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On non-holonomic irreducible D-modules

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Introduction

Let $D=D(\mathbf{C}^n)$ be the Weyl algebra, i.e., algebra of polynomial differential operators on $\mathbf{C}^n$. Until 1983 it was widely believed (for reasons which in retrospect are difficult to understand) that any irreducible D-module is automatically holonomic. In 1983–1984 J.T. Stafford constructed a counterexample (see [St]). Namely, he exhibited a concrete operator $d\in D$ and by direct, rather involved, computations showed that d generates a maximal left ideal $I=D\cdot d\subset D$. This shows that D-module $F=D/I$ is irreducible. On the other hand, it is clear that $\dim \mathrm{Ch}(F)=2n-1$, i.e., for $n>1$, $\dim \mathrm{Ch}(F)>n$ and F is not holonomic.

In this paper we want to show that "most" of irreducible D-modules are not holonomic. More precisely, we will show for $n=2$ that "almost each" operator $d\in D$ generates a maximal left ideal.

We deduce this result from the fact that a certain system of first order differential equations does not have algebraic solutions – a fact which is interesting in itself.

1.

The algebra D is generated by operators $\left\{u_i, \partial_i=\frac{\partial}{\partial u_i} \mid i=1, \ldots, n\right\}$. We will consider a filtration on D, $D=\bigcup_{k=0}^{\infty} D^k$, where D^k consists of polynomials in $\{u_i, \partial_i\}$ of degree $\leqq k$. This filtration has the advantage that all spaces D^k are finite-dimensional. Let $\Sigma=\bigoplus \Sigma^k$ be the associated graded algebra,

$$\sigma_k\colon\ D^k\to\Sigma^k=D^k/D^{k-1}$$

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be the symbol map. It is well known that Σ is isomorphic to the polynomial algebra $\mathbf{C}[x_1, \ldots, x_{2n}]$, where $x_i = \sigma_1(u_i)$, $x_{i+n} = \sigma_1(\partial_i)$ for $i = 1, \ldots, n$.

Our main result is

Theorem A. *Suppose $n=2$, i.e., $D = D(\mathbf{C}^2)$, and $k \geqq 4$. Let $P \in \Sigma^k$ be a generic polynomial. Then each operator $d \in D^k$ with symbol $\sigma_k(d) = P$ generates a maximal left ideal in D.*

Conjecture A. *An analogous statement is true for any $n \geqq 2$, $k \geqq 3$.*

Remarks. 1. Here the term "generic" is used in its naive sense. This means that the set $N \subset \Sigma^k$ of polynomials P for which the conclusion of the theorem holds satisfies the following condition:

Its complement $\Sigma^k \backslash N$ can be covered by a countable number of hypersurfaces (see discussion in § 5).

2. We also prove a slightly more general result: if d is an operator in Theorem A, then any left ideal $J \subset D$, satisfying $J d \subset J$, is generated by a polynomial in d (see § 3, Thm. 3.4).

2.

Our proof is based on the study of symplectic geometry of characteristic varieties of D-modules.

Namely, Σ has a natural interpretation as the algebra of polynomial functions on a $2n$-dimensional symplectic vector space X and to each finitely generated D-module F corresponds its characteristic variety $\mathrm{Ch}(F)$, which is a homogeneous subvariety of X. The basic geometric fact about such varieties is that they are always involutive; in particular, all their components have dimension $\geqq n$. We will use the abbreviation "i.h.-variety" for an involutive homogeneous subvariety of X.

In the case when F is holonomic, i.e., $\dim \mathrm{Ch}(F) = n$, each irreducible component of $\mathrm{Ch}(F)$ is a minimal i.h.-variety, i.e., it does not have proper i.h.-subvarieties. In fact, this is the property which is used in proving most of the nice facts about holonomic D-modules. Our main new observation is that there exist higher-dimensional minimal i.h.-varieties – and there are a lot of them.

Conjecture A′. *Let $n \geqq 2$, $k \geqq 3$. Then for a generic polynomial $P \in \Sigma^k$ the variety $Y = V(P) \subset X$ of its zeroes is a minimal i.h.-variety.*

We can prove only a particular case of this conjecture.

Theorem A′. *The conjecture holds for $n=2$, $k \geqq 4$.*

Let us show how this geometric conjecture A′ implies Conjecture A (and Theorem A′ implies Theorem A). Let $d \in D^k$ be an operator with the symbol $\sigma_k(d) = P$. Suppose the ideal $I = Dd \subset D$ is not maximal, i.e., there exists an ideal J such that $D \supsetneqq J \supsetneqq I$. Then $\Sigma \supsetneqq \sigma(J) \supsetneqq \sigma(I)$.

Since $\sigma(I) = \Sigma \cdot P$ and P is irreducible, this implies that $\emptyset \neq V(\sigma(J)) \subsetneqq V(\sigma(I)) = V(P)$. But $V(\sigma(J)) = \mathrm{Ch}(D/J)$ is an i.h.-variety, which contradicts Conjecture A′. This contradiction proves that the ideal $I = Dd$ is maximal.

3.

Let us examine the geometry behind Conjecture A′. Each function $f \in \Sigma$ defines a Hamiltonian vector field ξ_f on X. If $Z \subset X$ is an involutive subvariety and $f|_Z = 0$, then ξ_f is tangent to Z, i.e., at each point $z \in Z$, $\xi_f(z) \in T_z Z$.

In particular, let $P \in \Sigma^k$ be a generic polynomial, $Y = V(P) \subset X$. Then each i.h.-variety $Z \subset Y$ should be tangent to the field ξ_P. It means that Z is a solution of some "generic" differential equation. But it is intuitively clear, that generic differential equation should not have algebraic solutions.

In order to make this precise let us pass to projectivization. Let $\bar{X} \approx \mathbf{P}^{2n-1}$, $\bar{Y}$, $\bar{Z}$ be projectivizations of X, Y, and Z. The vector field ξ_P induces a direction field ρ_P on $\bar{X}$ (for generic P it has finite number of singular points).

The direction field ρ_P defines a differential equation on $\bar{Y}$ and $\bar{Z}$ is its solution in the sense that ρ_P is tangent to $\bar{Z}$.

Conjecture A″. *Let $n \geqq 2$, $k \geqq 3$. Let $P \in \Sigma^k$ be a generic polynomial, $Y = V(P) \subset X$, ρ_P the direction field on $\bar{Y}$, corresponding to ξ_P. Then there are no algebraic subvarieties $\bar{Z} \subsetneqq \bar{Y}$, with* $\dim \bar{Z} > 0$, *which are tangent to ρ_P.*

As was explained above, this conjecture, which looks quite plausible, implies Conjectures A′ and A. In case $n = 2$ we will see that it is equivalent to Conjecture A′. In this case $\dim \bar{Z} = 1$, i.e., $\bar{Z}$ is a solution of the differential equation ρ_P in a standard sense. We will prove Conjecture A″ for $n = 2$, $k \geqq 4$ using the fact that any algebraic curve $\bar{Z} \subset \bar{Y}$ can be given by one equation $Q = 0$, where $Q \in \Sigma$.

4.

The examples of operators $d \in D$ which we described are quite different from the original Stafford's example. He considered (for $n = 2$) a first order operator $d = \partial_1 + (1 + \lambda u_1 u_2)\,\partial_2 + u_2$.

Using methods described above it is not difficult to show that for $n = 2$ a generic first order operator with coefficients of degree $\geqq 2$ generates a maximal left ideal in D.

In §4 we will prove the following

Proposition 1. *Let ξ be a polynomial vector field on $\mathbf{C}^2$. For each singular point s of ξ (i.e., a point $s \in \mathbf{C}^2$ such that $\xi(s) = 0$), denote by $A(\xi, s)$ the endomorphism of $T_s \mathbf{C}^2$, given by 1-jet of ξ and by $\Lambda(\xi, s)$ the subgroup of* $\mathbf{C}$, *generated by eigenvalues of $A(\xi, s)$. Suppose ξ satisfies the following conditions.*

(*) (i) *There are no algebraic curves in $\mathbf{C}^2$, which are tangent to ξ*

(ii) *ξ has at least one singular point in $\mathbf{C}^2$.*

Choose a polynomial function f on $\mathbf{C}^2$ such that

(**) *for each points $s \in \mathbf{C}^2$ singular for ξ, $f(s) \notin \Lambda(\xi, s)$.*

Then the operator $d = \xi + f$ generates a maximal left ideal in D.

The proof uses the same methods as before but applied to the standard filtration in D by the order of an operator, instead of filtration by degree in u_i and ∂_i.

It is clear that condition (**) on function f is generic. Condition (*) (ii) is generic and not really important for the proof. The important condition is (*) (i). This condition for ξ can be either checked directly (e.g., in Stafford's example), or one can use the following general result, which we prove in the Appendix.

Statement. Generic direction field ρ on $\mathbf{P}^2$ does not have algebraic solutions.

5.

The paper is arranged as follows.

§ 1 recalls basic notions, connected with the Involutivity Theorem.

In § 2 we prove geometric theorems A′ and A″.

In § 3 we discuss some elementary corollaries for the theory of $\mathcal{D}$-modules of the fact that there are many minimal i.h.-varieties. In particular, we prove Theorem A and some related results.

