SUBCONVEXITY BOUNDS FOR TRIPLE L-FUNCTIONS AND REPRESENTATION THEORY

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Abstract. We describe a new method to estimate the trilinear period on automorphic representations of $PGL_2(\mathbb{R})$. Such a period gives rise to a special value of the triple L-function. We prove a bound for the triple period which amounts to a subconvexity bound for the corresponding special value of the triple L-function. Our method is based on the study of the analytic structure of the corresponding unique trilinear functional on unitary representations of $PGL_2(\mathbb{R})$.

1. Introduction

1.1. **Maass forms.** Let $\mathbb{H}$ denote the upper half plane equipped with the standard Riemannian metric of constant curvature -1. We denote by dv the associated volume element and by Δ the corresponding Laplace-Beltrami operator on $\mathbb{H}$.

Fix a discrete group Γ of motions of $\mathbb{H}$ and consider the Riemann surface $Y = \Gamma\backslash\mathbb{H}$. For simplicity we assume that Y is compact (the case of Y of finite volume is discussed at the end of the introduction). According to the uniformization theorem, any compact Riemann surface Y with the metric of constant curvature -1 is a special case of this construction.

Consider the spectral decomposition of the operator Δ in the space $L^2(Y, dv)$ of functions on Y. It is known that the operator Δ is non-negative and has a purely discrete spectrum; we will denote the eigenvalues of Δ by $0 = \mu_0 < \mu_1 \leq \mu_2 \leq \dots$.

For these eigenvalues, we always use a natural (from the representation-theoretic point of view) parametrization $\mu_i = \frac{1-\lambda_i^2}{4}$, where $\lambda_i \in \mathbb{C}$. We denote by $\phi_i = \phi_{\lambda_i}$ the corresponding eigenfunctions (normalized to have L^2-norm one).

In the theory of automorphic forms, the functions ϕ_{λ_i} are called automorphic functions or *Maass forms* (after H. Maass, [M]). The study of Maass forms plays an important role in analytic number theory, analysis and geometry. We are interested in their analytic properties and will present a new method of bounding some important quantities arising from functions ϕ_i.

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A specific problem that we are going to address in this paper belongs to an active area of research in the theory of automorphic functions that studies an interplay between periods, special values of automorphic L-functions and representation theory. One of the central features of this interplay is the uniqueness of invariant functionals associated to corresponding periods. The discovery of this interplay goes back to classical works of E. Hecke and H. Maass.

It is well-known that uniqueness plays a central role in the modern theory of automorphic functions (see [PS]). The impact that uniqueness has on the analytic behavior of periods and L-functions is yet another manifestation of this principle.

1.2. **Triple products.** For any three Maass forms $\phi_i,\ \phi_j,\ \phi_k$, we define the following *triple product* or *triple period*:

$$c_{ijk} = \int_Y \phi_i \phi_j \phi_k dv \ .$$

We would like to estimate the coefficient c_{ijk} as a function of parameters $\lambda_i,\ \lambda_j,\ \lambda_k$. In particular, we would like to find bounds for these coefficients as one or more of the indices $i,\ j,\ k$ tend to infinity.

The bounds on the coefficient c_{ijk} are related to bounds on automorphic L-functions as can be seen from the following beautiful formula of T. Watson (see [Wa], [Ic]):

$$\left|\int_Y \phi_i \phi_j \phi_k dv\right|^2 = \frac{\Lambda(1/2, \phi_i \otimes \phi_j \otimes \phi_k)}{\Lambda(1, \phi_i, Ad)\Lambda(1, \phi_j, Ad)\Lambda(1, \phi_k, Ad)} \ . \tag{1.1}$$

Here the ϕ_t are the so-called cuspidal Hecke-Maass functions of norm one on the Riemann surface $Y = \Gamma \setminus \mathbb{H}$ arising from the full modular group $\Gamma = SL_2(\mathbb{Z})$ or from the group of units of a quaternion algebra. The functions $\Lambda(s, \phi_i \otimes \phi_j \otimes \phi_k)$ and $\Lambda(s, \phi, Ad)$ are appropriate *completed* automorphic L-functions.

