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Models of Representations of Lie Groups*

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Starting with the classical works of E. Cartan and H. Weyl, representations of compact Lie groups have been studied in sufficient detail. Yet although the characters of the irreducible representations have been described with exhaustive thoroughness and lucidity, the construction of the representations themselves is in a less satisfactory state. Just about the only general construction, inspired by the theory of infinite-dimensional representations of groups, is as follows. Let U be a compact Lie group. Consider the so-called principal affine space $N\backslash G$, where G is the complex Lie group corresponding to U and N is a maximal unipotent subgroup of G. Then one can realize the irreducible representations of the group U in the space of homogeneous analytic functions on the principal affine space (in other words, in the space of analytic sections of some one-dimensional fiber space over the quotient space of the group U by a maximal torus).

We are dealing here with a model of the representations; namely one can introduce a scalar product on the space of analytic functions on the principal affine space so that in the decomposition of the resulting unitary representation of U into irreducible factors, all the irreducible representations of U occur with multiplicity one (see [1, 4]). Granted the naturalness of this approach, just about the only shortcoming of this construction is that in this model we require the functions to be analytic.

Let us consider an example to clarify this. Let $U = SO_3$ be the group of rotations of 3-space. Then the model associated with the principal affine space is as follows. Consider the space of even analytic functions $f(z_1, z_2)$ in two complex variables with the scalar product

$$\langle f, g\rangle = \int f\bar{g}e^{-|z_1|^2-|z_2|^2}\,dz_1\,d\bar{z}_1\,dz_2\,d\bar{z}_2.$$

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Decomposing the functions in this space into homogeneous functions, we get every irreducible representation of SO_3 exactly once. However, long before this construction one knew how to construct the representations of SO_3 in the space of all (not just analytic) square-integrable functions on the 2-sphere. This also gives a model of the representations of SO_3; i.e., every irreducible representation of SO_3 occurs once.

In this article a similar model of its representations is constructed for every semisimple compact Lie group U; i.e., a homogeneous space is described, such that every irreducible representation of U is contained exactly once in the space of all (not just analytic) suitably chosen square-integrable vector functions on the homogeneous space. For the group SO_3 this realization coincides with the one described above.

In our realization the homogeneous space is the compact symmetric space X corresponding to the group U (see [5]). Let $K_{\mathbf{R}}$ be the stationary subgroup of some point of $x_0 \in X$. We will construct a representation τ of the group $K_{\mathbf{R}}$ such that the induced representation of U contains every irreducible representation of U exactly once. For a simply connected group U we have $\dim \tau = 2^l$, where l is the rank. (The space $L^2(X)$ contains one 2^lth part of all the irreducible representations of U, the representations with even highest weight.)

In this article the necessary representations of the stationary subgroup $K_{\mathbf{R}}$ are constructed separately for each of the simple Lie groups. This method has its advantages, because the resulting spaces and representations of $K_{\mathbf{R}}$ are of independent interest.

As an example, let us show what our models look like for the classical Lie groups. The representations given here differ in appearance from those constructed in Section 4, because simply connected simple groups will be considered there. For this reason we have had to use the apparatus of Clifford algebras in Section 4. Recall that we are considering the representation of the group U induced by the representation τ of the stationary subgroup $K_{\mathbf{R}}$:

1. $U = U_n$, $K_{\mathbf{R}} = O_n$, τ is the natural representation of O_n in the space $\oplus_{i=0}^{n} \Lambda^i(\mathbf{R}^n)$.
2. $U = O_{2n+1}$, $K_{\mathbf{R}} = O_{n+1} \times O_n$, τ is the representation of $K_{\mathbf{R}}$ in the space $\oplus_{i=0}^{n+1} \Lambda^i(\mathbf{R}^{n+1})$, trivial on the second factor.
3. $U = U\mathrm{Sp}_{2n}$, $K_{\mathbf{R}} = U_n$, τ is the natural representation of $K_{\mathbf{R}}$ in $\oplus_{i=0}^{n} \Lambda^i(\mathbf{C}^n)$.
4. $U = O_{2n}$, $K_{\mathbf{R}} = O_n \times O_n$, τ is the representation of $K_{\mathbf{R}}$ in the space $\oplus_{i=0}^{n} \Lambda^i(\mathbf{R}^n)$, trivial on the second factor.

This paper is an introduction to the study, from a unified viewpoint, of the representations of noncompact real Lie groups and of semisimple algebraic groups over various fields (see [2]).

1. Statement of the main results

1. Let G be a connected algebraic reductive group over the field of complex numbers. Fix a Cartan subgroup H of G. Denote by R_H the lattice of weights of the group H, consisting of the algebraic homomorphisms $H \to \mathbf{C}^*$. Let $\Delta \subset R_H$ be the root system of the group G relative to H; for every root γ, denote by N_γ the one-parameter unipotent subgroup of G which corresponds to the root γ. Fix a system of positive roots $\Delta_+ \subset \Delta$.

Let θ be a fixed Cartan involution of the group G, i.e., an algebraic antiautomorphism such that $\theta^2 = \mathrm{id}$ and $\theta(h) = h$ for all $h \in H$ (it is easy to check that any two such involutions are conjugate by an inner automorphism corresponding to some element of H). Clearly, $\theta(N_\gamma) = N_{-\gamma}$ for all $\gamma \in \Delta$.

The group $K = \{ g \in G | \theta(g) = g^{-1}\}$ will be called an *involutory subgroup* of G. For example, if $G = \mathrm{GL}_n(\mathbf{C})$, and H is the subgroup of diagonal matrices, then θ can be taken to be the map $g \mapsto g^T$ (transposition); in this case $K = O_n(\mathbf{C})$.

Let i be an antilinear automorphism of the group G that maps H to itself and preserves the lattice of weights R_H (i.e., $\chi(i(h)) = \overline{\chi(h)}$ for all $\chi \in R_H$, $h \in H$). Then $i(N_\gamma) = N_\gamma$ for all $\gamma \in \Delta$, so that the subgroup $G_R = \{ g \in G | i(g) = g\}$ is a split real form of the group G. Assume that i commutes with θ or, equivalently, $\theta(G_R) = G_R$. Then the subgroup $U = \{ g \in G | i(\theta(g)) = g^{-1}\}$ is compact; it is the compact form of the group G. The compact subgroup $K_R = U \cap G_R = \{u \in U | \theta(u) = u^{-1}\}$ will play an important role. Let us call it the *involutory subgroup* of the group U.

The group $S = K \cap H = \{h \in H | h^2 = 1\}$ is also essential here. Since every element $h \in H$ is determined by the numbers $\chi(h)$, $\chi \in R_H$, it follows that $i(s) = s$ for all $s \in S$, so that $S \subset K_{\mathbf{R}}$. From the definition it follows that S is a finite commutative group and $\operatorname{card} S = 2^{\operatorname{rk} G}$, where $\operatorname{rk} G$ is the rank of G.

Example. $G = GL_n(\mathbf{C})$, H is the diagonal subgroup, θ is the transposition of matrices, i is the passage to the complex conjugate matrix. Then

$$K = O_n(\mathbf{C}); \qquad G_{\mathbf{R}} = GL_n(\mathbf{R}), \qquad K_{\mathbf{R}} = O_n,$$

and S consists of diagonal matrices with ± 1 on the diagonal.

2. Let us state the main results of this work. Let τ be a finite-dimensional representation of the group $K_{\mathbf{R}}$. For every irreducible representation π of the group U, we are interested in the multiplicity with which the representation π occurs in the representation $\operatorname{Ind}_{K_{\mathbf{R}}}^U(\tau)$, i.e., the number

$$\dim \operatorname{Hom}_U\big(\pi, \operatorname{Ind}_{K_{\mathbf{R}}}^U(\tau)\big).$$

Proposition 1. *Let $m \in R_H$ be the highest weight of the irreducible representation π and $m|_S$ the restriction of m to S, the corresponding one-dimensional* *representation of S. Then*

$$\dim \operatorname{Hom}_U\left(\pi, \operatorname{Ind}_{K_{\mathbf{R}}}^U(\tau)\right) \leqslant \dim \operatorname{Hom}_S(m|_S, \tau|_S). \tag{1}$$

We say that a representation of a compact group is multiplicity-free if it can be decomposed into the sum of pairwise inequivalent irreducible representations.

Corollary 1. *If $\tau|_S$ is multiplicity-free then so is $\operatorname{Ind}_{K_{\mathbf{R}}}^U(\tau)$.*

It turns out that for "almost all" irreducible representations of the group U, strict equality holds in formula (1).

More precisely, let $C \in R_H$ be the collection of highest weights of all irreducible representations of the group U (i.e., the Weyl chamber in R_H relative to the ordering given by the system Δ_+).

Proposition 2. *Let τ be a fixed representation of the group $K_{\mathbf{R}}$. Then there exists a weight $l \in C$ such that for all weights $m \in C + l$ one has*

$$\dim \operatorname{Hom}_U\left(\pi, \operatorname{Ind}_{K_{\mathbf{R}}}^U(\tau)\right) = \dim \operatorname{Hom}_S(m_S, \tau|_S), \tag{2}$$

where $m_S = m|_S$ and π is the irreducible representation of U with highest weight m.