§ 4 contains the proof of Proposition 0.4 which describes first order operators, generating maximal ideals.

§ 5 contains some miscellaneous remarks.

In the Appendix we discuss properties of algebraic solutions of direction fields on $\mathbf{P}^2$.

1. Involutivity theorem

1.1

We denote by D the algebra of polynomial differential operators on $\mathbf{C}^n$. It is generated by operators $\left\{u_i, \partial_i = \frac{\partial}{\partial u_i} \mid i=1, \ldots, n\right\}$. We introduce a filtration $D = \bigcup_{k=0}^{\infty} D^k$, where $D^k = \{\text{polynomials in } u_i, \partial_i \text{ of degree } \leqq k\}$. Put $\Sigma^k = D^k/D^{k-1}$ and denote by $\sigma_k \colon D^k \to \Sigma^k$ the corresponding symbol map. The associated graded algebra $\Sigma = \bigoplus_{k=0}^{\infty} \Sigma^k$ is isomorphic, as a graded algebra, to $\mathbf{C}[x_1, \ldots, x_{2n}]$, where $x_i = \sigma_1(u_i)$, $x_{i+n} = \sigma_1(\partial_i)$ for $i=1, \ldots, n$. We will interpret Σ as the algebra of polynomial functions on a linear space X with coordinates $\{x_i \mid i=1, \ldots, 2n\}$.

The commutator $[,]$ on D induces an operation $\{,\}\colon \Sigma \times \Sigma \to \Sigma$, which is called the *Poisson bracket*, such that $\sigma_{k+l-2}([d, d']) = \{\sigma_k(d), \sigma_l(d')\}$. This is a skew symmetric operation, satisfying the Leibnitz rule $\{f, gh\} = \{f, g\}\, h + g\{f, h\}$. This implies that there exists a 2-coform ω^*, i.e., a skew symmetric form on the cotangent bundle T^*X, such that $\{f, g\}(x) = \omega^*_x(df, dg)$. In fact, it is easy to see that this coform ω^* is nondegenerate and the corresponding form ω on the tangent bundle is equal to $\omega = \sum_{i=1}^{n} dx_i dx_{i+n}$.

The form ω defines an isomorphism between spaces of 1-forms and vector fields on X. Namely, to a vector field ξ corresponds 1-form $i(\xi)\,\omega$, where i is the interior product. Conversely, a 1-form α corresponds to a vector field ξ_a such that $\omega(\xi_a, \eta) = \alpha(\eta)$.

In particular, each function $f \in \Sigma$ defines a Hamiltonian vector field ξ_f, corresponding to the form df. It is characterized by the property $\xi_f(h) = \{f, h\}$.

1.2

Let $L \subset X$ be a linear subspace of dimension l, $L^\perp \subset X$ its orthogonal complement. It is clear that $rk(\omega|_L) = l - \dim(L \cap L^\perp)$. Since $\dim L^\perp = 2n - l$, we have $rk(\omega|_L) \geqq 2l - 2n$, and the equality holds iff $L^\perp \subset L$, i.e., $L^\perp$ is isotropic. If $L^\perp$ is isotropic, we call L an *involutive* subspace of X. In this case $l \geqq n$.

Lemma 1. *Let $Z \subset X$ be an algebraic subvariety. Then the following conditions are equivalent.*

(i) *The ideal $I_Z = \{f \in \Sigma \mid f|_Z = 0\}$ is closed under Poisson bracket.*
(ii) *For each point $z \in Z$ the tangent space $T_z Z \subset T_z X = X$ is involutive.*
(iii) *For almost each point $z \in Z$ the tangent space $T_z Z \subset X$ is involutive.*
(iv) *For each $f \in I_Z$ the vector field ξ_f is tangent to Z.*

The proof follows from the formula $\{f, h\} = \omega^*(df, dh)$ and the fact that the orthogonal complement $(T_z Z)^\perp \subset X^*$ is spanned by differentials $\{df \mid f \in I_Z\}$.

The subvarieties $Z \subset X$, satisfying these conditions we call *involutive*. Irreducible components of involutive variety are involutive and have dimension $\geqq n$.

1.3

Let F be a finitely generated D-module, $\{F^k\}$ be a good filtration on F, $F_\Sigma = \oplus F^k/F^{k-1}$ the corresponding Σ-module. The support in X of Σ-module F_Σ is denoted by $\mathrm{Ch}(F)$ (see [Gin, § 1]). In particular, if $F = D/I$ for some left ideal $I \subset D$ and $\sigma(I) = \sum_k \sigma_k(D^k \cap I)$ the corresponding symbol ideal in Σ, then $\mathrm{Ch}(F) = V(\sigma(I))$ – the set of common zeroes of functions in $\sigma(I)$.

Involutivity theorem (see [Gab, § 3, Th. I_k']). *For each finitely generated D-module F, the subvariety $\mathrm{Ch}(F) \subset X$ is an involutive homogeneous variety (i.h.-variety).*

2. Proof of theorems A′, A″

2.1

Fix $n \geqq 2$, $k \geqq 3$. A homogeneous polynomial $P \in \Sigma^k$ is called *nondegenerate* if dP does not vanish on $X \setminus 0$, or, equivalently, ξ_P does not vanish on $X \setminus 0$.

In this case, P is irreducible and the variety $Y_P = V(P) \subset X$ is nonsingular outside of 0. In particular, its projectivization $\bar{Y}_P \subset \bar{X} = \mathbf{P}(X)$ is a nonsingular irreducible variety.

Non degenerate polynomials form a Zariski open subset $\Sigma^k_{\text{reg}} \subset \Sigma^k$. This subset is not empty, since it contains $P = \Sigma x_i^k$. In particular, a generic polynomial is nondegenerate.

2.2

Let $E = \sum_{i=1}^{2n} x_i \frac{\partial}{\partial x_i}$ be the Euler vector field on X. We will often use the following identity

$$\omega(\xi_P, E) = dP(E) = EP = k \cdot P. \qquad (*)$$

Put $S_P = \{x \in X \backslash 0 \mid \xi_P(x) \text{ is proportional to } E(x)\}$ and denote by $\bar{S}_P$ the corresponding subvariety in $\bar{X} = \mathbf{P}(X)$.

Lemma 2. (i) $S_P \subset Y_P$

(ii) *If P is nondegenerate, then* $\dim S_P = 1$, *i.e., $\bar{S}_P$ is a finite set.*

Proof. By $(*)$ $\omega(\xi_P(x), E(x)) = 0$ iff $x \in Y_P$, which implies (i).

Put $S'_P = \{x \in X \backslash 0 \mid \xi_P(x) = E(x)\}$. Since ξ_P and E are homogeneous of different degrees and do not vanish on $X \backslash 0$, we have $S_P = \mathbf{C}^* \cdot S'_P$.

Let $\xi_P = \Sigma Q_i \frac{\partial}{\partial x_i}$. Then Q_i are homogeneous functions of degree $k-1$, which do not have common zeroes on $X \backslash 0$. Hence for some constant $C > 0$ we have $\Sigma |Q_i(x)|^2 \geqq C(\Sigma |x_i|^2)^{k-1}$.

The variety S'_P is defined by a system of equations $\{Q_i(x) = x_i \mid i = 1, \ldots, 2n\}$. Since $k > 2$, S'_P lies inside the ball $\Sigma |x_i|^2 \leqq C^{\frac{1}{k-2}}$. This implies that S'_P, and hence $\bar{S}_P$, are finite sets.

2.3

Let us fix a nondegenerate polynomial $P \in \Sigma^k$ and consider the variety $Y_P = V(P) \subset X$. The vector field ξ_P is tangent to Y_P and hence gives a vector field on Y_P. Since ξ_P is homogeneous, it induces a direction field ρ_P on the projectivization $\bar{Y}_P$. This direction field is well defined outside of the finite set $\bar{S}_P$.

We will consider ρ_P as a differential equation on $\bar{Y}_P$. By definition, a solution of this equation is a subvariety $\bar{Z} \subset \bar{Y}_P$, $\dim \bar{Z} > 0$, tangent to ρ_P. In affine terms a solution is a homogeneous subvariety $Z \subset Y_P$, $\dim Z > 1$, tangent to ξ_P.

For the rest of this section we assume $n = 2$. Then we have $\dim Y_P = 3$, so the only interesting case is when $\dim Z = 2$, i.e., $\dim \bar{Z} = 1$.

Let us consider the 1-form $\alpha = i(E)\, \omega$, corresponding to E.

Lemma 3. *Suppose $n=2$. Let $Z \subset Y_P$ be a 2-dimensional homogeneous subvariety. Then the following conditions are equivalent.*

(i) *Z is tangent to ξ_P,*
(ii) $\alpha|_Z \equiv 0$,
(iii) *Z is involutive.*

Proof. Since $\dim S_P = 1$, $Z \backslash S_P$ is dense in Z. Let $z \in Z \backslash S_P$ be a nonsingular point of Z, $L = T_z Z \subset X$. Then $L \ni E(z)$ since Z is homogeneous, $dP|_L = 0$ since $Z \subset Y_P$, $E(z)$ and $\xi_P(z)$ are linearly independent, α and dP are linearly independent and $\omega(\xi_P, E) = \alpha(\xi_P) = dP(E) = \omega^*(\alpha, dP) = 0$.