It was first discovered by R. Rankin [Ra] and A. Selberg [Se] that the special case of above mentioned triple product gives rise to an automorphic L-function (namely, they considered the case where one of the Maass forms is replaced by an Eisenstein series). That allowed them to obtain analytic continuation and effective bounds for these L-functions and, as an application, to obtain one of the first non-trivial bounds for Fourier coefficients of cusp forms towards Ramanujan's conjecture. The relation (1.1) can be viewed as a far reaching generalization of the original Rankin-Selberg formula. The relation (1.1) was motivated by the work of M. Harris and S. Kudla ([HK]) on a conjecture of H. Jacquet.

1.3. **Results.** In this paper we consider the following problem. We fix two Maass forms $\phi = \phi_\tau$ and $\phi' = \phi_{\tau'}$ as above and consider the coefficients defined by the triple period:

$$c_i = \int_Y \phi \phi' \phi_i dv \tag{1.2}$$

as the ϕ_i run over an orthonormal basis of Maass forms.

Thus we see from (1.1) that the estimates of the coefficients c_i are essentially equivalent to the estimates of the corresponding L-functions. One would like to have a general method of estimating the coefficients c_i and similar quantities. This problem was raised by Selberg in his celebrated paper [Se].

The first non-trivial observation is that the coefficients c_i have exponential decay in $|\lambda_i|$ as $i \to \infty$. Namely, as we have shown in [BR2], it is natural to introduce the normalized coefficients

$$d_i = \gamma(\lambda_i)|c_i|^2 \ . \tag{1.3}$$

Here $\gamma(\lambda)$ is given by an explicit rational expression in terms of the standard Euler Γ-function (see [BR2]) and, for purely imaginary λ, $|\lambda| \to \infty$, it has an asymptotic $\gamma(\lambda) \sim \beta|\lambda|^2 \exp(\frac{\pi}{2}|\lambda|)$ with some explicit $\beta > 0$. It turns out that the normalized coefficients d_i have at most *polynomial growth* in $|\lambda_i|$, and hence the coefficients c_i decay exponentially. This is consistent with (1.1) and general experience from the analytic theory of automorphic L-functions (see [BR2], [Wa]). In Section 5 we explain a more conceptual way to introduce the coefficients d_i which is based on considerations from representation theory.

In [BR2] we proved the following mean value bound

$$\sum_{|\lambda_i| \leq T} d_i \leq AT^2 \ , \tag{1.4}$$

for arbitrary $T > 1$ and some effectively computable constant A.

The constant A depends on the geometry of Γ and on parameters τ, τ' of eigenfunctions ϕ, ϕ'.

According to Weyl's law for the spectrum of the Laplace-Beltrami operator Δ on Y, the number of terms in this sum is of order CT^2. So this formula says that on average the coefficients d_i are bounded by some constant.

More precisely, let us fix an interval $I \subset \mathbb{R}$ centered at the point T and consider the finite set of all Maass forms ϕ_i with parameter $|\lambda_i|$ inside this interval. Then the average value of coefficients d_i in this set is bounded by a constant *provided* the interval I is long enough (i.e., of size $\approx T$).

Note that the best individual bound which we can get from this formula is $d_i \leq A|\lambda_i|^2$. For Hecke-Maass forms this bound corresponds to the convexity bound for the corresponding L-function via Watson formula (1.1).

The central result of this paper is the bound for the sum of the coefficients d_i over a shorter interval. Namely, we prove the following

Theorem. *There exist effectively computable constants B, $b > 0$ such that, for an arbitrary $T > 1$, we have the following bound*

$$\sum_{|\lambda_i| \in I_T} d_i \leq BT^{\frac{5}{3}} \ , \tag{1.5}$$

where I_T is the interval of size $bT^{1/3}$ centered at T.

The exponent 5/3 above appears for the reason similar to the appearance of the exponent 1/3 in the asymptotic of the Airy integral (namely, a degenerate critical point in the phase of an oscillatory integral; see Remark 2.7.2).

The constant B depends on the geometry of X and on parameters τ, τ' (see Remark 6.6). The constant b depends on parameters τ, τ' only.

Note that the theorem gives an individual bound $d_i \leq B|\lambda_i|^{\frac{5}{3}}$ (for $|\lambda_i| > 1$). Thanks to the Watson formula (1.1) and a lower bound of H. Iwaniec $|L(1, \phi_{\lambda_i}, Ad)| \gg |\lambda_i|^{-\varepsilon}$ (see [I]), this leads to the following *subconvexity* bound for the triple L-function (for more on the relation between triple period and special values of L-functions, see [Wa], [Ic]).

