then the categories $\mathcal{M}_{\mathcal{A}}$ and $\mathcal{M}^r_{\mathcal{A}^{\mathrm{op}}}$ are naturally isomorphic. Namely, let $M \in \mathcal{M}_{\mathcal{A}}$ be a *left* $\mathcal{A}$–module. We define on M the structure of a *right* $\mathcal{A}^{\mathrm{op}}$–module as follows

$$m \circ a := (-1)^{\deg(a)\cdot\deg(m)} am.$$

One checks that this establishes an isomorphism of categories $\mathcal{M}_{\mathcal{A}} \tilde{\leftrightarrow} \mathcal{M}^r_{\mathcal{A}^{\mathrm{op}}}$.

10.7. A DG–algebra is called **supercommutative** if $ab = (-1)^{\deg(a)\cdot\deg(b)} ba$. In other words $\mathcal{A}$ is supercommutative of $\mathcal{A} = \mathcal{A}^{\mathrm{op}}$.

10.8. *Hom.*

Let $M, N \in \mathcal{M}_{\mathcal{A}}$. Define the complex $Hom\ (M,N)$ of $\mathbb{Z}$–modules as follows:

$Hom^n(M,N) := \{\text{morphisms of A–modules } M \to N[n]\}$; if $f \in Hom^n(M,N)$, then

$$df = d_N f - (-1)^n f d_M.$$

Note that by definition $\mathrm{Hom}_{\mathcal{K}_{\mathcal{A}}}(M,N) = H^0 Hom(M,N)$.

10.8.1. One can check that the bifunctor $Hom(\cdot,\cdot)$ preserves homotopies and defines an exact bifunctor

$$Hom(\cdot,\cdot) : \mathcal{K}^0_{\mathcal{A}} \times \mathcal{K}_{\mathcal{A}} \to \mathcal{K}_{\mathbb{Z}}.$$

10.8.2. In case $\mathcal{A}$ is supercommutative the complex $Hom\ (M,N)$ has a natural structure of a DG–module over $\mathcal{A}$. Namely, for $f \in Hom(M,N)$ put $(af)(m) = af(m)$. In this case $Hom(\cdot,\cdot)$ descends to an exact bifunctor

$$Hom(\cdot,\cdot) : \mathcal{K}^0_{\mathcal{A}} \times \mathcal{K}_{\mathcal{A}} \to \mathcal{K}_{\mathcal{A}}.$$

10.9. $\otimes_{\mathcal{A}}$.

Let $M \in \mathcal{M}^r_{\mathcal{A}}$, $N \in \mathcal{M}_{\mathcal{A}}$ be a left and right DG–modules. Then the graded $\mathbb{Z}$–module $M \otimes_A N$ is a complex of abelian groups with the differential

$$d(m \otimes n) = d_M m \otimes n + (-1)^{\deg(m)} m \otimes d_N n$$

We denote this complex by $M \otimes_{\mathcal{A}} N$.

10.9.1. The bifunctor $\otimes_{\mathcal{A}}$ preserves homotopies and descends to an exact bifunctor

$$\otimes_{\mathcal{A}} : \mathcal{K}^r_{\mathcal{A}} \times \mathcal{K}_{\mathcal{A}} \to \mathcal{K}_{\mathbb{Z}}.$$

10.9.2. In case $\mathcal{A}$ is supercommutative the complex $M \otimes_{\mathcal{A}} N$ has a natrual structure of a DG–module over $\mathcal{A}$. Namely, put $a(m \otimes n) = (-1)^{\deg(a)\cdot\deg(m)} ma \otimes n$. Then $\otimes_{\mathcal{A}}$ descends to an exact bifunctor

$$\otimes_{\mathcal{A}} : \mathcal{K}^r_{\mathcal{A}} \times \mathcal{K}_{\mathcal{A}} \to \mathcal{K}_{\mathcal{A}}.$$

10.10. If $\mathcal{A}$ is supercommutative we have the following functorial isomorphisms ($M, N, K \in \mathcal{M}_{\mathcal{A}}$):

$$M \otimes_{\mathcal{A}} (N \otimes_{\mathcal{A}} K) = (M \otimes_{\mathcal{A}} N) \otimes_{\mathcal{A}} K$$

$$Hom(M, Hom(N,K)) = Hom(M \otimes_{\mathcal{A}} N, K)$$

$$\mathrm{Hom}_{\mathcal{M}_{\mathcal{A}}}(M, Hom(N,K)) = \mathrm{Hom}_{\mathcal{M}_{\mathcal{A}}}(M \otimes_{\mathcal{A}} N, K)$$

$$\mathrm{Hom}_{\mathcal{K}_{\mathcal{A}}}(M, Hom(N,K)) = \mathrm{Hom}_{\mathcal{K}_{\mathcal{A}}}(M \otimes_{\mathcal{A}} N, K).$$

10.11. Let $\phi : \mathcal{A} \to \mathcal{B}$ be a **homomorphism** of DG–algebras, that is a unitary homomorphism of graded algebras $\phi : A \to B$ which commutes with the differential. Consider $\mathcal{B}$ as a right DG–module over $\mathcal{A}$ via ϕ.

The assignment $M \mapsto \mathcal{B} \otimes_{\mathcal{A}} M$ defines the **extension of scalars** functor

$$\phi^* : \mathcal{M}_{\mathcal{A}} \to \mathcal{M}_{\mathcal{B}}$$

which descends to the exact functor

$$\phi^* : \mathcal{K}_{\mathcal{A}} \to \mathcal{K}_{\mathcal{B}}.$$

On the other hand, if $N \in \mathcal{M}_{\mathcal{B}}$ we can view N as a left $\mathcal{A}$–module via ϕ. This defines the **restriction of scalars** functor

$$\phi_* : \mathcal{M}_{\mathcal{B}} \to \mathcal{M}_{\mathcal{A}}$$

and the exact functor

$$\phi_* : \mathcal{K}_{\mathcal{B}} \to \mathcal{K}_{\mathcal{A}}.$$

The functors ϕ^* and ϕ_* are **adjoint**. Namely, for $M \in \mathcal{M}_{\mathcal{A}}$, $N \in \mathcal{M}_{\mathcal{B}}$ we have

$$\mathrm{Hom}_{\mathcal{M}_{\mathcal{B}}}(\phi^*(M), N) = \mathrm{Hom}_{\mathcal{M}_{\mathcal{A}}}(M, \phi_*(N))$$

$$\mathrm{Hom}_{\mathcal{K}_{\mathcal{B}}}(\phi^*(M), N) = \mathrm{Hom}_{\mathcal{K}_{\mathcal{A}}}(M, \phi_*(N)).$$

In case $\mathcal{A}$ and $\mathcal{B}$ are supercommutative we have also

$$\mathcal{B} \otimes_{\mathcal{A}} (M \otimes_{\mathcal{A}} N) = (\mathcal{B} \otimes_{\mathcal{A}} M) \otimes_{\mathcal{B}} (\mathcal{B} \otimes_{\mathcal{A}} N)$$

for $M, N \in \mathcal{M}_{\mathcal{A}}$, that is ϕ^* is a tensor functor.

10.12. Derived functors.

Our goal is to define derived functors in the sense of Deligne [D3] of *Hom* and $\otimes_{\mathcal{A}}$. In order to do that we will construct for each DG–module M a quasiisomorphism $P(M) \to M$, where $P(M)$ is the "bar resolution" of M. Then we show that $P(M)$ can be used for the definition of the derived functors. We use the results of N. Spaltenstein [Sp].

Let us recall the notion of a $\mathcal{K}$–projective complex of $\mathbb{Z}$–modules (see [Sp]).

10.12.1. Definition. Let $C \in \mathcal{M}_{\mathbb{Z}}$. We say that C is $\mathcal{K}$**–projective** if one of the following equivalent properties holds:

(i) For each $B \in M_{\mathbb{Z}}$

$$\mathrm{Hom}_{\mathcal{K}_{\mathbb{Z}}}(C,B) = \mathrm{Hom}_{D_{\mathbb{Z}}}(C,B)$$

(ii) For each $B \in \mathcal{M}_{\mathbb{Z}}$, if $H(B) = 0$, then

$$H(Hom(C,B)) = 0$$

The equivalence of (i) and (ii) is shown in [Sp]. We will repeat the argument in lemma 10.12.2.2 below.

Theorem. *For every complex $B \in \mathcal{M}_{\mathbb{Z}}$ there exists a $\mathcal{K}$–projective $C \in \mathcal{M}_{\mathbb{Z}}$ and a quasiisomorphism $C \to B$.*

Proof: see [Sp].

10.12.2. We need to extend the definition and the theorem in 10.12.1 to arbitrary DG–modules.

10.12.2.1. Definiton. Let $P \in \mathcal{M}_{\mathcal{A}}$. Then P is called $\mathcal{K}$–**projective** if one of the equivalent conditions in the following lemma holds.

10.12.2.2. Lemma. *Let $P \in \mathcal{M}_{\mathcal{A}}$. Then the following conditions are equivalent:*

(i) $\mathrm{Hom}_{\mathcal{K}_{\mathcal{A}}}(P,\cdot) = \mathrm{Hom}_{D_{\mathcal{A}}}(P,\cdot)$

(ii) For every acyclic $C \in \mathcal{M}_{D_{\mathcal{A}}}$ (that is $H(C) = 0$), the complex $Hom(P,C)$ is also acyclic.

Proof: (i) $\Longrightarrow$ (ii). Let C be acyclic. Then $\mathrm{Hom}_{\mathcal{K}_{\mathcal{A}}}(P,C) = \mathrm{Hom}_{D_{\mathcal{A}}}(P,C) = 0$. But $\mathrm{Hom}_{\mathcal{K}_{\mathcal{A}}}(P,C) = H^0 Hom\,(P,C)$. So $Hom(P,C)$ is acyclic in degree 0. Using the isomorphism $Hom(P,C[i]) = Hom\,(P,C)[i]$ we conclude that $Hom(P,C)$ is acyclic.

(ii) $\Longrightarrow$ (i). By the definition of morphisms in $\mathcal{K}_{\mathcal{A}}$ and $D_{\mathcal{A}}$ it suffices to prove the following: For a map $s \in \mathrm{Hom}_{\mathcal{K}_{\mathcal{A}}}(T,P)$ that is a quasi-isomorphism, there exists a map

$$t \in \mathrm{Hom}_{\mathcal{K}_{\mathcal{A}}}(P,T), \quad \text{s.t.} \quad s \cdot t = Id_P.$$

Consider the cone of s in $\mathcal{K}_{\mathcal{A}}$:

$$T \xrightarrow{s} P \to C(s).$$

Then by (ii) $\mathrm{Hom}_{\mathcal{K}_{\mathcal{A}}}(P,C(s)) = 0$. Hence from the long exact sequence of $\mathrm{Hom}_{\mathcal{K}_{\mathcal{A}}}(P,\cdot)$ it follows that there exists $t \in \mathrm{Hom}_{\mathcal{K}_{\mathcal{A}}}(P,T)$ s.t. $s \cdot t = Id_P$. This proves the lemma.

10.12.2.3. Remark. One checks directly that the $\mathcal{A}$-module $\mathcal{A}$ is $\mathcal{K}$-projective.

10.12.2.4. Bar construction. For a DG-module $M \in \mathcal{M}_{\mathcal{A}}$ we will now define its bar resolution $B(M) \in \mathcal{M}_{\mathcal{A}}$ together with a quasiisomorphism

$$B(M) \to M.$$

Then we will prove that $B(M)$ is K-projective, hence there are enough K-projective objects in $\mathcal{K}_{\mathcal{A}}$.

So let $M \in \mathcal{M}_{\mathcal{A}}$. Consider M as just a complex of abelian groups $M \in \mathcal{M}_{\mathbb{Z}}$. Let $S_0 = S(M) \xrightarrow{\varepsilon} M$ be its K-projective resolution in $\mathcal{K}_{\mathbb{Z}}$ (which exists by theorem 10.12.1). We may (and will) assume that ε is surjective. Consider the induced $\mathcal{A}$-module $P_0 = \mathcal{A} \otimes_{\mathbb{Z}} S_0$ corresponding to the natural homomorphism $\mathbb{Z} \to \mathcal{A}$.. There is a natural map of $\mathcal{A}$-modules

$$\delta_0 : P_0 \to M, \quad \delta_0(a \otimes s) = a \cdot \varepsilon(s).$$

We claim that δ_0 induces a surjection on the cohomology $H(P_0) \to H(M)$. Indeed, the map $\varepsilon : S_0 \to M$ is a quasiisomorphism and for the cycles $s \in S_0$ and $1 \otimes s \in P_0$ we have $\varepsilon(s) = \delta_0(1 \otimes s)$.

Let $\mathcal{K} = \operatorname{Ker}(\delta_0)$. Then the exact sequence

$$0 \to K \to P_0 \xrightarrow{\delta_0} M \to 0$$

in $\mathcal{M}_{\mathcal{A}}$ induces the exact sequence on cohomology

$$0 \to H(K) \to H(P_0) \to H(M) \to 0$$

We now repeat the preceding construction with K instead of M, etc. This produces a complex of $\mathcal{A}$-modules

$$(*) \qquad \xrightarrow{\delta_{-3}} P_{-2} \xrightarrow{\delta_{-2}} P_{-1} \xrightarrow{\delta_{-1}} P_0 \to 0.$$

Define a new $\mathcal{A}$-module $B(M) = \oplus_{i=-\infty}^{0} P_{-i}[i]$, where the A-module structure on $P_{-i}[i]$ is the *same* as on P_{-i} and the differential

$$d : P_{-i}[i] \to P_{-i}[i] \oplus P_{-i+1}[i-1] \text{ is}$$

$$d(p) = (d_{P_{-i}}(p), (-1)^{\deg(p)} \delta_{-i}(p)).$$

There is an obvious morphism of $\mathcal{A}$-modules $B(M)\xrightarrow{\delta}M$, where $\delta|_{P_0} = \delta_0$ and $\delta|_{P_{-i}} = 0$ for $i > 0$. We call $B(M)$ the **bar resolution** of M, which is justified by the following claim.

10.12.2.5. Claim. *$\delta : B(M) \to M$ is a quasi-isomorphism.*

Indeed, $B(M)$ is the total complex, associated to the double complex (*) of abelian groups. Hence $H(B(M))$ can be computed using the spectral sequence of the double complex (*). The E_1 term is the complex

$$\to H(P_{-2}) \to H(P_{-1}) \to H(P_0) \to 0,$$

which is exact except at P_0 by the construction of $B(M)$. Hence the spectral sequence degenerates at E_2 and $H(B(M)) = E_2 = H(M)$.

10.12.2.6. Proposition. *The $\mathcal{A}$-module $B(M)$ as constructed above is $\mathcal{K}$-projective.*

Proof. We will prove property (ii) of the lemma 10.12.2.2: for an acyclic $\mathcal{A}$-module C the complex $Hom(B(M), C)$ is acyclic. Since $H^0 Hom(B(M), C) = \mathrm{Hom}_{\mathcal{K}_\mathcal{A}}(B(M), C)$ it suffices to prove

$$\mathrm{Hom}_{\mathcal{K}_\mathcal{A}}(B(M), C) = 0.$$

So let $f : P \to C$ be a morphism of $\mathcal{A}$-modules, where $H(C) = 0$. We will construct a homotopy $h : f \sim 0$, defining h inductively on the increasing sequence of submodules $B_n = \oplus_{i=0}^{n} P_{-i}[i] \subset B(M)$.

$n = 0$. Recall that $B_0 = P_0 = \mathcal{A} \otimes_{\mathbb{Z}} S_0$ where $S_0 \in \mathcal{M}_{\mathbb{Z}}$ is $\mathcal{K}$-projective. By the adjunction properties in 1.11 the morphism $f|_{P_0} : P_0 \to C$ of $\mathcal{A}$-modules comes from a morphism $g : S_0 \to C$ of $\mathbb{Z}$-modules. But g is homotopic to zero, because S_0 is $\mathcal{K}$-projective. Hence by the same adjunction property there exists a homotopy $h_0 : P_0 \to C[-1]$ s.t. $f|_{P_0} = dh_0 + h_0 d$.

Suppose we have constructed a homotopy $h_{n-1} : B_{n-1} \to C[-1]$ s.t.

$$f|_{B_{n-1}} = dh_{n-1} + h_{n-1}d.$$

We will extend h_{n-1} to a homotopy $h_n : B_n \to C[-1]$. So we need to define h_n on $P_{-n}[n]$.

Let us introduce a local notation. For $M \in \mathcal{M}_\mathcal{A}$ (resp. $M \in \mathcal{M}_{\mathbb{Z}}$) we denote by $M[\bar{n}] \in \mathcal{M}_\mathcal{A}$ (resp. $M[\bar{n}] \in \mathcal{M}_{\mathbb{Z}}$) the appropriately shifted module, where the differential and the A–module structure (resp. the differential) are the *same* as in M.

Let $K \subset P_{-n+1}$ be the kernel of δ_{-n+1}. Let $\alpha : S(K) \to K$ be the $\mathcal{K}$–projective resolution in $\mathcal{M}_{\mathbb{Z}}$ used in the construction of $B(M)$. Then $P_{-n} = \mathcal{A} \otimes_{\mathbb{Z}} S(K)$ and denote by $i : S(K)[\bar{n}] \to P_{-n}[\bar{n}]$, $i(s) = 1 \otimes s$ the map of $\mathbb{Z}$–complexes. Recall that the differential $d_{B(M)}$ acts on $i(s) = 1 \otimes s \in P_{-n}[\bar{n}]$ as

$$d_{B(M)}(i(s)) = 1 \otimes d_{S(K)}(s) + (-1)^{\deg(s)}\alpha(s)$$

Put $\tilde{\alpha}(s) = (-1)^{\deg(s)}\alpha(s)$

Define an additive map $g : S(K)[\bar{n}] \to C$ as follows:

$$g = f \cdot i - h_{n-1} \cdot \tilde{\alpha}.$$

Note that g has degree zero.