(i)$\Rightarrow$(ii). $\xi_P(z) \in L \Rightarrow L = \operatorname{span}(E, \xi_P) \Rightarrow \alpha|_L = 0$.

(ii)$\Rightarrow$(iii). $\alpha|_L = 0 \Rightarrow L^\perp = \operatorname{span}(\alpha, dP) \Rightarrow L$ is involutive.

(iii)$\Rightarrow$(i). L is involutive$\Rightarrow$a vector ξ_P, corresponding to the form $dP \in L^\perp$, lies in L

Remark. This lemma shows that for $n=2$ conjectures A′ and A″ are equivalent.

2.4

Theorem 1. *Let $n=2$, $k \geqq 4$. Then for a generic polynomial P the differential equation given by the direction field ρ_P does not have (one-dimensional) algebraic solutions on the surface $\bar{Y}_P$.*

Proof. Let P be a nondegenerate polynomial. We assume that there exists an irreducible homogeneous subvariety $Z \subset Y_P$ such that $\dim Z = 2$ and Z is tangent to ξ_P and deduce a contradiction, making additional assumptions about P. Then we check that these assumptions hold for generic P. Our crucial assumption is

Ass. 1. Any curve $C \subset \bar{Y}$ is defined (as a scheme) by a homogeneous function Q on X.

We apply it to $C = \bar{Z} \subset \bar{Y}$. It implies that C is defined by some function $Q \in \Sigma^l$, i.e., the ideal $I_Z \subset \Sigma$ is generated by P and Q.

Denote by Z^0 the nonsingular part of Z. Since the ideal I_Z is generated by P and Q at each point $z \in Z^0$ the space $(T_z Z)^\perp \subset (T_z X)^*$ is generated by linearly independent vectors dP and dQ. In particular, since $\alpha|_Z \equiv 0$, $\alpha(z)$ can be uniquely written as

$$\alpha(z) = f(z)\, dP(z) + h(z)\, dQ(z). \tag{†}$$

Since α, dP and dQ have homogeneous degrees 2, k and l, respectively, the functions f and h on Z^0 have homogeneous degrees $2-k$ and $2-l$.

Let us denote $\mathcal{O}_C(i)$ the restriction of the sheaf $\mathcal{O}(i)$ on $\bar{X}$ to C (see [Hart], p. 117). Then f and h can be interpreted as sections of sheaves $\mathcal{O}_{C^0}(2-k)$ and $\mathcal{O}_{C^0}(2-l)$ on $C^0 = \bar{Z}^0$.

Let us first consider the case when C is nonsingular, i.e., $C = C^0$. Note that the sheaf $\mathcal{O}_C(1)$ on C has positive degree, since it has sections with zeroes. This implies that sheaves $\mathcal{O}_C(i)$ do not have nonzero global sections for $i<0$. In particular, $f(z)\equiv 0$. Since $\alpha(z)\not\equiv 0$, we have $h(z)\not\equiv 0$, which implies that $2-l\geqq 0$.

Thus we have proved the following

Statement. Q is either linear or quadratic form and at each point $z\in Z$ 1-forms α and dQ are proportional, i.e., vectors ξ_Q and E are proportional.

2.5

Now let us use the following simple lemma from linear algebra.

Lemma 4. *Let $Q\neq 0$ be a quadratic form. Put $S_Q=\{x\in X\,|\,\xi_Q(x)$ is proportional to $E(x)\}$. Then S_Q lies in a union of a finite number of hyperplanes.*

Proof. To each linear operator $A\in\operatorname{End}X$ corresponds a vector field ξ_A given by $\xi_A(x)=Ax$. Clearly $E=\xi_{Id}$, $\xi_Q=\xi_A$ for some operator A. Thus S_Q is the set of eigenvectors of the operator A. Since $\xi_Q(Q)=0$ and $E(Q)=2Q\not\equiv 0$, we see that A is not a scalar operator. Hence S_Q is a union of proper linear subspaces, corresponding to different eigenvalues of A. Q.E.D.

Since Z is irreducible and $Z\subset S_Q$, it lies in a hyperplane, i.e., there exists a linear form φ on X such that $\varphi|_Z=0$. Since Z is involutive, it is tangent to the constant vector field $\xi=\xi_\varphi$. Choose a vector $\eta\in Z$ which is not proportional to $\xi\in X$. Since Z is homogeneous and tangent to ξ, it contains the plane $L=\operatorname{span}(\xi,\eta)$. Being irreducible Z coincides with L. But then $C=\bar{Z}=\bar{L}\approx\mathbf{P}^1$ is a curve of degree 1 in $\bar{X}$. On the other hand, since C is a complete intersection, given by P and Q, its degree equals kl. This contradiction proves that Z cannot exist.

2.6

Now let us consider the general case, when C can be singular. We need to make an additional assumption.

Ass. 2. Let $p\in\bar{Y}$ and $C\subset\bar{Y}$ be a local or formal solution of the equation ρ_P, which contains p. Then at p, C is a divisor with normal crossings. In other words, there exists a formal coordinate system (y_1,y_2) at p on $\bar{Y}$ such that $y_1\cdot y_2|_C\equiv 0$.

Let $\tilde{C}$ and $\tilde{Z}$ be normalizations of C and $Z\backslash 0$, respectively.

Proposition 2. *Functions f and h, defined by formula* 2.4 (†) *on nonsingular part $Z^0\subset Z\backslash 0$, can be extended to regular functions on $\tilde{Z}$.*

Assuming this proposition, we see that f and h define global sections of invertible sheaves $\mathcal{O}_{\tilde{C}}(2-k)$ and $\mathcal{O}_{\tilde{C}}(2-l)$ on the nonsingiular complete curve $\tilde{C}$. Since the sheaf $\mathcal{O}_{\tilde{C}}(1)$ on $\tilde{C}$ has positive degree, the same arguments as before show that $f\equiv 0$, $l=1$ or 2, i.e., prove the statement in 2.4 for arbitrary C. The arguments in 2.5 then finish the proof of Theorem 2.4.

Proof of proposition. Let us recall the situation. We have a subvariety $Z \subset X$ and 2 functions P, Q which generate the ideal I_Z. We also have a 1-form α on X, which restricts to Z as 0 (as a 1-form on Z). We consider α, dP and dQ as sections of the sheaf $\Omega^1 X$. Denote by $\mathscr{F}$ the restriction of this sheaf to Z and consider α, dP, dQ as sections of $\mathscr{F}$ on Z. On nonsingular part $Z^0 \subset Z$ we can uniquely write $\alpha = f dP + h dQ$ and we have to show that f and h extend to regular functions on $\tilde{Z}$.

Clearly, it is enough to check this statement in a formal neighbourhood of a singular point $z \in Z \backslash 0$. By assumption 2 we can choose a formal coordinate system $\{y_i, i=1, \ldots, 4\}$ on X at z, such that $y_3 = P$ and Z is given by equations $\{y_1 \cdot y_2 = 0 = y_3\}$. Then $Q = \varepsilon P + \delta y_1 \cdot y_2$, where ε and δ are some power series and δ is invertible. The equation (†) $\alpha = f dP + h dQ$ becomes $\alpha = (f + h\varepsilon)\, dP + h\delta d(y_1 y_2)$. Therefore we may assume that $Q = y_1 y_2$.

Let $\alpha = \sum_{i=1}^{4} \alpha_i d y_i$. Then equation (†) reads

(‡) $$\sum_{i=1}^{4} \alpha_i d y_i = f d y_3 + h y_2 d y_1 + h y_1 d y_2 \quad \text{(as sections of } \mathscr{F}\text{)}.$$

We see that $\alpha_4|_Z = 0$ and $\alpha_3|_Z = f$, which proves that f extends to Z.

Let $Z = Z_1 \cup Z_2$, where Z_i is given by equations $\{y_i = y_3 = 0\}$, $i = 1, 2$. We have to prove that h extends to $\tilde{Z}$ = disjoint union of Z_1 and Z_2. According to (‡) the function h on $Z_2^0 = Z^0 \cap Z_2$ equals α_2 / y_1. Since α restricts to Z_1 as 0, the coefficient α_2 lies in the ideal (y_1, y_3), which shows that h extends to a regular function on Z_2. Similarly it extends to a regular function on Z_1.

This finishes the proof of Theorem 2.4 for polynomials P satisfying assumptions 1, 2.

2.7. Verification of assumptions for generic polynomials P

Ass 1 for generic P is Noether's theorem (see [Del]). This is the place in our proof when we use that $k \geqq 4$.

In order to check assumption 2 we introduce an invariant of a direction field. Let (M, m) be a germ of a nonsingular surface, ρ a germ of a direction field on M, defined outside of m. The field ρ can be given by a vector field θ, which does not vanish outside of m. The field θ is defined up to multiplication by a function f which is invertible on $M \backslash m$ and hence invertible on M. We say that m is a *regular* point of ρ if $\theta(m) \neq 0$.