Corollary. *Let ϕ and ϕ' be fixed Hecke-Maass cusp forms. For any $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that the bound*

$$L(\tfrac{1}{2} \ , \phi \otimes \phi' \otimes \phi_{\lambda_i}) \leq C_\varepsilon |\lambda_i|^{\frac{5}{3}+\varepsilon} \tag{1.6}$$

holds for any Hecke-Maass form ϕ_{λ_i}.

The convexity bound for the triple L-function corresponds to (1.6) with the exponent 5/3 replaced by 2. We refer to [IS] for a discussion of the subconvexity problem which is at the core of modern analytic number theory. We note that the above bound is the first subconvexity bound for an L-function of degree 8 which does not split in a product of smaller degree L-functions. All previous subconvexity results were obtained for L-functions of degree at most 4.

In [V] A. Venkatesh obtained a subconvexity bound for the triple L-function in the level aspect (i.e., with respect to a tower of congruence subgroups $\Gamma(N)$ as $N \to \infty$). His method is quite different from the method we present in this paper and is based on ergodic theory.

We formulate a natural

Conjecture. *For any $\varepsilon > 0$ we have $d_i \ll |\lambda_i|^\varepsilon$.*

For Hecke-Maass forms on congruence subgroups, this conjecture is consistent with the Lindelöf conjecture for the triple L-functions (for more details, see [BR2] and [Wa]).

1.3.1. *Remarks.* 1. Our results can be extended to the case of a general finite co-volume lattice $\Gamma \subset G$ (see Remark 7.2.2 for more detail).

2. First results on the exact exponential decay of triple products for a general lattice Γ and holomorphic forms were obtained by A. Good [Go] using Poincaré series. P.

Sarnak [Sa] discovered ingenious analytic continuation of Maass forms to the complexification of the Riemann surface Y to obtain somewhat weaker results for Maass forms (for representation-theoretic approach to this method and generalizations, see [BR1] and [KS]). Our present method seems to be completely different and avoids analytic continuation.

3. We would like to stress that the bound for the triple product in Theorem 1.3 is valid for a general lattice Γ, including *non-arithmetic* lattices. In fact, in our method we do not use Fourier coefficients or Hecke eigenvalues through which one usually accesses values of L-functions for congruence subgroups. Our method gives estimates for *periods* of automorphic functions directly and L-functions appear only through the Watson formula (1.1) (the same is true for the method of Venkatesh [V]).

The paper is organized as follows. The next section is devoted to a detailed explanation of ideas behind the method of the proof of Theorem 1.3. The main body of the paper (Sections 3-10) is devoted to the proof. Two Appendices containing technical calculations conclude the paper. The numbering in the paper is organized as follows. Each subsection has a unique Theorem, Proposition, Lemma etc., and these are numbered by the corresponding section. Equations are numbered continuously within each section.

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2. Outline of the proof

We describe now the general ideas behind our proof. It is based on ideas from representation theory (for a detailed account of the corresponding setting, see [BR2] and Section 4 below). In what follows we sketch the method of the proof whose technical details appear in the rest of the paper.

2.1. **Automorphic representations.** Let G denote the group of all motions of $\mathbb{H}$. This group is naturally isomorphic to $PGL_2(\mathbb{R})$ and as a G-space $\mathbb{H}$ is naturally isomorphic to G/K, where $K = PO(2)$ is the standard maximal compact subgroup of G.

By definition, Γ is a subgroup of G. The space $X = \Gamma\backslash G$ with the natural right action of G is called an *automorphic space*. We will identify the Riemann surface $Y = \Gamma\backslash\mathbb{H}$ with $X/K = \Gamma \setminus G/K$.

We use the standard language of automorphic representations (see [G6] and Section 3 below). Let (π, G, V) be an irreducible smooth representation of G. An *automorphic structure* on V is a continuous G-morphism $\nu : V \to C^\infty(X)$.

The pair (π, ν) consisting of an abstract representation (π, V) and the automorphic structure ν will be called an *automorphic representation*. This terminology is slightly more precise then the standard one. We find it more convenient for our purposes.

We always assume that (π, V) is unitary (i.e., V is equipped with a positive definite G-invariant Hermitian form P), and that the automorphic structure ν is compatible with the invariant Hermitian form P.