Claim. *g is a map of $\mathbb{Z}$-complexes, i.e.*

$$d_C \cdot g = g \cdot d_{S(K)}.$$

Proof.

$$\begin{aligned}
d_C \cdot g &= d_C(f \cdot i - h_{n-1} \cdot \tilde{\alpha}) \\
&= d_C \cdot f \cdot i - d_C \cdot h_{n-1} \cdot \tilde{\alpha} \\
&= f \cdot d_{B(M)} \cdot i - [d_C \cdot h_{n-1} + h_{n-1} \cdot d_{B(M)} - h_{n-1} \cdot d_{B(M)}] \cdot \tilde{\alpha} \\
&= f[i \cdot d_{S(K)} + \tilde{\alpha}] - f \cdot \tilde{\alpha} + h_{n-1} \cdot d_K \cdot \tilde{\alpha} \\
&= f \cdot i \cdot d_{S(K)} + h_{n-1} \cdot [-\tilde{\alpha} \cdot d_{S(K)}] \\
&= [f \cdot i - h_{n-1} \cdot \tilde{\alpha}]d_{S(K)} = g \cdot d_{S(K)}.
\end{aligned}$$

Since $S(K)$ is $\mathcal{K}$–projective there exists a homotopy of $\mathbb{Z}$–complexes

$$h : S(K)[\bar{n}] \to C[-1], \quad \text{s.t.}$$

$$hd_{S(K)} + d_C h = g$$

We get

$$hd_{S(K)} + d_C h = f \cdot i - h_{n-1} \cdot \tilde{\alpha}$$

$$hd_{S(K)} + h_{n-1}\tilde{\alpha} + d_C h = f \cdot i$$

Now there is a unique map of $\mathcal{A}$–modules $h_n : P_{-n}[\bar{n}] = \mathcal{A} \otimes_{\mathbb{Z}} S(K)[\bar{n}] \to C[-1]$ which extends the homotopy $h : S(K)[\bar{n}] \to C[-1]$. We claim that $h_n d_{B(M)} + d_C h_n = f$ on $P_{-n}[\bar{n}]$, that is h_n is the desired extension of h_{n-1}.

Indeed,

$$(h_n d_{B(M)} + d_C h_n)(a \otimes s) = h_n d_{B(M)}(a \otimes s) + d_C h_n(a \otimes s)$$

$$= h_n(da \otimes s + (-1)^{\deg(a)} a \otimes d_{S(K)} s + (-1)^{\deg(s)+\deg(a)} a\alpha(s))$$

$$+ d_C((-1)^{\deg(a)} ah(s))$$

$$= (-1)^{\deg(a)+1} da \cdot h(s) + (-1)^{\deg(a)+\deg(a)} a\ h(d_{S(K)} s)$$

$$+ (-1)^{\deg(s)+\deg(a)+\deg(a)} a\ h_{n-1}(\alpha(s)) + (-1)^{\deg(a)} da \cdot h(s)$$

$$+ (-1)^{\deg(a)+\deg(a)} a d_C h(s)$$

$$= a[hd_{S(K)} + h_{n-1}\tilde{\alpha} + d_C h](s)$$

$$= af \cdot i(s) = f(a \otimes s).$$

So we have extended the homotopy $h : f \sim 0$ from B_{n-1} to B_n. This completes the induction step and the proof of the proposition.

10.12.2.7 Remark. In case the algebra A contains a field the bar construction can be simplified. Namely, we do not need the intermediate complexes $S(K)$, since every complex over a field is $\mathcal{K}$–projective.

10.12.2.8. Denote by $\mathcal{KP}_{\mathcal{A}}$ the full triangulated subcategory of $\mathcal{K}_{\mathcal{A}}$ consisting of $\mathcal{K}$-projectives. The following corollary follows immediately from 10.12.2.1-6.

10.12.2.9. Corollary. *The localization functor $\mathcal{K}_{\mathcal{A}} \to D_{\mathcal{A}}$ induces an equivalence of triangulated categories $\mathcal{KP}_{\mathcal{A}} \simeq D_{\mathcal{A}}$.*

10.12.3. In sections 10.12.2.4–6 above we proved that the category $\mathcal{K}_{\mathcal{A}}$ has enough $\mathcal{K}$–projective objects. This allows us to define the derived functor of $\mathcal{H}om(\cdot, \cdot)$.

10.12.3.1. Definition. For $M, N \in \mathcal{M}_{\mathcal{A}}$ we define the **derived functor**

$$RHom(M,N) := Hom(B(M),N),$$

where $B(M)$ is the bar resolution of M as in 10.12.2.4.

The results in 10.12.2.2–6 above show that $RHom$ is a well defined exact bifunctor

$$RHom : D^0_{\mathcal{A}} \times D_{\mathcal{A}} \to D_{\mathbb{Z}},$$

which is a right derived functor of Hom in the sense of Deligne ([D3]).

In case $\mathcal{A}$ is supercommutative we get the exact bifunctor

$$RHom : D^0_{\mathcal{A}} \times D_{\mathcal{A}} \to D_{\mathcal{A}}$$

10.12.4. Next we want to define the derived functor of $\otimes_{\mathcal{A}}$. Let us recall the following definition, which is again due to Spaltenstein [Sp].

10.12.4.1. Definition. A DG–module $P \in \mathcal{M}_{\mathcal{A}}$ is called $\mathcal{K}$–**flat** if the complex $N \otimes_{\mathcal{A}} P$ is acyclic for every acyclic $N \in \mathcal{M}^r_{\mathcal{A}}$.

10.12.4.2. Lemma. *A $\mathcal{K}$-projective complex $S \in \mathcal{M}_{\mathbb{Z}}$ is $\mathcal{K}$-flat.*

Proof. See [Sp].

10.12.4.3. Proposition. *For every $M \in \mathcal{M}_{\mathcal{A}}$ its bar resolution $B(M) \in \mathcal{M}_{\mathcal{A}}$ is $\mathcal{K}$-flat.*

Proof. Let $C \in \mathcal{M}^r_{\mathcal{A}}$ be acyclic. Recall that $B(M)$ is an $\mathcal{A}$–module associated with a complex of $\mathcal{A}$–modules

$$(*) \qquad \ldots \to P_{-2} \xrightarrow{\delta_{-2}} P_{-1} \xrightarrow{\delta_{-1}} P_0 \to 0.$$

Each term P_{-i} is of the form $\mathcal{A} \otimes_{\mathbb{Z}} S$ where S is a $\mathcal{K}$–projective $\mathbb{Z}$–complex. By the previous lemma 10.12.4.2 S is $\mathcal{K}$–flat, so

$$C \otimes_{\mathcal{A}} P_{-i} = C \otimes_{\mathcal{A}} \mathcal{A} \otimes_{\mathbb{Z}} S = C \otimes_{\mathbb{Z}} S$$

is acyclic.

But $C \otimes_{\mathcal{A}} B(M)$ is a complex associated with the double complex

$$\ldots C \otimes_{\mathcal{A}} P_{-2} \xrightarrow{\pm 1 \otimes \delta_{-2}} C \otimes_{\mathcal{A}} P_{-1} \xrightarrow{\pm 1 \otimes \delta_{-1}} C \otimes_{\mathcal{A}} P_0 \to 0,$$

where the "columns" $C \otimes_{\mathcal{A}} P_{-i}$ are acyclic. Hence $C \otimes_{\mathcal{A}} B(M)$ is acyclic. This proves the proposition.

10.12.4.4. Corollary. *A $\mathcal{K}$-projective object in $\mathcal{M}_{\mathcal{A}}$ is $\mathcal{K}$-flat.*

Proof. Let P be $\mathcal{K}$-projective and let $B(P) \xrightarrow{\delta} P$ be its bar resolution. Then δ is a homotopy eqinvalence. Suppose now that $C \in \mathcal{M}^r_{\mathcal{A}}$ is acyclic. Then $C \otimes_{\mathcal{A}} B(P)$ is acyclic by the last proposition. But $C \otimes_{\mathcal{A}} B(P) \xrightarrow{1\otimes\delta} C \otimes_{\mathcal{A}} P$ is a homotopy equivalence, hence also $C \otimes_{\mathcal{A}} P$ is acyclic. This proves the corollary.

10.12.4.5. Definition. Let $M \in \mathcal{M}_{\mathcal{A}}$, $N \in \mathcal{M}^r_{\mathcal{A}}$. We define the derived functor

$$N \overset{L}{\otimes}_{\mathcal{A}} M := N \otimes_{\mathcal{A}} B(M),$$

where $B(M)$ is the bar resolution as in 10.12.2.4.

The fact that $B(M)$ is $\mathcal{K}$–projective and $\mathcal{K}$–flat implies that $\overset{L}{\otimes}_{\mathcal{A}}$ is a well defined exact bifunctor

$$\overset{L}{\otimes}_{\mathcal{A}} : D^r_{\mathcal{A}} \times D_{\mathcal{A}} \to D_{\mathbb{Z}},$$

which is the left derived functor of $\otimes_{\mathcal{A}}$ in the sense of Deligne ([D3]).

In case $\mathcal{A}$ is supercommutative we get the exact bifunctor

$$\overset{L}{\otimes}_{\mathcal{A}} : D_{\mathcal{A}} \times D_{\mathcal{A}} \to D_{\mathcal{A}}.$$

10.12.5. Let $\phi : \mathcal{A} \to \mathcal{B}$ be a homomorphism of DG–algebras. We now define the derived functor of the extension of scalars functor $\mathcal{B} \otimes_{\mathcal{A}} = \phi^* : \mathcal{K}_{\mathcal{A}} \to \mathcal{K}_{\mathcal{B}}$ to be

$$\mathcal{B} \overset{L}{\otimes}_{\mathcal{A}} = \phi^* : D_{\mathcal{A}} \to D_{\mathcal{B}}$$

We also have the restriction functor

$$\phi_* : D_{\mathcal{B}} \to D_{\mathcal{A}}$$

obtained by restriction of scalars from $\mathcal{B}$ to $\mathcal{A}$. The above functors are adjoint:

$$\mathrm{Hom}_{D_{\mathcal{B}}}(\phi^*(M), N) = \mathrm{Hom}_{D_{\mathcal{A}}}(M, \phi_*(N))$$

for $M \in D_{\mathcal{A}}, N \in D_{\mathcal{B}}$.

10.12.5.1. Theorem. *Let $\phi : \mathcal{A} \to \mathcal{B}$ be a homomorphism of DG–algebras which induces an isomorphism on cohomology $H(\mathcal{A}) \xrightarrow{\sim} H(\mathcal{B})$. Then the extension and the restriction functors*

$$\phi^* : D_{\mathcal{A}} \to D_{\mathcal{B}},$$

$$\phi_* : D_{\mathcal{B}} \to D_{\mathcal{A}}$$

are mutually inverse equivalences of categories.

Proof. Let $M \in D_{\mathcal{A}}$, and $B(M) \xrightarrow{\delta} M$ its bar resolution. Then $\phi_* \cdot \phi^*(M) = \mathcal{B} \otimes_{\mathcal{A}} B(M)$ considered as a left $\mathcal{A}$–module. Define a morphism of functors

$$\alpha : Id_{D_{\mathcal{A}}} \to \phi_* \cdot \phi^*$$

where $\alpha : M \to \mathcal{B} \otimes_{\mathcal{A}} B(M)$ is a composition of δ^{-1} with the map f:

$$f : B(M) \to \mathcal{B} \otimes_{\mathcal{A}} B(M)$$

$$f : e \mapsto 1 \otimes e.$$

To show α is a quasiisomorphism it suffices to show that f is so. But this is immediate since $B(M)$ is $\mathcal{K}$–flat (proposition 10.12.4.3) and hence

$$f = \phi \otimes id : \mathcal{A} \otimes_{\mathcal{A}} B(M) \to \mathcal{B} \otimes_{\mathcal{A}} B(M)$$

is a quasiisomorphism.

Let us define a morphism of functors

$$\beta : \phi^* \cdot \phi_* \to Id_{D_{\mathcal{B}}}.$$

For $N \in D_{\mathcal{B}}$, $\phi^* \cdot \phi_*(N) = \mathcal{B} \otimes_{\mathcal{A}} B(N)$, where $B(N) \xrightarrow{t} N$ is the bar resolution of N considered as an $\mathcal{A}$–module. Put

$$\beta : \mathcal{B} \otimes_{\mathcal{A}} B(N) \to N$$

$$\beta : b \otimes e \mapsto bt(e)$$

We claim that β is a quasiisomorphism. Indeed, consider the commutative diagram

$$\begin{tikzcd}
B(N) = \mathcal{A} \otimes_{\mathcal{A}} B(N) \arrow[d, "\phi \otimes 1"'] \arrow[dr, "1 \otimes t"] & \\
\mathcal{B} \otimes_{\mathcal{A}} B(N) \arrow[r, "\beta"] & \mathcal{A} \otimes_{\mathcal{A}} N = N,
\end{tikzcd}$$

where the maps $\phi \otimes 1$ and $1 \otimes t$ are quasiisomorphisms.

This proves the theorem.

10.12.6. If $\mathcal{A}$ is supercommuntative, there are the following functional identities $(M, N, K \in D_{\mathcal{A}})$:

$$M \overset{L}{\otimes}_{\mathcal{A}} (N \overset{L}{\otimes}_{\mathcal{A}} K) = (M \overset{L}{\otimes}_{\mathcal{A}} N) \overset{L}{\otimes}_{\mathcal{A}} K$$

$$RHom\ (M, RHom(N, K)) = RHom(M \overset{L}{\otimes}_{\mathcal{A}}, K)$$

$$\mathrm{Hom}_{D_{\mathcal{A}}}(M, RHom(N, K)) = \mathrm{Hom}_{D_{\mathcal{A}}}(M \overset{L}{\otimes}_{\mathcal{A}} N, K)$$

If in addition $\mathcal{B}$ is supercommutative then $\phi^* : D_{\mathcal{A}} \to D_{\mathcal{B}}$ is a tensor functor, that is

$$\mathcal{B} \overset{L}{\otimes}_{\mathcal{A}} (M \overset{L}{\otimes}_{\mathcal{A}} N) = (\mathcal{B} \overset{L}{\otimes}_{\mathcal{A}} M) \overset{L}{\otimes}_{\mathcal{B}} (\mathcal{B} \overset{L}{\otimes}_{\mathcal{A}} N).$$

11. Categories $D^f_{\mathcal{A}}$, $D^+_{\mathcal{A}}$.

In this work we will be interested in a very special DG–algebra $\mathcal{A} = (A, d = 0)$, where $A = \mathbb{R}[X_1, \ldots, X_n]$ is the commutative polynomial ring, and the generators x_i have various *even* degrees. This is a supercommutative (even commutative) DG–algebra. Denote by $m := (X_1, \ldots, X_n)$ the maximal ideal in A. The algebra A appears as the cohomology ring $H(BG, \mathbb{R})$ of the classifying space BG of a connected Lie group G. Eventually, we will describe the derived category $D_G(pt)$ of G–equivariant sheaves on a point using the derived category $D_{\mathcal{A}}$ (see next section).

11.1. For the remaining part of this section let us fix a DG–algebra $\mathcal{A}$ as above.

Consider the following triangulated full subcategories of $\mathcal{M}_{\mathcal{A}}$:

$$\mathcal{M}^f_{\mathcal{A}} = \{M \in \mathcal{M}_{\mathcal{A}} | M \text{ is a finitely generated } A-\text{module}\},$$

$$\mathcal{M}^+_{\mathcal{A}} = \{M \in \mathcal{M}_{\mathcal{A}} | M^i = 0 \text{ for } i << 0\}.$$

We repeat the construction of the homotopy category and the derived category in section 10 above replacing the original abelian category $\mathcal{M}_{\mathcal{A}}$ by $\mathcal{M}^f_{\mathcal{A}}$ (resp. $\mathcal{M}^+_{\mathcal{A}}$). Denote the resulting categories by $\mathcal{K}^f_{\mathcal{A}}$, $D^f_{\mathcal{A}}$, (resp. $\mathcal{K}^+_{\mathcal{A}}$, $D^+_{\mathcal{A}}$). We have the obvious fully faithful inclusions of categories

$$\mathcal{M}^f_{\mathcal{A}} \subset \mathcal{M}^+_{\mathcal{A}} \subset \mathcal{M}_{\mathcal{A}},$$
$$\mathcal{K}^f_{\mathcal{A}} \subset \mathcal{K}^+_{\mathcal{A}} \subset \mathcal{K}_{\mathcal{A}}$$

The following proposition implies that there are similar inclusions of the derived categories (see 11.1.3 below).

11.1.1 Proposition. *Let $M \in \mathcal{M}_{\mathcal{A}}$. Assume that $M \in \mathcal{M}^f_{\mathcal{A}}$ (resp. $M \in \mathcal{M}^+_{\mathcal{A}}$). Then there exists a $\mathcal{K}$-projective $P \in \mathcal{M}_{\mathcal{A}}$ and a quasiisomorphism $P \to M$, such that $P \in \mathcal{M}^f_{\mathcal{A}}$ (resp. $P \in \mathcal{M}^+_{\mathcal{A}}$).*

Proof. Consider the exact sequence of $\mathcal{A}$-modules

$$(*) \qquad 0 \to \mathrm{Ker} d_M \to M \to M/\mathrm{Ker} d_M \to 0,$$

where the $\mathcal{A}$-modules $\mathrm{Ker} d_M$ and $M/\mathrm{Ker} d_M$ have zero differential and belong to $\mathcal{M}^f_{\mathcal{A}}$ (resp. $\mathcal{M}^+_{\mathcal{A}}$). The sequence (*) is an exact triangle in $D_{\mathcal{A}}$. Since $\mathcal{K}$-projective objects form a triangulated subcategory in $\mathcal{K}_{\mathcal{A}}$, we may assume that the differential d_M in the module M is zero.

Let

$$0 \to P_{-n} \xrightarrow{d} \ldots \xrightarrow{d} P_0 \xrightarrow{\varepsilon} M \to 0$$

be a graded resolution of M (as an A-module) by finitely generated (resp. bounded below) projective (hence free) A-modules. This resolution defines an $\mathcal{A}$-module

$P = \oplus P_{-i}[i]$ with the differential $d : P_{-i}[i] \to P_{-i+1}[i-1]$ and a quasiisomorphism $\varepsilon : P \tilde{\to} M$ $(\varepsilon|_{P_{-i}[i]} = 0,\ i > 0)$. Clearly $P \in \mathcal{M}_{\mathcal{A}}^f$ (resp. $P \in \mathcal{M}_{\mathcal{A}}^+$).

Claim. *The $\mathcal{A}$-module P is $\mathcal{K}$-projective.*

Proof of the claim. One immitates the proof of the corresponding statement for complexes (see [Hart]). Namely, the following statement is easily verified: Let $C \in \mathcal{M}_{\mathcal{A}}$, s.t. $H(C) = 0$, then $\mathrm{Hom}_{\mathcal{K}_{\mathcal{A}}}(P, C) = 0$.