Corollary 2. *Let τ be a representation of the group $K_{\mathbf{R}}$ such that $\tau|_S$ is the regular representation of the group S. Then for any irreducible representation π of the group U, $\dim \operatorname{Hom}_U(\pi, \operatorname{Ind}_{K_{\mathbf{R}}}^U(\tau)) \leqslant 1$, and there exists a weight $l \in C$ such that $\dim \operatorname{Hom}_U(\pi, \operatorname{Ind}_K^U(\tau)) = 1$ for all irreducible representations π with highest weight $m \in C + l$.*

We want to find the representations τ such that equality (2) holds for all irreducible representations of the group U. It is easy to point out one such example.

Proposition 3. *If $\tau = 1$ is the trivial representation of the group $K_{\mathbf{R}}$, an irreducible representation π of group U occurs in $\operatorname{Ind}_{K_{\mathbf{R}}}^U(1)$ if and only if $m|_S = 1$, i.e., m is an even weight ($m \in 2R_H$).*

Propositions 1–3 are proved in Section 2.

Definition. A representation of a compact group is called a model if any irreducible representation occurs in it exactly once.

One could say that if τ is any representation of the group $K_{\mathbf{R}}$ such that $\tau|_S$ is a regular representation of S, then $\operatorname{Ind}_{K_{\mathbf{R}}}^U(\tau)$ is "almost a model." An important result of our work is the construction of a representation τ of the group $K_{\mathbf{R}}$ for every group G such that $\operatorname{Ind}_{K_{\mathbf{R}}}^U(\tau)$ is a model for the group U. Namely, we have:

Theorem 1 (on models). *There exists a representation τ of the group $K_{\mathbf{R}}$ such that the representation $\operatorname{Ind}_{K_{\mathbf{R}}}^U(\tau)$ is a model; i.e., every irreducible representation of U occurs exactly once in the decomposition.*

This theorem is proved in Sections 3 and 4. In Section 3 we state a condition on the representation τ under which $\mathrm{Ind}_{K_{\mathbf{R}}}^{U}(\tau)$ is a model (Theorem 1), and reduce the problem to the case of simple groups. In Section 4 we construct the representation τ for each simple group separately.

2. Proof of Propositions 1–3

1. The proof of the propositions formulated above is based on an analysis of the dimensions

$$\dim \mathrm{Hom}_U\big(\pi, \mathrm{Ind}_{K_{\mathbf{R}}}^{U}(\tau)\big) = \dim \mathrm{Hom}_{K_{\mathbf{R}}}(\pi|_{K_{\mathbf{R}}}, \tau) \quad \text{and} \quad \dim \mathrm{Hom}_S(m|_S, \tau|_S).$$

First of all, by Frobenius' duality

$$\dim \mathrm{Hom}_U\big(\Pi, \mathrm{Ind}_{K_{\mathbf{R}}}^{U}(\tau)\big) = \dim \mathrm{Hom}_{K_{\mathbf{R}}}(\Pi|_{K_{\mathbf{R}}}, \tau).$$

Further, every finite-dimensional representation of the group U can be extended to an algebraic representation of the group G; we thus get a one-to-one correspondence between the representations of the groups U and G (in the following we will consider only the algebraic representations of G). Since $K_{\mathbf{R}}$ is a compact form of the group K, a similar assertion also holds for the groups $K_{\mathbf{R}}$ and K. Thus our problem is reduced to comparing the numbers $\dim \mathrm{Hom}_K(\pi|_K, \tau)$ with the numbers $\dim \mathrm{Hom}_S(m|_S, \tau|_S)$ in a purely algebraic setting. In the following we will only be concerned with algebraic groups and representations.

2. We introduce some additional notation. Let $N = \prod_{\gamma \in \Delta_+} N_\gamma$ be the maximal unipotent subgroup, $N^- = \prod_{\gamma \in \Delta_+} N_{-\gamma}$ the opposite subgroup, $B = HN$ the Borel subgroup, and let $B^- = HN^-$. Clearly, $\theta(B) = B^-$, $\theta(N) = N^-$.

Lemma 1. *$B \cap K = S$, and KB is an open dense subset of G.*

Proof. Clearly, $B \cap K = \{g \in B \mid \theta(g) = g^{-1}\}$. Since $\theta(B) = B^-$ and $B \cap B^- = H$, we have $B \cap K = H \cap K = S$. Considering the tangent spaces at the identity, it is easy to convince oneself that KB is an open and (because G is connected) dense subset of G.

Suppose we are given a representation τ of the group K on the space L. We will say that the rational function f on the group G with values in L is τ-equivariant if $f(kg) = \tau(k)f(g)$ for all $k \in K$, $g \in G$.

Suppose m is a weight of the group H. Extend it to a character of the group B, setting $m(N) = 1$; we say that the function f on G has weight m if $f(gb) = m(b)f(g)$ for all $b \in B$, $g \in G$.

Since KB is dense in G, every τ-equivariant rational function f of weight m is regular on KB.

3. The following lemma reduces the study of the multiplicities we are interested in to the study of the dimensions of certain spaces of algebraic functions on G.

Lemma 2.

a. *Let τ be a representation of the group K in the space L and $m \in R_H$. Then the space* $\mathrm{Hom}_S(m|_S, \tau|_S)$ *is isomorphic to the space of rational τ-equivariant functions of weight m.*

b. *Let m be the highest weight of an irreducible representation π of the group G. Then the space* $\mathrm{Hom}_K(\pi|_K, \tau)$ *is isomorphic to the space of regular τ-equivariant functions of weight m.*

Proof.

a. Let L be the space of the representation τ, and $\varphi: \mathbf{C} \to L$ be a homomorphism from $\mathrm{Hom}_S(m|_S, \tau|_S)$. Define a regular function $\tilde{f}$ on $K \times B$ by the formula $\tilde{f}(k,b) = m(b) \cdot \tau(k)\varphi(1)$.
Clearly, $\tilde{f}(ks, b) = \tilde{f}(k, sb)$; i.e., $\tilde{f}$ depends only on the product kb; therefore, $\tilde{f}(k,b) = f(kb)$, where f is a certain regular τ-equivariant function of weight m on KB. f can be viewed as a rational function on G. Conversely, for every such function one constructs a homomorphism $\varphi \in \mathrm{Hom}_S(m|_S, \tau|_S)$ by the formula $\varphi(1) = f(1)$.

b. Let V be the space of the representation π, and let v^+ be the highest weight vector in V. For every $\varphi \in \mathrm{Hom}_K(\pi|_K, \tau)$ one constructs a regular τ-equivariant function of weight m on G by

$$f(g) = \varphi(\pi(g)v^+).$$

To construct the inverse mapping, consider the representation π^* on the space V^* that is dual to π. Define the mapping ψ from the space V^* to the space of complex regular functions of weight m on G by the formula $\psi(v^*)(g) = (v^*, \pi(g)v^+)$. The mapping ψ is an isomorphism [7, 9].

Now let f be a regular τ-equivariant function of weight m. Then to every vector $l^* \in L^*$ there corresponds a regular function of weight m $u_{l^*}(g) = (l^*, f(g))$ and therefore the element $\psi^{-1}(u_{l^*}) \in V^*$. In this way we have obtained a mapping $\varphi^*: L^* \to V^*$ and the dual mapping $\varphi: V \to L$. Clearly, $\varphi \in \mathrm{Hom}_K(\pi|_K, \tau)$. The mappings just constructed define an isomorphism between the space of regular τ-equivariant functions of weight m and the space $\mathrm{Hom}_K(\pi|_K, \tau)$. The lemma is proved.

Proposition 1 immediately follows from Lemma 2.

4. *Proof of Proposition 3.* It follows from Lemma 2 that if a representation π with highest weight m occurs in $\mathrm{Ind}_K^U(1)$, then $m|_S = 1$; i.e., m is an even weight. Conversely, suppose that m is an even weight; i.e., $m = 2l$ where $l \in C$. We have to prove that there exists a nonzero regular function $q_m(g)$ such that $q_m(kgb) = m(b)q_m(g)$ for all $k \in K$, $g \in G$, $b \in B$.

Let ρ be an irreducible representation of the group U with highest weight

l, V the space of the representation ρ, V^* the dual space, v^+ a vector of highest weight in V, v^{*-} a vector of lowest weight in V^*. Set $a_l(g) = (v^{*-}, \rho(g)v^+)$. Then

$$a_l(\theta(b_1)gb_2) = l(b_1b_2)a(g).$$

Indeed,

$$\begin{aligned} a_l(\theta(b_1)gb_2) &= (v^{*-}, \rho(\theta(b_1))\rho(g)\rho(b_2)v^+) \\ &= (\rho^*(\theta(b_1))^{-1}v^{*-}, \rho(g)\rho(b_2)v^+) \\ &= l(b_1)l(b_2)(v^{*-}, \rho(g)v^+) = l(b_1b_2)a_l(g). \end{aligned}$$

Now set $q_{2l}(g) = a_l(\theta(g)g)$. Then $q_{2l}(kg) = q_{2l}(g)$ for all $k \in K$, $g \in G$ and

$$q_{2l}(gb) = a_l(\theta(b)gb) = l(b^2)q_{2l}(g) = m(b)q_{2l}(g),$$

as required.