We will usually present the abstract representation (π, V) by an explicit model. We will deal mostly with class one irreducible representations of G (i.e., those with a non-zero K-fixed vector). If (π, V) is a non-trivial class one representation we use for it the model $V = V_\lambda$, where $\lambda \in i\mathbb{R} \cup (0, 1)$ and V_λ is the space of smooth even homogeneous functions on $\mathbb{R}^2 \setminus 0$ of the homogeneous degree $\lambda - 1$ (see [G5], [BR2]). We denote by $e_\lambda \in V_\lambda$ the function taking constant value 1 on $S^1 \subset \mathbb{R}^2 \setminus 0$. This gives a K-invariant vector in the representation V_λ which we call the standard K-fixed vector in V_λ. We normalize the invariant Hermitian form P on V_λ by the condition $P(e_\lambda) = 1$.

The theorem of Gelfand and Fomin states that all Maass forms (or more generally automorphic functions) could be obtained as special vectors in appropriate automorphic representations (see [G6]). Namely, a Maass form $\phi = \nu(e_\lambda)$ corresponding to an automorphic structure ν on a representation with a model V_λ has the eigenvalue $\mu = \frac{1-\lambda^2}{4}$.

We translate various questions about Maass forms into corresponding questions about associated automorphic representations. This allows us to employ powerful methods of representation theory.

2.2. Let us fix two (nontrivial) automorphic representations (π, ν) and (π', ν'). We assume that both are representations of class one (i.e., $V \simeq V_\tau$ and $V' \simeq V_{\tau'}$, $\tau,\ \tau' \in i\mathbb{R} \cup (0, 1)$). These give rise to Maass forms $\phi = \nu(e_\tau)$ and $\phi' = \nu'(e_{\tau'})$. Let (π_i, V_i, ν_i) be a third automorphic representation (which we a going to vary) with the parameter λ_i (i.e., $V_i \simeq V_{\lambda_i}$).

The triple product $c_i = \int_Y \phi\phi'\phi_i dv$ extends to a G-equivariant trilinear functional on the corresponding automorphic representations $l_i^{aut} : V \otimes V' \otimes V_i \to \mathbb{C}$.

Next we use a general result from representation theory that such a G-equivariant trilinear functional is unique up to a scalar, i.e., that $\dim \mathrm{Mor}_G(V \otimes V' \otimes V'', \mathbb{C}) \leq 1$ for any smooth irreducible representations $V,\ V',\ V''$ of G (see [O], [P], [Lo] and the discussion in [BR2]). This implies that the automorphic functional l_i^{aut} is proportional to some explicit *model* functional $l^{mod}_{\lambda_i}$. In [BR2] we gave a description of such a model functional $l^{mod}_{\lambda} : V \otimes V' \otimes V_\lambda \to \mathbb{C}$ for any λ using explicit realizations of representations V, V' and V_λ of the group G in spaces of homogeneous functions; it is important that the model functional knows nothing about the automorphic picture and carries no arithmetic information.

Thus we can write $l^{aut}_i = a_i \cdot l^{mod}_{\lambda_i}$ for some constant a_i, and hence

$$c_i = l^{aut}_i(e_\tau \otimes e_{\tau'} \otimes e_{\lambda_i}) = a_i \cdot l^{mod}_{\lambda_i}(e_\tau \otimes e_{\tau'} \otimes e_{\lambda_i}) \ , \tag{2.1}$$

where $e_\tau,\ e_{\tau'},\ e_{\lambda_i}$ are the standard K-invariant unit vectors in representations V, V' and V_{λ_i} corresponding to the automorphic forms ϕ, ϕ' and ϕ_i.

It turns out that the proportionality coefficient a_i in (2.1) carries important "automorphic" information while the second factor carries no arithmetic information and can be computed in terms of Γ-functions using explicit realizations of representations V_τ, $V_{\tau'}$ and V_λ (see Appendix in [BR2] where this computation is carried out). This second factor is responsible for the exponential decay, while the first factor a_i has a polynomial behavior in parameter λ_i. An explicit computation shows (see loc. cit.) that $|c_i|^2 = \frac{1}{\gamma(\lambda_i)}|a_i|^2$, and hence $d_i = |a_i|^2$ (where the function $\gamma(\lambda)$ was described in Section 1.3).

So, from now on we will deal with coefficients d_i and no longer refer to coefficients a_i and c_i at all.

2.3. Hermitian forms. In order to estimate the quantities d_i, we consider the space $E = V_\tau \otimes V_{\tau'}$ and use the fact that the coefficients d_i appear in the spectral decomposition of the following *geometrically defined* non-negative Hermitian form H_Δ on E (for a detailed discussion, see [BR2]).

