Part III. Equivariant cohomology of toric varieties.

15. Toric varieties.

15.0. In this part we present some applications of the theory developed in parts I and II. Namely, we work out the "simpliest" case of toric varieties. It turns out that there exists a natural complex which is a resolution of the equivariant intersection cohomology (with compact supports) of a toric variety. As a byproduct we obtain some applications to combinatorics (see theorem 15.7 below).

We start by recalling the notion of a toric variety. Then we define some combinatorial structure (the minimal complex) on the fan corresponding to a toric variety. After that we state our main result about this structure and its relation with the equivariant intersection cohomology. The rest of the text is devoted to the proof of the main theorem 15.7.

15.1. Review of toric varieties.

We recall some basic notions and results in the theory of toric varieties. For the proofs the reader is referred to [Dan] or [KKMS-D].

Let $T = (\mathbb{C}^*)^n$ be a complex torus of dimension n. Let $\Lambda := \operatorname{Hom}(\mathbb{C}^*, T) \simeq \mathbb{Z}^n$ be the group of 1–parameter subgroups in T. Denote by $N = \Lambda \otimes_{\mathbb{Z}} \mathbb{R}$ the corresponding real vector space of dimension n. Note that N can be identified with the Lie algebra of the compact torus $(S^1)^n \subset (\mathbb{C}^*)^n$.

Let $Y = Y^n$ be an algebraic variety. We say that Y is a **toric variety** (or a T–toric variety) if T acts on Y and Y contains a dense orbit isomorphic to T (this implies that the number of T–orbits in Y is finite).

A normal toric variety Y is described combinatorially by a **fan** $\Phi_Y = \Phi$ in N. Recall that a fan Φ is a collection $\Phi = \{\sigma\}$ of finitely many **rational** (with respect to the lattice $\Lambda \subset N$) **convex polyhedral cones** σ which intersect along common faces, such that if $\sigma \in \Phi$ and τ is a face of σ then also $\tau \in \Phi$.

In that description Y is **affine** if Φ consists of a unique cone σ together with its faces and X is **complete** if $\cup_{\sigma \in \Phi} \sigma = N$.

The orbits $\mathcal{O}$ of T in Y are in $1-1$ correspondence with the cones $\sigma \in \Phi$. A cone $\sigma \in \Phi$ corresponding to an orbit $\mathcal{O}$ will be denoted by $\sigma_{\mathcal{O}}$, and vice versa, we denote by $\mathcal{O}_\sigma$ the orbit corresponding to a cone $\sigma \in \Phi$. We have $\dim_{\mathbb{C}} \mathcal{O} = n - \dim_{\mathbb{R}}(\sigma_{\mathcal{O}})$. More precisely, the subspace of N spanned by $\sigma_{\mathcal{O}}$ is the (real part of) Lie algebra of the stabilizer of $\mathcal{O}$. We have $\mathcal{O}' \subset \overline{\mathcal{O}}$ if and only if $\sigma_{\mathcal{O}} \subset \sigma_{\mathcal{O}'}$.

15.2. Fix a torus $T = (\mathbb{C}^*)^n$. Let $A = \mathbb{R}[x_1, \dots, x_n]$ be the ring of polynomial functions on N. We consider A as a graded ring, where $\deg(x_i) = 2$. Let $m \subset A$ be the maximal ideal. For a cone $\sigma \subset N$ denote by A_σ the (graded) ring of polynomial functions on σ. The restriction of functions defines the ring homomorphism $A \to A_\sigma$. Fix a fan $\Phi = \{\sigma\}$ in N (not necessarily rational).

Consider a complex

$$\mathcal{K} : 0 \to \mathcal{K}^{-n} \xrightarrow{\partial^{-n}} \mathcal{K}^{-n+1} \xrightarrow{\partial^{-n+1}} \cdots \xrightarrow{\partial^{-1}} \mathcal{K}^0 \to 0$$

of graded A–modules, where $\mathcal{K}^{-i}$ is a direct sum of terms $\mathcal{K}_\sigma$ for $\sigma \in \Phi$, dim$\sigma = i$:

$$\mathcal{K}^{-i} = \oplus_{dim\sigma=i}\mathcal{K}_\sigma.$$

Assume that each $\mathcal{K}_\sigma$ is a free A_σ–module, i.e., $\mathcal{K}_\sigma$ is a direct sum of (shifted) topics of A_σ.