Suppose m is a singular point of ρ. Then the 1-jet at m of the vector field θ is an endomorphism A_θ: $T_m M \to T_m M$. Up to a constant A_θ depends only on ρ, so we denote it $A(\rho, m)$. We say that m is *nondegenerate* point of ρ if $\det A(\rho, m) \neq 0$.

For nondegenerate point m let us introduce an invariant $\lambda(\rho, m)$ which is the ratio of 2 eigenvalues of $A(\rho, m)$. A point m is called a *Poincaré* point of ρ if it is nondegenerate and $\lambda(\rho, m) \notin \mathbf{Q}$.

Proposition 3. *Let ρ be a germ of a direction field on (M, m), C a germ of a solution of ρ, containing m.*

(i) *If ρ is regular at m, then C is nonsingular at m.*

(ii) *If m is a Poincaré point of ρ, then C is a divisor with normal crossings at m.*

Proof. We can assume that (M, m) is a formal germ, i.e., it is given by an algebra $R \approx \mathbf{C}|[[y_1, y_2]]$. The direction field ρ is given by a vector field θ, i.e., a derivation of R. The solution C is given by a principal radical ideal $J \subset R$, which is θ invariant. In the case of regular θ, we can choose coordinates (y_1, y_2) such that $\theta = \frac{\partial}{\partial y_1}$. In this case it is easy to see that $J = R y_2$.

Let m be a Poincaré point of ρ. By Poincaré's theorem (see [Poin] or [Arn]), we can choose coordinates (y_1, y_2) such that $\theta = \lambda_1 y_1 \frac{\partial}{\partial y_1} + \lambda_2 y_2 \frac{\partial}{\partial y_2}$, where $\lambda_1, \lambda_2 \in \mathbf{C}$, $\lambda(\rho, m) = \frac{\lambda_1}{\lambda_2} \notin \mathbf{Q}$.

Each $f \in R$ is a sum of monomials, $f = \sum_\beta f_\beta, f_\beta = a_\beta y^\beta$, $a_\beta \in \mathbf{C}$, $\beta = (p, q)$, $y^\beta = y_1^p y_2^q$.

Lemma 5. *If $f \in J$ then all monomials f_β lie in J.*

Indeed, since monomials are eigenvectors of θ with distinct eigenvalues, for a given β and each $N \in \mathbf{Z}^+$ we can find a polynomial Q_β such that $Q_\beta(\theta) f_\beta = f_\beta$ and $Q_\beta(\theta) f_\gamma = 0$ for all $\gamma \neq \beta$, $|\gamma| \leqq N$. This implies that $f_\beta = Q_\beta(\theta) f (\operatorname{mod} m^N)$, where m is the maximal ideal of R. Since θ preserves J, $f_\beta \in J + m^N$. But by Artin-Rees lemma (see [AM]), J is closed in R in m-adic topology, which proves that $f_\beta \in J$.

Now, let us apply the lemma to the case when f is a generator of J and $f_\beta \neq 0$ be a monomial of f of minimal degree. Then by the lemma, $f_\beta \in J$, i.e., $f_\beta = hf$. Comparing monomials of minimal degree we see that $h(0) \neq 0$. This implies that h is invertible and J is generated by the monomial f_β. Since J is radical, the only possibilities for β are $(0,0)$, $(0,1)$, $(1,0)$ and $(1,1)$. In all cases $C = V(J)$ is a divisor with normal crossings. This finishes the proof of the proposition.

2.8

Let $P \in \Sigma^k$ be a nondegenerate polynomial, $Y = V(P)$, and let ρ be the direction field on $\bar{Y}$, corresponding to the vector field ξ_P. Proposition 2.7 shows that Ass 2 follows from the following.

Ass. 2′. Each point $p \in \bar{Y}$ is either regular or Poincaré point of ρ.

Let us check that Ass 2′ holds for a generic polynomial P if $k \geqq 3$. Denote by $\Sigma^k_{reg} \subset \Sigma^k$ the subset of nondegenerate polynomials and put

$$\Xi = \{(P, s) \mid P \in \Sigma^k_{\mathrm{reg}}, s \in S_P\},$$
$$\bar{\Xi} = \{(P, \bar{s}) \mid P \in \Sigma^k_{\mathrm{reg}}, \bar{s} \in \bar{S}_P\}.$$

Note that Ξ is irreducible since the projection $\Xi \to X \backslash 0, (P, s) \mapsto s$ is a fibration and each fiber is an open subset of a linear space. Since the projection $\bar{\Xi} \to \Sigma^k_{\mathrm{reg}}$, $(P, \bar{s}) \mapsto P$ has finite fibers, it is enough to prove that for generic point $(P, s) \in \Xi, \bar{s}$ is a Poincaré point of the direction field $\rho = \rho_P$ on $\bar{Y} = \bar{Y}_P$. Let $\lambda(P, s) = \lambda(\rho_P, \bar{s})$ be the Poincaré's invariant. It is an algebraic function on Ξ, so it is enough to check that it is not constant.

One can compute the invariant by direct computations, but they are a little bit messy. We will use another method, based on the description of ρ as the kernel of a 1-form.

Let α be a 1-form on (M, m), whose kernel is ρ. It can be constructed in the following way: choose a nondegenerate volume form v on the surface M, which we consider as a map v: $TM \to T^* M$, and put $\alpha = v(\theta)$, where θ is a vector field, generating ρ. Conversely, given α we can define $\theta = v^{-1}(\alpha)$. We can assume that $\alpha(m) = 0$ and α does not vanish outside of m. Since $\alpha(m) = 0$ its 1-jet at m is a morphism B: $T_m M \to T^*_m M$. Since $A(\rho, m)$ is a 1-jet of θ, we have $A(\rho, m) = v_m^{-1} \circ B$, where v_m: $T_m M \to T^*_m M$ is the value of v at m.

Let $(P, s) \in \Xi$. Without loss of generality we can assume $s = (1, 0, 0, 0) \in X$. Near s we will identify $\bar{X}$ with a hyperplane $H = \{(1, x_2, x_3, x_4)\}$. Then $\bar{Y}$ is given by the equation $P = 0$ and the direction field ρ on $\bar{Y}$ is given by the 1-form $\alpha = i(E)\,\omega = x_1\,dx_3 - x_3\,dx_1 + x_2\,dx_4 - x_4\,dx_2 = dx_3 + x_2\,dx_4 - x_4\,dx_2$.

Since $s \in S_P$, $dP(s)$ is proportional to $\alpha(s) = dx_3$. Hence we can choose coordinates (y_2, y_4) on $\bar{Y}$ such that $x_2 = y_2$, $x_4 = y_4$, $x_3 = f(y_2, y_4)$ and $df(0, 0) = 0$. In these coordinates $\alpha = df + (y_2\,dy_4 - y_4\,dy_2)$. Choosing form $v = 2\,dy_2\,dy_4$ on $\bar{Y}$, we see that the matrix $A(\rho, s)$ is given by

$$A(\rho, s) = v^{-1} \cdot \operatorname{Hess} f + 1_{T_s \bar{Y}}$$

where Hess f is the Hessian of f.

It is easy to check that by choosing appropriate P we can make Hess f an arbitrary symmetric matrix Q (for instance, put $P = x_1^{k-2}(x_1 x_3 - Q(x_2, x_4)/2)$. This shows that Poincaré's invariant $\lambda(P, s)$ can be made arbitrary, which proves that assumptions 2′ and 2 hold for generic P.

3. Applications to D-modules, $D = D(\mathbf{C}^n)$

3.1

Let F be a finitely generated D-module, $W \subset X$ an irreducible subvariety. We define the number $m_W(F) \in \mathbf{Z}^+ \cup \infty$ as follows. Consider a good filtration F^k on F, and corresponding Σ-module F_Σ. Consider the algebra Σ_W – localization of Σ at W – and Σ_W-module $F_W = \Sigma_W \otimes_\Sigma F_\Sigma$. Then by definition

$$m_W(F) = \text{length of } \Sigma_W\text{-module } F_W.$$

It is easy to see that

(§) $m_W(F) = 0$ if $W \not\subset \mathrm{Ch}(F)$
$m_W(F) = \infty$ if W is a proper subset of an irreducible component of $\mathrm{Ch}(F)$.
$m_W(F) =$ multiplicity of F at W if W is an irreducible component of $\mathrm{Ch}(F)$.

Standard arguments (see [Gin], § 1, Cor. 1.3) show that $m_W(F)$ is well defined (does not depend on a good filtration) and additive, i.e., for an exact sequence $0 \to F' \to F \to F'' \to 0$ one has $m_W(F) = m_W(F') + m_W(F'')$.

Put $m(F) = \sum_W m_W(F)$, where the sum is over all minimal i.h.-varieties $W \subset X$.

Proposition 4. *For each finitely generated D-module F*

$$\operatorname{length}(F) \leqq m(F).$$

Proof. Since the function m is additive, it is enough to check that for $F \neq 0$, $m(F) > 0$. Let W' be an irreducible component of $\mathrm{Ch}(F)$. By the involutivity theorem it is an i.h.-variety. Choose a minimal i.h.-subvariety $W \subset W'$. Then by (§)

$$m_W(F) > 0 \quad \text{and hence } m(F) > 0.$$

3.2

Let us call a finitely generated D-module F singular if $m(F) < \infty$. From (†) it is clear that F is singular iff all components of $\mathrm{Ch}(F)$ are minimal i.h.-varieties. The category $\mathcal{M}_S(D)$ of singular D-modules is a full subcategory of category $\mathcal{M}(D)$ of all D-modules, closed with respect to subquotients an extensions. The above proposition shows that its properties are similar to properties of category $\mathrm{Hol}(D) \subset \mathcal{M}_S(D)$ of holonomic D-modules.