This proves the proposition.

11.1.2. Consider the full subcategory $\mathcal{KP}_{\mathcal{A}} \subset \mathcal{K}_{\mathcal{A}}$ consisting of $\mathcal{K}$-projectives and denote by $\mathcal{KP}_{\mathcal{A}}^f \subset \mathcal{KP}_{\mathcal{A}}$ the full subcategory consisting of objects $M \in \mathcal{M}_{\mathcal{A}}^f$. We have a natural commutative diagram of functors

$$\begin{array}{ccc} \mathcal{KP}_{\mathcal{A}}^f & \xrightarrow{a} & D_{\mathcal{A}}^f \\ \downarrow c & & \downarrow d \\ \mathcal{KP}_{\mathcal{A}} & \xrightarrow{b} & D_{\mathcal{A}}. \end{array}$$

The functor b is an equivalence (10.12.2.9). It follows from the above proposition that the functor a is also an equivalence. Hence d is fully faithful. Similarly for $\mathcal{KP}_{\mathcal{A}}^+$ and $D_{\mathcal{A}}^+$. So we proved the following

11.1.3. Corollary. *The natural functor $D_{\mathcal{A}}^f \to D_{\mathcal{A}}$ (resp. $D_{\mathcal{A}}^+ \to D_{\mathcal{A}}$) is fully faithful.*

11.1.4. Actually the proof of 11.1.1 gives more. Namely, the $\mathcal{K}$-projective DG-module P constructed in this proof is an iterated extension of finite (shifted) direct sums of $\mathcal{A}$ (resp. of bounded below direct sums $\oplus_{i>\mu}(\oplus\mathcal{A}[-i])$) in case of the category $\mathcal{M}_{\mathcal{A}}^f$ (resp. $\mathcal{M}_{\mathcal{A}}^+$). This shows that the triangulated category $\mathcal{KP}_{\mathcal{A}}^f$ (resp. $\mathcal{KP}_{\mathcal{A}}^+$) is generated by $\mathcal{A}$ (resp. by bounded below direct sums $\oplus_{i>\mu}(\oplus\mathcal{A}[-i])$). Let us write for short $\oplus^+\mathcal{A}[-i]$ for $\oplus_{i>\mu}(\oplus\mathcal{A}[-i])$. Using the equivalence a in 11.1.2 we obtain

11.1.5. Corollary. *The triangulated category $D_{\mathcal{A}}^f$ (resp. $D_{\mathcal{A}}^+$) is generated by $\mathcal{A}$ (resp. by bounded below direct sums $\oplus^+\mathcal{A}[-i]$).*

11.1.6. The categories $\mathcal{M}_{\mathcal{A}}^f$, $\mathcal{M}_{\mathcal{A}}^+$ (resp. $\mathcal{K}_{\mathcal{A}}^f$, $\mathcal{K}_{\mathcal{A}}^+$) are closed under the tensor product $\otimes_{\mathcal{A}}$, and $\mathcal{M}_{\mathcal{A}}^f$, $\mathcal{K}_{\mathcal{A}}^f$ are closed under $Hom(\cdot,\cdot)$. Using the proposition 11.1.1 we can define the derived functors of $Hom(\cdot,\cdot)$ and $\otimes_{\mathcal{A}}$ on the categories $D_{\mathcal{A}}^f, D_{\mathcal{A}}^+$, using the $\mathcal{K}$-projective resolutions (see definitions 10.12.3,4). So we obtain the exact functors

$$RHom(\cdot,\cdot) : D_{\mathcal{A}}^f \times D_{\mathcal{A}}^f \to D_{\mathcal{A}}^f$$

$$RHom(\cdot,\cdot) : D^f_{\mathcal{A}} \times D^+_{\mathcal{A}} \to D^+_{\mathcal{A}}$$

$$\cdot \overset{L}{\otimes} \cdot : D^f_{\mathcal{A}} \times D^f_{\mathcal{A}} \to D^f_{\mathcal{A}}$$

$$\cdot \overset{L}{\otimes} \cdot : D^+_{\mathcal{A}} \times D^+_{\mathcal{A}} \to D^+_{\mathcal{A}}$$

Clearly, all relations in 10.12.6 hold for the above functors.

Our main interest lies in the category $D^f_{\mathcal{A}}$. We proceed to define some additional structures on $D^f_{\mathcal{A}}$.

11.2. Duality.

11.2.1. Definition. The $\mathcal{A}$-module $D_{\mathcal{A}} := \mathcal{A} \in D^f_{\mathcal{A}}$ is called the **dualizing** module.

Next we define the **duality** functor

$$D : D^f_{\mathcal{A}} \to D^f_{\mathcal{A}}$$

$$D(M) := RHom(M, D_A)$$

11.2.2. Proposition. $D^2 = Id.$

11.2.2.1. Lemma. *For a $\mathcal{K}$-projective $P \in \mathcal{K}^f_{\mathcal{A}}$ the $\mathcal{A}$-module $Hom(P, D_A)$ is also $\mathcal{K}$-projective.*

Proof. Since the category of $\mathcal{K}$-projectives is generated by $\mathcal{A}$ (see 11.1.4), it suffices to prove the lemma if $P = \mathcal{A}$ in which case it is obvious.

Proof of proposition. Let $M \in D^f_{\mathcal{A}}$ be $\mathcal{K}$-projective. Then using the above lemma we have

$$DD(M) = Hom(Hom(M, D_A), D_A).$$

Define a map of $\mathcal{A}$-modules

$$\alpha : M \to DD(M)\ ,\ \alpha(m)(f) = (-1)^{\deg(m)\deg(f)} f(m)$$

We claim that α defines an isomorphism of functors $Id \xrightarrow{\sim} D^2$. By corollary 11.1.5 above it suffices to prove that α is an isomorphism in case $M = \mathcal{A}$, which is obvious. This proves the proposition.

11.3. Relations with Ext_A and Tor_A.

Let us point out some relations between the operations $RHom$ and $\overset{L}{\otimes}_{\mathcal{A}}$ in $D^f_{\mathcal{A}}$ and the operations Ext_A and Tor_A in the category Mod_A of graded A-modules.

Let $M, N \in \mathcal{M}^f_{\mathcal{A}}$ be $\mathcal{A}$-modules with **zero differentials.** Recall the construction of a $\mathcal{K}$-projective resolution P of M as in the proof of proposition 11.1.1. We considered a projective resolution

$$(*) \qquad 0 \to P_{-n} \xrightarrow{d} \ldots \to P_{-1} \xrightarrow{d} P_0 \xrightarrow{\varepsilon} M \to 0$$

of M in the category Mod_A. Then the $\mathcal{A}$ - module P was defined as $P = \oplus_i P_{-i}[i]$ with the differential $d : P_{-i}[i] \to P_{-i+1}[i-1]$ and the quasiisomorphism $\varepsilon : P \xrightarrow{\sim} M$ ($\varepsilon(P_{-i}[i]) = 0$, for $i > 0$).

By definition,

$$(1) \qquad RHom(M, N) = Hom(P, N).$$

On the other hand, the complex of A-modules

$$(2) \qquad \ldots \to \mathrm{Hom}_A(P_{-i}, N) \to \mathrm{Hom}_A(P_{-i-1}, N) \to \ldots$$

computes the modules $\mathrm{Ext}^i_A(M, N)$

Comparing (1) and (2) we find

11.3.1. Proposition. *If $M, N \in D^f_{\mathcal{A}}$ have zero differentials, then*

$$(i) \qquad H(RHom(M, N)) = \oplus_i Ext^i_A(M, N)[-i].$$

In particular, if $Ext^i_A(M, N) = 0$, $i \neq k$, then the $\mathcal{A}$-module $RHom(M, N)$ is quasi-isomorphic to its cohomology $H(RHom(M, N))$ and hence

$$RHom(M, N) = Ext^k_A(M, N)[-k].$$

In case $N = D_A = A$ the last equality becomes

$$D(M) = Ext^k_A(M, A)[-k],$$

which shows the close relation between the duality in $D^f_{\mathcal{A}}$ and the coherent duality in Mod_A.

(ii) There exists a natural morphism in $D^f_{\mathcal{A}}$

$$RHom(M, N) \to Ext^n_A(M, N)[-n],$$

which induces a surjection on the cohomology (the differential in the second A-module is zero).

In the previous notations we also have

$$(3) \qquad M \overset{L}{\otimes}_{\mathcal{A}} N = P \otimes_{\mathcal{A}} N.$$

On the other hand the complex of A-modules

$$(4) \qquad \ldots \to P_{-i} \otimes_A N \to P_{-i+1} \otimes_A N \to \ldots$$

computes the modules $\mathrm{Tor}_A^i(M,N)$. Comparing (3) and (4) we find

11.3.2. Proposition. *If $M, N \in D_A^f$ have zero differentials, then*

$$(i) \qquad H(M \overset{L}{\otimes}_{\mathcal{A}} N) = \oplus_i Tor_A^i(M,N)[i].$$

In particular, if $Tor_A^i(M,N) = 0$, $i \neq k$, then the $\mathcal{A}$-module $M \overset{L}{\otimes}_{\mathcal{A}} N$ is quasiisomorphic to its cohomology $H(M \overset{L}{\otimes}_{\mathcal{A}} N)$ and hence

$$M \overset{L}{\otimes}_{\mathcal{A}} N = Tor_A^k(M,N)[k].$$

(ii) There exists a natural morphism in D_A^f

$$M \overset{L}{\otimes}_{\mathcal{A}} N \to M \otimes_A N,$$

which induces a surjection on the cohomology (the differential in the second module is zero).

11.3.3. For a given $M \in D_{\mathcal{A}}$ it is useful to know if M is quasiisomorphic to its cohomology, i.e. if $M \simeq (H(M), d = 0)$ (see, for example, the previous propositions 11.3.1, 11.3.2).

Proposition. *Let $M \in D_{\mathcal{A}}$ be such that the A-module $H(M)$ has cohomological dimension 0 or 1. Then $M \simeq (H(M), d = 0)$.*

Proof. Choose $\{c_i\} \subset \mathrm{Ker} d_M \subset M$ such that $\{c_i\}$ generate the cohomology $H(M)$ as on A-module. Let $P_0 = \oplus_i A c_i$ be the free A-module on generators c_i with the natural map of A-modules

$$\varepsilon : P_0 \to \mathrm{Ker} d_M$$

$$c_i \mapsto c_i.$$

Let $P_{-1} \subset P_0$ be the kernel of the composed surjective map

$$P_0 \xrightarrow{\varepsilon} \mathrm{Ker} d_M \to \mathrm{Ker} d_M / \mathrm{Im} d_M = H(M).$$

By our assumption P_{-1} is a free A-module, and hence

$$0 \to P_{-1} \xrightarrow{d} P_0 \to H(M) \to 0$$

is a projective resolution of the A-module $H(M)$.

As in the proof of 11.1.1 consider the $\mathcal{K}$-projective $\mathcal{A}$-module $P = P_0 \oplus P_{-1}[1]$ with the differential $d : P_{-1}[1] \to P_0$. Then P is quasiisomorphic to $H(M)$ (from the exact sequence above). Let us construct a quasiisomorphism $P \tilde{\to} M$, which will prove the proposition.

We already have the map $P_0 \xrightarrow{\varepsilon} M$. It remains to find a map of A-modules $\varepsilon' : P_{-1} \to M$ which makes the following diagram commutative

$$\begin{array}{ccc} P_{-1} & \xrightarrow{\varepsilon'} & M \\ \downarrow d & & \downarrow d_M \\ P_0 & \xrightarrow{\varepsilon} & M. \end{array}$$

Such a map exists since $\varepsilon \circ d(P_{-1}) \subset Imd_M$ and P_{-1} is a projective A-module. This proves the proposition.

11.4. t-structure.

Let us recall the definition of a t-category in [BBD].

11.4.1. Definition. A **t-category** is a triangulated category D together with two full subcategories $D^{\leq 0}, D^{\geq 0}$ s.t. if $D^{\leq n} := D^{\leq 0}[-n]$ and $D^{\geq n} := D^{\geq 0}[-n]$, then

(i) For $X \in D^{\leq 0}, Y \in D^{\geq 1}, \mathrm{Hom}_D(X,Y) = 0$

(ii) $D^{\leq 0} \subset D^{\leq 1}$ and $D^{\geq 0} \supset D^{\geq 1}$

(iii) For $X \in D$ there exists an exact triangle $A \to X \to B$ s.t. $A \in D^{\leq 0}, B \in D^{\geq 1}$.

Let us introduce a t-structure an $D^f_{\mathcal{A}}$.

11.4.2. Definition.

$D^{f,\geq 0}_{\mathcal{A}} := \{M \in D^f_{\mathcal{A}} |$ there exists $N \in D^f_{\mathcal{A}}$ quasiisomorphic to M such that $N^i = 0, i < 0\}$.

$D^{f,\leq 0}_{\mathcal{A}} := \{N \in D^f_{\mathcal{A}} | \mathrm{Hom}_{D^f_{\mathcal{A}}}(N,M) = 0$ for all $M \in D^{f,\geq 1}_{\mathcal{A}}\}$.

11.4.3. Theorem. *The triple $(D^f_{\mathcal{A}}, D^{f,\geq 0}_{\mathcal{A}}, D^{f,\leq 0}_{\mathcal{A}})$ is a t-category.*

Proof. The properties (i),(ii) of definition 11.4.1 are obvious. In order to prove (iii) we need some preliminaries. The proof will be finished in 11.4.11 below.

11.4.4. Definition. Let $N \in D^f_{\mathcal{A}}$ be a free A-module. Denote by $rk_A N$ its rank as an A-module. Let now $M \in D^f_{\mathcal{A}}$ be arbitrary. Then we define the **rank** of M as follows

$$rkM := \min_P \{rk_A P | P \text{ is } \mathcal{K} - \text{projective, free as}$$

$$\text{an } A\text{-module, quasiisomorphic to } M\}.$$

11.4.5. Remark. The function $rk(M)$ satisfies the "triangle inequality". Namely, if $M \to N \to K$ is an exact triangle in $D^f_{\mathcal{A}}$, then $rkN \leq rkM + rkK$.

11.4.6. Definition. Let $P \in D^f_{\mathcal{A}}$ be a $\mathcal{K}$-projective, free as an A-module. Then P is called a **minimal** $\mathcal{K}$-projective, if $rk_A P = rkP$.

11.4.6.1. Let $P = (\oplus A[i], d_P)$ be a free A-module. Define the following A-submodules of P:

$$P_{\leq i} = \oplus_{j \geq -i} A[j]$$

$$P_{\geq i} = \oplus_{j \leq -i} A[j],$$

so that $P = P_{\leq i} \oplus P_{\geq i+1}$.

11.4.7. Lemma. *Let (P, d_P) be a $\mathcal{K}$-projective $\mathcal{A}$-module, free as an A-module. Then the following statements are equivalent.*

(i) $d_P(P) \subset mP$;

(ii) P is a minimal $\mathcal{K}$-projective $\mathcal{A}$-module;

(iii) for all k, $P_{\leq k}$ is an $\mathcal{A}$-submodule of P, i.e. $d_P(P_{\leq k}) \subset P_{\leq k}$.

Proof. Let $e_1, \ldots, e_n$ be a graded A-basis of P, s.t. deg$(e_{i+1}) \geq$ deg(e_i). Then the differential d_P is an endomorphism of P given by a matrix $M = (a_{ij})$, where

$$d_P(e_j) = \sum_{i=1}^{n} a_{ij} e_i.$$

Then clearly, (i)$\Leftrightarrow$ $a_{ij} \in m, \ \forall i, j \ \Leftrightarrow M$ is upper triangular $\Leftrightarrow$(iii).

(i)$\Rightarrow$(ii). Assume that P satisfies (i). Consider the complex of $\mathbb{R}$-vector spaces

$$\mathbb{R} \overset{L}{\otimes}_{\mathcal{A}} P = P/mP.$$

By our assumption this complex has zero differential. Hence P is minimal.

(ii)$\Rightarrow$(i). Induction on the A-rank of P.

Suppose that $d_P(e_1) = 0$. Then we have a short exact sequence of $\mathcal{A}$- modules

$$Ae_1 \to P \to P/Ae_1,$$

where all modules are free. This sequence is A-split, hence defines an exact triangle in $\mathcal{K}$ (10.3.4). The first two terms are $\mathcal{K}$-projective, hence the third one is also such. Moreover, from the triangle inequality (11.4.5) it follows that P/Ae_1 is minimal. By the induction hypothesis (i) holds for P/Ae_1. Hence it also holds for P.

Now suppose that $d_P(e_1) \neq 0$. Then $d_P(e_1)$ is an $I\!R$-linear combination of e_i's. We may (and will) assume that $d_P(e_1) = e_i$ for some i. Denote by E the $\mathcal{A}$-submodule of P spanned by $e_1,\ e_i$. Note that E is $\mathcal{K}$-projective as the cone of the identity morphism $id : Ae_i \to Ae_i$ and $H(E) = 0$. Consider the short exact sequence of $\mathcal{A}$-modules

$$E \to P \to P/E,$$

where all modules are free. This sequence is A-split, hence defines an exact triangle in $\mathcal{K}_{\mathcal{A}}$ (10.3.4). The first two terms are $\mathcal{K}$-projective, hence the third one is also such. The map $P \to P/E$ is a quasiisomorphism, which contradicts the minimality of P. This proves the lemma.

11.4.8. Lemma. *Let P be a minimal $\mathcal{K}$-projective. Then the $\mathcal{A}$-submodules $P_{\leq k} \subset P$ (11.4.7) are $\mathcal{K}$-projective for all k. Hence also $P/P_{\leq k}$ are $\mathcal{K}$-projective and actually $P_{\leq k},\ P/P_{\leq k}$ are minimal.*

Proof. Induction on k.

11.4.9. Remark. Let $M \in D^f_{\mathcal{A}}$. Let $e_1, \dots, e_n$ be a graded A-basis for a minimal $\mathcal{K}$-projective module P quasiisomorphic to M. The previous lemma implies that there is an isomorphism of graded $I\!R$-vector spaces

$$H(I\!R \overset{L}{\otimes}_{\mathcal{A}} M) = \oplus I\!R e_i.$$

In particular

$$rkM = dim_{I\!R} H(I\!R \overset{L}{\otimes}_{\mathcal{A}} M).$$

11.4.10. Proposition. *Let $P, Q \in \mathcal{M}_{\mathcal{A}}$ be two minimal $\mathcal{K}$-projectives. If P, Q are quasiisomorphic then they isomorphic.*

Proof. Let $a : P \to Q$ be a quasiisomorphism. Since P is $\mathcal{K}$-projective, a is an actual morphism of modules. Applying the functor $I\!R \otimes_{\mathcal{A}} \cdot$ we find that a induces an isomorphism of the vector spaces

$$P/mP = Q/mQ.$$

Hence a is an isomorphism by the Nakayama lemma.