We assume that the differential ∂ maps $\mathcal{K}_\sigma$ to $\oplus_{\tau\subset\sigma}\mathcal{K}_\tau$. Let $\sigma \in \Phi$ be a dimension i. Denote by

$$\mathcal{K}(\sigma) : \mathcal{K}_\sigma \xrightarrow{\partial_\sigma^{-i}} \bigoplus_{\substack{\tau\subset\sigma \\ dim\tau=i-1}} \mathcal{K}_\tau \xrightarrow{\partial_\sigma^{-i+1}} \cdots \to \mathcal{K}^0 \to 0$$

the "restriction" of the complex $\mathcal{K}$ to the cone σ with its faces.

15.3. Definition. A complex $\mathcal{K}$ as above is called **minimal** if it satisfies the following conditions

(a) $\mathcal{K}^0 = \mathbb{R}[n]$, i.e. it is the A–module $\mathbb{R} = A/m$ placed in degree $-n$.

(b) Let $I_\sigma = ker\partial_\sigma^{-i+1}$ in the complex $\mathcal{K}(\sigma)$. Then the differential ∂_σ^{-i} induces an isomorphism of $\mathbb{R}$–vector spaces

$$\partial_\sigma^{-i} : \mathcal{K}_\sigma/m\mathcal{K}_\sigma \tilde{\to} I_\sigma/mI_\sigma.$$

One can construct a minimal complex by induction on the dimension of σ.

15.4. Lemma. *Let $\mathcal{K}', \mathcal{K}''$ be two minimal complexes. Then they are isomorphic (noncanonically).*

Proof. The proof is essentially the same as that of the uniqueness of a minimal projective resolution. Namely, one constructs an isomorphism between $\mathcal{K}'$ and $\mathcal{K}''$ step by step by induction on the dimension of σ.

15.5. Remark. A minimal complex $\mathcal{K}$ is by definition "locally" exact (except in degree $-n$), i.e., $im\,\partial_\sigma^{-i} = ker\,\partial_\sigma^{-i+1}$ for all σ. Hence the exactness of $\mathcal{K}$ at $\mathcal{K}^{-i}$ is equivalent to the kernel $ker\,\partial^{-i}$ being the sum of local kernels $ker\,\partial_\sigma^{-i}$, dim$\sigma = i$.

15.6. Note that the ring A is canonically identified with the cohomology ring $A_T = H(BT, \mathbb{R})$ (13.10). Hence in case of a rational fan Φ one may hope that the minimal complex has a meaning in terms of the T–equivariant cohomology of the corresponding toric variety X. This is so indeed.

15.7. Theorem. *Let Φ be a rational fan in N, corresponding to a normal toric variety X. Assume that Φ is complete or consists of a single cone of dimension n together with its faces. Then the following hold.*

(i) The minimal complex is **exact** *except in degree $-n$.*

(ii) There is an isomorphism of A-modules

$$Ker\,\partial^{-n} \simeq IH_{T,c}(X)$$

(iii) The free A_σ-module $\mathcal{K}_\sigma$ has a graded basis isomorphic to the stalk $IC(X)|_{O_\sigma}$ of the intersection cohomology sheaf $IC(X)$ on the corresponding orbit $O_\sigma \subset X$.

15.8. Remarks. 1. The assumptions on the fan Φ in the theorem mean that the corresponding variety X is complete or affine with a (unique) T–fixed point. In these cases we know that $IH_{T,c}(X)$ is a **free** A–module with a basis $IH_c(X)$ (see 13.4 and 14.3(ii')). Hence by (ii) in the theorem we get

$$Ker\,\partial^{-n} \simeq A \otimes IH_c(X).$$

2. Assume that the fan Φ is simplicial. Then the variety X has only quotient singularities, i.e., $IC(X) = C_X$. Then from (iii) we get $\mathcal{K}_\sigma \simeq A_\sigma$. Hence a minimal complex is isomorphic to the complex of functions, i.e. $\mathcal{K}_\sigma = A_\sigma$ and ∂ is the restriction of functions with $\pm$ sign depending on some chosen orientation of the cones σ. Can the reader check directly that this complex of functions is exact except at $\mathcal{K}^{-n}$?

3. We will prove the theorem in two steps. First of all we will construct a certain **canonical** "geometric" complex $\mathcal{L}$ of A–module coming from the variety X and prove that it is a resolution of $IH_{T,c}(X)$. Then we will prove that $\mathcal{L}$ is a minimal complex. We know no other way of proving that a minimal complex is exact (away from $\mathcal{K}^{-n}$) except by interpreting it geometrically as the complex $\mathcal{L}$.