5. *Proof of Proposition 2.* Let τ be a representation of the group K. By Lemma 2 it suffices to prove that there exists a weight $l_0 \in C$ such that every τ-equivariant rational function of weight $m \in l_0 + C$ is regular. In Section 2.4 we constructed, for every even weight $m \in 2R_H$, a regular function q_m on G satisfying $q_m(kgb) = m(b)q_m(g)$. Clearly, the mapping $f \to q_m f$ establishes an isomorphism between the space of τ-equivariant functions of weight l and the space of τ-equivariant functions of weight $l + m$.

Note that $R_H/2R_H$ is a finite set, and the space of τ-equivariant rational functions of any given weight is finite-dimensional. Since any τ-equivariant weight function f is regular on KB, Proposition 2 follows from the following lemma.

Lemma 3. *There exists a weight* $m \in 2R_H$ *such that* $q_m(g) = 0$ *for all* $g \in G$, $g \notin KB$.

Proof. We first prove that $g \in KB$ if and only if $\theta(g)g \in B^-B = N^-HN$. Indeed, if $g = kb \in KB$, then $\theta(g)g = \theta(b)\theta(k)kb = \theta(b)b \in B^-B$. Conversely, let

$$g_1 = \theta(g)g, \quad \text{and} \quad g_1 = u^-hu, \qquad u^- \in N^-, \quad h \in H, \quad u \in N. \quad (3)$$

Clearly, $\theta(g_1) = g_1$; i.e., $g_1 = \theta(u)\theta(h)\theta(u^-)$. Since $\theta(u) \in N^-$, $\theta(h) = h \in H$, $\theta(u^-) \in N$, and the decomposition (3) is unique, we have $\theta(u) = u^-$. Since the group G is connected, there exists an element $h_1 \in H$ such that $h_1^2 = h$. Set $b = h_1u$ and $k = gb^{-1}$. Then it is clear that $\theta(b)b = g_1$ and $\theta(k)k = 1$; i.e., $k \in K$. Therefore, $g = kb \in KB$.

To complete the proof we use the formula $q_{2l}(g) = a_l(\theta(g)g)$, where a_l is the function introduced in Section 2.4.

Let l be an arbitrary regular highest weight, i.e., $wl \neq l$ for any element w of the Weyl group W of the group G. Then it is known (see, for example,

[7]) that $a_l(g)=0$ when $g \notin N^-HN$. Therefore, $q_{2l}(g)=a_l(\theta(g)g)=0$ when $g \notin KB$.

3. Shallow representations

This section and Section 4 are devoted to proving the theorem on models. Here we present a method that allows us to determine when a representation τ of the group $K_{\mathbf{R}}$ is a model (Theorem 1). In Section 4 we will turn to the construction of models using this method.

1. We first reduce the problem of constructing a model to the case of a simply connected group G. Let $p: G_1 \to G$ be the universal cover; θ, θ_1 and i, i_1 agree with p. Then $pU_1 = U$, $pK_{1\mathbf{R}} \subset K_{\mathbf{R}}$. Suppose τ_1 defines a model for $K_{1\mathbf{R}}$ (i.e., $\mathrm{Ind}_{K_{1\mathbf{R}}}^{U_1}(\tau_1)$ is a model for U_1) and $\tau' \subset \tau_1$ is the largest subrepresentation which is trivial on $\mathrm{Ker}\, p \cap K_{1\mathbf{R}}$. Then τ' can be viewed as a representation of $pK_{1\mathbf{R}} \subset K_{\mathbf{R}}$. Now it is easy to show that $\tau = \mathrm{Ind}_{pK_{1\mathbf{R}}}^{K_{\mathbf{R}}}(\tau')$ defines a model for $K_{\mathbf{R}}$.

2. By Propositions 1 and 2 and Lemma 2 the representation τ of the group K defines a model if and only if the following two properties hold:

1. $\tau|_S$ is a regular representation of S.
2. Suppose that $m \in C$ and f is a τ-equivariant rational function on G of weight m, then f is a regular function.

In this section we will find a condition on the representation τ that guarantees condition 2.

It is convenient for us to consider the simplest case first—$G = SL_2(\mathbf{C})$. Let i be complex conjugation, $i(g)=\bar{g}$, and θ transposition. Then

$$U = SU_2, \quad K = SO_2(C), \quad K_{\mathbf{R}} = SU_2 \cap SL_2(\mathbf{R}) = SO_2, \quad S = \{\pm e\},$$

$$H = \left\{ \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \right\}, \qquad B = \left\{ \begin{pmatrix} t & x \\ 0 & t^{-1} \end{pmatrix} \right\}.$$

The subgroups H and K are isomorphic to the multiplicative group $\mathbf{C}^*$. Hence their one-dimensional (algebraic) representations are given by an integer, the degree n of the representation $t \to t^n$.

Lemma 4. *If $g = \theta(g)$, then $g = \theta(g_1)g_1$ for some $g_1 \in G = SL_2(\mathbf{C})$.*

The proof is by direct verification.

Lemma 5. *Let τ be a one-dimensional representation of K of degree $j=0, +1$, or -1; and let m be the highest weight of B, $m\begin{pmatrix} t & x \\ 0 & t^{-1} \end{pmatrix} = t^n$, $n \geqslant 0$. Let f be a rational τ-equivariant function on G. Then f is a regular function.*

Proof. Since KB is dense in G, $f(e) \neq 0$. Now, setting $b = k = -e$, $g = e$ in

$$f(kgb) = \tau(k)m(b)f(g), \tag{4}$$

we see that the weight m must be even if $i=0$ and odd if $i = \pm 1$.

In the space of all regular functions on G that satisfy $f(gb) = m(b)f(g)$, left translations realize the $(n+1)$-dimensional representation of the group $G = SL_2(\mathbf{C})$ (n is the degree of weight m). But the $(n+1)$-dimensional representation of $SL_2(\mathbf{C})$ contains all the weights of H of degrees between $-n$ and n whose parity coincides with that of n. Therefore, the existence of a nonzero regular function f satisfying the condition follows from the fact that the groups K and H are conjugate in $SL_2(\mathbf{C})$. The lemma now follows from the fact that a rational function f is determined uniquely up to a factor by (4).

Corollary. *Let τ be a representation of K which is a direct sum of representations of degrees 0, $+1$, and -1, and f a rational τ-equivariant function on G of weight $\geqslant 0$ with respect to B. Then f is a regular function.*

3. Let us return to the general case. For every simple root α of the group G consider the corresponding homomorphism $\varphi_\alpha : SL_2(\mathbf{C}) \to G$ which agrees with the choice of i and θ. Since the group G is simply connected, φ_α is an embedding [7]. Set

$$G_\alpha = \varphi_\alpha(SL_2(\mathbf{C})) \quad \text{and} \quad K_\alpha = \varphi_\alpha(SO_2(\mathbf{C})) \subset K.$$

A representation τ of the group K is called shallow if for any simple root α the restriction of τ to $K_\alpha \cong SO_2(\mathbf{C})$ contains only one-dimensional representations of K_α of degrees 0, $+1$, and -1.

Theorem 2. *Let τ be a shallow representation of the group K, $m \in C$, and f a τ-equivariant rational function on G of weight m. Then f is a regular function.*

Proof. Since KB is dense in G, the function f is regular on KB. Further, $G_\alpha \cap KB$ is dense in G_α, and therefore the function f can be restricted to G_α. By the corollary to Lemma 5, $f|_{G_\alpha}$ is a regular function on G_α. Therefore, the function f is regular on the set $P = \cup_\alpha KG_\alpha B$. Therefore, Theorem 1 follows from the following lemma:

Lemma 6. $\dim(G \setminus P) \leqslant \dim G - 2$.