Another similarity can be seen in the following result.

Let $\mathcal{M}_S^d(D)$ be the subcategory of singular D-modules F such that all components of $\mathrm{Ch}(F)$ have codimension d (i.e., $\mathcal{M}_S^n(D) = \mathrm{Hol}(D)$). For each D-module F and $i \geqq 0$ denote by $E_r^i(F)$ the right D-module $E_r^i(F) = \mathrm{Ext}_D^i(F, D)$ and by $E^i(F)$ the corrsponding left D-module (see [Bor], Ch. VI, §3).

Proposition 5. *Suppose $F \in \mathcal{M}_S^d(D)$. Then $E^i(F) = 0$ for $i \neq d$. The contravariant functor E^d: $\mathcal{M}_S^d(D) \to \mathcal{M}_S^d(D)$ is a duality; in particular, $E^d(E^d F) = F$.*

Proof. It is known that for any F, $\mathrm{Ch}(E^i(F)) \subset \mathrm{Ch}(F)$, codim $\mathrm{Ch}(E^i(F)) \geqq i$ and $E^i(F) = 0$ if codim $\mathrm{Ch}(F) > i$ (see [Ehl]). This implies that $E^i(F) = 0$ for $i < d$. For $i > d$, $\mathrm{Ch}(E^i(F)) \subset \mathrm{Ch}(F)$ has dimension strictly less than dim $\mathrm{Ch}(F)$. Since $\mathrm{Ch}(F)$ consists of minimal i.h.-varieties, $\mathrm{Ch}(E^i(F)) = \emptyset$, i.e., $E^i(F) = 0$. The rest is proved in the same way as in ([Bor], Ch. VI, Thm. 3.7).

Our definition of a singular D-module has an obvious fault – it depends on the choice of a filtration in D. One can give analogous definitions for other choices of filtration, e.g., for the filtration by degree of operator. It is not clear how these different notions are connected with each other.

3.3

Let $P \in \Sigma^k$. Consider the following properties of P.

Prop. 1. P is irreducible and $Y_P = V(P)$ is a minimal i.h.-variety.

Prop. 2. P is irreducible and Y_P does not have proper homogeneous subvarieties of dimension > 1 tangent to ξ_P.

By Lemma 1.2, Prop. 2 implies Prop. 1. Conjectures A′ and A″ (which we proved for $n=2$, $k\geqq 4$) claim that Prop. 1 and Prop. 2 hold for a generic polynomial P if $n\geqq 2$, $k\geqq 3$.

Theorem 2. *Let $d\in D^k$ be an operator whose symbol $P=\sigma_k(d)$ satisfies Prop. 1. Then the left ideal $I=Dd\subset D$ is maximal.*

Proof. Suppose there exists an ideal J such that $D\supsetneqq J\supsetneqq I$. Then $\sigma(D)=\Sigma\supsetneqq\sigma(J)\supsetneqq\sigma(I)=\Sigma\cdot P$. Since P is irreducible this implies that $\emptyset\neq V(\sigma(J))\subsetneqq Y_P$. But $V(\sigma(J))=\mathrm{Ch}(D/J)$ is an i.h.-variety, which contradicts Prop. 1.

3.4

Theorem 3. *Let $d\in D^k$ be an operator whose symbol $P=\sigma_k(d)$ satisfies Prop. 2. Let $J\subset D$ be a left ideal such that $Jd\subset J$. Then $J=D\cdot d'$, where d' is a polynomial in d, i.e., $d'\in\mathbf{C}[d]$.*

Proof. Put $\mathcal{J}=\sigma(J)$, $Z=V(\mathcal{J})=\mathrm{Ch}(D/J)$. The condition $[d,J]\subset J$ implies $\{P,\mathcal{J}\}\subset\mathcal{J}$, i.e., $\xi_P(\mathcal{J})\subset\mathcal{J}$.

Step 1. $\xi=\xi_P$ is tangent to Z, i.e., $\xi(I_Z)\subset I_Z$. Let us choose $f\in I_Z$ and show that $\xi f\in I_Z$. By Nullstellensatz $f^p\in\mathcal{J}$ for large p. Since $\xi\mathcal{J}\subset\mathcal{J}$ we have $\xi^p(f^p)\in\mathcal{J}$. For each point $z\in Z$, $\xi^p(f^p)(z)=p!((\xi f)(z))^p$. Since $\xi^p(f^p)\in\mathcal{J}$ we have $\xi^p(f^p)(z)=0$, i.e., $\xi(f)(z)=0$. This proves that $\xi f\in I_Z$.

Step 2. Let Z be an i.h.-variety tangent to ξ_P. Then $Z=\emptyset$ or $Z=Y$ or $Z=X$. Indeed, suppose not. Then, replacing Z by one of its irreducible components we can assume that Z is a nonempty irreducible i.h.-variety tangent to ξ_P and $Z\not\supset Y$.

Put $Z'=Z\cap Y$. Then $Z'\neq Y$ is homogeneous and tangent to ξ_P. The property Prop. 2 implies that $\dim Z'\leqq 1$. On the other hand, since Z and Y are homogeneous and $\operatorname{codim} Y=1$, we have $\dim Z'\geqq\dim Z-1$, i.e., $\dim Z\leqq 2$. Since Z is involutive, $\dim Z\geqq n$, which shows that $\dim Z=n=2$. This implies that Z is Lagrangian. Since vector fields ξ_P and E are tangent to Z, we have $\omega(\xi_P,E)|_Z=0$, i.e., $kP|_Z=0$ (see 2.2 (*)). In other words, this shows that $Z\subsetneqq Y_P$, which contradicts Prop. 2.

Step 3. If $Z=X$ or $Z=\emptyset$, then $J=0=D\cdot 0$ or $J=D=D\cdot 1$, respectively. So let us consider the only possible case $Z=Y$. Let $m=m_Y(D/J)$ be the multiplicity of D-module $F=D/J$ at component Y. The Proposition 3.1 implies that $\mathrm{length}(F)\leqq m$.

Consider the operator $\tilde{d}\in End_D(F)$ given by $\tilde{d}(a)=a\cdot d$, where $a\in D$; it is well defined since $Jd\subset J$. Since $\mathrm{length}(F)\leqq m$ by Shur's lemma (see [Dix], Lemma 2.6.4) there exists a polynomial $R(t)\in\mathbf{C}[t]$ of degree $r\leqq m$, such that $R(\tilde{d})=0$. Put $d'=R(d)\in D$. Then $R(\tilde{d})=0$ implies that $d'\in J$.

Consider an ideal $J'=Dd'\subset J$ and put $F'=D/J'$. Clearly $m_Y(F')=r$. Since $r\leqq m$ and F is a quotient of F', we see that $r=m$ and $F=F'$, i.e., $J=J'=Dd'$. Q.E.D.

4. Proof of Proposition 0.4

4.1

Let $D=\bigcup_{j=0}^{\infty}\tilde{D}^j$ be the filtration of $D=D(\mathbf{C}^n)$ given by the order of an operator, $\tilde{\Sigma}$ the associated graded algebra, $\tilde{\sigma}_j\colon \tilde{D}^j\to\tilde{\Sigma}^j$ the symbol map. Then $\tilde{\Sigma}=\mathbf{C}[u_1,\ldots,u_n,\eta_1,\ldots,\eta_n]$, where $u_i=\tilde{\sigma}_0(u_i)$, $\eta_i=\tilde{\sigma}_1(\partial_i)$. We consider $\tilde{\Sigma}$ as the algebra of polynomial function on the space $\tilde{X}=T^*\mathbf{C}^n$ – cotangent bundle to $\mathbf{C}^n$. The commutator in D defines the standard symplectic structure on cotangent bundle $\tilde{X}$, given by the form $\omega=\sum_{i=1}^{n}du_i d\eta_i$.

Note that the space $\tilde{X}$ is the same as the space X before, but the geometry is quite different, since now we consider homoteties which act only in η direction. In particular, by i.h.-variety we now mean an involutive subvariety of $\tilde{X}$, invariant with respect to homoteties in η direction. The involutivity theorem claims that for each finitely generated D-module F its characteristic variety $\mathrm{Ch}(F)$ is an i.h.-variety (see [Gab]).

4.2

In case of $\tilde{X}$ an i.h.-variety of dimension $>n$ is rarely minimal. The reason is that it usually contains i.h.-subvarieties of the following types

(i) $\mathbf{C}^n\subset T^*\mathbf{C}^n$ – the zero section

(ii) $T_s^*\mathbf{C}^n$ for some points $s\in\mathbf{C}^n$.

We will call such i.h.-varieties "trivial". We say that an i.h.-variety $Y\subset\tilde{X}$ is almost minimal if it does not have proper nontrivial irreducible i.h.-subvarieties.