4. The complex $\mathcal{L}$ mentioned above provides a **natural** resolution of $IH_{T,c}(X)$. Assume, for example, that a group Γ acts on the space N by automorphisms of the lattice Λ and preserves the fan Φ. Then Γ also acts on the toric variety X and hence on its cohomology $IH_c(X)$. It is sometimes easy to compute the (graded) character of Γ on each term $\mathcal{L}^{-i}$ of the complex $\mathcal{L}$ (for example, if the fan Φ is simplicial). This provides the charater of Γ on the equivariant intersection cohomology $IH_{T,c}(X)$. But $IH_{T,c}(X)$ is a *free* A–module with the basis $IH_c(X)$. Hence from the character of Γ on $IH_{T,c}(X)$ one can find the character of Γ on $IH_c(X)$.

Here is an example. Let T be a maximal torus in an algebraic group G. We have the (simplicial) fan Φ of the Weyl chambers in N. The Weyl group W acts on the fan Φ and hence on the corresponding toric variety X. The above method allows one to compute the character of W on the cohomology $H(X)$ (see [DL]).

15.9. Conjecture. The statement (i) in the theorem still holds if we drop the assumption on the rationallity of the fan Φ.

15.10. Proof of Theorem 15.7.

Consider the following filtration of X by closed subsets

$$\emptyset = X^{-1} \subset X^0 \subset X^1 \subset \ldots \subset X^n = X,$$

where $X^k := \coprod_{dim\mathcal{O} \leq k} \mathcal{O}$, $k = -1, 0, 1, \ldots, n$.

Put $U^{k+1} := X - X^k$, so that $U^{n+1} = \emptyset$, $U^0 = X$ and $U^n = T$ is the dense orbit in X. Let $Z^k := X^k - X^{k-1} = \coprod_{dim\mathcal{O}=k} \mathcal{O}$.

Denote by

$$j_k : U^k \hookrightarrow X,$$

$$i_k : Z^k \hookrightarrow X$$

the open and the locally closed embeddings.

Put $F = IC_T(X)$ - the T-equivariant intersection cohomology sheaf on X, and

$$F_k := j_{k!} j_k^* F.$$

Consider the collection of exact triangles in $D^b_{T,c}(X)$:

$$F_k[k] \to i_{k!} i_k^* F[k] \to F_{k+1}[k+1] \tag{1}$$

for $k = 0, \ldots, n$.

Apply the functor $H_{T,c}$ – equivariant cohomology with compact supports – to triangles (1). This produces complexes of A_T-modules

$$0 \to H_{T,c}(F_k)[k] \xrightarrow{a_k} H_{T,c}(i_k^* F)[k] \xrightarrow{b_k} H_{T,c}(F_{k+1})[k+1] \to 0 \tag{2}$$

Consider the induced complex

$$0 \to H_{T,c}(i_0^* F) \xrightarrow{d^0} H_{T,c}(i_1^* F)[1] \xrightarrow{d^1} \cdots \xrightarrow{d^{n-1}} H_{T,c}(i_n^* F)[n] \to 0 \tag{3}$$

where $d^i = a_{i+1} \cdot b_i$.

15.11. Theorem. *(a) The complex (3) above is exact except at* $H_{T,c}(i_0^* F)$.
(b) $Ker d^0 = IH_{T,c}(X)$.

This theorem is equivalent to the following

15.12. Claim. *Complexes (2) are short exact sequences.*

It is easier to prove a more precise statement. We need the following

15.13. Definition. An A-module $P \neq 0$ is called (cohomologically) pure of codimension k if

$$\mathrm{Ext}^i_A(P, A) = 0 \quad \text{for} \quad i \neq k.$$

The theorem follows from the following

15.14. Proposition. *The sequences (2) are short exact and the A-module $H_{T,c}(F_k)$ is pure of codimension k.*

Proof of proposition 15.14.