Proof. Let $G_\theta = \{g \in G \mid g = \theta(g)\}$. Consider the mapping $r: G \to G_\theta$ given by $r(g) = \theta(g)g$. The inverse image of any point $x \in G_\theta$ is either empty or homeomorphic to K. Since $\dim G = \dim K + \dim C_\theta$, to prove the lemma it suffices to show that $\dim(G_\theta \setminus r(P)) \leqslant \dim G_\theta - 2$. To this end we use the Bruhat decomposition of the group G. Let $WH \subset G$ be the normalizer of H in G, and for each $w \in W$ pick a representative $x_w \in wH$. Define the subgroups

$$N_w^+ = N^+ \cap wN^+w^{-1}, \qquad \tilde{N}_w^+ = N^+ \cap wN^-w^{-1},$$

$$N_w^- = N^- \cap w^{-1}N^-w, \qquad \tilde{N}_w^- = N^- \cap w^{-1}N^+w,$$

so that $N^- = N_w^-\tilde{N}_w^-$, $N_w^- \cap \tilde{N}_w^- = \{e\}$. The Bruhat decomposition asserts that every element $g \in G$ can be uniquely represented as $g = u^-hx_wu^+$, where $w \in W$, $u^+ \in N_w^+$, $h \in H$, $u^- \in N^-$. Set $G^w = N^- H x_w N_w^+$; now consider $G^w \cap G_\theta$ for each w.

a. If $n \in wH$ for some w, then $\theta(n) \in w^{-1}H$. Since $\theta(N^-) = N^+$, $\theta(N^+) = N^-$, we have $\theta(G^w) = G^{w^{-1}}$. Therefore, $G^w \cap G_\theta \neq \emptyset$ only if $w \neq w^{-1}$.

b. Let $w = w^{-1}$. Then $\theta(N_w^+) = N_w^-$, $\theta(\tilde{N}_w^+) = \tilde{N}_w^-$. Let

$$g \in G^w \cap G_\theta \quad \text{and} \quad g = u^- \tilde{u}^- n u^+$$

be its Bruhat decomposition. We have

$$\theta(g) = \theta(u^+)\theta(n)\theta(\tilde{u}^-)\theta(u^-),$$

and the equality $g = \theta(g)$ is equivalent to the following set of equalities:

$$\theta(n) = n; \qquad \theta(u^-) = u^+; \qquad \theta(\tilde{u}^-) = n^{-1}\tilde{u}^- n.$$

Therefore

$$\dim G^w \cap G_\theta = d_1^w + d_2^w,$$

where

$$d_2^w = \dim(G_\theta \cap wH) \leqslant \dim H, \qquad d_2^w = \dim\{\tilde{u}^- \in \tilde{N}_w^-, \theta(\tilde{u}^-) = n^{-1}\tilde{u}^- n\}$$

(one can prove that d_1^w depends only on w, not on $n \in wH$). Let us estimate d_1^w and d_2^w separately.

c. Let n_0 be a fixed element of $G_\theta \cap wH$ and $n = hn_0 \in G_\theta \cap wH$. Then $hn_0 = n = \theta(n) = \theta(n_0)\theta(h) = whw^{-1} \cdot n_0$, i.e., $h = whw^{-1}$. Therefore, $d_1^w = \dim H$ {multiplicity of the eigenvalue $+1$ of the element w}. It is well known [7] that if $w \neq w_\gamma$ for some root γ, then w is not a reflection in a hyperplane, and hence $d_1^w \leqslant \dim H - 2$. Therefore,

$$\dim(G^w \cap G_\theta) \leqslant \dim H + \dim N^- - 2 = \dim G_\theta - 2$$

where

$$w \neq w_\gamma, w \neq e.$$

d. Let $w = w_\gamma$ for some positive root γ. Then $d_1^w = \dim H - 1$. Let us estimate d_2^w. Fix $h \in wH \cap G_\theta$. Then d_2^w equals the dimension of the space X_f of fixed points of the map $f(\tilde{u}) = n^{-1}\theta(\tilde{u})n$ of the subgroup $\tilde{N}_w^-$ into itself. Since for every root subgroup N_β of G, $\theta(N_\beta) = N_{-\beta}$, we have $f(N_\beta) = N_{-w\beta}$. Among the negative roots β there is exactly one for which $-w_\gamma\beta = \beta$, namely $\beta = -\gamma$, it is clear that $N_{-\gamma} \in \tilde{N}_w^-$. For all the other negative roots we have $-w_\gamma\beta \neq \beta$. Since whenever $-w_\gamma\beta \neq \beta$ the component of the element $\tilde{u} \in X_f$ in $N_{-w\beta}$ is determined by the component of $\tilde{u}$ in N_β, it follows that $\dim X_f = 1/2(\dim \tilde{N}_w^- + 1)$. If $w = w_\gamma$ and γ is not a simple root, then $\dim N_w^- \geqslant 3$; therefore $d_1^w \leqslant \dim \tilde{N}_w^- - 1$. Hence in this case too, one has $\dim(G^w \cap G_\theta) \leqslant \dim G_\theta - 2$.

e. To prove the lemma it remains to show that $G^e \cap G_\theta \subset r(P)$ and that for any simple root α, $G^{w_\alpha} \cap G_\theta \subset r(P)$.

First, suppose that $g \in G^e \cap G_\theta$. Then

$$g = u^- hu^+, \quad u^- \in N^-, \quad h \in H, \quad u^+ \in N^+ \quad \text{and} \quad \theta(u^+) = u^-.$$

Since H is a connected torus, there exists an element $h_1 \in H$ such that $h = h_1^2 = h_1\theta(h_1)$. Then

$$g = u^- h_1^2 u^+ = r(hu^+) \in r(P),$$

because $hu^+ \in B$.

f. Now suppose that $w = w_\alpha$, and α is a simple root. Any element $x \in G^w$ can be represented as $x = u^- hg_\alpha u^+$, where $u^- \in U_w^-$, $u^+ \in U_w^+$, $h \in Z_H^0(HG_\alpha)$ (connected component), $g_\alpha \in G_\alpha$. Here u^+, u^-, and hg_α are uniquely defined. If $x = \theta(x)$, then $u_1 = \theta(u_2)$ and $\theta(hg_\alpha) = hg_\alpha$. But $\theta(hg_\alpha) = \theta(g_\alpha)h = h\theta(g_\alpha)$ and therefore $g_\alpha = \theta(g_\alpha)$. Since $G_\alpha \cong SL_2(\mathbf{C})$, $g_\alpha = \theta(g_1)g_1$ for some $g_1 \in G_2$ (Lemma 4). Further, $Z_H^0(HG_\alpha)$ is a connected torus; therefore, there exists an $h_1 \in Z_H^0(HG_\alpha)$ such that $h_1^2 = h$. Then

$$x = r(g_1 h_1 u_1^+) \in r(P),$$

because $g_1 h_1 u^+ \in G_\alpha B$.

Corollary 3. *Let τ be a shallow representation of the group K that is regular when restricted to S. Then* $\operatorname{Ind}_{K_R}^U(\tau)$ *is a model.*

4. Constructing models

1. Using Theorem 1 for every semisimple group G we construct a model representation of the corresponding subgroup K. Clearly, it suffices to construct such a representation for a simple, simply connected group G. We cannot yet offer a general construction for such a representation, so we will have to consider each case separately.

We first list the simple groups G and the corresponding subgroups $K \subset G$ (see [8]). (For the exceptional types, G is taken to be the simple, simply connected group of the type given. See text for the precise description of the subgroup $\mathbf{Z}_2$):

$$A_n : G = SL_n(\mathbf{C}), K = SO_n(\mathbf{C});$$

$$B_n : G = \mathrm{Spin}_{2n+1}, K = (\mathrm{Spin}_{n+1} \times \mathrm{Spin}_n)/\mathbf{Z}_2;$$

$$C_n : G = \mathrm{Sp}_{2n}(\mathbf{C}), K = GL_n(\mathbf{C});$$

$$D_n : G = \mathrm{Spin}_{2n}, K = (\mathrm{Spin}_n \times \mathrm{Spin}_n)/\mathbf{Z}_2;$$

$$E_6 : K = \mathrm{Sp}_8(\mathbf{C})/\mathbf{Z}_2;\ E_7 : K = SL_8(\mathbf{C})/\mathbf{Z}_2;\ E_8 : K = \mathrm{Spin}_{16}/\mathbf{Z}_2;$$

$$F_4 : K = (SL_2(\mathbf{C}) \times \mathrm{Sp}_6(\mathbf{C}))/\mathbf{Z}_2;\ G_2 : K = (SL_2(\mathbf{C}) \times SL_2(\mathbf{C}))/\mathbf{Z}_2.$$

2. We will use Clifford algebras in constructing the representations that define a model. We recall the essential facts about these algebras [6].

Let E be an n-dimensional complex space, $e_1, \ldots, e_n$ a basis for E.

Consider the form Σx_i^2 on E, and let $O(E)$ and $SO(E)$ be the orthogonal and the special orthogonal groups with respect to this form. Denote by $C_n = C(E)$ the algebra with generators $e_1, \ldots, e_n$ and relations $e_i^2 = -1$, $e_i e_j = -e_j e_i$, $i, j = 1, \ldots, n$; $i \neq j$. The group $O(E)$ acts on the algebra $C(E)$ by changing coordinates in the space generated by $e_1, \ldots, e_n$.

We list the basic properties of Clifford algebras.

a. The elements $e_{i_1} \cdots e_{i_k}$, where $1 \leqslant i_1 < i_2 < \cdots < i_k \leqslant n$ form a basis for C_n; in particular $\dim C_n = 2^n$.

b. If $n = 2k$, then C_n is isomorphic to the matrix algebra of order 2^k and the center C_n consists of scalars.