11.4.11. Now we can finish the proof of theorem 11.4.3. Let $M \in D^f_{\mathcal{A}}$. Let P be the minimal $\mathcal{K}$-projective quasiisomorphic to M (which is unique by 11.4.10). By lemma 11.4.7 the A-submodule $P_{\leq 0} \subset P$ is actually an $\mathcal{A}$-submodule. Consider the exact triangle

$$P_{\leq 0} \to P \to P/P_{\leq 0}.$$

We claim that this is the desired triangle. Indeed, $P/P_{\leq 0} \in D_{\mathcal{A}}^{f,\geq 1}$. Since $P_{\leq 0}$ has generators in negative degrees and is $\mathcal{K}$-projective (11.4.8), it lies in $D_{\mathcal{A}}^{f,\leq 0}$. This proves the theorem.

11.4.12. Recall that a t–structure on a triangulated category D defines the **truncation functors**

$$\tau_{\leq n} : D \to D^{\leq n}$$

$$\tau_{\geq n} : D \to D^{\geq n},$$

which are respectively right and left adjoint to the inclusions $D^{\leq n} \subset D, D^{\geq n} \subset D$. Then for $X \in D$ the exact triangle

$$\tau_{\leq 0}X \to X \to \tau_{\geq 1}X$$

is the unique triangle (up to a unique isomorphism) satisfying condition (iii) of definition 11.4.1 (see [BBD]).

In our case the truncation functors are made explicit by the argument in 11.4.11.above. Namely if P is a minimal $\mathcal{K}$-projective then $\tau_{\leq i}P = P_{\leq i}$ and $\tau_{\geq i+1}P = P/P_{\leq i}$.

11.4.13. Given a t-structure on D, its **heart** is the full subcategory $\mathcal{C} := D^{\geq 0} \cap D^{\leq 0}$. It is known ([BBD]) that $\mathcal{C}$ is **abelian**.

Claim. *The abelian category $D_{\mathcal{A}}^{f,\geq 0} \cap D_{\mathcal{A}}^{f,\leq 0}$ is equivalent to $Vect_{\mathbb{R}}$ – the category of finite dimensional vector spaces over $\mathbb{R}$.*

Proof. Let $P \in D_{\mathcal{A}}^{f,\geq 0} \cap D_{\mathcal{A}}^{f,\leq 0}$. We may assume that P is minimal $\mathcal{K}$-projective. Since $P \in D_{\mathcal{A}}^{f,\leq 0}$, $\tau_{\geq 1}P = 0$. Since $P \in D_{\mathcal{A}}^{f,\geq 0}$, $\tau_{\leq -1}X = 0$. So by 11.4.12 we find that $P = \oplus A$, $d_P = 0$. Hence $D_{\mathcal{A}}^{f,\geq 0} \cap D_{\mathcal{A}}^{f,\leq 0}$ is equivalent to the category of free A-modules of finite rank, placed in degree zero, which in turn is equivalent to $\mathrm{Vect}_{\mathbb{R}}$.

11.4.14. One can characterize the subcategories $D_{\mathcal{A}}^{f,\geq 0}, D_{\mathcal{A}}^{f,\leq 0} \subset D_{\mathcal{A}}^{f}$ in the following way.

Proposition. *Let $M \in D_{\mathcal{A}}^{f}$.*

1. The following conditions are equivalent

(i) $M \in D_{\mathcal{A}}^{f,\geq 0}$.

(ii) There exists a $\mathcal{K}$-projective $P \in D_{\mathcal{A}}^{f}$ quasiisomorphic to M such that $P^i = 0, i < 0$.

(iii) If $P \in D_{\mathcal{A}}^{f}$ is a minimal $\mathcal{K}$-projective quasiisomorphic to M, then $P^i = 0, i < 0$.

2. The following conditions are equivalent
(iv) $M \in D_{\mathcal{A}}^{f,\leq 0}$
(v) There exists a $\mathcal{K}$*-projective* $P \in D_{\mathcal{A}}^{f}$ *quasiisomorphic to* M *such that* P *is generated as an* A*-module by elements in nonpositive degrees.*
(vi) If $P \in D_{\mathcal{A}}^{f}$ *is a minimal* $\mathcal{K}$*-projective quasiisomorphic to* M*, then* P *is generated as an* A*-module by elements in nonpositive degrees.*

Proof. 1. Clearly, (iii) $\Rightarrow$ (ii) $\Rightarrow$ (i).
(i) $\Rightarrow$ (iii). Let $M \in D_{\mathcal{A}}^{f,\geq 0}$ and let P be a minimal $\mathcal{K}$-projective quasiisomorphic to M. Since $P \in D_{\mathcal{A}}^{f,\geq 0}$, $\tau_{\leq -1}P = 0$. But then by 11.4.12, $P^i = 0, i < 0$ which proves (iii).

2. Clearly, (vi) $\Rightarrow$ (v) $\Rightarrow$ (iv).
(iv) $\Rightarrow$ (vi). Let $M \in D_{\mathcal{A}}^{f,\leq 0}$ and let P be a minimal $\mathcal{K}$-projective quasiisomorphic to M. Then $\tau_{\geq 1}P = 0$, so by 11.4.12 $P = \tau_{\leq 0}P = P_{\leq 0}$, which proves (vi).

12. DG–modules and sheaves on topological spaces.

This is a fairly technical section whose only purpose is to prove the main theorem 12.7.2. Otherwise, it is never used later.

12.0 In this section we show how DG–modules are connected with sheaves on topological spaces. Namely, to a topological space X one can associate a canonical DG–algebra $\mathcal{A}_X$, so that a continuous map $X \xrightarrow{f} Y$ defines a homomorphism of DG–algebras $\phi : \mathcal{A}_Y \to \mathcal{A}_X$. Let $D_{\mathcal{A}_X}$ be the derived category of left $\mathcal{A}_X$-modules and $D(X)$ be the derived category of sheaves on X. We define the localization functor

$$\mathcal{L}_X : D_{\mathcal{A}_X} \to D(X)$$

and the global sections functor

$$\gamma_X : D^+(X) \to D_{\mathcal{A}_X}.$$

These functors establish an equivalence between certain natural subcategories of $D_{\mathcal{A}_X}$ and $D(X)$. Then we study the compatibility of the localization functor with the inverse image $f^* : D(Y) \to D(X)$ and the direct image $f_* : D^+(X) \to D^+(Y)$. These results will be applied to the derived category of equivariant sheaves $D_G(pt)$.

12.1. DG–algebras associated to a topological space.

Let X be a topological space, C_X – the constant sheaf of R–modules on X (later on we will stick to the reals $R = \mathbb{R}$).

Definition. Let $0 \to C_X \to \mathcal{F}^\cdot$ be a resolution of the constant sheaf. We say that it is **multiplicative** if there is given a map of complexes $m : \mathcal{F}^\cdot \otimes \mathcal{F}^\cdot \to \mathcal{F}^\cdot$ which is associative and induces the ordinary multiplication on the subsheaf C_X. The resolution $\mathcal{F}^\cdot$ is called **acyclic** if all sheaves $\mathcal{F}^n$ are acyclic, i.e. $H^i(X, \mathcal{F}^n) = 0,\ i > 0$.

Given a multiplicative resolution $C_X \to \mathcal{F}^\cdot$ the complex of global sections $\Gamma(\mathcal{F}^\cdot)$ has a structure of a DG–algebra. This algebra makes sense if $\mathcal{F}^\cdot$ is in addition acyclic; then, for example, $H^i(\Gamma(\mathcal{F}^\cdot)) = H^i(X, C_X)$.

12.1.1. Examples. 1. The canonical Godement resolution $C_X \to C^\cdot$ (see [Go], 4.3).

2. The canonical simplicial Godement resolution $C_X \to \mathcal{F}^\cdot$ (see [Go], 6.4).
3. The resolution by localized singular cochains $C_X \to CS^\cdot$ (see [Go], 3.9).
4. If X is a manifold, one can take the resolution by the de Rham complex of smooth forms $C_X \to \Omega^\cdot_X$.

All resolutions in above examples are acyclic at least if X is paracompact. Notice that the first three resolutions are functorial with respect to continuous maps. Namely, given a continuous map $f : X \to Y$, we have a natural map $f^* B^\cdot_Y \to B^\cdot_X$

(where $B^{\cdot}$ is a resolution from examples 1-3) which induces the homomorphism of DG–algebras $\phi : \Gamma(B_Y) \to \Gamma(B_X)$. The de Rham complex $\Omega^{\cdot}_X$ is functorial with respect to smooth maps.

The next proposition shows that the choice of a particular acyclic resolution is not important.

12.1.2. Proposition. *Let X be a topological space and $C_X \to B^{\cdot}$ be an acyclic multiplicative resolution. Then the DG–algebra $\Gamma(B^{\cdot})$ is canonically quasiisomorphic to the DG–algebra $\Gamma(\mathcal{F}^{\cdot})$, where $C_X \to \mathcal{F}^{\cdot}$ is the simplicial Godement resolution. More precisely, there exists an acyclic multiplicative resolution $C_X \to \mathcal{F}^{\cdot}(B^{\cdot})$ and canonical morphisms $\mathcal{F}^{\cdot} \to \mathcal{F}^{\cdot}(B^{\cdot})$ and $B^{\cdot} \to \mathcal{F}^{\cdot}(B^{\cdot})$, which induce quasiisomorphisms on DG–algebras of global sections. In particular, any two DG–algebras coming from acyclic multiplicative resolutions of C_X are canonically quasiisomorphic, and hence the corresponding derived categories of DG–modules are canonically equivalent (10.12.5.1).*

Proof. Let us recall the simplicial Godement resolution $\mathcal{F}^{\cdot}$ (see [Go], 6.4). Let $A \in Sh(X)$. There exists a canonical resolution $\mathcal{F}^{\cdot}(A)$ of A:

$$0 \to A \to \mathcal{F}^0(A) \xrightarrow{d} \mathcal{F}^1(A) \xrightarrow{d} \dots,$$

where $\mathcal{F}^0(A) = C^0(A)$ – the sheaf of discontinuous sections of A (see [Go], 4.3) and $\mathcal{F}^n(A) = C^0(\mathcal{F}^{n-1}(A))$. We denote the resolution $\mathcal{F}^{\cdot}(C_X)$ simply by $\mathcal{F}^{\cdot}$.

Recall that local sections $s^n \in \mathcal{F}^n(A)(U)$ are represented by functions

$$s^n(x_0, \dots, x_n) \in A_{x_n}$$

defined on U^{n+1}. Two such functions define the same section if they satisfy certain equivalence relation (see [Go], 6.4).

Following Godement we will use the following convention. Let $u \in A_x$. Then we denote by $y \mapsto u(y) \in A_y$ any local (continuous) section of A which is equal to u when $y = x$. Using these notations we can write the differential

$$d : \mathcal{F}^n(A) \to \mathcal{F}^{n+1}(A)$$

as follows

$$(ds^n)(x_0, \dots, x_{n+1}) = \sum_{i=0}^{n} (-1)^i s^n(x_0, \dots, \hat{x}_i, \dots, x_{n+1})$$

$$+(-1)^{n+1} s^n(x_0, \dots, x_n)(x_{n+1}).$$

The functor $A \mapsto \mathcal{F}^{\cdot}(A)$ has the following properties.

(1) It is exact.

(2) Each sheaf $\mathcal{F}^n(A)$ is flabby, hence acyclic.

(3) If A is a sheaf of rings, then $\mathcal{F}^\cdot(A)$ has a multiplicative structure $\times$ defined by the formula

$$s^p \times s^q(x_0, \dots, x_{p+q}) = s^p(x_0, \dots, x_p)(x_{p+q}) \circ s^q(x_p, \dots, x_{p+q}),$$

where $s^p \in \mathcal{F}^p(A), s^q \in \mathcal{F}^q(A)$ and $\circ$ denotes the multiplication in A.

Hence in particular $\mathcal{F}^\cdot$ is an acyclic multiplicative resolution of C_X.

Let $C_X \to B^\cdot$ be an acyclic multiplicative resolution. Consider the double complex

$$\begin{array}{ccccccccc}
0 & \to & \mathcal{F}^1 & \to & \cdots & & \cdots & & \\
 & & \uparrow & & \uparrow & & & & \\
0 & \to & \mathcal{F}^0 & \to & \mathcal{F}^0(B^0) & \to & \mathcal{F}^0(B^1) & & \cdots \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \to & C_X & \to & B^0 & \to & B^1 & \to & \cdots \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

Denote by $\mathcal{F}^\cdot(B^\cdot)$ the total complex of the inside part

$$\begin{array}{ccccc}
 & & \cdots & & \\
 & & \uparrow & & \\
0 & \to & \mathcal{F}^1(B^0) & \to & \cdots \\
 & & \uparrow & & \uparrow \\
0 & \to & \mathcal{F}^0(B^0) & \to & \mathcal{F}^0(B^1) \quad \cdots \\
 & & \uparrow & & \uparrow \\
 & & 0 & & 0
\end{array}$$

The complexes $\mathcal{F}^\cdot$ and $B^\cdot$ embed naturally in $\mathcal{F}^\cdot(B^\cdot)$, and properties (1), (2) above imply that these embeddings induce quasiisomorphisms between the global sections.

So it remains to construct a multiplicative structure on $\mathcal{F}^\cdot(B^\cdot)$ which will agree with the given ones on $\mathcal{F}^\cdot$ and $B^\cdot$. Let $\circ : B^\cdot \otimes B^\cdot \to B^\cdot$ denote the given multiplication on $B^\cdot$. Let us define the multiplication $\times : \mathcal{F}^\cdot(B^\cdot) \otimes \mathcal{F}^\cdot(B^\cdot) \to \mathcal{F}^\cdot(B^\cdot)$ as follows. Given $s^{p,i} \in \mathcal{F}^p(B^i), s^{q,j} \in \mathcal{F}^q(B^j)$, the section $s^{p,i} \times s^{q,j} \in \mathcal{F}^{p+q}(B^{i+j})$ is defined by the formula

$$s^{p,i} \times s^{q,j}(x_0, \dots, x_{p+q}) = (-1)^{qi} s^{p,i}(x_o, \dots, x_p)(x_{p+q}) \circ s^{q,j}(x_p, \dots, x_{p+q}).$$

One checks immediately that $\times$ is a morphism of complexes and that it induces the given multiplication on $\mathcal{F}^\cdot$ and $B^\cdot$. This proves the proposition.

As was mentioned above the de Rham algebra is functorial only with respect to smooth maps. However, it has the advantage of being supercommutative, and hence its category of DG–modules has more structure. Since in this work we are interested only in very special topological spaces – the classifying spaces for Lie groups – we will stick to the de Rham algebra.

12.2. The de Rham complex of an ∞–dimensional manifold.

For the remaining part of this section 12 let us put $R = \mathbb{R}$.

12.2.1. Definition. An ∞–**dimensional manifold** M is a paracompact topological space with a fixed homeomorphism,

$$M \simeq \varinjlim M_n,$$

where $M_1 \overset{j}{\hookrightarrow} M_2 \overset{j}{\hookrightarrow} M_3 \ldots$ is a sequence of smooth (paracompact) manifolds of increasing dimensions $d_1 < d_2 < \ldots$, and j is an embedding of a closed submanifold. (A subset $U \subset \varinjlim M_n$ is open iff $U \cap M_n$ is open in M_n for each n).

Let $\Omega^{\cdot}_{M_n} := 0 \to \Omega^0_{M_n} \to \Omega^1_{M_n} \to \cdots \to \Omega^{d_n}_{M_n} \to 0$ be the de Rham complex of smooth differential forms on M_n. It is known (Poincare lemma), that $\Omega^{\cdot}_{M_n}$ is a resolution of the constant sheaf C_{M_n}.

Extend the complex $\Omega^{\cdot}_{M_n}$ by zero to M via the closed embedding $M_n \hookrightarrow M$ and denote this extension again by $\Omega^{\cdot}_{M_n}$. Then the restriction of forms from M_{n+1} to M_n produces the inverse system of complexes on M:

$$\cdots \to \Omega^{\cdot}_{M_2} \to \Omega^{\cdot}_{M_1}.$$

12.2.2. Definition. The **de Rham complex** on M is the inverse limit

$$\Omega^{\cdot}_M := \varprojlim \Omega^{\cdot}_{M_n}$$

12.2.3. Proposition. *(i) The complex $\Omega^{\cdot}_M$ is a resolution of the constant sheaf C_M.*

(ii) Each sheaf $\Omega^k_M = \varprojlim \Omega^k_{M_n}$ is soft. Since M is paracompact, it follows that Ω^k_M is acyclic, i.e. $H^i(M, \Omega^k_M) = 0$, $i > 0$ (see [Go],3.5.4).

Proof. (i) Fix a point $x \in M$, say $x \in M_n$. Let us show that the sequence of stalks $0 \to C_{M,x} \to \Omega^0_{M,x} \to \Omega^1_{M,x} \to \cdots$ is exact. It suffices to show that for a small open subset $U \ni x$ the complex of global sections

$$(*) \qquad 0 \to \Gamma(U, C_M) \to \Gamma(U, \Omega^0_M) \to \cdots$$

is exact. There is a fundamental system of neighborhoods of x consisting of open subsets U s.t. $U \cap M_k \simeq \mathbb{R}^{d_k}$, $k \geq n$, and $U \cap M_k \hookrightarrow U \cap M_{k+1}$ is the embedding of a plane $\mathbb{R}^{d_k} \hookrightarrow \mathbb{R}^{d_{k+1}}$. Then by the Poincare lemma the complex

$$0 \to \Gamma(U, C_{M_k}) \to \Gamma(U, \Omega^0_{M_k}) \to \cdots$$

is exact. Now the exactness of (*) follows since in the inverse system

$$\Gamma(U, \Omega^s_M) = \varprojlim_k \Gamma(U, \Omega^s_{M_k})$$

all maps

$$\Gamma(U, \Omega^s_{M_{k+1}}) \to \Gamma(U, \Omega^s_{M_k})$$

are surjective.