Consider the space $C^\infty(X \times X)$. The diagonal $\Delta : X \to X \times X$ gives rise to the restriction morphism $r_\Delta : C^\infty(X \times X) \to C^\infty(X)$. We define a non-negative Hermitian form H_Δ on $C^\infty(X \times X)$ by setting $H_\Delta = (r_\Delta)^*(P_X)$, where P_X is the standard L^2 Hermitian form on $C^\infty(X)$ i.e.,

$$H_\Delta(w) = P_X(r_\Delta(w)) = \int_X |r_\Delta(w)|^2 d\mu_X$$

for any $w \in C^\infty(X\times X)$. We call the restriction of the Hermitian form H_Δ to the subspace $E \subset C^\infty(X \times X)$ the *diagonal* Hermitian form and denote it by the same letter.

We will describe the spectral decomposition of the Hermitian form H_Δ in terms of Hermitian forms corresponding to trilinear functionals. Namely, if L is a pre-unitary representation of G with G-invariant Hermitian norm $||\ ||_L$, then every G-invariant trilinear functional $l : V \otimes V' \otimes L \to \mathbb{C}$ defines a Hermitian form H^l on E by $H^l(w) = \sup_{||u||_L=1} |l(w \otimes u)|^2$.

Here is another description of this form (see [BR2]). The functional $l : V \otimes V' \otimes L \to \mathbb{C}$ gives rise to a G-intertwining morphism $T^l : E \to L^*$ which image lies in the smooth part $\tilde{L}^*$ of L^*. Then the form H^l is just the pull back of the Hermitian form on $\tilde{L}^*$ corresponding to the inner product on L.

Consider the orthogonal decomposition $L^2(X) = (\bigoplus_i V_i) \oplus (\bigoplus_\kappa V_\kappa)$ where V_i correspond to Maass forms and V_κ correspond to representations of discrete series. Every G-invariant subspace $L \subset L^2(X)$ defines a trilinear functional $l : E \otimes L \to \mathbb{C}$ and hence a Hermitian form H^l on E. Hence, the decomposition of $L^2(X)$ gives rise to the corresponding decomposition

$$H_\Delta = \sum H_i^{aut} + \sum H_\kappa^{aut}$$

of Hermitian forms (see [BR2]).

We denote by H_λ the *model* Hermitian form corresponding to the *model* trilinear functional $l_\lambda^{mod} : V \otimes V' \otimes V_\lambda \to \mathbb{C}$. The uniqueness of trilinear functionals mentioned in Section 2.2 (i.e., the formula (2.1)) implies that $H_i^{aut} = d_i H_{\lambda_i}$. This leads us to

The basic spectral identity

$$H_\Delta = \sum_i d_i H_{\lambda_i} + \sum_\kappa H_\kappa^{aut}, \tag{2.2}$$

Of course, one can introduce similar model trilinear functionals for the discrete series representations V_κ and the corresponding coefficients d_κ via $H_\kappa^{aut} = d_\kappa H_\kappa$. We will not need these in this paper (in fact, in this paper we are trying to avoid computations with the discrete series representations; see Remark 8.1).

We will mostly use the fact that for every vector $w \in E$ this basic spectral identity gives us an inequality

$$\sum_i d_i H_{\lambda_i}(w) \leq H_\Delta(w) \tag{2.3}$$

which turns into an *equality* if the vector $r_\Delta(w)$ does not have projection to discrete series representations (for example, if the vector w is *invariant* with respect to the diagonal action of K on E).

We can use this inequality to bound coefficients d_i. Namely, for a given vector $w \in E$ we usually can compute the values of the weight function $H_\lambda(w)$ by explicit computations in the model of representations $V,\ V',\ V_\lambda$. It is usually much more difficult to get reasonable estimates of the right hand side $H_\Delta(w)$ since it refers to the automorphic picture. In cases when we manage to do this we get some bounds for the coefficients d_i.

2.4. Mean-value estimates. In [BR2], using the geometric properties of the diagonal form and explicit estimates of forms H_λ, we established the mean-value bound (1.4): $\sum_{|\lambda_i| \leq T} d_i \leq AT^2$. Roughly speaking, the proof of this bound is based on the fact that while the value of the form H_Δ on a given vector $w \in E$ is very difficult to control, we can show that for many vectors w the value $H_\Delta(w)$ can be bounded by $P_E(w)$, where P_E is the Hermitian form which defines the standard unitary structure on E.