If $n = 2k+1$, then C_n is isomorphic to the direct sum of two matrix algebras of order 2^k; its center consists of elements $x + ye$ where $e = e_1 e_2 \cdots e_n$; $x, y \in \mathbf{C}$.

c. Let C_n^+ and C_n^- be the subspaces of C_n generated by the monomials with an even or an odd number of factors, respectively. Then

$$C^+ \cdot C^+ = C^- \cdot C^- = C^+, \qquad C^+ \cdot C^- = C^- \cdot C^+ = C^-.$$

d. Consider the mapping $\varphi_n : C_{n-1} \to C_n$ defined on the generators by $\varphi_n(e_i) = e_i \cdot e_n$ $(i = 1, \ldots, n-1)$. This mapping defines an isomorphism of C_{n-1} onto C_n^+.

e. Let $* : C_n \to C_n$ be the antiautomorphism which is the identity on the generators $e_i^* = e_j$. Then

$$(e_{i_1} \cdots e_{i_k})^* = e_{i_1} \cdots e_{i_k} \cdot (-1)^{k(k-1)/2}.$$

If $x = \Sigma x_i e_i$, then $x \cdot x = -\Sigma x_i^2$; in particular, if $\Sigma x_i^2 = 1$, then x is invertible in C_n. For each vector $x = \Sigma x_i e_i \in E$ such that $\Sigma x_i^2 = 1$, denote by $\varphi(x)$ the transformation of the space E given by $\varphi(x)y = xyx^*$; this is the reflection in the plane orthogonal to x. In particular, $\varphi(x) \in O(E)$. Denote by pin_n the subgroup of the multiplicative group of the algebra C_n generated by all the elements $x \in E$ for which $x \cdot x = -1$, and extend the mapping φ to a homomorphism $\varphi : \mathrm{pin}_n \to O(E)$. Then set

$$\mathrm{Spin}_n = \{ u \in \mathrm{pin}_n | uu^* = 1 \}.$$

It is easy to check that

$$\mathrm{Spin}_n = \{ u \in C_n^+ | uu^* = 1, uEu^* = E \}.$$

f. The homomorphism $\varphi = \varphi_n$ maps Spin_n to $SO(E) \approx SO_n(\mathbf{C})$. The kernel of the mapping φ is $\{\pm 1\}$. The group Spin_n is connected, and for $n > 2$ it is also simply connected.

Every element $u \in \mathrm{Spin}_n$ defines an automorphism of the algebra $C_n (x \to uxu^{-1})$ which coincides with the action of the element $\varphi(u) \in SO_n$.

g. Denote by δ_{2k} the standard action of the algebra C_{2k} on the space of dimension 2^k.

If $n = 2k + 1$, then the mapping $\delta_{n-1}\varphi_n^{-1}$ defines a 2^k-dimensional representation of the algebra C_n^+ and therefore a 2^k-dimensional representation of the group Spin_n, called the spinor representation; it is an irreducible representation.

If $n = 2k$, then δ_n defines a 2^k-dimensional representation of the group Spin_n, which is also called a spinor representation. This representation is reducible: it can be decomposed as the direct sum of two irreducible 2^{k-1}-dimensional representations (which correspond to the different eigenvalues of the operator $\delta_n(e_1, \ldots, e_n)$. These representations are called the half-spinor representations.

3. Type $\mathbf{A}_n$. Let $G = SL_n(\mathbf{C})$; let H be the diagonal subgroup of G; and θ transposition. Then $K = SO_n(\mathbf{C})$, and S is the group of diagonal matrices (of determinant 1) with ± 1 on the diagonal.

Let E be an n-dimensional space on which the group G acts; $e_1, \ldots, e_n$ a basis of it. Let $e = e_1 \cdots e_n \in C(E)$. Then $e^2 = (-1)^{n(n-1)/2} = \epsilon^2$.

We split up the space $C(E)$ into the direct sum of the subspaces C_ϵ and $C_{-\epsilon}$ annihilated by left multiplication by $e - \epsilon$ and $e + \epsilon$ respectively. Since $k(e) = e$ for all $k \in K = SO_n(E)$ and $k'(e) = -e$ for all $k' \in O(E) - SO(E)$, it follows that the subspaces C_ϵ and $C_{-\epsilon}$ are invariant under $SO(E)$ and are interchanged by the elements of $O(E) \backslash SO(E)$.

Let us take τ to be the representation of the group $SO(E)$ in the space C_ϵ and prove that it defines a model.

a. Let S_0 be the subgroup of all diagonal matrices with ± 1 on the diagonal. Then $\mathrm{card}(S_0/S) = 2$, and all the elements of $S_0 - S$ interchange C_ϵ and $C_{-\epsilon}$. To prove that $\tau|_S$ is a regular representation it then suffices to check that the representation τ' of the entire group S_0 in $C(E) = C_\epsilon \oplus C_{-\epsilon}$ is regular. This is verified directly.

b. It suffices to prove that τ' is a shallow representation of SO_n. But $C(E)$ can be naturally identified with $\Lambda^*(E)$, and the action of $SO(E)$ on $C(E) \cong \Lambda^*(E)$ can be extended to an action of the whole group G. It remains to check that for all $\gamma \in \Delta$ the representation in the space $\Lambda^*(E)$ is shallow with respect to H_γ. This is also easy to verify directly.

4. Type $\mathbf{B}_n$. $G = \mathrm{Spin}_{2n+1}$. Let us realize the group G as a subgroup of the multiplicative group of the algebra $C_{2n+1} = C(E)$; let $e_1, \ldots, e_{2n+1}$ be a basis for E.

For every pair of indices $i, i + n$ $(i = 1, \ldots, n)$ set

$$H_i = \{\alpha + \beta e_i e_{i+n}, \alpha^2 + \beta^2 = 1\};$$

H_i is a commutative subgroup of G, and $H = \prod_{i=1}^n H_i$ is a Cartan subgroup of G.

Set $\hat{e} = e_1 \cdots e_n$, and define an antiautomorphism θ of the group G by $\theta(g) = (\hat{e}g\hat{e}^{-1})^* = (\hat{e}g\hat{e}^{-1})^{-1}$; since $\theta|_H = \mathrm{id}$, θ is a Cartan involution.

The subgroup K corresponding to θ has the form

$$K = \{ g \in G \mid \hat{e}g\hat{e}^{-1} = g^{-1} \}.$$

In particular, $\varphi : G \to SO_n$ maps K to the operators on E that commute with $\varphi(e)$, i.e., preserve the subspaces E' and E'' generated by the vectors $(e_1, \ldots, e_n)$ and $(e_{n+1}, \ldots, e_{2n+1})$, respectively. Since $\operatorname{Ker}\varphi$ lies in the center of G, it follows that $\varphi(K) = \{$connected component of the group $SO(E) \times SO(E')\}$. Therefore, K is generated by the groups Spin_n and Spin_{n+1} corresponding to E' and E''; i.e., $K = (K_1 \times K_2)/\{1, \rho\}$, where $K_1 \cong \mathrm{Spin}_n$, $K_2 \cong \mathrm{Spin}_{n+1}$, $\rho = (\rho_1, \rho_2)$, and $\rho_1 \in K_1$ and $\rho_2 \in K_2$ are the nontrivial elements of the kernel of the map $\mathrm{Spin} \to SO$.

The group S consists of the elements of the form

$$\pm e_{i_1} e_{i_2} \cdots e_{i_k} \cdot e_{i_1+n} e_{i_2+n} \cdots e_{i_k} + n,$$

where k is even; $\operatorname{ord} S = 2^n$.

Let us construct the representation τ. Denote by τ_1' the 2^{n-1}-dimensional representation of the group SO_n on the space $C_{n,\epsilon}$; and by τ_1 the representation of the group K obtained from τ_1' by means of the projection $K \to SO_n$.

Let ρ be the spinor representation of the group Spin_{2n+1} in the space V, $\dim V = 2^n$. Let $z = e_1 \cdots e_n$ for even n, and let $z = e_{n+1} \cdots e_{2n+1}$ for odd n. Then $z \in \mathrm{Spin}_{2n+1}$, z centralizes K, and $z^2 = \epsilon^2$ is a scalar. Therefore V can be written as the sum $V_\epsilon \oplus V_{-\epsilon}$ of K-invariant subspaces, where

$$V_\epsilon = \operatorname{Ker}(\rho(z) - \epsilon), \qquad V_{-\epsilon} = \operatorname{Ker}(\rho(z) + \epsilon).$$

Set $\tau_2 = \rho|_{V_\epsilon}$ and then $\tau = \tau_1 \oplus \tau_2$. We prove that τ generates a model.

a. Clearly, $S = \{ i, \nu \} \cdot S'$, where $\nu = -1$ and S' is the subgroup of S consisting of the monomials $e_{i_1} \cdots e_{i_k} e_{i_1+n} \cdots e_{i_k+n}$, taken with the plus sign. We have $\tau_1(\nu) = 1$, $\tau_2(\nu) = -1$. Therefore, to prove that $\tau|_S$ is regular we must check that $\tau_1|_{S'}$ and $\tau_2|_{S'}$ are regular representations of S'.