(ii) Since Ω^s_M is a module over the sheaf of ring Ω^0_M ("smooth functions" on M), it suffices to prove the following

12.2.4. Lemma. *The sheaf Ω^0_M is soft.*

Indeed, the lemma implies that Ω^s_M is soft (as a module over a soft sheaf of rings) (see [Go],3.7.1), and since M is paracompact, it is acyclic ([Go],3.5.4).

Proof of lemma. Since M is paracompact it is enough to prove the following statement. Each point $x \in M$ has a neighborhood U such that for any disjoint closed subsets $S, T \subset M$ which are contained in U there is $f \in \Omega^0_M(U)$ s.t. $f \equiv 1$ in some neighborhood of S and $\equiv 0$ in some neighborhood of T.

Fix $x \in M$. Let us choose $U \ni x$ such that $U_k := U \cap M_k \simeq \mathbb{R}^{d_k}$ is relatively compact in M_k and $U_k \hookrightarrow U_{k+1}$ is the embedding of the plane $\mathbb{R}^{d_k} \subseteq \mathbb{R}^{d_{k+1}}$. Let $S, T \subset M$ be disjoint closed subsets contained in U. Then the intersections $S_k := S \cap U_k$, $T_k := T \cap U_k$ are *compact* subsets in U_k. It is shown in [Go], 3.7 that there exists a smooth function f_k on U_k such that $f_k \equiv 1$ is a neighborhood of S_k and $f_k \equiv 0$ in a neighborhood of T_k. So it remains to choose f_{k+1} on U_{k+1} so that $f_{k+1}|_{U_k} = f_k$.

Suppose f_k is chosen. Denote again by f_k its extension to U_{k+1} using the product structure $U_{k+1} = U_k \times \mathbb{R}^{d_{k+1}-d_k}$. Let $\tilde{f}_{k+1}$ be a smooth function on U_{k+1} such that $\tilde{f}_{k+1} \equiv 1$ near S_{k+1} and $\tilde{f}_{k+1} \equiv 0$ near T_{k+1}. Since S_{k+1} and T_{k+1} are compact we can choose a small open neighborhood V of U_k in U_{k+1} with the following property. Put $W = U_{k+1} \backslash U_k$, and let φ_V, φ_W be a parition of 1 subject to the covering $U_{k+1} = V \cup W$. Then the function

$$f_{k+1} := \varphi_V \cdot f_k + \varphi_W \cdot \tilde{f}_{k+1}$$

will be equal to 1 in a neighborhood of S_{k+1} and equal to 0 in a neighborhood of T_{k+1}. Clearly $f_{k+1}|_{U_k} = f_k$ which proves the lemma and the proposition.

12.2.5. Since the restriction of forms $\Omega^{\cdot}_{M_{n+1}} \to \Omega^{\cdot}_{M_n}$ commutes with the wedge product, the de Rham complex $\Omega^{\cdot}_M$ inherits a natural multiplicative structure. By the above proposition, $\Omega^{\cdot}_M$ is an acyclic multiplicative resolution of C_M (see definition 12.1). Denote by $\mathcal{A}_M$ the corresponding DG–algebra of global sections

$$\mathcal{A}_M := \Gamma(\Omega^{\cdot}_M).$$

12.2.6. Definition. Let $M' = \varinjlim M'_n$ be another ∞–dimensional manifold and $f : M \to M'$ be a continuous map. We say that f is **smooth** if for each n there exists n' such that $f(M_n) \subset M'_{n'}$, and the restriction

$$f|_{M_n} : M_n \to M'_{n'},$$

is smooth.

Let $f : M \to M'$ be a smooth map. Then we have a natural morphism $f^*\Omega^{\cdot}_M \to \Omega^{\cdot}_{M'}$ which preserves the product structure and hence defines the homomorphism of DG–algebras

$$\phi : \mathcal{A}_{M'} \to \mathcal{A}_M.$$

12.3. Localization and global sections.

In this section and in sections 12.4-6 below all spaces $X, Y, \ldots$ are smooth paracompact manifolds (possibly ∞–dimensional) and all maps $f : X \to Y$ are smooth. For a space X, $\mathcal{A}_X$ denotes the de Rham DG–algebra defined in 12.2.5.

12.3.1. Let us define the localization functor

$$\mathcal{L}_X : D_{\mathcal{A}_X} \to D(X),$$

where $D_{\mathcal{A}_X}$ is the derived category of (left) DG–modules over $\mathcal{A}_X$ (10.4.1) and $D(X)$ is the derived category of sheaves on X. Let $M \in D_{\mathcal{A}_X}$. Put

$$\mathcal{L}_X(M) := \Omega^{\cdot}_X \overset{L}{\otimes}_{\mathcal{A}_X} M.$$

In other words, let $P \to M$ be a $\mathcal{K}$–projective resolution of M (10.12.1, 10.12.4.5).

Then

$$\mathcal{L}_X(M) = \Omega^{\cdot}_X \otimes_{\mathcal{A}_X} P$$

which is the sheaf of complexes (or the complex of sheaves) on X associated to the presheaf of complexes

$$U \mapsto \Omega^{\cdot}_X(U) \otimes_{\mathcal{A}_X} P.$$

Note that $\mathcal{L}_X(\mathcal{A}_X) \simeq C_X$. Indeed, $\mathcal{A}_X$ is $\mathcal{K}$–projective as an $\mathcal{A}_X$–module (10.12.2.3), hence

$$\mathcal{L}_X(\mathcal{A}_X) = \Omega^{\cdot}_X \otimes_{\mathcal{A}_X} \mathcal{A}_X = \Omega^{\cdot}_X \simeq C_X.$$

12.3.2. Definition. Let D be a triangulated category, $S \in D$.

(i) Denote by $D(S) \subset D$ the full triangulated subcategory generated by S.

(ii) Consider "bounded below" direct sums $\oplus_{i>\mu}(\oplus S[-i])$. As in 11.1.4 we denote them by $\oplus^+ S[-i]$. Denote by $D^+(\oplus S) \subset D$ the full triangulated category generated by sums $\oplus^+ S[-i]$.

12.3.2.1. Remark. Let $\mathcal{A}$ be the DG-algebra studied in section 11. Take $D = D_{\mathcal{A}}$ and $S = \mathcal{A}$. Then $D(\mathcal{A}) = D^f_{\mathcal{A}}$ and $D^+(\oplus\mathcal{A}) = D^+_{\mathcal{A}}$ (11.1.5).

12.3.3. Proposition. *The localization functor induces an equivalence of categories*

$$\mathcal{L}_X : D(\mathcal{A}_X) \tilde{\to} D(C_X)$$

and

$$\mathcal{L}_X : D^+(\oplus\mathcal{A}_X) \tilde{\to} D^+(\oplus C_X).$$

Proof. We have $\mathcal{L}_X(\mathcal{A}_X) = C_X$. So to prove the first assertion we only have to check that

$$\mathrm{Hom}_{D_{\mathcal{A}_X}}(\mathcal{A}_X, \mathcal{A}_X[i]) = \mathrm{Hom}_{D(X)}(C_X, C_X[i]).$$

But

$$\mathrm{Hom}_{D_{\mathcal{A}}}(\mathcal{A}_X, \mathcal{A}_X[i]) = H^i(\mathcal{A}_X) = H^i(X, C_X) = \mathrm{Hom}_{D(x)}(C_X, C_X[i]).$$

Let us prove the second assertion. Let

$$M = \oplus^+\mathcal{A}_X[-i],\ N = \oplus^+\mathcal{A}_X[-j] \in D^+(\oplus\mathcal{A}_X).$$

Then

$$\mathcal{L}_X M = \oplus^+ C_X[-i],\ \mathcal{L}_X N = \oplus^+ C_X[-j] \in D^+(\oplus C_X),$$

since $\mathcal{L}_X$ preserves direct sums and the DG-modules M, N are $\mathcal{K}$-projective. It suffices to check that

$$\mathrm{Hom}_{D_{\mathcal{A}_X}}(M, N) = \mathrm{Hom}_{D(X)}(\mathcal{L}_X M, \mathcal{L}_X N).$$

Obviously, the left hand side is

$$\prod_i (\oplus_j H^{i-j}(\mathcal{A}_X)),$$

and the right hand side is

$$\prod_i (H^i(X, \oplus_j C_X[-i])).$$

So we need to show that

$$(*)\qquad H^i(X, \oplus_j C_X[-j]) = \oplus_j H^{i-j}(X, C_X).$$

Since X is a paracompact locally contractible space, we may use the singular cohomology to compute the groups in (*). Namely,

$$H^i(X, C_X) = H^i(X, \mathbb{R}),$$

and then the equality (*) follows from the universal coefficients formula. This proves the proposition.

The last argument also proves the following

12.3.4. Lemma. *Let $\oplus^+ C_X[-i] \in D^+(\oplus C_X)$. Consider its canonical soft resolution*

$$\oplus^+ C_X[-i] \to \oplus^+ \Omega^\cdot_X[-i].$$

Then the natural map of $\mathcal{A}_X$-modules

$$\oplus^+ \mathcal{A}_X[-i] \to \Gamma(X, \oplus^+ \Omega^\cdot_X[-i])$$

is a quasiisomorphism.

12.3.5. In order to define the functor of global sections

$$\gamma_X : D(X) \to D_{\mathcal{A}_X}$$

on the whole category $D(X)$ we need the notion of a $\mathcal{K}$-injective resolution (see [Sp]). To avoid the use of these resolutions and some other technical problems we prefer to work with the bounded below derived category $D^+(X)$. So we define the functor of global sections

$$\gamma_X : D^+(X) \to D_{\mathcal{A}_X}$$

as follows. Let $S^\cdot \in D^+(X)$ be a complex of sheaves. Then the complex of global sections of the tensor product $\Omega^\cdot_X \otimes_{C_X} S^\cdot$ has a natural structure of a left module over $\mathcal{A}_X = \Gamma(\Omega^\cdot_X)$. Put

$$\gamma_X(S^\cdot) := \Gamma(\Omega^\cdot_X \otimes_{C_X} S^\cdot).$$

We must check that γ_X is well defined on $D^+(X)$, that is γ_X preserves quasiisomorphisms. Note that the functor $\Omega^\cdot_X \otimes_{C_X} (\cdot)$ is exact, since we work with sheaves of $\mathbb{R}$–vector spaces. Also, the complex $\Omega^\cdot_X \otimes_{C_X} S^\cdot$ is bounded below and consists of sheaves $(\Omega^\cdot_X \otimes_{C_X} S^\cdot)^m = \oplus_{k\geq 0} \Omega^k_X \otimes_{C_X} S^{m-k}$ which are modules over the soft sheaf of rings Ω^0_X (12.2.4). Hence they are also soft and therefore acyclic for Γ, since X is paracompact ([Go],3.5.4). So γ_X is well defined.

Note that

$$\gamma_X(C_X) = \Gamma(\Omega^{\cdot}_X \otimes_{C_X} C_X) = \Gamma(\Omega^{\cdot}_X) = \mathcal{A}_X,$$

and by lemma 12.3.4

$$\gamma_X(\oplus^+ C_X[-i]) = \Gamma(\oplus^+ \Omega^{\cdot}_X[-i]) \simeq \oplus^+ \mathcal{A}_X[-i].$$

Hence γ_X maps subcategories $D(C_X), D^+(\oplus C_X) \subset D^+(X)$ to subcategories $D(\mathcal{A}_X)$ and $D^+(\mathcal{A}_X)$ respectively.

12.3.6. Proposition. *The functor*

$$\gamma_X : D^+(\oplus C_X) \to D^+(\oplus \mathcal{A}_X)$$

is an equivalence, which is the inverse to the equivalence

$$\mathcal{L}_X : D^+(\oplus \mathcal{A}_X) \to D^+(\oplus C_X)$$

of proposition 12.3.4. More precisely, there exist canonical isomorphisms of functors

$$\sigma : Id_{D^+(\oplus \mathcal{A}_X)} \to \gamma_X \cdot \mathcal{L}_X,$$

$$\tau : \mathcal{L}_X \cdot \gamma_X \to Id_{D^+(\oplus C_X)}.$$

Similarly, for $D(C_X)$ and $D(\mathcal{A}_X)$.

Proof. Let us define the morphism σ.

Let $M \in D^+(\oplus \mathcal{A}_X)$ be $\mathcal{K}$-projective. Since direct sums $\oplus^+ \mathcal{A}_X[-i]$ are $\mathcal{K}$-projective, we may (and will) assume that $M^i = 0,\ i << 0$. Then

$$\mathcal{L}_X(M) = \Omega^{\cdot}_X \otimes_{\mathcal{A}_X} M \in D^+(X),$$

and

$$\gamma_X \cdot \mathcal{L}_X(M) = \Gamma(\Omega_X \otimes_{C_X} (\Omega_X \otimes_{\mathcal{A}_X} M)).$$

Consider the map of complexes

$$\Omega^{\cdot}_X \otimes_{C_X} (\Omega^{\cdot}_X \otimes_{\mathcal{A}_X} M) \xrightarrow{m \otimes 1} \Omega^{\cdot}_X \otimes_{\mathcal{A}_X} M,$$

where $m : \Omega^{\cdot}_X \otimes \Omega^{\cdot}_X \to \Omega^{\cdot}_X$ is the multiplication. We claim that $m \otimes 1$ is a quasiisomorphism. This follows from the following lemma.

12.3.7. Lemma. *Let $S^{\cdot} \in D(X)$ be a complex of sheaves and*

$$t : \Omega^{\cdot}_X \otimes_{C_X} S^{\cdot} \to S^{\cdot}$$

be a morphism of complexes, such that $t(1 \otimes s) = s$. Then t is a quasiisomorphism.

Proof of lemma. Indeed, the inclusion

$$i : S^{\cdot} \to \Omega^{\cdot}_X \otimes_{C_X} S^{\cdot}, \ s \mapsto 1 \otimes s$$

is a quasiisomorphism, and $t \cdot i = id_S$.

Since the map $m \otimes 1$ is a quasiisomorphism of bounded below complexes consisting of soft (hence acyclic) sheaves, it induces the quasiisomorphism of left $\mathcal{A}_X$-modules

$$\alpha : \Gamma(\Omega_X \otimes_{C_X} (\Omega_X \otimes_{\mathcal{A}_X} M)) \xrightarrow{\alpha} \Gamma(\Omega^{\cdot}_X \otimes_{\mathcal{A}_X} M).$$

On the other hand there is the obvious morphism of left $\mathcal{A}_X$-modules

$$\beta : M \to \Gamma(\Omega^{\cdot}_X \otimes_{\mathcal{A}_X} M), \ m \mapsto 1 \otimes m.$$

Finally we define $\sigma = \alpha^{-1} \cdot \beta$.

Assume that $M = \oplus^+ \mathcal{A}_X[-i]$. Then β is a quasiisomorphism by lemma 12.3.4. Hence σ is a quasiisomorphism if $M \in D^+(\oplus \mathcal{A}_X)$.

Let us define the morphism τ.

Choose $S^{\cdot} \in D^+(\oplus C_X)$. Then $\gamma_X(S^{\cdot}) = \Gamma(\Omega^{\cdot}_X \otimes_{C_X} S^{\cdot})$. Choose a quasiisomorphism $P \to \gamma_X(S^{\cdot})$, where $P \in D_{\mathcal{A}_X}$ is a $\mathcal{K}$-projective $\mathcal{A}_X$-module. We have

$$\mathcal{L}_X \cdot \gamma_X(S^{\cdot}) = \mathcal{L}_X(P) = \Omega^{\cdot}_X \otimes_{\mathcal{A}_X} P$$

with the morphism

$$1 \otimes a : \Omega_X \otimes_{\mathcal{A}_X} P \to \Omega_X \otimes_{\mathcal{A}_X} \Gamma(\Omega^{\cdot}_X \otimes_{C_X} S^{\cdot}).$$

Compose it with the multiplication map

$$b : \Omega^{\cdot}_X \otimes_{\mathcal{A}_X} \Gamma(\Omega^{\cdot}_X \otimes_{C_X} S^{\cdot}) \to \Omega^{\cdot}_X \otimes_{C_X} S^{\cdot}$$

$$w \otimes w' \otimes s \mapsto ww' \otimes s.$$

to get the morphism

$$b \cdot (1 \otimes a) : \mathcal{L}_X \cdot \gamma_X(S^{\cdot}) \to \Omega^{\cdot}_X \otimes_{C_X} S^{\cdot}$$

On the other hand we the obvious quasiisomorphism

$$c : S^{\cdot} \to \Omega^{\cdot}_X \otimes_{C_X} S^{\cdot}$$

$$s \mapsto 1 \otimes s.$$

So we define the morphism τ as the composition

$$\tau = c^{-1} \cdot b \cdot (1 \otimes a) : \mathcal{L}_X \cdot \gamma_X \to Id.$$

Assume that $S^{\cdot} = \oplus^{+} C_X[-i]$. Then $\gamma_X(S^{\cdot})$ is quasiisomorphic to $\oplus^{+}\mathcal{A}_X[-i]$ (lemma 12.3.4), which is $\mathcal{K}$- projective, so we may take $P = \oplus^{+}\mathcal{A}_X[-i]$. Then the map

$$b \cdot (1 \otimes a) : \Omega^{\cdot}_X \otimes_{\mathcal{A}_X} (\oplus^{+}\mathcal{A}_X[-i]) \to \Omega^{\cdot}_X \otimes_{C_X} (\oplus^{+} C_X[-i])$$

is an isomorphism. Hence τ is a quasiisomorphism if $S^{\cdot} \in D^{+}(\oplus C_X)$. This proves the proposition.

12.3.8. Remark. All the results in this section 12.3 are valid for general paracompact locally contractible spaces X and a DG–algebra $\Gamma(X, \mathcal{F}^{\cdot})$ for a multiplicative acyclic resolution $\mathcal{F}^{\cdot}$ of C_X (the sheaf $\mathcal{F}^0$ must be soft and the basic ring R must be a field). In particular we never used the fact that $\mathcal{A}_X$ was supercommutative.

12.4. Applications to classifying spaces.

We want to apply the results of previous sections 12.1–12.3 to "smooth models" of classifying spaces.

Let G be a Lie group.

12.4.1. Defintion. A **smooth classifying sequence** for G is a sequence of closed embeddings

$$M_0 \subset M_1 \subset \dots,$$

where M_k is a free k–acyclic smooth paracompact G–space, $M_k \subset M_{k+1}$ is an embedding of a submanifold and $\dim(M_{k+1}) > \dim(M_k)$.