Let $\xi=\Sigma R_i(u)\,\partial_i$ be a polynomial vector field on $\mathbf{C}^n$, $S_\xi=\{s\in\mathbf{C}^n\,|\,\xi(s)=0\}$ the set of its singular points. Consider the symbol $\tilde{\sigma}_1(\xi)$ and put $Y_\xi=V(\tilde{\sigma}_1(\xi))$ – the zero set of the symbol. By definition $Y_\xi=\{(u,\eta)\in\tilde{X}\,|\,u\in\mathbf{C}^n,\eta\in\xi(u)^\perp\subset T_u^*\mathbf{C}^n)$. In particular, Y_ξ contains the zero section $\mathbf{C}^n$ and spaces $T_s^*\mathbf{C}^n$ for $s\in S_\xi$.

Conjecture 1. *Let ξ be a generic vector field with coefficients $\{R_i\}$ of degree $k\geqq 2$. Then Y_ξ is an almost minimal i.h.-variety.*

4.3.

For $n=2$ the condition that Y_ξ is an almost minimal i.h.-variety means that ξ is not tangent to any algebraic curve. Indeed, in this case $\dim Y_\xi=3$, so any proper i.h.-subvariety $Z\subset Y_\xi$ has dimension 2, i.e., is Lagrangian.

It is easy to check (see [Kash, Th. 5.1.6, Lemma 1]) that any irreducible homogeneous Lagrangian subvariety $Z\subset T^*\mathbf{C}^n$ is the conormal bundle $T_M^*\mathbf{C}^n$ of an irreducible subvariety $M\subset\mathbf{C}^n$ (by definition $T_M^*\mathbf{C}^n$ is the closure of the set $\{(u,\eta)\,|\,u$ is a nonsingular point of M, $\eta\in T_uM^\perp\subset T_u^*\mathbf{C}^n\})$.

In case $n=2$ an i.h.-variety $T_M^* \mathbf{C}^2$ is trivial if $\dim M=0$ or 2. So we are left with the possibility $Z=T_M^* \mathbf{C}^2$ for some curve $M \subset \mathbf{C}^2$. The condition $T_M^* \mathbf{C}^2 \subset Y_\xi$ means that for nonsingular points $u \in M$, $T_u M^\perp \subset \xi(u)^\perp$, i.e., $\xi(u) \in T_u M$, which means that ξ is tangent to M.

Thus for $n=2$ the conjecture follows from results in the appendix.

4.4

Proposition 6. *Let $n \geqq 2$ and ξ be a polynomial vector field on $\mathbf{C}^n$. Suppose ξ satisfies*

(*) (i) *The i.h.-variety Y_ξ is almost minimal.*

(ii) *The set S_ξ of singular points of ξ is not empty.*

Choose a polynomial function f on $\mathbf{C}^n$ such that

(**) *For each point $s \in S_\xi$, $f(s) \notin \Lambda(\xi, s)$ – the subgroup of $\mathbf{C}$ generated by eigenvalues of the operator $A(\xi, s)$: $T_s \mathbf{C}^n \to T_s \mathbf{C}^n$, given by the 1-jet of ξ.*

Then the operator $d=\xi+f$ generates a maximal left ideal in D.

Proof. Note that (*) (i) implies that S_ξ is finite and that $\tilde{\sigma}_1(\xi)$ is irreducible.

Step 1. Suppose that the ideal $I=Dd$ is not maximal and choose a maximal ideal J such that $D \supsetneqq J \supsetneqq I=Dd$. Then $\Sigma \supsetneqq \tilde{\sigma}(J) \supsetneqq \tilde{\sigma}(I)=\Sigma \cdot \tilde{\sigma}_1(\xi)$. Since the polynomial $\tilde{\sigma}_1(\xi)$ is irreducible, it implies

$$\emptyset \neq V(\tilde{\sigma}(J)) \subsetneqq Y_\xi .$$

Consider D-modules $F=D/I, H=D/J$; note that H is irreducible. Then $V(\tilde{\sigma}(J))=\operatorname{Ch}(H)$ is an i.h.-variety. So by (*) (i) it is a union of some of the trivial components $\mathbf{C}^n$ and $T_s^* \mathbf{C}^n$, $s \in S_\xi$.

Step 2. Suppose $Z=\operatorname{Ch}(H)$ contains the zero section $\mathbf{C}^n$. Consider H as $\mathscr{D}$-module, i.e., a sheaf of $\mathscr{D}$-modules on $\mathbf{C}^n$ (see [Bor], Ch. VI, § 1) and denote by H' its restriction to $V=\mathbf{C}^n \backslash S_\xi$. Then $\operatorname{Ch}(H')=V \subset T^* V$, which implies that H' is $\mathscr{O}$-coherent and hence locally free as $\mathscr{O}_V$-module ([Bor], Ch. VI, Prop. 10.4, Prop. 1.7). Denote by H'' the direct image of H' from V to $\mathbf{C}^n$. Since H' is locally free as $\mathscr{O}_V$-module, and $\operatorname{codim}_{\mathbf{C}^n}(S_\xi) \geqq 2$, H'' is $\mathscr{O}_{\mathbf{C}^n}$-coherent (indeed, we can imbed H' in a free $\mathscr{O}_V$-module $\mathscr{O}_V^l$, and then H'' will be a submodule of $\mathscr{O}_{\mathbf{C}^n}^l$, which is the direct image of $\mathscr{O}_V^l$). Again applying [Bor, Ch. VI, Prop. 1.7] we see that H'' is locally free as $\mathscr{O}_{\mathbf{C}^n}$-module.

We have the natural morphism i: $H \to H''$, which is an isomorphism on V. Since H'' does not have quotients supported on S_ξ, i: $H \to H''$ is an epimorphism. But H is irreducible, so $H=H''$ is $\mathscr{O}$-coherent.

Step 3. Fix a point $s \in S_\xi$. For each $\mathscr{D}$-module $\mathscr{F}$ define spectrum $\operatorname{Sp}_{\xi,s}(\mathscr{F})$ to be the set of all possible eigenvalues of ξ in all finite-dimensional ξ-invariant $\mathscr{O}$-module subquotients of $\mathscr{F}$, supported at s. In other words, we consider pairs $(\mathscr{L}, \mathscr{L}')$, where $\mathscr{F} \supset \mathscr{L} \supset \mathscr{L}'$, $\mathscr{L}, \mathscr{L}'$ are ξ-invariant finitely generated $\mathscr{O}$-modules,

$\dim \mathscr{L}/\mathscr{L}' < \infty$ and $\mathscr{L}/\mathscr{L}'$ is annihilated by some power of the maximal ideal m_s, and put $Sp_{\xi,s}(\mathscr{F}) = \bigcup_{\mathscr{L},\mathscr{L}'} Sp(\xi, \mathscr{L}/\mathscr{L}')$. It is easy to check that for each subquotient $\mathscr{F}'$ of $\mathscr{F}$, $Sp_{\xi,s}(\mathscr{F}') \subset Sp_{\xi,s}(\mathscr{F})$.

Step 4. Let us prove that $Sp_{\xi,s}(H) \subset \Lambda(\xi, s)$. Indeed ξ acts trivially on $H/m_s H$, which easily implies that its eigenvalues on $m_s^k H/m_s^{k+1} H$ and hence on $H/m_s^k H$ lie in $\Lambda(\xi, s)$ for each k. Since H is $\mathcal{O}$-coherent, Artin-Rees lemma implies that for each pair $\mathscr{L} \supset \mathscr{L}'$ the quotient $\mathscr{L}/\mathscr{L}'$ is a subquotient of $H/m_s^k H$ for large k, i.e., $Sp(\xi, \mathscr{L}/\mathscr{L}') \subset \Lambda(\xi, s)$.

On the other hand, consider an element $h \in H$, which is the image of $1 \in D$ and put $\mathscr{L} = \mathcal{O} \cdot h$, $\mathscr{L}' = m_s \cdot h$. Since $H = Dh$, $h \neq 0$ and since H is locally free as $\mathcal{O}$-module, h is nonzero in $\mathscr{L}/\mathscr{L}'$. Since $dh = 0$, we have $\xi h = -f(s)\, h$ in $\mathscr{L}/\mathscr{L}'$, i.e., $f(s) \in Sp_{\xi,s}(H)$. This contradicts (*) (ii) and (**).

Step 5. We have shown that Z does not contain the zero section. Then $Z \subset \bigcup_{s \in S_\xi} T_s^* C^n$, i.e., as a $\mathscr{D}$-module H is supported in S_ξ. Since H is irreducible, it is supported at one point $s \in S_\xi$ and by Kashiwara's theorem (see [Bor], Ch. VI, Thm 7.11) it is isomorphic to the standard module δ_s, supported at s.

It is easy to check directly that $Sp_{\xi,s}(\delta_s) \subset \Lambda(\xi, s)$ and for each $h \in \delta_s$, $\mathcal{O} \cdot h \notin m_s h$. Now the same arguments as before lead to a contradiction with (**). This ends the proof of the proposition.