For the representation τ_1 this has already been proved in Section 4.3. Consider τ_2. Let

$$S_0' = \{ e_{i_1} \cdots e_{i_k} e_{i_1+n} \cdots e_{i_k+n,} \text{ where } 0 < i_1 < i_2 < \cdots < i_k \leqslant n \}.$$

Then S' is a subgroup of S_0' of index 2, and for all elements $s \in S_0' \setminus S'$, we have $-sz = zs$. Therefore, all the elements of $S_0' \setminus S'$ interchange the spaces V_ϵ and $V_{-\epsilon}$, and it suffices to prove that all the characters of S_0' occur in the restriction of ρ to S_0'. This follows from the fact that elements of S_0' generate in $\operatorname{End} V = C_{2n+1}^+$ a subspace of dimension $\operatorname{card} S_0' = 2^n$.

b. Since τ_2 occurs in the representation $\rho|_{V_\epsilon}$, it suffices to check that the representation ρ is shallow with respect to all the subgroups $H_\gamma \subset \mathrm{Spin}_{2n+1}$, but this follows immediately from the description of its highest weight [3].

Consider the representation τ_1. We can consider a larger representation τ_1'; this is a representation in the space of all forms in n-space.

Let $G' \cong \mathrm{Spin}_3$ be the subgroup of G generated by the linear combinations of e_1, e_{n+1} and e_{2n+1}. The short root α of the group G is realized in G'. Since $G' \cap K \subset K_2$, it follows that $\tau_1|_{K_2} \equiv 1$.

Consider the space $\tilde{E}$ with basis $\tilde{e}_1$, $\tilde{e}_2$, $\tilde{e}_3$, $\tilde{e}_4$, and the corresponding group $\tilde{G} \cong \mathrm{Spin}_4$. Embed $\tilde{G}$ in G by the map $\tilde{e}_1 \to e_1$, $\tilde{e}_2 \to e_2$, $\tilde{e}_3 \to e_{n+1}$, $\tilde{e}_4 \to e_{n+2}$. Then the group $\tilde{K} = K(\tilde{G})$ is isomorphic to $(\tilde{K}_1 \times \tilde{K}_2)/(1, \rho)$, where $\tilde{K}_1 \cong \tilde{K}_2 \cong \mathrm{Spin}_2$.

Since $\tilde{G}$ contains the long root α, it suffices to prove that $\tau_1|_{\tilde{G}}$ is a shallow representation. But the restriction of τ_1 to $\tilde{G}$ is proportional to the four-dimensional representation σ obtained in the following way. Consider the standard representation of the group SO_2 on the two-dimensional space $\tilde{E}$, and consider the corresponding representation in $\Lambda^*\tilde{E}$ from which σ is obtained by means of the projection $K_1 \times K_2 \to SO_2$ (where $K_1 \to SO_2$ is the natural projection and K_2 is mapped to 1); here we are using the fact that

$$\Lambda^*(E_1 \oplus E_2) = \Lambda^*(E_1) \otimes \Lambda^*(E_2)).$$

One has to prove that the weights of σ restricted to K_α equal 0, ± 1.

Let $\hat{K}_\alpha$ be a two-sheeted covering of the group K_α. Then the embedding of K_α into K can be lifted to a mapping from $\hat{K}_\alpha$ to $K_1 \times K_2$; let k and l be the degrees of the respective projections of $\hat{K}_\alpha$ to K_1 and to K_2. Note that neither k nor l is zero, since the dimension of the centralizer of K_α is 4 (because $\mathrm{Spin}_4 \cong SL_2(\mathbf{C}) \times SL_2(\mathbf{C})$) but the centralizers of K_1 and K_2 coincide with K.

Consider the spinor representation ρ of the group $\tilde{G}$. It can be decomposed into the sum of two half-spinor representations ρ_1 and ρ_2. Under the isomorphism $\tilde{G} \cong SL_2 \times SL_2$, ρ_1 and ρ_2 become the standard representations of the two components of the group $\tilde{G}$. In particular, in the representation ρ the weights of the group K_α coincide with the weights of the group H_α and equal 0, 0, $+1$, -1, and the weights of $\hat{K}_2$ are 0, 0, ± 2.

On the other hand, the weights of groups K_1 and K_2 are $\{\pm 1, \pm 1\}$; i.e., the weights of $\hat{K}_\alpha$ are $\pm k \pm l$. Hence $|k| = |l| = 1$. Since K_1 has weights 0, 0, ± 2 in the representation σ, it follows that $\hat{K}_\alpha$ has weights 0, 0, ± 2, and K_α has weights 0, 0, ± 1, as required.

5. *Type* $\mathbf{C}_n$. Let us realize the group $G = \mathrm{Spin}_{2n}$ as the group of transformations of a $2n$-dimensional space E, which preserve the form

$$I = \begin{vmatrix} 0 & 1 & & & & & & \\ -1 & 0 & 0 & 1 & & & & \\ & & -1 & 0 & & & & \\ & & & & \ddots & & & \\ & & & & & & 0 & 1 \\ & & & & & & -1 & 0 \end{vmatrix},$$

i.e., $gIg^* = I$. Pick H to be the subgroup of diagonal elements in G:

$$\left|\begin{matrix} \lambda_1 & & & & & & \\ & \lambda_1^{-1} & & & & & \\ & & \lambda_2 & & & & \\ & & & \lambda_2^{-1} & & & \\ & & & & \ddots & & \\ & & & & & \lambda_n & \\ & & & & & & \lambda_n^{-1} \end{matrix}\right|$$

Set $\theta(g) = g^*$. In this case

$$K = \{ g \in GL_{2n} | g = g^{*-1}, gIg^{-1} = I \} = \{ g \in \mathrm{Sp}_{2n} | g = g^{*-1} \}.$$

The operator I has eigenvalues $+i$ and $-i$; denote by E_+ and E_- the corresponding n-dimensional spaces. Clearly, E_+ and E_- are invariant under K. To every element k of K we associate the corresponding transformation $\tau'(k)$ on the space E_+ (this gives an isomorphism $K \to GL(E_+)$). Denote by τ the resulting representation of the group K on the space $\Lambda^*(E)$. We prove that it defines a model.

a. Clearly, S is the subset of matrices in H whose weights λ_i are ± 1. Choose a basis $e_1^+, \ldots, e_n^+$ for E_+, where $e_j^+ = e_{2j-1} + ie_{2j}$; then, in this basis, S is identified with the set of diagonal matrices with ± 1s on the diagonal. Therefore $\tau|_S$ is a regular representation.

b. Let α be a long root. Then it is realized in the group $\mathrm{Sp}_2 \cong SL_2$, embedded in Sp_{2n}. Since $E_+ = E'_+ \oplus E''_+$, where E'_+ is generated by e_1^+ and E''_+ by $e_2^+, \ldots, e_n^+$, it follows that $\Lambda^*(E_+) = \Lambda^*(E'_+) \otimes \Lambda(E''_+)$; i.e., it is a multiple of $\Lambda^*(E'_+)$. In the space generated by e_1 and e_2, K_α can be written as

$$\left\{ \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix} \middle| \alpha^2 + \beta^2 = 1 \right\}.$$

Therefore K_α has weight 1 in the one-dimensional space E'_+ and weights 0 and 1 in $\Lambda^*(E'_+)$.

Now suppose that α is a short root. Then (just as before) everything can be reduced to the case $G = \mathrm{Sp}_4$.

Let E' be the space generated by e_1 and e_3, and E'' the space generated by e_2 and e_4. The form I defines a pairing of E' and E''. If g is any unimodular transformation of E' and g^{*-1} is one of E'', then the resulting transformation of E preserves the form I. We thus get an embedding

$\varphi_\alpha : SL_2 \to G;$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \to \begin{bmatrix} a & 0 & b & 0 \\ 0 & d & 0 & -c \\ c & 0 & d & 0 \\ 0 & -b & 0 & d \end{bmatrix}.$$

This embedding corresponds to the short root α, so that K_α has the form

$$\begin{bmatrix} a & 0 & b & 0 \\ 0 & a & 0 & b \\ -b & 0 & a & 0 \\ 0 & -b & 0 & a \end{bmatrix}.$$

In the space E_+, K_α has weights ± 1; i.e., in $\Lambda^*(E_+)$ it has weights 0, ± 1, as required.

6. *Type* $\mathbf{D}_n$. Consider the natural embedding of the group $G = \mathrm{Spin}_{2n}$ into the group $\tilde{G} = \mathrm{Spin}_{2n+1}$ of type $\mathbf{B}_n$. This mapping defines an isomorphism of Cartan subgroups. Therefore, the restriction of the representation τ constructed for $\tilde{G}$ in Section 4.4 is the required representation on G.

7. *Type* $\mathbf{G}_2$. Let G be a group of type $\mathbf{G}_2$. Then $K = (K_1 \times K_2)/\{1, \rho\}$, where K_1 and K_2 are isomorphic to $SL_2(\mathbf{C})$, ρ_1 and ρ_2 are the nontrivial elements in the centers of K_1 and K_2, and $\rho = (\rho_1, \rho_2)$ (see Tits [8]). Consider the root decomposition of the group G with respect to a Cartan subgroup H contained in K. Clearly, the roots corresponding to the unipotent subgroups of K_1 and K_2 are orthogonal. Assume that the roots $\pm\beta$ correspond to K_1; and the roots $\pm(\beta + 2\alpha)$, to K_2.