More precisely, consider the natural representation $\sigma = \pi \otimes \pi'$ of the group $G \times G$ on the space E. Then for a given compact neighborhood $U \subset G \times G$ of the identity element, there exists a constant C such that for any vector $w \in E$, the inequality $H_\Delta(\sigma(g)w) \leq CP_E(w)$ holds for at least half of the points $g \in U$. This follows from the fact that the average over U of the quantity $H_\Delta(\sigma(g)w)$ is bounded by $CP_E(w)/2$.

This allows us, for every $T \geq 1$, to show the *existence* of a vector $w \in E$ such that $H_\Delta(w) \leq CT^2$ and $H_\lambda(w) \geq c$ for all λ satisfying $|\lambda| \leq T$. The bound (2.3) then implies the mean-value bound (1.4).

2.5. Bounds for sums over shorter intervals. The main starting point of our approach to the subconvexity bound is the inequality (2.3) for Hermitian forms. For a given $T > 1$, we construct a test vector $w_T \in E$ such that the weight function $\lambda \mapsto H_\lambda(w_T)$ has a sharp peak near $|\lambda| = T$ (i.e., a vector satisfying the condition (2.6) below).

The problem is how to estimate effectively $H_\Delta(w_T)$. The idea is that the Hermitian form H_Δ is geometrically defined and, as a result, satisfies some non-trivial bounds, symmetries, etc. None of the explicit *model* Hermitian forms H_λ satisfies similar properties. By applying these symmetries to the vector w_T, we construct a new vector $\tilde{w}_T$ and from the geometry of the automorphic space X, we deduce the bound $H_\Delta(w_T) \leq H_\Delta(\tilde{w}_T)$.

On the other hand, the weight function $H_\lambda(\tilde{w}_T)$ in the spectral decomposition $H_\Delta(\tilde{w}_T) = \sum d_i H_{\lambda_i}(\tilde{w}_T)$ for $\tilde{w}_T$ behaves quite differently from the weight function $H_\lambda(w_T)$ for w_T. Namely, the function $H_\lambda(\tilde{w}_T)$ behaves regularly (i.e., satisfies condition (2.11) below), while the weight function $H_\lambda(w_T)$ has a sharp peak near $|\lambda| = T$.

The regularity of the function $H_\lambda(\tilde{w}_T)$ coupled with the mean-value bound (1.4) allows us to prove a sharp upper bound on the value of $H_\Delta(\tilde{w}_T)$ by purely spectral considerations (in the cases that we consider there is no contribution from discrete series). We do not see how to get such sharp bound by geometric considerations working on the automorphic space $X \times X$.

Using this bound for $H_\Delta(\tilde{w}_T)$ and the inequality $H_\Delta(w_T) \leq H_\Delta(\tilde{w}_T)$, we obtain a non-trivial bound for $H_\Delta(w_T)$ and, as a result, the desired bound for the coefficients d_i.

We now describe this strategy in more detail.

2.6. Proof of Theorem 1.3. We only consider the case of representations of the principal series, i.e., we assume that $V = V_\tau$, $V' = V_{\tau'}$ for some τ, $\tau' \in i\mathbb{R}$; the case of representations of the complementary series can be treated similarly.

We denote by ν and ν' the corresponding automorphic realizations of V and V'. We choose an orthonormal basis $\{e_n\}_{n \in 2\mathbb{Z}}$ in V consisting of K-types and similarly an orthonormal basis $\{e'_n\}$ in V'.

Vectors $w_n = e_n \otimes e'_{-n} \in E = V \otimes V'$ will play an important role in our computations.

Let us set

$$\mathcal{S} = 2(|\tau| + |\tau'|) + 1 \tag{2.4}$$

the constant depending on parameters of representations V and V' only. For a given $T \geq \mathcal{S}$, we choose an even integer n such that $|T - 2n| \leq 10$ and set

$$w_T = w_n = e_n \otimes e'_{-n} \ . \tag{2.5}$$

In fact, all we need is that $|T - 2n|$ remain bounded as $T \to \infty$.

By a direct computation involving stationary phase method, we show in Section 9.2 that the following lower bound holds

First spectral bound:

There exist constants $b, c > 0$ such that

$$H_\lambda(w_T) \geq c\, T^{-5/3} \text{ for } |\lambda| \in I_T \ , \tag{2.6}$$

where I_T is the interval of length $bT^{1/3}$ centered at the point T.