5. Miscellaneous remarks

5.1. Basic facts about generic points

Let $K \subset \mathbf{C}$ be a countable subfield, $\Gamma = \mathrm{Aut}_K \mathbf{C}$ the group of all field automorphisms of $\mathbf{C}$ over K. Let M be an affine algebraic variety defined over K and $M(\mathbf{C})$ be the set of its $\mathbf{C}$-points. Clearly Γ acts on $M(\mathbf{C})$.

For simplicity we assume that M is irreducible over K, i.e., that the algebra $K[M]$ of regular K-functions on M does not have zero divisors. We denote by $K(M)$ its quotient field.

By definition, points $x \in M(\mathbf{C})$ correspond to K-algebra morphisms $v_x : K[M] \to \mathbf{C}$. We call a point $x \in M(\mathbf{C})$ *generic* if v_x is an imbedding, i.e., v_x extends to a morphism $K(M) \to \mathbf{C}$. The set of generic points we denote by $M(\mathbf{C})^*$.

Proposition 7. *Let $N \subset M(\mathbf{C})$ be a Γ-invariant subset. Then there are two mutually exclusive possibilities.*

I) *N does not intersect $M(\mathbf{C})^*$ and then N can be covered by a countable number of hypersurfaces in $M(\mathbf{C})$.*

II) *N contains $M(\mathbf{C})^*$ and then N cannot be covered by a countable number of hypersurfaces in $M(\mathbf{C})$. Moreover, the intersection of N with any nonempty subset $U \subset M(\mathbf{C})$, open in usual topology, cannot be covered by a countable number of hypersurfaces.*

Proof. The proposition follows from the following 3 facts:

F1. Γ acts transitively on the set $M(\mathbf{C})^* = Mor_K(K(M), \mathbf{C})$. This easily follows from the fact that for each imbedding $K(M) \hookrightarrow \mathbf{C}$, $\mathbf{C}$ is an algebraic closure of a transcendent extension of $K(M)$ with continuum generators.

F2. Any nonempty subset $U \subset M(\mathbf{C})$ open in usual topology cannot be covered by a countable number of hypersurfaces. This is a consequence of Baire's theorem.

F3. $M(\mathbf{C}) \backslash M(\mathbf{C})^*$ can be covered by a countable number of hypersurfaces. Indeed, for each $f \in K[m] \backslash 0$ consider a hypersurface

$$H_f = \{x \in M(\mathbf{C}) \mid f(x) = 0\} = \{x \in M(\mathbf{C}) \mid v_x(f) = 0\}.$$

By definition the union of countable number of hypersurfaces $H_f, f \in K[M] \backslash 0$, is equal to $M(\mathbf{C}) \backslash M(\mathbf{C})^*$.

5.2. Case $k=2$ in Theorem A

Let us check that for $n \geqq 2$, $k=2$ the operator d never generates a maximal ideal. Let $P = \sigma_2(d) \in \Sigma^2$. For simplicity we assume that P is generic. P is a quadratic form, i.e., a morphism $P\colon X \to X^*$. We also can interpret the form ω on X as an antisymmetric morphism $\omega\colon X \to X^*$. Consider the operator $A = \omega^{-1} \circ P\colon X \to X$. It is easy to check that for generic P the operator A^2 has all eigenvalues of multiplicity 2. Using eigenspaces decomposition for A^2 we can decompose $X = \bigoplus_{i=1}^{n} X_i$, where $\dim X_i = 2$ and all X_i are orthogonal both with respect to P and with respect to ω.

It is known that the group $G = Sp(2n) \rhd\!\!< X$ acts on the algebra $D(\mathbf{C}^n)$, preserving the filtration $\{D^k\}$. Using an automorphism from G we can transform P to the polynomial

$$P = \sum_{i=1}^{n} \lambda_i (x_i^2 + x_{i+n}^2), \qquad \lambda_i \in \mathbf{C}.$$

Using an additional transformation from the translation part of G we can kill the first order terms in d, i.e., we can write d in the form

$$d = \sum_{i=1}^{n} d_i, \qquad d_i = \lambda_i (u_i^2 + \partial_i^2) + r_i, \qquad \lambda_i, r_i \in \mathbf{C}.$$

This implies that the ideal $I = Dd$ can be imbedded in a larger ideal $J = D(d_1, d_2, \ldots, d_n) \neq D$.

Appendix: Algebraic solutions of a direction field on $\mathbf{P}^2$

A1. Let ρ be a direction field on $\mathbf{P}^2$, defined on a dense open subset. Choose a differential 1-form β with rational coefficients, which vanishes on ρ and consider its divisor $\mathrm{Div}(\beta)$. We define $\deg \rho = -\deg \mathrm{Div}(\beta)$.

Let p: $\mathbf{C}^3 \backslash 0 \to \mathbf{P}^2$ be the standard projection. Choose a rational function φ on $\mathbf{C}^3$ of homogeneous degree $\deg \rho$ such that $\mathrm{Div}(\varphi) = -\mathrm{Div}(p^*\beta)$ and put $\alpha = \varphi \cdot p^*\beta$. Then α is a rational 1-form on $\mathbf{C}^3$ with $\mathrm{Div}(\alpha) = 0$. This implies that α is regular.

The form α has the following properties:

(*) α is homogeneous of degree $k = \deg \rho$ and $\alpha(E) \equiv 0$, where E is the Euler vector field.

(**) The singular set $S_\alpha = \{s \in \mathbf{C}^3 \mid \alpha(s) = 0\}$ has codimension > 1.

Clearly α is defined by ρ uniquely up to a scalar. Conversely, any 1-form α, satisfying (*), (**) defines a direction field ρ, by $p^*\rho = \mathrm{Ker}\,\alpha$. The set $S_\rho = \bar{S}_\alpha \subset \mathbf{P}^2$ of singular points of ρ is finite.

Denote by Ω_k the space of regular 1-forms on $\mathbf{C}^3$ of homogeneous degree k, and by Ω'_k the subspace of 1-form, orthogonal to E. As we saw, direction fields of degree k are parametrized by an open subset of Ω'_k.

Let $\mathscr{F}_k$ be the space of homogeneous polynomials of degree k on $\mathbf{C}^3$.. Then for $k \geqq 0$, $\dim \mathscr{F}_k = (k+1)(k+2)/2$. This implies

$$\dim \Omega_k = 3 \dim \mathscr{F}_{k-1} = 3k(k+1)/2 \qquad \text{for } k \geqq 0,$$

$$\dim \Omega'_k = \dim \Omega_k - \dim \mathscr{F}_k = k^2 - 1 \qquad (k>0), \qquad \dim \Omega'_0 = 0.$$

Thus direction fields of degree $k > 1$ are parametrized by $k^2 - 2$ parameters.

A2. An algebraic curve $C \subset \mathbf{P}^2$, tangent to ρ we call an *algebraic solution* of ρ.

Proposition 8. *Any algebraic solution C of ρ passes through one of the singular points of ρ.*

Proof. Suppose C does not intersect $\bar{S}_\rho$. Then C is nonsingular. Denote by $\mathscr{F}$ the algebra of regular functions on $\mathbf{C}^3$ and by $I_Z \subset \mathscr{F}$ the ideal of functions which vanish on subvariety $Z = p^{-1}(C) \subset \mathbf{C}^3$. This ideal is generated by one homogeneous function. We denote this function by Q and put $l = \deg Q$. Also denote by $\alpha = \alpha_\rho$ a 1-form of degree $k = \deg \rho$ on $\mathbf{C}^3$, which defines ρ.

For each point $z \in Z$ we have

$\alpha(z) \neq 0$ (since $z \notin S(\rho)$),
$dQ(z) \neq 0$ (since Z is nonsingular at z),
$\alpha(z)$ and $dQ(z)$ vanish on $T_z(Z)$.

This implies that there exists an invertible function f on Z such that $\alpha = f\,dQ$. Clearly f is homogeneous of degree $k - l$, i.e., f is an invertible section of the sheaf $\mathcal{O}_C(k-l) = \mathcal{O}(k-l)|_C$. Since the sheaf $\mathcal{O}_C(1)$ on C has positive degree, this implies that $k = l$ and f is constant.

Let $f=f_0 \in \mathbf{C}$. Consider the form $\alpha'=\alpha-f_0\,dQ$. For each $z\in Z, \alpha'(z)=0$, so α' is divisible by Q, i.e., $\alpha'=Q\gamma$, where γ is a 1-form of homogeneous degree 0. Since there are no such forms, $\gamma=0$, i.e., $\alpha=f_0\,dQ$.

However, this implies $0=\alpha(E)=f_0\,dQ(E)=f_0 l\cdot Q$ – a contradiction which proves the proposition.

A 3. When a solution passes through a singular point, it is difficult to tell anything about the solution, unless this point happened to be a Poincaré point (see 2.7). So we will pay special attention to direction fields ρ for which all singular points are Poincaré points (we call such a direction field and the corresponding 1-form α Poincaré). It is easy to check that a generic 1-form α of degree $\geqq 3$ is Poincaré (compare with 2.8).

Theorem 4. *Fix $k>3$. Then a generic direction field ρ on $\mathbf{P}^2$ of degree k does not have algebraic solutions. Moreover, there exists an open dense subset $V\subset \Omega'_k$ such that any Poincaré 1-form $\alpha\in V$ corresponds to a direction field with no algebraic solution.*

Proof. Step 1. Let $\alpha\in\Omega'_k$ be a Poincaré 1-form, ρ corresponding direction field, $C\subset\mathbf{P}^2$ an irreducible algebraic solution of ρ, $Z=p^{-1}(C)$, $Q\in\mathscr{F}$ a generator of the ideal I_Z, $l=\deg Q$.