Let $M_0 \subset M_1 \subset \dots$ be a smooth classifying sequence for G. Denote the quotient $G \backslash M_k = BG_k$. Then we get a sequence of closed embeddings of smooth manifolds $BG_0 \subset BG_1 \subset \dots$. The classifying space $BG = \varinjlim BG_k$ is a smooth ∞–dimensional manifold (12.2.1). We call is a **smooth model** or a smooth classifying space.

12.4.2. Lemma. *Assume that the Lie group G has one of the following properties*

(a) G is a linear group, i.e. a closed subgroup of $GL(n, \mathbb{R})$ for some n.

(b) G has a finite number of connected components.

Then there exists a smooth classifying sequence for G.

Proof. (a) Let M_k denote the Stiefel manifold of n–frames in $\mathbb{R}^{n+k}$. Then the sequence

$$M_0 \subset M_1 \subset \dots$$

is a smooth classifying sequence for G.

(b) Let $K \subset G$ be a maximal compact subgroup.

By a theorem of G. Mostow G/K is contractible. By the Peter–Weyl theorem K is linear and so by (a) there exists a smooth classifying sequence for K

$$M_0 \subset M_1 \subset \dots .$$

Then

$$G \times_K M_0 \subset G \times_K M_1 \subset \dots$$

is a smooth classifying sequence for G. This proves the lemma.

12.4.3. In the rest of this section 12.4 we will consider only **connected** Lie groups. Let G be such a group, and BG be its smooth classifying space (12.4.1). The derived category $D^b_{G,c}(pt)$ of G–equivariant constructible sheaves on pt is canonically equivalent to the full subcategory of $D^b(BG)$ consisting of complexes with constant cohomology sheaves of finite rank (see 2.7.2,2.8). This last category is generated by the constant sheaf C_{BG}. In other words

$$D^b_{G,c}(pt) = D(C_{BG}). \tag{1}$$

But by (12.3.3) $D(\mathcal{A}_{BG}) \simeq D(C_{BG})$, where $\mathcal{A}_{BG}$ is the de Rham algebra of the smooth space BG. So we obtain an equivalence of triangulated categories

$$D(\mathcal{A}_{BG}) \simeq D^b_{G,c}(pt) \tag{2}$$

We will go one step further and make the left hand side of (2) more accessible.

It is known that the cohomology ring $H^*(BG, \mathbb{R})$ is isomorphic to a polynomial ring $\mathbb{R}[X_1, \dots, X_n]$, where generators X_i have various *even* degrees. Denote this ring by A_G and consider the DG–algebra

$$\mathcal{A}_G := (A_G, d = 0)$$

as in section 11.

12.4.4. Proposition. *There exists a homomorphism of DG–algebras $\mathcal{A}_G \to \mathcal{A}_{BG}$ which is a quasiisomorphism; hence it induces an equivalence of categories $D_{\mathcal{A}_G} \simeq D_{\mathcal{A}_{BG}}$ (10.12.5.1). This equivalence is unique up to a canonical isomorphism.*

Proof. Choose differential forms $\phi_1, \dots, \phi_n \in \mathcal{A}_{BG}$ which represent cohomology classes $X_1, \dots, X_n$. Since the degrees of X_i's are even the forms ϕ_i generate a *commutative* subalgebra in $\mathcal{A}_{BG}$. Hence we may define a homomorphism

$$\phi : \mathcal{A}_G \to \mathcal{A}_{BG} \ , \ X_i \mapsto \phi_i \ ,$$

which is clearly a quasiisomorphism. It defines an equivalence of categories

$$\phi^* : D_{\mathcal{A}_G} \tilde{\to} D_{\mathcal{A}_{BG}}.$$

Let $\psi_1, \dots, \psi_n \in \mathcal{A}_{BG}$ be a difference choice of forms which induces the corresponding equivalence

$$\psi^* : D_{\mathcal{A}_G} \tilde{\to} D_{\mathcal{A}_{BG}}.$$

We will show that functors ϕ^*, ψ^* are canonically isomorphic.

Since ϕ_i, ψ_i represent the same cohomology class we can choose $w_i \in \mathcal{A}_{BG}$ such that $dw_i = \phi_i - \psi_i$. Let $C \subset \mathcal{A}_{BG}$ be the DG–subalgebra generated by $\{\varphi_i, \psi_i, w_i\}$, and $\gamma : C \to \mathcal{A}_{BG}$ be the inclusion. We have two natural embeddings

$$\alpha, \beta : \mathcal{A}_G \to C,$$

where $\alpha(X_i) = \phi_i,\ \beta(X_i) = \psi_i$. Consider the induced equivalences of categories

$$\alpha^*, \beta^* : D_{\mathcal{A}_G} \to D_C, \quad \gamma^* : D_C \to D_{\mathcal{A}_{BG}}.$$

Since $\phi^* = \gamma^* \cdot \alpha^*$, $\psi^* = \gamma^* \cdot \beta^*$ it suffices to construct an isomorphism of functors $\alpha^* \tilde{\to} \beta^*$.

Let $\delta : C \to \mathcal{A}_G$ be the homomorphism $\delta(\phi_i) = \delta(\psi_i) = X_i,\ \delta(w_i) = 0$, and let $\delta^* : D_C \to D_{\mathcal{A}_G}$ be the corresponding equivalence. Note that $\delta^* \cdot \alpha^* = id = \delta^* \cdot \beta^*$. Hence functors α^*, β^* are canonically isomorphic.

To complete the proof of the proposition we must show that a different choice of forms w_i will produce the same isomorphism $\phi^* \tilde{\to} \psi^*$. We will only sketch the argument since it is similar to the one just given.

Let w_i' be a different choice of forms that produces the DG–subalgebra $C' \subset \mathcal{A}_{BG}$. Since $d(w_i - w_i') = 0$ and the odd cohomology of BG vanishes we can find η_i such that $d\eta_i = w_i - w_i'$. Let $E \subset \mathcal{A}_{BG}$ be the subalgebra generated by $\{\phi_i, \psi_i, w_i, w_i', \eta_i\}$. Now all algebras $\mathcal{A}_G, C, C'$ embed in E and it suffices to prove the equality of two morphisms in D_E, which is done similarly. This proves the proposition.

12.4.5. Composing the equivalence of 12.4.4 with the localization functor of 12.4.3 we obtain the functor

$$\mathcal{L}_G := D_{\mathcal{A}_G} \to D(BG)$$

which induces an equivalence

$$\mathcal{L}_G : D(\mathcal{A}_G) \tilde{\to} D^b_{G,c}(pt).$$

But $D(\mathcal{A}_G)$ is the category $D^f_{\mathcal{A}_G}$ studied in detail in section 11 (see 11.1.5). We call the obtained equivalence

$$\mathcal{L}_G : D^f_{\mathcal{A}_G} \tilde{\to} D^b_{G,c}(pt)$$

the localization.

12.4.6. Proposition. *The localization functor*

$$\mathcal{L}_G : D^f_{\mathcal{A}_G} \tilde{\to} D^b_{G,c}(pt)$$

is an equivalence of t–categories, which commutes with functors $\overset{L}{\otimes}$, *RHom, D and the cohomological functor* $H : D^f_{\mathcal{A}_G} \to Mod_{\mathcal{A}_G}$.

Proof. Let $P \in D^f_{\mathcal{A}_G}$ be a minimal $\mathcal{K}$–projective (11.4.6). Then $P \in D^{f,\geq 0}_{\mathcal{A}_G} \Leftrightarrow P^i = 0,\ i < 0$ and $P \in D^{f,\leq 0}_{\mathcal{A}_G} \Leftrightarrow P$ is generated by elements in nonpositive degrees (see prop. 11.4.14). Note that $\mathcal{L}_G(\mathcal{A}_G) = C_{BG}$. Hence $\mathcal{L}_G$ preserves the subcategories $D^{\geq 0}$ and $D^{\leq 0}$ and so is an equivalence of t–categories.

Let $M, N \in D^f_{\mathcal{A}_G}$ be two $\mathcal{K}$–projective DG–modules. Then

$$\mathcal{L}_G(M\overset{L}{\otimes}N) = \Omega_{BG} \otimes_{\mathcal{A}_G} (M \otimes_{\mathcal{A}_G} N).$$

Define a morphism of complexes

$$\theta : \mathcal{L}_G(M) \otimes_{C_{BG}} \mathcal{L}_G(N) \to \mathcal{L}_G(M\overset{L}{\otimes}N)$$

by the formula

$$\theta : (w \otimes m) \otimes (w' \otimes n) \mapsto (-1)^{\deg(m)\deg(w')} ww' \otimes m \otimes n.$$

Since θ is a quasiisomorphism if $M = N = \mathcal{A}_G$, it is an isomorphism of functors. Hence $\mathcal{L}_G$ commutes with $\overset{L}{\otimes}$.

To prove the statement for $RHom$ we need the following.

12.4.7. Lemma. *Let* $M, N \in D^f_{\mathcal{A}_G}$ *be* $\mathcal{K}$*–projective. Then Hom* (M, N) *is also so.*

Proof. Since the subcategory of $\mathcal{K}$–projectives in $\mathcal{K}^f_{\mathcal{A}_G}$ is generated by $\mathcal{A}_G$, it suffices to prove the lemma for $M = \mathcal{A}_G[i], N = \mathcal{A}_G[j]$, in which case it is obvious.

Let $M, N \in D^f_{\mathcal{A}_G}$ be $\mathcal{K}$–projective. Then by the lemma

$$\mathcal{L}_G(RHom(M, N)) = \Omega_{BG} \otimes_{\mathcal{A}_G} Hom(M, N).$$

Let $i : \mathcal{L}_G(N) \to I$ be an injective resolution. Then

$$RHom(\mathcal{L}_G(M), \mathcal{L}_G(N)) = Hom(\Omega_{BG} \otimes_{\mathcal{A}_G} M, I).$$

Define a morphism of complexes

$$\delta : \mathcal{L}_G(RHom(M, N)) \to RHom(\mathcal{L}_G(M)\mathcal{L}_G(N))$$

by the formula

$$\delta(w \otimes f)(w' \otimes m) = (-1)^{\deg(f)\deg(w')} i(ww' \otimes f(m)).$$

It is a quasiisomorphism if $M = N = \mathcal{A}_G$, hence is an isomorphism of functors. So $\mathcal{L}_G$ commutes with $RHom$. Since $\mathcal{L}_G(\mathcal{A}_G) = C_{BG}$, it also commutes with the duality D. It remains to treat the cohomological functor H.

Let $M \in D^f_{\mathcal{A}_G}$ be $\mathcal{K}$–projective. We have a map of DG–modules

$$\gamma : M \to \Gamma(\Omega_{BG} \otimes_{\mathcal{A}_G} M) = \Gamma(\mathcal{L}_G(M))$$

$$m \mapsto 1 \otimes m.$$

It is a quasiisomorphism if $M = \mathcal{A}_G$, hence is so in general. This proves the proposition.

12.4.8. Proposition. *The localization functor $\mathcal{L}_G : D_{\mathcal{A}_G} \to D(BG)$ induces an equivalence of full subcategories*

$$\mathcal{L}_G : D^+_{\mathcal{A}_G} \xrightarrow{\sim} D^+_G(pt)$$

(see 11.1,11.1.5). It commutes with $\overset{L}{\otimes}$ and H.

Proof. We know that $D^+_{\mathcal{A}_G}$ is generated by bounded below direct sums $\oplus^+ \mathcal{A}_G[-i]$ (11.1.5). If $M = \oplus^+ \mathcal{A}_G[i], N = \oplus^+ \mathcal{A}_G[j] \in D^+_{\mathcal{A}_G}$ are two such modules then

$$\mathrm{Hom}_{D_{\mathcal{A}_G}}(M, N) = \mathrm{Hom}_{D(BG)}(\mathcal{L}_G(M), \mathcal{L}_G(N)),$$

by proposition 12.3.3, so $\mathcal{L}_G$ is an equivalence of $D^+_{\mathcal{A}_G}$ with its essential image in $D(BG)$. Since $\mathcal{L}_G(D^+_{\mathcal{A}_G}) \subset D^+_G(pt)$ it remains to show that any complex $S \in D^+(BG)$ with constant cohomology sheaves lies in $\mathcal{L}_G(D^+_{\mathcal{A}_G})$.

Let S be such a complex. Then S is the direct limit

$$S = \varinjlim \tau_{\leq n} S.$$

Each $\tau_{\leq n} S$ lies in $\mathcal{L}_G(D^+_{\mathcal{A}_G})$, say $\tau_{\leq n} S \simeq \mathcal{L}_G(M_n)$ for a $\mathcal{K}$-projective M_n. The modules M_n form a corresponding direct system in $D^+_{\mathcal{A}_G}$ and if we put

$$M := \varinjlim M_n \in D_{\mathcal{A}_G},$$

then $\mathcal{L}_G(M) = S$, since $\mathcal{L}_G$ preserves direct limits. It only remains to show that M lies in $D^+_{\mathcal{A}_G}$.

Note that $\tau_{\leq n+1} S = C(H^{n+1}(S)[-1] \to \tau_{\leq n} S)$. Hence we can put M_{n+1} to be

$$M_{n+1} = C(H^{n+1}(S)[-1] \otimes_{\mathbb{R}} \mathcal{A}_G \to M_n),$$

so that if $M^i_n = 0$ for $i < m$, then the same is true for M_{n+1}. But then clearly

$$M = \varinjlim M_n = \cup_n M_n \in D^+_{\mathcal{A}_G}$$

This proves the first part of the proposition.

The second part is prove similarly to proposition 12.4.6 above using lemma 12.3.4.

12.4.9. Corollary. *The subcategory $D^+_G(pt) \subset D(BG)$ coincides with $D^+(\oplus C_{BG})$.*

Proof. This follows from 11.1.5, 12.3.3, 12.4.8.

12.4.10. To conclude this section we want to show that the constructed equivalences

$$\mathcal{L}_G : D^f_{\mathcal{A}_G} \to D^b_{G,c}(pt)$$

$$\mathcal{L}_G : D^+_{\mathcal{A}_G} \to D^+_G(pt)$$

do not depend on the choice of a smooth model for BG.

Let

$$M_1 \subset M_2 \subset \dots$$

$$M'_1 \subset M'_2 \subset \dots$$

be two smooth classifying sequences for G giving rise to smooth models BG, BG'. Consider the product sequence

$$M_1 \times M'_1 \subset M_2 \times M'_2 \subset \dots$$

which produces the smooth model BG''. We have the diagram of smooth maps

$$\begin{array}{ccccc} & & BG'' & & \\ & p \swarrow & & \searrow p' & \\ BG & & & & BG' \end{array}$$

and the corresponding homomorphisms of DG–algebras

$$\begin{array}{ccccc} & & \mathcal{A}_{BG''} & & \\ & \phi \nearrow & & \phi' \nwarrow & \\ \mathcal{A}_{BG} & & & & \mathcal{A}_{BG'} \end{array}$$

So we may assume that $BG' = BG''$. Then the required result follows from the commutativity of the functorial diagram

$$\begin{array}{ccc} D_{\mathcal{A}_{BG''}} & \xrightarrow{\mathcal{L}} & D(BG'') \\ \phi^* \uparrow & & \uparrow p^* \\ D_{\mathcal{A}_{BG}} & \xrightarrow{\mathcal{L}} & D(BG), \end{array}$$

which is a special case of proposition 12.5.1 below.

12.5. Localization and inverse image.

We keep the notations of section 12.3. Let $f : X \to Y$ be a (smooth) map. It induces functors of inverse and direct image $f^* : D(Y) \to D(X), f_* : D(X) \to D(Y)$ (see [Sp]). On the other hand we have the corresponding homomorphism of DG–algebras $\psi : \mathcal{A}_Y \to \mathcal{A}_X$ which induces functors $\psi^* : D_{\mathcal{A}_Y} \to D_{\mathcal{A}_X}, \psi_* : \mathcal{A}_X \to D_{\mathcal{A}_Y}$. It is natural to ask if the above functors are compatible with the localization (see 12.3), i.e. if the following diagrams are commutative.

$$
\begin{array}{ccc}
D_{\mathcal{A}_X} & \xrightarrow{\mathcal{L}_X} & D(X) \\
\psi^* \uparrow & & \uparrow f^* \\
D_{\mathcal{A}_Y} & \xrightarrow{\mathcal{L}_Y} & D(Y)
\end{array} \tag{1}
$$

$$
\begin{array}{ccc}
D_{\mathcal{A}_X} & \xrightarrow{\mathcal{L}_X} & D(X) \\
\psi_* \downarrow & & \downarrow f_* \\
D_{\mathcal{A}_Y} & \xrightarrow{\mathcal{L}_Y} & D(Y)
\end{array} \tag{2}
$$

Here we discuss the inverse image. The direct image is discussed in the next section 12.6.

12.5.1. Proposition. *The diagram (1) is commutative. More precisely, there exists a canonical isomorphism of functors $f^* \cdot \mathcal{L}_Y \tilde{\to} \mathcal{L}_X \cdot \psi^*$.*

Proof. Let $N \in D_{\mathcal{A}_Y}$ be $\mathcal{K}$–projective. Then $\psi^*(N) = \mathcal{A}_X \otimes_{\mathcal{A}_Y} N \in D_{\mathcal{A}_X}$ is also $\mathcal{K}$–projective. So

$$\mathcal{L}_X \cdot \psi^*(N) = \Omega_X \otimes_{\mathcal{A}_X} (\mathcal{A}_X \otimes_{\mathcal{A}_Y} N) = \Omega_X \otimes_{\mathcal{A}_Y} N.$$

Also $\mathcal{L}_Y(N) = \Omega_Y \otimes_{\mathcal{A}_Y} N$ and

$$f^* \cdot \mathcal{L}_Y(N) = f^*(\Omega_Y \otimes_{\mathcal{A}_Y} N)$$

Given open subsets $U \subset X, V \subset Y$ such that $f(U) \subset V$ we have the natural map of right $\mathcal{A}_Y$–modules $\Omega_Y(V) \to \Omega_X(U)$ which induces a quasiisomorphism on the stalks

$$\Omega_{Y,f(x)} \tilde{\to} \Omega_{X,x}$$

for each $x \in X$. We get the corresponding map of complexes

$$f^*(\Omega_Y \otimes_{\mathcal{A}_Y} N) \to \Omega_X \otimes_{\mathcal{A}_Y} N$$

which is also a quasiisomorphism on stalks since N is $\mathcal{K}$–projective. This proves the proposition.