15.15. Lemma. *Let $\mathcal{O} \subset X$ be an orbit. Then the restriction $F|_{\mathcal{O}}$ is a direct sum of (shifted) constant equivariant sheaves $C_{\mathcal{O},T}$.*

Proof of lemma. Let $U \subset X$ be the star of the orbit $\mathcal{O}$, that is, U is an open set consisting of orbits $\mathcal{O}'$ such that $\mathcal{O} \subset \overline{\mathcal{O}'}$. Let $T_{\mathcal{O}} = Stab(\mathcal{O}) \subset T$ be the stabilizer of the orbit $\mathcal{O}$. Since X in normal, one can show that $T_{\mathcal{O}}$ is connected hence a subtorus in T. Let $X_{\mathcal{O}} = \overline{T}_{\mathcal{O}} \subset U$ be the closure in U of the subtorus $T_{\mathcal{O}} \subset T \subset U$. Then the T-space U is the induced one from the $T_{\mathcal{O}}$-space $X_{\mathcal{O}}$

$$U = T \times_{T_{\mathcal{O}}} X_{\mathcal{O}}.$$

The space $X_{\mathcal{O}}$ is an affine $T_{\mathcal{O}}$-toric variety with a fixed point $q = X_{\mathcal{O}} \cap \mathcal{O}$. Hence it satisfies assumptions of theorem 14.2. Thus the stalk $IC_{X_{\mathcal{O}},T_{\mathcal{O}}}|_q$ is a direct sum of shifted constant sheaves $C_{q,T_{\mathcal{O}}}$ (14.3(i')). Hence the corresponding statement is true on the induced space U, which proves the lemma.

15.16. Remark. Notice that $H_{T,c}(i_k^*F) = \oplus_{dim\mathcal{O}=k} H_{T,c}(F|_{\mathcal{O}})$. Moreover, it follows from the above lemma and from theorem 13.11.4 that

$$H_{T,c}(F|_{\mathcal{O}})[k] = A_\sigma \otimes_{\mathbb{R}} IC(X)|_{\mathcal{O}},$$

where k=dim$\mathcal{O}$, $IC(X)|_{\mathcal{O}}$ is the stalk of $IC(X)$ at a point on $\mathcal{O}$ and σ is the cone in Φ corresponding to the orbit $\mathcal{O}$.

15.17. Corollary. *The A-module $H_{T,c}(i_k^*F)$ is pure of codimension k.*

15.18. Corollary. *The codimension of the support of the A-module $H_{T,c}(F_k)$ in SpecA is $\leq k$. Hence*

$$Ext^i_A(H_{T,c}(F_k), A) = 0, \quad \textit{for } i < k.$$

Proof of corollary. The A-module $H_{T,c}(F_k)$ may be computed using the spectral sequence associated to the filtration

$$U^n \subset \ldots \subset U^{k+1} \subset U^k.$$

The E_1 term consists of A-modules $H_{T,c}(i_s^* F), s \geq k$, which are pure of codimension s (lemma 15.17) and hence have support of codimension s. Hence the codimension of the support of $H_{T,c}(F_k)$ is $\geq k$ (in fact $= k$). This proves the corollary.

Now we can prove proposition 15.14. We use induction on k. For $k = 0$ we have the sequence

$$0 \to H_{T,c}(F) \to H_{T,c}(i_0^* F) \to H_{T,c}(F_1)[1] \to 0.$$

This is a piece of the long exact sequence of cohomology arising from the triangle (1) (for $k = 0$). Hence it suffices to prove that the connecting homomorphism $H_{T,c}(F_1)[1] \xrightarrow{\delta} H_{T,c}(F)[1]$ is zero. We know that $H_{T,c}(F)$ is a *free* A-module (remark 15.8(1)) and that the support of $H_{T,c}(F_1)$ has codimension 1 in $SpecA$. Hence there exists no nonzero map δ. This shows that the sequence above is short exact. By considering the long exact sequence of $\mathrm{Ext}_A(\cdot, A)$ applied to this short sequence we find that $H_{T,c}(F_1)$ is pure of codimension 1 (use corollary 15.17). This finishes the proof for $k = 0$.

Suppose we proved the lemma for $k-1$. Consider the sequence

$$0 \to H_{T,c}(F_k)[k] \to H_{T,c}(i_k^* F)[k] \to H_{T,c}(F_{k+1})[k+1] \to 0$$

By induction we know that $H_{T,c}(F_k)$ is pure of codimension k. On the other hand $H_{T,c}(F_{k+1})$ has support of codimension $k+1$. Hence the connecting homomorphism $H_{T,c}(F_{k+1})[k+1] \xrightarrow{\delta} H_{T,c}(F_k)[k+1]$ is zero and the sequence is short exact. Applying $\mathrm{Ext}_A(\cdot, A)$ to this sequence we find

$$0 \to \mathrm{Ext}_A^k(H_{T,c}(i_k^* F)[k]) \to \mathrm{Ext}_A^k(H_{T,c}(F_k)[k]) \to \mathrm{Ext}_A^{k+1}(H_{T,c}(F_{k+1})[k+1]) \to 0$$

so that $H_{T,c}(F_{k+1})$ is pure of codimension $k+1$. This finishes the induction step and proves proposition 15.14 and theorem 15.11.