Let $\tilde{C}$ and $\tilde{Z}$ be normalizations of C and Z, respectively. In the same way as in 2.8 and 2.6, one checks that C is a divisor with normal crossings and there exists a regular function f on $\tilde{Z}$ of homogeneous degree $k-l$ such that on a dense subset of Z, $\alpha=f\,dQ$.

We can interpret f as a global section of the invertible sheaf $\mathcal{O}_{\tilde{C}}(k-l)$ on $\tilde{C}$. Since $f\not\equiv 0$, we have $k\geqq l$.

Step 2. Fix l, $1\leqq l\leqq k$ and denote by Ξ_l a subset of forms $\alpha\in\Omega'_k$ which satisfy the following property.

(*) There exists an irreducible divisor $C\subset\mathbf{P}^2$ with normal crossings, given by a polynomial $Q\in\mathscr{F}_l$ and a regular function f on $\tilde{Z}$ (where $\tilde{Z}$ is the normalization of $Z=p^{-1}(C)$), such that on a dense subset of Z, $\alpha=f\,dQ$.

Clearly Ξ_l is a constructible subset of Ω'_k. We denote by V the complement in Ω'_k of the closure of $\bigcup_{l=1}^{k}\Xi_l$. Step 1 shows that each Poincaré 1-form $\alpha\in V$ does not have algebraic solutions. Hence it is enough to check that $V\neq\emptyset$, i.e., to check that for each l, $\dim\Xi_l<\dim\Omega'_k$.

Step 3. Let us estimate $\dim\Xi_l$. Since $Q\in\mathscr{F}_l$ it depends on $\dim\mathscr{F}_l=(l+1)(l+2)/2$ parameters.

For a fixed Q, f is a section of $\Gamma(\tilde{C},\mathcal{O}_{\tilde{C}}(k-l))$. Since the curve $\tilde{C}$ is irreducible,

$$\dim\Gamma(\tilde{C},\mathcal{O}_{\tilde{C}}(k-l))\leqq 1+\deg\mathcal{O}_{\tilde{C}}(k-l)=1+(k-l)\deg\mathcal{O}_{\tilde{C}}(1)=1+(k-l)\,l,$$

i.e., f depends on $1+(k-l)\,l$ parameters.

For fixed Q and f, α is uniquely defined on Z by $\alpha=f\,dQ$, so α is defined up to addition of a form $Q\gamma$, with $\gamma\in\Omega'_{k-l}$. This gives $\dim Q'_{k-l}=(k-l)^2-1$ additional parameters (zero if $k=l$). Since transformation $Q\mapsto\lambda Q, f\mapsto\lambda^{-1}f$ does not change α, we see that

$$\dim \Xi_l \leqq (l+1)(l+2)/2 + (1+(k-l)\,l) + ((k-l)^2-1) - 1$$
$$= (l+1)(l+2)/2 + k(k-l) - 1 = \varphi(l).$$

We should check that for $1 \leqq l \leqq k$, $\varphi(l) < \dim \Omega'_k = k^2 - 1$. Since φ is a convex function, it is enough to check the inequality for $l=1$ and $l=k$.

$$\varphi(1) = 3 + k^2 - k - 1 < k^2 - 1 \quad \text{since } k > 3$$
$$\varphi(k) = (k+1)(k+2)/2 - 1 < k^2 - 1 \quad \text{for } k > 3.$$

Since for $l=k$, $\dim \Omega'_{k-l} = 0 \neq (k-l)^2 - 1$, this case should be treated separately. In this case $\alpha = f_0\, dQ$ on Z with $f_0 \in \mathbf{C}$ and the same arguments as in A2 show that $\alpha \equiv 0$, i.e., $\Xi_k = 0$.

This proves Theorem A3.

A4. Let V_k be the space of vector fields ξ on $\mathbf{C}^2$ of the form $\xi = R_1 \partial_1 + R_2 \partial_2$, where R_1, R_2 are polynomials of degree $\leqq k-2$ (we will see that they correspond to direction fields on $\mathbf{P}^2$ of degree k).

Theorem 5. *Fix $k > 3$. Then a generic field $\xi \in V_k$ is not tangent to any algebraic curve.*

Proof. Let ρ be the direction field, defined by ξ. We imbed $\mathbf{C}^2$ into $\mathbf{P}^2$ and consider ρ as a direction field on $\mathbf{P}^2$. Let us describe explicitly the corresponding 1-form α on $\mathbf{C}^3$. We imbed $\mathbf{C}^2 \to \mathbf{C}^3$ by $u_1, u_2 \to 1, u_1, u_2$. Let t_0, t_1, t_2 be coordinates on $\mathbf{C}^3$. For each polynomial R in u_1, u_2 of degree $\leqq k-2$ we denote by $\tilde{R} \in \mathscr{F}_{k-2}$ the homogeneous polynomial on $\mathbf{C}^3$, whose restriction to $\mathbf{C}^2$ gives R. The direction field ρ on $\mathbf{C}^2$ is given by a 1-form $\alpha' = R_2\, du_1 - R_1\, du_2$, which extends to a 1-form $\alpha \in \Omega'_k$ given by

$$(*) \qquad \alpha = (t_2 \tilde{R}_1 - t_1 \tilde{R}_2)\, dt_0 + t_0 \tilde{R}_2\, dt_1 - t_0 \tilde{R}_1\, dt_2.$$

Unfortunately, we cannot directly apply Theorem A3, since the morphism i: $V_k \to \Omega'_k$ defined by $(*)$ is not onto. Namely, denote by Z_0 the hyperplane $t_0 = 0$ in $\mathbf{C}^3$ and consider a subspace $\Omega''_k \subset \Omega'_k$, consisting of forms α such that $\alpha | Z_0 = 0$. It is easy to see that i: $V_k \tilde{\to} \Omega''_k$.

What we can do is just to repeat the proof of Theorem A3 with changing Ω' by Ω''. Let us sketch the proof.

Step 1. Generic form $\alpha \in \Omega''_k$ is Poincaré. It is enough to find an example of a form $\alpha \in \Omega'_k$ and a point $s \in C_0 = \mathbf{P}(Z_0)$ which is a Poincaré singular point for α. The rest is like in 2.8. In other words, we should find some direction field ρ of degree k, which has a solution $C \approx \mathbf{P}^1 \subset \mathbf{P}^2$ and has a Poincaré point s on C. We can take ρ to be the direction field, corresponding to the vector field $\xi = (1+u_1)^{k-3}(\lambda_1 u_1 \partial_1 + \lambda_2 u_2 \partial_2))$, C be the line $u_1 = 0$ and s be the point $u_1 = u_2 = 0$.

Step 2. Now let us repeat steps 2 and 3 of the proof of Theorem A3.

We denote by Ξ'_l the set of forms α such that

($*$) There exists a divisor $C \subset \mathbf{P}^2$ with normal crossings, such that $C \supset C_0$ and $C \backslash C_0$ is irreducible, given by a polynomial $Q \in \mathscr{F}_l$, and a regular function f on $\tilde{Z}$ (normalization of $Z = p^{-1}(C)$), such that on a dense subset of Z, $\alpha = f\, dQ$.

Let us estimate $\dim \Xi'_l$ for $2 < l < k$ (note that $l = \deg Q > 1$, since $Q = t_0 Q'$). Since $Q \in t_0 \mathscr{F}_{l-1}$, it depends on $\dim \mathscr{F}_{l-1} = l(l+1)/2$ parameters. For a fixed Q, f depends on $\dim \Gamma(\tilde{C}, \mathcal{O}_{\tilde{C}}(k-l))$ parameters. Since C has 2 irreducible components,

$$\dim \Gamma(\tilde{C}, \mathcal{O}_{\tilde{C}}(k-l)) \leqq 2 + (k-l) \cdot l.$$

For fixed Q, f the form α is defined up to $Q \cdot \Omega'_{k-l}$, i.e., has $\dim \Omega'_{k-l} = (k-l)^2 - 1$ additional parameters. Hence

$$\dim \Xi'_l \leqq l(l+1)/2 + ((k-l)\, l + 2) + ((k-l)^2 - 1) - 1 = l(l+1)/2 + k(k-l) = \varphi'(l).$$

It is enough to check the inequality

$$\varphi'(l) < \dim \Omega''_k = k^2 - k \quad \text{for } l = 2 \quad \text{and} \quad l = k.$$
$$\varphi'(2) = 3 + k^2 - 2k = (k^2 - k) + (3 - k) < k^2 - k \quad \text{for } k > 3$$
$$\varphi'(k) = k(k+1)/2 = (k^2 - k) - \tfrac{1}{2}(k^2 - 3k) < k^2 - k \quad \text{for } k > 3.$$

This proves the Theorem A4.

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Note added in proof

Conjecture A was proved for $n \geqq 2$, $k \geqq 4$ in paper "Algebraic varieties preserved by generic flows" by V. Lunts