Consider the representation of the group $K_1 \times K_2$ of the form $\tau = 1 \otimes 1 \oplus 1 \otimes \tau_2$, where τ_2 is the three-dimensional representation of the group K_2. Let us prove that τ defines a model.

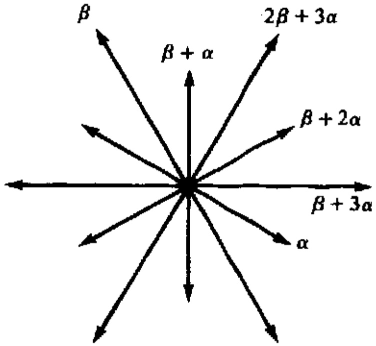
a. Consider the root decomposition with respect to the group H, and construct the mapping $\varphi : \tilde{G} \to G$ (where $\tilde{G} = SL_3(\mathbf{C})$ is mapped to the subgroup generated by the long roots). It is easy to check that this is an embedding that defines an isomorphism of Cartan subgroups; assume that it agrees with the involution θ.

Therefore, φ defines an isomorphism $\tilde{S} \to S$, where $\tilde{S}$ is the subgroup of elements of order 2 in the Cartan subgroup of G.

Consider the involutory subgroup $\tilde{K}$ in $\tilde{G}$. $\tilde{K} \cong PSL_2(\mathbf{C})$, and φ is an embedding of $\tilde{K}$ into K. Any such embedding is conjugate to the embedding induced by the diagonal map

$$SL_2(\mathbf{C}) \to SL_2(\mathbf{C}) \times SL_2(\mathbf{C}).$$

As verified in Section 4.3, $\tau|_{\tilde{S}}$ is a regular representation.



b. We check that τ is a shallow representation. The group $G_\beta \subset \varphi(\tilde{G})$, and it easily follows from Section 4.3 that all the weights in τ with respect to K are 0, ± 1.

Consider the group K_α. Let $\tilde{K}_\alpha$ be a two-sheeted covering of K_α, and let $\psi : \tilde{K}_\alpha \to K_1 \times K_2$ be the mapping which corresponds to the embedding

$$K_\alpha \to K = (K_1 \times K_2)/\{1, \rho\}.$$

If H_1 and H_2 are Cartan subgroups of K_1 and K_2, we can assume that $\psi(\tilde{K}_\alpha) \subset H_1 \times H_2$ and that the mapping ψ is determined by the two integers k and l, the degrees of the mappings of $\tilde{K}$ to H_1 and H_2. Note that k and l have the same parity, and they may be assumed to be nonnegative. Let us find these numbers.

For this purpose we consider the weights of the adjoint action of the group $\tilde{K}_\alpha$ on the Lie algebra $\mathfrak{g}$ of the group G. On the one hand, K_α is conjugate to H_α, so that the weights of K_α in $\mathfrak{g}$ are 0, 0, 0, 0, ± 1, ± 1, ± 2, ± 3, ± 3; on the other hand, those weights are $\pm k \pm l$, $\pm k \pm 3l$, $\pm 2k$, $\pm 2l$, 0, 0 (this can be seen from the computation of weights with respect to the groups H_1, $H_2 \subset \hat{H}$). If one of the numbers k or l were zero, then K_α would be contained in one of the subgroups K_1 or K_2; then the centralizer of K_α in K would have dimension 4. However, the centralizers of K_α in K and in G have different dimensions, since the centralizer of K_α in G contains the subgroup $G_{3\alpha+2\beta}$, so that $k, l > 0$. Hence $k = 3$, $l = 1$. Therefore, the weights of the group $\tilde{K}_\alpha$ in the representation τ are 0, ± 2, so that the weights of the group K_α in this representation are 0, ± 1.

8. *Type* $\mathbf{F}_4$. Let G be a group of type $\mathbf{F}_4$. Then $K = (K_1 \times K_2)/\{1, \rho\}$, where $K_1 \cong SL_2(\mathbf{C})$, $K_2 = \mathrm{Sp}_6(\mathbf{C})$, ρ_1, ρ_2 are the nontrivial elements in the centers of K_1 and K_2, and $\rho = (\rho_1, \rho_2)$ (see Tits [8]). Set $\tau = \tau_1 \oplus \tau_2$. Here τ_1 is trivial on K_2; on K_1 it is the sum of the trivial and the 3-dimensional representations, and $\tau_2 = \Phi_1 \otimes \Phi_2$, where Φ_1, Φ_2 are the standard representations of K_1 and K_2 on 2-dimensional and 6-dimensional spaces. We prove that τ generates a model.

The root system of $\mathbf{F}_4$ in a 4-dimensional space with basis ϵ_i ($i = 1, \ldots, 4$) has the form $\pm\epsilon_i$, $\pm\epsilon_i \pm \epsilon_j$, $\frac{1}{2}(\pm\epsilon_1 \pm \epsilon_2 \pm \epsilon_3 \pm \epsilon_4)$ (see [3]). The roots $\pm\epsilon_i$, $\pm\epsilon_i \pm \epsilon_j$ generate a root system of type $\mathbf{B}_4$. This defines a morphism $\varphi : \tilde{G} \to G$, where $G = \mathrm{Spin}_9$ is a group of type $\mathbf{B}_4$. Here $\tilde{H}$ (a Cartan subgroup of $\tilde{G}$) is mapped to H. Since the lattices of weights for the system of type $\mathbf{F}_4$ and for the subsystem $\mathbf{B}_4$ coincide [3], φ is an isomorphism between $\tilde{H}$ and H and defines an embedding $\tilde{G} \to G$. This induces an embedding of $\tilde{K} = K(\tilde{G})$ into K.

The group $\tilde{K}$ equals $(\tilde{K}_1 \times \tilde{K}_2)/\{1, \tilde{\rho}\}$, where $\tilde{K}_1 = \mathrm{Spin}_4$, $\tilde{K}_2 = \mathrm{Spin}_5$. But $\mathrm{Spin}_5 \cong \mathrm{Spin}_4$, $\mathrm{Spin}_4 = \tilde{K}_1' \times \tilde{K}_1''$, where $\tilde{K}_1'$ and $\tilde{K}_1''$ are isomorphic to SL_2. Therefore $\tilde{K} = (\tilde{K}_1' \times \tilde{K}_1'' \times \tilde{K}_2)/\{1, \tilde{\rho}\}$, where $\tilde{\rho} = (\tilde{\rho}_1', \tilde{\rho}_1'', \tilde{\rho}_2)$ and $\tilde{\rho}_1'$, $\tilde{\rho}_1''$, $\tilde{\rho}_2$ are the nontrivial elements of the centers of $\tilde{K}_1'$, $\tilde{K}_1''$, and $\tilde{K}_2$.

Consider the embedding $\tilde{K} \to K$ and lift it to a mapping

$$\psi : \tilde{K}_1' \times \tilde{K}_1'' \times \tilde{K}_2 \to K_1 \times K_2.$$

Let ψ_1, ψ_2 be the projections of ψ to K_1 and K_2. Then $\psi_1(\tilde{K}_2) = 1$. Further, ψ is nontrivial on one of the groups $\tilde{K}_1'$ and $\tilde{K}_1''$; for instance, $\tilde{K}_1'$. Then $\psi_1 : \tilde{K}_1' \to K$ is an isomorphism; since $\tilde{K}_1''$ commutes with $\tilde{K}_1'$, it follows that $\psi_1(K_1'') = 1$.

Therefore, $\mathrm{Ker}\, \psi_2|_{\tilde{K}_1'' \times \tilde{K}_2}$ has at most two elements. But we then have an embedding (since all the 6-dimensional representations of the group $SL_2 \times \mathrm{Sp}_4$ can easily be enumerated). Since $\psi_2(\tilde{K}_1')$ commutes with $\psi_2(\tilde{K}_1'' \times \tilde{K}_2)$, it follows that $\psi_2(\tilde{K}_1') = 1$.

Thus, ψ maps $\tilde{K}_1'$ isomorphically to K_1, and embeds $\tilde{K}_1'' \times \tilde{K}_2 = \mathrm{Sp}_2 \times \mathrm{Sp}_4$ in a natural way into $K_2 = \mathrm{Sp}_6$. Recall the structure of the representation $\tilde{\tau}$ for the group $\tilde{K}$ (see Section 4.4). It had the form $\tilde{\tau}_1 \oplus \tilde{\tau}_2$, with $\tilde{\tau}_1$ and $\tilde{\tau}_2$ 8-dimensional representations, where $\tilde{\tau}_1$ is trivial on $\tilde{K}_2$ and $\tilde{\tau}_2(\tilde{\rho}_2) = -1$. This immediately implies that $\tilde{\tau}_2$ is the tensor product of the standard representations of $\tilde{K}_2 = \mathrm{Sp}_4$ and the standard representation of one of the groups $\tilde{K}_1'$ and $\tilde{K}_1''$. Since these groups are interchanged by an inner automorphism of the group G corresponding to some element of H (any element in $S_0 - S$ in the notation of Section 4.4), we may assume that $\tilde{\tau}_2 = \tilde{\Phi}_1' \oplus \tilde{\Phi}_2$ where $\tilde{\Phi}_2'$ and $\tilde{\Phi}_2$ are the standard representations of the groups $\tilde{K}_1'$ and $\tilde{K}_2$.