12.6. Localization and the direct image.

We keep notations of sections 12.3, 12.5. Let $f : X \to Y$ be a map, and $\psi : \mathcal{A}_Y \to \mathcal{A}_X$ be the correspondong homomorphism of DG-algebras. Recall (12.3.3) the equivalences

$$\mathcal{L}_X : D^+(\oplus\mathcal{A}_X) \to D^+(\oplus C_X),$$

$$\mathcal{L}_Y : D^+(\oplus\mathcal{A}_Y) \to D^+(\oplus C_Y).$$

12.6.1. Proposition. *Assume that the direct image f_* maps $D^+(\oplus C_X)$ to $D^+(\oplus C_Y)$. Then ψ_* maps $D^+(\oplus\mathcal{A}_X)$ to $D^+(\oplus\mathcal{A}_Y)$ and the following functorial giagram is commutative*

$$\begin{array}{ccc} D^+(\oplus\mathcal{A}_X) & \overset{\mathcal{L}_X}{\to} & D^+(\oplus C_X) \\ \psi_* \downarrow & & \downarrow f_* \\ D^+(\oplus\mathcal{A}_Y) & \overset{\mathcal{L}_Y}{\to} & D^+(\oplus C_Y) \end{array}$$

Proof. Recall that the functor

$$\mathcal{L}_Y : D^+(\oplus\mathcal{A}_Y) \to D^+(\oplus C_Y)$$

has the inverse

$$\gamma_Y : D^+(\oplus C_Y) \to D^+(\oplus\mathcal{A}_Y)$$

with canonical isomorphisms

$$\sigma : Id \to \gamma_Y \cdot \mathcal{L}_Y,$$

$$\tau : \mathcal{L}_Y \cdot \gamma_Y \to Id$$

(see 12.3.6). So it suffices to construct an isomorphism of functors

$$\alpha : \psi_* \to \gamma_Y \cdot f_* \cdot \mathcal{L}_X$$

from $D^+(\oplus\mathcal{A}_X)$ to $D^+(\oplus\mathcal{A}_Y)$.

Let $M \in D^+(\oplus\mathcal{A}_X)$ be $\mathcal{K}$-projective. We may (and will) assume that $M^i = 0,\ i << 0$. Then $\psi_* M = M$ considered as an $\mathcal{A}_Y$ -module and

$$\gamma_Y \cdot f_* \cdot \mathcal{L}_X(M) = \gamma_Y(f_*(\Omega_X \otimes_{\mathcal{A}_X} M))$$

$$= \Gamma(\Omega_Y \otimes_{C_Y} f_*(\Omega_X \otimes_{\mathcal{A}_X} M))$$

(Here we use the fact that $\Omega_X \otimes_{\mathcal{A}_X} M$ is bounded below and consists of soft sheaves on a paracompact space, hence acyclic for f_*).

The multiplication map

$$\Omega_Y \otimes f_*\Omega_X \to f_*\Omega_X$$

induces a quasiisomorphism of bounded below complexes of soft sheaves

$$\Omega_Y \otimes f_*(\Omega_X \otimes_{\mathcal{A}_X} M) \to f_*(\Omega_X \otimes_{\mathcal{A}_X} M)$$

(lemma 12.3.7) and hence a quasiisomorphism of left $\mathcal{A}_Y$-modules

$$a : \gamma_Y \cdot f_* \cdot \mathcal{L}_X(M) \tilde{\to} \Gamma(f_*(\Omega_X \otimes_{\mathcal{A}_X} M)$$

$$= \Gamma(\Omega_X \otimes_{\mathcal{A}_X} M).$$

On the other hand the canonical morphism of $\mathcal{A}_X$-modules

$$b : M \to \Gamma(\Omega_X \otimes_{\mathcal{A}_X} M)\ ,\ m \mapsto 1 \otimes m$$

is a quasiisomorphism, since $M \in D^+(\oplus\mathcal{A}_X)$ (12.3.4). Hence we may put

$$\alpha = a^{-1} \cdot b.$$

This proves the proposition.

12.7. Applications to $D_G(pt)$.

2.7.0. Let $\phi : H \to G$ be a homomorphism of **connected** Lie groups. Let

$$M_0 \subset M_1 \subset \dots$$

$$N_0 \subset N_1 \subset \dots$$

be smooth classifying sequences for H and G respectively (12.4.1). Then

$$M_0 \times N_0 \subset M_1 \times N_1 \subset \dots$$

is also a smooth classifying sequence for H and projections

$$M_i \times N_i \to N_i$$

induce a smooth map of the corresponding smooth models

$$f : BH \to BG$$

Recall that we have canonical identifications of $D^b_{H,c}(pt), D^+_H(pt)$ as certain full subcategories of $D^+(BH)$, and similar for G (2.7.2, 2.9.5). Consider

$$pt \to pt$$

as a ϕ–map. Then we get the functors

$$Q^* : D^b_{G,c}(pt) \to D^b_{H,c}(pt)$$

$$Q^* : D^+_G(pt) \to D^+_H(pt)$$

$$Q_* : D^+_H(pt) \to D^+_G(pt)$$

which under the above identification coincide with

$$f^* : D^+(BG) \to D^+(BH)$$

$$f_* : D^+(BH) \to D^+(BG)$$

respectively (6.11).Recall that we have the equivalences of categories

$$\mathcal{L}_H : D^+_{\mathcal{A}_H} \tilde{\to} D^+_H(pt)$$

$$\mathcal{L}_G : D^+_{\mathcal{A}_G} \tilde{\to} D^+_G(pt)$$

(12.4.8). The map f induces a homomorphism of the cohomology rings $A_G \to A_H$ and hence two functors $\psi^* : D^+_{\mathcal{A}_G} \to D^+_{\mathcal{A}_H}, \psi_* : D^+_{\mathcal{A}_H} \to D^+_{\mathcal{A}_G}$.

Consider the functional diagrams

$$(1) \qquad \begin{array}{ccc} D^+_{\mathcal{A}_H} & \xrightarrow{\mathcal{L}_H} & D^+_H(pt) \\ \psi^* \uparrow & & \uparrow Q^* \\ D^+_{\mathcal{A}_G} & \xrightarrow{\mathcal{L}_G} & D^+_G(pt) \end{array}$$

and

$$(2) \qquad \begin{array}{ccc} D^+_{\mathcal{A}_H} & \xrightarrow{\mathcal{L}_H} & D^+_H(pt) \\ \psi_* \downarrow & & \downarrow Q_* \\ D^+_{\mathcal{A}_G} & \xrightarrow{\mathcal{L}_G} & D^+_G(pt) \end{array}$$

12.7.1. Proposition. *The above diagrams (1) and (2) are commutative.*

Proof. Recall (12.4.5) that the localization functor $\mathcal{L}_H$ is the composition of the equivalence $D_{\mathcal{A}_H} \tilde{\to} D_{\mathcal{A}_{BH}}$ (12.4.4) with the localization $\mathcal{L}_{BH} : D_{\mathcal{A}_{BH}} \to D(BH)$ (and similar for G).

We have the obvious commutative diagrams

$$\begin{array}{ccc} D_{\mathcal{A}_H} & \tilde{\to} & D_{\mathcal{A}_{BH}} \\ \psi^* \uparrow & & \uparrow \psi^* \\ D_{\mathcal{A}_G} & \tilde{\to} & D_{\mathcal{A}_{BG}} \end{array}$$

and

$$\begin{array}{ccc} D_{\mathcal{A}_H} & \tilde{\to} & D_{\mathcal{A}_{BH}} \\ \psi_* \downarrow & & \downarrow \psi_* \\ D_{\mathcal{A}_G} & \tilde{\to} & D_{\mathcal{A}_{BG}}. \end{array}$$

Hence the commutativity of (1) follows from 12.5.1.

By the corollary 12.4.9 the category $D^+(pt)$ coincides with $D^+(\oplus C_{BH})$ (and similar for G). Hence f_* maps $D^+(\oplus C_{BH})$ to $D^+(\oplus C_{BG})$ and therefore by proposition 12.6.1 the following diagram is commutative

$$\begin{array}{ccc} D^+(\oplus \mathcal{A}_{BH}) & \xrightarrow{\mathcal{L}_{BH}} & D^+_H(pt) \\ \psi_* \downarrow & & \downarrow Q_* \\ D^+(\oplus \mathcal{A}_{BG}) & \xrightarrow{\mathcal{L}_{BG}} & D^+_G(pt) \end{array}$$

is commutative. But $D^+_{\mathcal{A}_H} = D^+(\oplus \mathcal{A}_H)$ (11.1.5) and similar for G. Hence the diagram (2) is also commutative which proves the proposition.

Let us now summarize the results of 12.4.6, 12.4.8, 12.7.1 in the following

12.7.2. Main Theorem. *Let G be a connected Lie group, $A_G = H(BG)$. Let $\mathcal{A}_G = (A_G, d = 0)$ be the corresponding DG-algebra.*

(i) There exists an equivalence of triangulated categories

$$\mathcal{L}_G : D^+_{\mathcal{A}_G} \tilde{\to} D^+_G(pt)$$

which is unique up to a canonical isomorphism. It commutes with $\overset{L}{\otimes}$ and the cohomology functor $(\cdot) \xrightarrow{H} Mod_{A_G}$.

(ii) The above equivalence restricts to the functor between the full subcategories

$$\mathcal{L}_G : D^f_{\mathcal{A}_G} \tilde{\to} D^b_{G,c}(pt),$$

which is an equivalence of t-categories commuting with $\overset{L}{\otimes}$, *RHom, D and H.*

(iii) Let $\phi : H \to G$ *be a homomorphism of connected Lie groups. It induces a homorphism of rings* $\psi : A_G \to A_H$ *and hence the functors of extension and restriction of scalars*

$$\psi^* : D^+_{A_G} \to D^+_{A_H}$$

$$\psi_* : D^+_{A_H} \to D^+_{A_G}.$$

Let $Q^* : D^+_G(pt) \to D^+_H(pt)$ *and* $Q_* : D^+_H(pt) \to D^+_G(pt)$ *be the functors of the inverse and direct image corresponding to the* ϕ*-map*

$$pt \to pt.$$

Then under the identification of (i) we have $\psi^* = Q^*, \psi_* = Q_*$.

The above theorem gives an algebraic interpretation of the category $D_G(pt)$ and is our main tool in applications.

13. Equivariant cohomology.

13.0. Let G be a Lie group. Let X be a G-space and $p : X \to pt$ be the map to a point. It induces the direct image functor

$$p_* : D_G^+(X) \to D_G^+(pt),$$

where the category $D_G^+(pt)$ can be naturally realized as a full subcategory of $D^+(BG)$ for the classifying space BG (2.9.5). Put $A_G = H(BG, R)$. Notice that for $S \in D(BG)$ its cohomology $H(S) = H(BG, S)$ is naturally a graded A_G-module.

13.1. Definition. Let $F \in D_G^+(X)$. The G-**equivariant cohomology** $H_G(X, F)$ of X with coefficients in F is by definition the graded A_G-module

$$H_G(X, F) := H(p_* F).$$

13.2. Definition. Assume that X is nice (1.4) and let $F \in D_G^b(X)$. The G-equivariant cohomology $H_{G,c}(X, F)$ with **compact supports** of X with coefficients in F is by definition the graded A_G-module

$$H_{G,c}(X, F) := H(p_! F).$$

13.3. Certainly the object $p_* F \in D_G^+(pt)$ (or $p_! F$) carries more information than the A_G-module $H_G(X, F)$ (or $H_{G,c}(X, F)$), and we usually prefer to work with the triangulated category $D_G^+(pt)$ rather than with the abelian one Mod_{A_G}. In particular, if X is a pseudomanifold and we work with the constructible category $D_{G,c}^b(X)$, then we interpret the formula

$$D \cdot p_! \simeq p_* \cdot D$$

as the **equivariant Poincare duality**. Notice that in case of a connected Lie group G this formula relates the Verdier duality in $D_{G,c}^b(X)$ with the "coherent" duality in $D_{A_G}^f \simeq D_{D,c}^b(pt)$ (see 12.7.2(ii)).

For the rest of this section 13 we put $R = \mathbb{R}$.

13.4. Example. Let G be a complex linear algebraic group acting algebraically on a complex algebraic variety X. Consider the G-equivariant intersection cohomology sheaf $IC_G(X)$. We denote by

$$IH_G(X) := H_G(X, IC_G(X))$$

$$IH_{G,c}(X) := H_{G,c}(X, IC_G(X))$$

the **equivariant intersection cohomology** (resp. with compact supports) of X.

Assume that X is proper. Then by the decomposition theorem (5.3) the direct image $p_*IC_G(X) = p_!IC_G(X)$ is a direct sum of (shifted) local systems on BG. If furthermore the group G is connected then each local system is constant, and we conclude that $IH_G(X)$ is a *free* A_G-module with the graded basis $IH(X)$, i.e.

$$IH_G(X) = A_G \otimes IH(X).$$

13.5. Let $\phi : H \to G$ be a homomorphism of Lie groups. It induces a map of classifying spaces $\overline{\phi} : BH \to BG$ and hence a map on cohomology $A_G \to A_H$. For an A_H-module M we denote by ${}_{A_G}M$ the corresponding A_G-module obtained via the restriction of scalars. It is clear that for $S \in D^+(BH)$ we have

$$(*) \qquad H(\overline{\varphi}_* S) = {}_{A_G}H(S).$$

Let $f : X \to Y$ be a ϕ-map and $F \in D^+_H(X)$. Let $Q_{f*}F \in D^+_G(Y)$ be its direct image. The following formula immediately follows from $(*)$ and 6.12.2

$${}_{A_G}H_H(X, F) = H_G(Y, Q_{f*}F).$$

13.6. Example. Let $f : X \to Y$ be a principal G-bundle. Then we know that $Q_* C_{X,G} = C_Y$. Hence by the above formula we have an isomorphism of graded groups

$$H_G(X) = H(Y).$$

Of course, $H_G(X)$ has more structure, namely the action of generators of A_G (the Chern classes of the G-bundle $X \to Y$).

13.7. Example. In the previous example assume that Y is a compact manifold and $G = S^1$. Assume moreover that the manifold X is orientable. We want to make the equivariant Poincare duality explicit in this case. Consider the map $p : X \to pt$. We have $A_{S^1} \simeq \mathbb{R}[x]$ and identify $D^b_{S^1,c}(pt) = D^f_{A_{S^1}}$ (12.7.2). Since X is compact we have $p_* C_{X,S^1} = p_! C_{X,S^1}$. Also $DC_{X,S^1} = C_{X,S^1}[d_X]$, $d_X = \dim X$, since X is orientable. Hence the Poincare duality formula

$$D \cdot p_! C_{X,S^1} = p_* \cdot DC_{X,S^1}$$

becomes

$$D \cdot p_* C_{X,S^1} = p_* C_{X,S^1}[d_X].$$

Put $p_* C_{X,S^1} = M \in D^f_{A_{S^1}}$. Since $H(M)$ has cohomological dimension ≤ 1 as an A_{S^1} - module, it follows from 11.3.3 that $M = H(M)$, i.e., the DG-module M has the zero differential. Note that in fact M is a torsion module, since $M = H(Y, I\!R)$. Hence by 11.3.1(i) we have

$$DM = \mathrm{Ext}^1_{A_{S^1}}(M, A_{S^1})[-1]$$

So the Poincare duality is a canonical isomorphism for the A_{S^1} – module $H_{S^1}(X)$

$$H_{S^1}(X) = \mathrm{Ext}^1_{A_{S^1}}(H_{S^1}(X), A_{S^1})[-d_X - 1]$$

13.8. Remark. Replace in the previous example S^1 by an arbitrary connected compact Lie group K of rank r. If we knew that the DG–module $p_* C_{X,K} \in D^f_{A_K}$ had zero differential (which is probably true), then we would obtain the similar formula

$$H_K(X) = \mathrm{Ext}^r_{A_K}(H_K(X), A_K)[-d_X - r]$$

using the same argument.

13.9. Example. Suppose we are in the situation of Theorem 9.1. Namely, let $0 \to K \to H \xrightarrow{\varphi} G \to 0$ be an exact sequence of complex linear reductive algebraic groups. Let $f : X \to Y$ be an algebraic morphism which is a ϕ–map. Assume that the following conditions hold.

(a) The group K acts on X with only finite stabilizers.

(b) The morphism f is affine and is the geometric quotient map by the action of K (all K–orbits on X are closed).

Then we know that $Q_* IC_H(X) = IH_G(Y)[d_K]$, where $d_K = \dim_{\mathbb{C}} K = d_X - d_Y$. Hence as in 13.5 above we obtain an isomorphism of A_G–modules

$$_{A_G}IH_H(X) = IH_G(Y)[d_X - d_Y].$$

13.10. Borel's interpretation of A_G.

Let G be a compact Lie group and T be a maximal torus in G. Let t be the Lie algebra of T and W be the Weyl group $W = N(T)/T$. The group W acts on t and hence also on the ring of polynomial functions $S(t^*)$ on t. By the classical result of A. Borel ([Bo3]) we have a canonical isomorphism of graded algebras

$$A_G = H(BG) \simeq S(t^*)^W,$$

where linear functions in $S(t^*)$ are assigned degree 2.

If G is connected then $S(t^*)^W$ is a polynomial ring and $S(t^*)$ is a *free* $S(t^*)^W$–module, since the group W is generated by reflections. Let $U \subset G$ be a closed subgroup with a maximal torus $T' \subset U$ and the Weyl group W'. The diagram of inclusions of groups

$$\begin{array}{ccc} T' & \to & T \\ \downarrow & & \downarrow \\ U & \to & G \end{array}$$

induces the diagram of classifying spaces

$$\begin{array}{ccc} BT' & \rightarrow & BT \\ \downarrow & & \downarrow \\ BU & \rightarrow & BG \end{array}$$

and the corresponding diagram of cohomology rings

$$\begin{array}{ccc} A_{T'} & \leftarrow & A_T \\ \uparrow & & \uparrow \\ A_U & \leftarrow & A_G. \end{array}$$

Borel showed that this diagram coincides with the natural diagram

$$\begin{array}{ccc} S(t'^*) & \leftarrow & S(t^*) \\ \uparrow & & \uparrow \\ S(t'^*)^{W'} & \leftarrow & S(t^*)^W \end{array}$$

under the above identification (the horizontal arrows in the last diagram are restrictions of functions).

Let G be a Lie group with finitely many components and $K \subset G$ be a maximal compact subgroup. Then topolically

$$G \simeq K \times \mathbb{R}^d.$$

Hence $A_G = A_K$ and the above picture can be applied to A_G.