15.19. Definition. Let

$$0 \to \mathcal{L}^{-n} \xrightarrow{\partial^{-n}} \mathcal{L}^{-n+1} \xrightarrow{\partial^{-n+1}} \cdots \xrightarrow{\partial^{-1}} \mathcal{L}^0 \to 0$$

be the complex (3) shifted by n, i.e.

$$\mathcal{L}^{-k} := H_{T,c}(i_{n-k}^* F)[n-k]$$

$$\partial^{-k} := d^{n-k}$$

Fix a cone $\sigma \in \Phi$ of dimension k. Consider the restriction $F|_{\mathcal{O}_\sigma}$ of F to the corresponding orbit $\mathcal{O}_\sigma \subset X$. By remark 15.16 above we know that $H_{T,c}(F|_{\mathcal{O}_\sigma})[n-k]$ is a free A_σ module with the basis $IC(X)|_{\mathcal{O}_\sigma}$. We put $\mathcal{L}_\sigma := H_{T,c}(F|_{\mathcal{O}_\sigma})[n-k]$, so that

$$\mathcal{L}^{-k} = \oplus_{dim\sigma=k}\mathcal{L}_\sigma.$$

Clearly, the differential

$$\partial^{-k} : \mathcal{L}^{-k} \to \mathcal{L}^{-k+1}$$

maps

$$\partial^{-k} : \mathcal{L}_\sigma \to \oplus_{\tau\subset\sigma}\mathcal{L}_\tau.$$

In view of theorem 15.11 above the theorem 15.7 follows from the following

15.20. Proposition. *The complex $\mathcal{L}$ is minimal.*

Proof. We have to check conditions (a), (b) of definition 15.3.

(a) $\mathcal{L}^0 = H_{T,c}(i_n^* F)[n] = H_{T,c}(C_T[n])[n] = \mathbb{R}[n]$ (13.11.4).

(b) Let $\sigma \in \Phi$ be a cone of dimension k. Let

$$\mathcal{L}(\sigma) : \mathcal{L}_\sigma \xrightarrow{\partial_\sigma^{-k}} \bigoplus_{\substack{\tau\subset\sigma \\ dim\tau=k-1}} \mathcal{L}_\tau \xrightarrow{\partial_\sigma^{-k+1}} \cdots \to \mathcal{L}_0 \to 0$$

be the restriction of the complex $\mathcal{L}$ to σ as on 15.2. Let $I_\sigma = Ker\partial_\sigma^{-k+1}$. We need to show that ∂_σ^{-k} induces an isomorphism.

$$(*) \qquad \partial_\sigma^{-k} : \mathcal{L}_\sigma/m\mathcal{L}_\sigma \tilde{\to} I_\sigma/mI_\sigma$$

Put $\mathcal{O} = \mathcal{O}_\sigma$. Let $U \subset X$ be the open subset as in the proof of lemma 15.15. Geometrically the complex $\mathcal{L}(\sigma)$ corresponds to the "restriction" of the complex (3) to the open set U. Since

$$U = T \times_{T_\mathcal{O}} X_\mathcal{O}$$

(see the proof of lemma 15.15 above), the complex $\mathcal{L}(\sigma)$ is obtained from the corresponding complex (3) for $X_\mathcal{O}$ by restricting scalars from A_σ to A and shifting (13.11.4).

Hence we may (and will) assume that $\mathcal{O} = q$ is the T-fixed point ($dim\sigma = k = n$) in the affine toric variety U. By theorem 15.11 the complex $\mathcal{L}(\sigma)$ is exact except at $\mathcal{L}_\sigma$. Put $V = U - \{q\}$. The kernell I_σ is equal to $H_{T,c}(F|_V[1])$ and the map

$$\partial_\sigma^{-n} : \mathcal{L}_\sigma \to I_\sigma$$

is the canonical boundary map

$$H_{T,c}(F_q) \to H_{T,c}(F|_V)[1],$$

which induces an isomorphism module the maximal ideal (corollary 14.8). This proves proposition 15.20 and theorem 15.7.