It is easy to check that $\tilde{\tau}_1 = \tilde{\tau}_1' \oplus \tilde{\tau}_2''$, where $\tilde{\tau}_1'$ is the sum of the trivial representations and the 3-dimensional representation T of $\tilde{K}_1'$, and $\tilde{\tau}_2'' = \tilde{\Phi}_1' \otimes \tilde{\Phi}_2''$ where $\tilde{\Phi}_1'$ and $\tilde{\Phi}_2''$ are the standard representations of $\tilde{K}_1'$ and $\tilde{K}_2''$ ($\tilde{K}_1'$ and $\tilde{K}_1''$ may have to be interchanged here). In this way,

$$\tilde{\tau} = 1 \oplus (T \otimes 1 \otimes 1) \oplus (\tilde{\Phi}_1' \otimes (\tilde{\Phi}_1'' \otimes 1 \oplus 1 \otimes \tilde{\Phi}_2)),$$

i.e., $\tilde{\tau} = \tilde{\tau}|_G$, as required.

9. *Type* $\mathbf{E}_8$. Let G be a group of type $\mathbf{E}_8$. Then K is a group of type $\mathbf{D}_8$; the kernel of the mapping $\mathrm{Spin}_{16} \to K$ has two elements and *does not coincide* with the kernel of the spinor representation $\mathrm{Spin}_{16} \to SO_{16}$ (see [8]).

Consider the spinor representation ρ of the group Spin_{16} on the space V, and denote by τ the resulting representation of Spin_{16} in End V. Since τ is trivial on the center, it can be viewed as a representation of the group K. We prove that it defines a model.

There exists an embedding of the root system $\mathbf{A}_8$ into $\mathbf{E}_8$, because the extended Dynkin diagram of $\mathbf{E}_8$, obtained by adjoining the lowest weight, contains a subdiagram of type $\mathbf{A}_8$. It induces a mapping $\varphi: \tilde{G} \to G$, where $\tilde{G} \cong SL_9$, which maps the Cartan subgroup $\tilde{H}$ of the group $\tilde{G}$ onto the Cartan subgroup H. Moreover, φ defines a mapping of $\tilde{K} = K(\tilde{G})$ to K. It suffices to check that $\tau|_{\tilde{K}}$ defines a model for $\mathbf{A}_8$.

Let us analize in greater detail the mapping of $\tilde{K} = SO_9$ to K. Let $\psi': \mathrm{Spin}_9 \to \mathrm{Spin}_{16}$ be a lifting of it, and let $\psi'': \mathrm{Spin}_9 \to SO_{16}$ be the composition of ψ' and the projection $\mathrm{Spin}_{16} \to SO_{16}$. The mapping ψ'' defines a 16-dimensional representation of the group Spin_9. Spin_9 has only three representations of dimension no greater than 16: the trivial one 1, the standard one Φ of dimension 9, and the spinor one ρ of dimension 16. Clearly ψ'' is nontrivial; nor can it be the sum of Φ and trivial representations, because in such a case $\psi': \mathrm{Spin}_9 \to \mathrm{Spin}_{16}$ would be the natural mapping, and the mapping $\psi''': \mathrm{Spin}_9 \to K$ would be nontrivial on the kernel of the mapping $\mathrm{Spin}_9 \to \mathrm{Spin}_9$; i.e., it would not factor through the mapping $SO_9 \to K$.

Thus ψ'' is a spinor representation on the space V, and $\tau|_K$ is given by the natural action of Spin_9 in End V. The space End V can be identified with C_9^+ (by definition of the spinor representation). But C_9^+ is isomorphic to the representation of the group SO_9 constructed in Section 4.2, as required.

10. Type $\mathbf{E}_7$. Let G be a simply connected group of type $\mathbf{E}_7$. Then K is isomorphic to the group $SL_8/\{\pm 1\}$ (see [8]).

Consider the standard representation Φ of the group SL_8 in the space E, $\dim E = 8$. Then the representation $\tau = \Phi \otimes (\Phi \oplus \Phi^*)$ can be viewed as a representation of the group K. We prove that it generates a model.

The root system of $\mathbf{E}_7$ can be embedded in the root system of $\mathbf{E}_8$. Let $\varphi: G \to \tilde{G}$ be the corresponding mapping of groups which agrees with θ, and $\varphi(H) \subset \tilde{H}$, $\varphi(K) \subset \tilde{K}$. The mapping φ defines a homomorphism $\varphi: K \to \tilde{K}$ with discrete kernel, and its lifting to the universal covers, $\varphi': SL_8 \to SO_{16}$. Since all the representations of SL_8 except Φ, Φ^*, and the trivial representation have dimension greater than 16, and there is no invariant bilinear form on the representations $\Phi \oplus \Phi$ and $\Phi^* \oplus \Phi^*$, it follows that the 16-dimensional representation of the group SL_8 defined by the mapping φ' has the form $\Phi \oplus \Phi^*$ and is defined on the space $E \oplus E^*$. In this way, the representation of the group SL_8 on the space $E \oplus E^*$ can be extended to a representation of the group Spin_{16}. It follows, in particular, that $\varphi: K \to \tilde{K}$ is an embedding, so that the group S can be viewed as a subgroup of index 2 in $\tilde{S} = \tilde{K} \cap \tilde{H}$. Let $S' \subset SL_8$ be the inverse image of the group S under the projection $SL_8 \to K$, $\tilde{S}' \subset \mathrm{Spin}_{16}$ the inverse image of $\tilde{S}$ under the projection $\mathrm{Spin}_{16} \to \tilde{K}$.

Since $\mathrm{End}(E \oplus E^*) = (E \otimes E) \oplus (E \otimes E^*) \oplus (E^* \otimes E) \oplus (E^* \otimes E^*)$, and the representation of $\tilde{K}$ in End $(E \oplus E^*)$ is shallow, the representation τ is also shallow.

We prove that $\tau|_S$ is a regular representation. Indeed, the representation of the group $\tilde{S}'$ on $E \oplus E^*$ is irreducible (since the space of invariants of the group $\tilde{S}'$ in $\mathrm{End}(E \oplus E^*)$ is 1-dimensional by the preceding section). Therefore, if $\tilde{s} \in \tilde{S}' \setminus S'$, then $\tilde{S}(E)$ does not intersect E (since $sS' \cup S' = \tilde{S}'$ and $\tilde{s}^2 \in S'$). Hence $\tilde{s}$ interchanges the spaces $E \otimes (E \oplus E^*)$ and $\tilde{s}E \otimes (E \oplus E^*)$, whose sum gives us

$$(E \oplus E^*) \otimes (E \oplus E^*) \cong \mathrm{End}(E \oplus E^*).$$

Since the representation of the group $\tilde{S}$ in $\mathrm{End}(E \oplus E^*)$ is regular, the representation of S on $E \otimes (E \oplus E^*)$ is also regular, as required.

11. Type $\mathbf{E}_6$. Let G be the simply connected group of type $\mathbf{E}_6$. Then $K \cong \mathrm{Sp}_8/\{\pm 1\}$ (see [8]). Let Φ be the standard 8-dimensional representation of the group Sp_8 on the space E. The representation $\tau = \Phi \otimes \Phi^*$ in the space End V can be viewed as a representation of the group K. Let us prove that it defines a model.

Embed the root system $\mathbf{E}_6$ into $\mathbf{E}_7$, and consider the corresponding mapping $\varphi: G \to \tilde{G}$. Here $\varphi(H) \subset \tilde{H}$ and $\varphi(K) \subset \tilde{K}$. The mapping $K \to \tilde{K}$ can be lifted to a mapping $\mathrm{Sp}_8 \to SL_8$ that obviously coincides with the standard representation. In particular, $\varphi: K \to \tilde{K}$ is an embedding. By means of this embedding we can identify S with a subgroup of $\tilde{S}$ of index 2. Since the representation τ of $\tilde{K}$ in the space $(E^* \otimes E) \oplus (E^* \otimes E)$ is shallow, it follows that it is also shallow with respect to K.

We now prove that $\tau|_S$ is a regular representation. Indeed, since S is a subgroup of $\tilde{S}$ of index 2 and the representation $\tilde{\tau}|_{\tilde{S}}$ is regular, $\tilde{\tau}|_S$ is the doubled regular representation. But $\Phi|_{S'} = \Phi^*|_{S'}$, where $S' \subset \mathrm{Sp}_8$ is the inverse image of the group S under the projection $\mathrm{Sp}_8 \to K$. Therefore, $\tilde{\tau}|_S$ is isomorphic to $\tau|_S \oplus \tau|_S$. It follows that $\tau|_S$ is a regular representation, as required.

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