13.11. Equivariant cohomology of induced spaces.

13.11.1. Let G be a group and $\phi : H \hookrightarrow G$ be an embedding of a closed subgroup. Let X be an H–space and $Y = G \times_H X$ be the induced G–space. The inclusion $f : X \hookrightarrow Y$ is a ϕ–map. Let $F \in D_H^+(X)$ and $Q_{f*}F \in D_G^+(Y)$. Then by 13.5 we have

$${}_{A_G}H_H(F) = H_G(Q_{f*}F).$$

We want to derive a similar relation for the cohomology with compact supports. So let us assume in addition that X is a constructible space (1.10) and that H, G are Lie groups with finitely many components.

Consider the commutative functorial diagram

$$\begin{array}{ccc} D^b_{H,c}(X) & \xrightarrow{Q_{f*}} & D^b_{G,c}(Y) \\ p_* \downarrow & & \downarrow p_* \\ D^b_{H,c}(pt) & \xrightarrow{Q_*} & D^b_{G,c}(pt). \end{array}$$

In order to find a relation between $H_{H,c}(F)$ and $H_{G,c}(Q_{f*}F)$ we need to know how functors Q_{f*}, Q_* behave with respect to duality.

We denote by d_M the dimension of a manifold M.

13.11.2. Recall (7.6.3) that if the group H is connected then there exists a canonical isomorphism of functors from $D^b_{H,c}(X)$ to $D^b_{G,c}(Y)$

$$Q_{f*} \cdot D = (D \cdot Q_{f*})[d_H - d_G].$$

Denote by $K(H) \subset H$ and $K(G) \subset G$ the maximal compact subgroups of H, G.

13.11.3. Proposition. *Assume that $K(H)$ is connected. Then there exists a canonical isomorphism of functors*

$$Q_* \cdot D = (D \cdot Q_*)[d_{K(H)} - d_{K(G)}].$$

Proof. We may (and will) assume that $K(H) = H, K(G) = G$.

Let E be an ∞–acyclic free G–space, hence also a free H–space. Then $G\backslash E = BG$, $H\backslash E = BH$ and we have the natural fibration $\pi : BH \to BG$ with the fiber G/H. Recall that categories $D^b_{H,c}(pt)$ and $D^b_{G,c}(pt)$ are naturally identified as certain full subcategories in $D^b(BH)$ and $D^b(BG)$ respectively. Under this identification the functor Q_* is π_*.

Consider the G–space $Z = G/H$ and its ∞–acyclic free resolution $P = E \times Z \to Z$. let $\overline{P} = G\backslash P$. Then $D^b_{G,c}(Z)$ is identified as a full subcategory in $D^b(\overline{P})$. Notice that $\overline{P} = H\backslash E = BH$ and the categories $D^b_{G,c}(Z)$ and $D^b_{H,c}(pt)$ are identified as the same full subcategory in $D^b(\overline{P}) = D^b(BH)$. Indeed, both categories consist of bounded complexes $S \in D^b(\overline{P})$ with constant cohomology sheaves of finite rank. Moreover, the direct image functor $\pi_* : D^b_{H,c}(pt) \to D^b_{G,c}(pt)$ is then identified with the direct image $p_* : D^b_{G,c}(Z) \to D^b_{G,c}(pt)$ for the G–map $p : Z \to pt$.

The space Z is compact, hence $p_* = p_!$ and so by 13.3

$$D \cdot p_* = p_* \cdot D,$$

where the duality D on the right takes place in $D^b_{G,c}(Z)$. So it remains to show that the equivalence of categories $D^b_{G,c}(Z) \to D^b_{H,c}(pt)$ commutes with the duality up to the shift by $d_Z = d_G - d_H$. Since the group H is connected this follows from 7.6.3. This proves the proposition.

We can now state our main result.

13.11.4. Theorem. *Assume that in the setup of 13.11.1 the group H is connected. There exists a natural isomorphism of functors*

$$Q_* \cdot p_! = p_! \cdot Q_{f*}[d_{K(H)} - d_{K(G)} + d_G - d_H]$$

from $D^b_{H,c}(X)$ to $D^b_{G,c}(pt)$.

In particular for $F \in D^b_{H,c}(X)$ we have a natural isomorphism of A_G–modules

$$_{A_G}H_{H,c}(F) = H_{G,c}(Q_{f*}F)[d_{K(H)} - d_{K(G)} + d_G - d_H]$$

Proof. Recall that $p_! = D \cdot p_* \cdot D$. So it remains to apply 13.11.2 and 13.11.3.

13.12. Relation with nonequivariant cohomology.

13.12.1. Let G be a Lie group and X be a G-space. Let $\phi : H \hookrightarrow G$ be an embedding of a closed subgroup. We want to compare the G-equivariant cohomology of X with the H-equivariant one.

Consider the commutative diagram

$$\begin{array}{ccc} X & \overset{f=id}{\rightarrow} & X \\ p\downarrow & & \downarrow p \\ pt & \overset{\pi}{\rightarrow} & pt \end{array}$$

where horizontal arrows are ϕ-maps. By theorem 7.3 we have a natural isomorphism of functors

$$p_* \cdot Q_f^* = Q_\pi^* \cdot p_*$$

from $D_G^+(X)$ to $D_H^+(pt)$.

13.12.2. Corollary. *In the setup of 13.12.1 assume that the groups G, H are connected and identify $D_G^+(pt) = D_{\mathcal{A}_G}^+, D_H^+(pt) = D_{\mathcal{A}_H}^+$ (12.7.2(i),(iii)). Then for $F \in D_G^+(X)$ we have*

$$p_* \cdot Res_{H,G} F = \mathcal{A}_H \overset{L}{\otimes}_{\mathcal{A}_G} (p_* F).$$

In particular, if $H = \{e\}$, then the nonequivariant cohomology $H(X, For(F))$ is computed from $p_* F \in D_{\mathcal{A}_G}^+$ by the formula

$$H(X, For(F)) = \mathbb{R} \overset{L}{\otimes}_{\mathcal{A}_G} (p_* F).$$

13.12.3. Remark. Let $F \in D_G^+(X)$. If we only know the equivariant cohomology $H_G(X, F)$ then we cannot in general recover the nonequivariant one $H(X, For(F))$. However, we have more information if we work with the DG-module $p_* F \in D_{\mathcal{A}_G}^+$. If, for example $p_* F \in D_{\mathcal{A}_G}^f$, then $H(X, For(F))$ is a graded basis of a minimal $\mathcal{K}$-projective $\mathcal{A}_G$-module P quasiisomorphic to $p_* F$ (11.4.6).

14. Fundamental example.

We analyze the stalk of the equivariant intersection cohomology sheaf at a point fixed by a 1-parameter subgroup.

14.1. Let $0 \to \mathbb{C}^* \to H \xrightarrow{\phi} G \to 0$ be an exact sequence of complex connected reductive groups. Let X be an affine complex variety with an algebraic action of the group H. Assume that X has an H-fixed point q which is the unique $\mathbb{C}^*$–fixed point. Assume that q is the attraction point under the $\mathbb{C}^*$–action on X, that is, the ring of functions on X is nonegatively graded by characters of $\mathbb{C}^*$.

Let $\{q\} \overset{i}{\hookrightarrow} X \overset{j}{\hookleftarrow} X_0 := X - \{q\}$ denote the corresponding closed and open embeddings. Put $F := IC_H(X)$ and $F_q^! := i^! F \in D_H^b(\{q\})$, $F_0 := j^* F \in D_H^b(X_0)$. Consider the exact triangle in $D_{H,c}^b(X)$

$$(1) \qquad i_* F_q^! \to F \to j_* F_0$$

and its direct image in $D_{H,c}^b(pt)$ under the map $p : X \to pt$

$$(2) \qquad p_* F_q^! \to p_* F \to p_* F_0.$$

We identify the categories $D_{H,c}^b(pt) = D_{\mathcal{A}_H}^f$ (12.7.2(ii)) and denote the DG–module $p_* F_0$ by M. Consider the canonical exact triangle in $D_{\mathcal{A}_H}^f$

$$(3) \qquad (\tau_{\geq 0} M)[-1] \to \tau_{<0} M \to M.$$

14.2. Theorem.
(i) The triangles (2), (3) above are isomorphic.

(ii) The objects $\tau_{<0} M$, $\tau_{\geq 0} M \in D_{\mathcal{A}_H}^f$ are free A_H–modules with zero differential and a basis given by $IH(X)$ and the costalk $i^! IC(X)$ respectively.

The proof of this theorem uses the decomposition theorem and the hard Lefschetz theorem. Let us first of all deduce an important corollary.

14.3. Corollary. *(i) The costalk $F_q^!$ is a* **direct sum** *of (shifted) constant equivariant sheaves $C_{q,H}$ at q:*

$$F_q^! = C_{q,H} \otimes i^! IC(X)$$

(i') Similarly for the stalk $F_q = i^* F$:

$$F_q = C_{q,H} \otimes i^* IC(X)$$

(ii) The equivariant intersection cohomology $IH(X)$ is a free A_H–module with a basis $IH(X)$, i.e.

$$IH_H(X) = A_H \otimes IH(X)$$

(ii') Similarly for $IH_{H,c}(X)$:

$$IH_{H,c}(X) = A_H \otimes IH_c(X)$$

Proof. (i) and (ii) follow immediately from the theorem; (i') and (ii') follow from (i) and (ii) by duality, since all basic functors commute with the forgetful functor.

14.4. Proof of theorem 14.2.

By our assumptions the action of the subgroup $\mathbb{C}^* \subset H$ defines on X the structure of an affine quasihomogeneous cone over the projective variety $\overline{X} = \mathbb{C}^* \backslash X_0$. Note that the group G acts naturally on $\overline{X}$ and the projection $f : X_0 \to \overline{X}$ is a ϕ-map.

(i) Let $Q_0, Q_{-1} \in D^f_{\mathcal{A}_H}$ be minimal $\mathcal{K}$-projective DG-modules quasiisomorphic to $p_* F$ and $p_* F^!_q$ respectively, so that the triangle (2) is isomorphic to a triangle

$$(2') \qquad Q_{-1} \xrightarrow{\varepsilon} Q_0 \to M$$

By remark 13.12.3 the free A_H-modules Q_0, Q_{-1} have bases $IH(X)$ and $i^! IC(X)$ respectively. It is known that $IH(X)[-1]$ and $i^! IC(X)[1]$ are isomorphic to the primitive and the coprimitive parts of $IH(\overline{X})$ with respect to the Lefschetz operator on $\overline{X}$. Hence is particular Q_0 is generated in degrees < 0 and Q_{-1} is generated in degrees > 0. This implies that in the $\mathcal{K}$-projective module $Q = \text{cone}(\varepsilon) = Q_0 \oplus Q_{-1}[1]$ we have

$$d_Q Q \subset mQ,$$

where $m \subset A_H$ is the maximal ideal. Therefore Q is the *minimal* $\mathcal{K}$-projective quasisomorphic to M (11.4.7). Moreover,

$$Q_0 = \tau_{<0} M$$

$$Q_{-1} = (\tau_{\geq 0} M)[-1].$$

Hence triangles (2') and (3) are isomorphic. This proves (i).

(ii) The homomorphism $\phi : H \to G$ induces an embedding $A_G \hookrightarrow A_H$. We have (non canonically)

$$A_H \simeq A_G[\lambda]$$

where λ has degree 2.

Consider the ϕ-map $f : X_0 \to \overline{X}$. Since $\mathbb{C}^*$ acts with only finite stabilizers, $Q_{f_*} F_0 = IC_G(\overline{X})[1]$ (9.1(iv)). Therefore

$$(*) \qquad {}_{A_G}M = IH_G(\overline{X})[1]$$

(13.5). The variety $\overline{X}$ is projective, so $IH_G(\overline{X})$ is a free A_G–module and

$$IH_G(\overline{X}) = A_G \otimes_{\mathbb{R}} IH(\overline{X})$$

(13.4). Therefore we obtain an isomorphism of $\mathbb{R}$–vector spaces

$$\mathbb{R} \otimes_{A_G} M \simeq IH(\overline{X})[1]. \tag{4}$$

Notice that the left hand side in (4) is naturally an $\mathbb{R}[\lambda]$–module.

14.5. Lemma. *Under the identification (4) the action of λ on $IH(\overline{X})$ coincides (up to a scalar) with the Lefschetz operator for the projective variety $\overline{X}$.*

Let us postpone the proof of the lemma and finish the proof of the theorem.

Since the A_G–module ${}_{A_G}H(M) = IH_G(\overline{X})[1]$ is free, ,the A_H–module $H(M)$ has cohomological dimension ≤ 1. So by proposition 11.3.3 the DG–module M has zero differential $M = H(M)$.

Let Pr and CPr be the primitive and the coprimitive parts of the cohomology $IH(\overline{X})$ with respect to the Lefschetz operator λ. By considering the Hilbert polynomial of the A_H–module M we find that it has a minimal projective resolution of the form

$$0 \to P_{-1} \xrightarrow{\delta} P_0 \to M \to 0,$$

where $P_0 = A_H \otimes_{\mathbb{R}} Pr[1]$, $P_{-1} = A_H \otimes_{\mathbb{R}} CPr[-1]$. Hence $\mathrm{Cone}(\delta) = P_0 \oplus P_{-1}[1]$ is a minimal $\mathcal{K}$–projective quasiisomorphic to M. Moreover

$$P_0 = \tau_{<0} M$$

$$P_{-1} = (\tau_{\geq 0} M)[-1]$$

(use Hard Lefschetz for $\overline{X}$). Hence $P_0 = Q_0, P_{-1} = Q_{-1}$ and hence Q_0, Q_{-1} have zero differential which proves part (ii) in the theorem.

Proof of Lemma 14.5.

Consider the commutative diagram of group homomorphisms

$$\begin{array}{ccc} \mathbb{C}^* & \to & H \\ \downarrow & & \downarrow \\ \{e\} & \to & G. \end{array}$$

Since $\phi : H \to G$ is surjective the assumption of theorem 7.3 is satisfied. Consider the diagram

$$\begin{array}{ccc} X_0 & \xrightarrow{f} & \overline{X} \\ p\downarrow & & \downarrow p \\ pt & \xrightarrow{\pi} & pt \end{array}$$

where the horizontal arrows are the ϕ-maps. Then by theorem 7.3 and proposition 7.2 the functors of the direct image $Q_{f*}, Q_{\pi*}, p_*$ commute with the restriction functors $Res_{\{e\},G}$,
$Res_{\mathbb{C}^*,H}$. We have to prove something about the object

$$Res_{\{e\},G} \cdot Q_{\pi*} \cdot p_* F_0 = \mathbb{R} \otimes_{A_G} M \in D^+_{\{e\}}(pt).$$

But by above mentioned results it is equal to

$$Q_{\pi*} \cdot p_* \cdot Res_{\mathbb{C}^*,H} F_0.$$

So we may (and will) assume that $G = \{e\}, H = \mathbb{C}^*$.

Let E be an ∞-acyclic free $\mathbb{C}^*$-space and $\mathbb{C}^* \backslash E = B\mathbb{C}^*$ be the classifying space. Put $X_{0\mathbb{C}^*} = \mathbb{C}^* \backslash (X_0 \times E)$. We have natural projections

$$\overline{X} \xleftarrow{\overline{f}} X_{0\mathbb{C}^*} \xrightarrow{p} B\mathbb{C}^*. \tag{1}$$

If we embed $D^+_{\mathbb{C}^*}(X_0) \subset D^+(X_{0\mathbb{C}^*})$ then the direct image $Q_{f^*} : D^+_{\mathbb{C}^*}(X_0) \to D^+(\overline{X})$ becomes $\overline{f}_* : D^+(X_{0\mathbb{C}^*}) \to D^+(\overline{X})$. We know that $\overline{f}_* C_{X_{0\mathbb{C}^*}} = C_{\overline{X}}$ (9.1(ii)), hence

$$H(\overline{X}) = H(X_{0\mathbb{C}^*}). \tag{2}$$

But $H(X_{0\mathbb{C}^*})$ is a module over $H(B\mathbb{C}^*) = \mathbb{R}[\lambda]$ via the projection p in (1); and the image of λ on $H(\overline{X})$ via the identification (2) above is the first Chern class of the (almost) principal $\mathbb{C}^*$-bundle $f : X_0 \to \overline{X}$. This proves the lemma.

14.6. Corollary. *In the previous setup the natural map of A_H-modules*

$$IH_H(X) \to IH_H(X_0)$$

induces an isomorphism modulo the maximal ideal

$$IH(X) = \mathbb{R} \otimes_{A_H} IH_H(X) \tilde{\to} \mathbb{R} \otimes_{A_H} IH_H(X_0)$$

Proof. Indeed, in the proof of the above theorem 14.2 we saw that this map is the minimal projective cover

$$P_0 \to M.$$

Hence the assertion follows.

14.7. Consider the dual picture.

Put $F_q := i^* F$ and consider the exact triangle

$$F[-1] \to i_* F_q[-1] \to j_! F_0 \tag{1'}$$

which is the dual to the triangle (1) in 14.1. Consider its direct image with compact supports in $D^b_{H,c}(pt)$

$$(2') \qquad p_!F[-1] \to p_!F_q[-1] \to p_!F_0.$$

Since $DF = F$ we find that the triangle (2′) is dual to the triangle (2) in 14.1.

By the above theorem 14.2 the triangle (2) in 14.1 is of the form

$$(3) \qquad (\tau_{\geq 0}M)[-1] \to \tau_{<0}M \to M$$

where all modules have zero differential. Moreover, the A_H–modules $\tau_{\geq 0}M, \tau_{<0}M$ are free and the diagram (3) is the minimal projective resolution of the A_H–module M. Hence the dual traingle is

$$(3') \qquad \tau_{>0}(DM)[-1] \to \tau_{\leq 0}(DM) \to DM,$$

which is also a minimal projective resolution of the A_H–module $DM = \mathrm{Ext}^1_{A_H}(M, A_H)$ (11.3.1(i)). Identifying terms in isomorphic triangles (2′) and (3′) we find the following

14.8. Corollary. *The natural map of A_H-modules*

$$(IC_{X,H})_q[-1] \to IH_{H,c}(X_0)$$

induces an isomorphism modulo the maximal ideal

$$IC_{X,q}[-1] = \mathbb{R} \otimes_{A_H} (IC_{X,H})_q[-1] \tilde{\to} \mathbb{R} \otimes_{A_H} IH_{H,c}(X_0).$$