This inequality together with the bound $\sum_i d_i H_{\lambda_i}(w_T) \leq H_\Delta(w_T)$ (see (2.3)) imply the bound

$$\sum_{|\lambda_i| \in I_T} d_i \leq CT^{5/3} H_\Delta(w_T) \ , \tag{2.7}$$

for some constant C.

Now we claim that the quantity $H_\Delta(w_T)$ is uniformly bounded by some constant D which does not depend on T. Namely we can write

$$H_\Delta(w_T) = \int_X |\nu(e_n)|^2 |\nu'(e'_{-n})|^2 \, d\mu_X \leq \frac{1}{2}\left(||\nu(e_n)||^4_{L^4(X)} + ||\nu'(e'_{-n})||^4_{L^4(X)} \right) \ . \tag{2.8}$$

Hence the necessary bound follows from the following result which, we feel, is of independent interest.

Theorem. *For a fixed class one automorphic representation $\nu : V \to C^\infty(X)$, there exists a constant $D > 0$ such that $||\nu(e_n)||_{L^4(X)} \leq D$ for all n.*

This finishes the proof of Theorem 1.3. □

Remark. One would expect that L^4-norms of K-types for representations of the discrete series are uniformly bounded as well. It is a very interesting and deep question to study dependence of the constant D in Theorem 2.6 on the parameter τ of the automorphic representation and on the subgroup Γ (for a discussion, see Remark 6.6). Moreover, it would be interesting to identify (as a norm on an abstract representation π_τ) the *invariant* (non-Hermitian) norm which the L^4-norm on X induces on the representation π_τ via automorphic isometry ν_τ (see a discussion in [BR1]).

Another interesting question is an analog of the above theorem for a cuspidal representation for a non-uniform Γ. Specifically, we would like ask if L^4-norm of K-types are bounded for a fixed cuspidal representation (compare to Remark 2, Section 7.2.2).

2.7. **L^4-norms of K-types.** We now explain the proof of the uniform bound for L^4-norm of K-types (i.e., Theorem 2.6).

Let $\bar{V}$ be the complex conjugate to V representation. The representation $\bar{V}$ is also an automorphic representation with the realization $\bar{\nu} : \bar{V} \to C^\infty(X)$ (see details in Section 6.1). For the proof of Theorem 2.6 it is enough to consider the setup described above (i.e., the space E, forms H_Δ, H_λ, etc.) for the special case when V' is isomorphic to the representation $\bar{V}$.

We only consider the case of representations of the principal series, i.e., we assume that $V = V_\tau$ and $V' = \bar{V} = V_{-\tau}$ for some $\tau \in i\mathbb{R}$; the case of representations of the complementary series can be treated similarly.

Choose an orthonormal basis $\{e_n\}_{n\in 2\mathbb{Z}}$ in V consisting of K-types. We denote by $\{e'_n = \overline{e_{-n}} = c(e_{-n})\}$ the complex conjugate basis in $\bar{V}$ (note that e'_n is of the K-type n).

For a given $n \in 2\mathbb{Z}$, we set

$$w_n = e_n \otimes e'_{-n} \quad \text{and} \quad \tilde{w}_n = w_n + w_{n+2}. \tag{2.9}$$

With such a choice of test vectors we have the following bounds.

Geometric bound:

$$H_\Delta(w_n) \le H_\Delta(\tilde{w}_n)\ . \tag{2.10}$$

Second spectral bound:

There exists a constant C' such that

$$H_\lambda(\tilde{w}_n) \le \begin{cases} C'(1+|n|)^{-1}|\lambda|^{-1} + C'|\lambda|^{-3} & \text{for all } \mathcal{S} \le |\lambda| \le 4|n|\ , \\ C'|\lambda|^{-3} & \text{for all } |\lambda| > 4|n|\ . \end{cases} \tag{2.11}$$

Here $\mathcal{S}$ is as in (2.4).

Using the bound (2.11) we will get the following *sharp* estimate of $H_\Delta(\tilde{w}_n)$ (see Proposition 6.5):

$$H_\Delta(\tilde{w}_n) \le D \tag{2.12}$$

with some explicit constant $D > 0$ (for the proof, see Section 7.1). Bounds (2.12) and (2.10) imply the bound for the L^4-norm of K-types since in this case $H_\Delta(w_n) = ||\nu(e_n)||^4_{L^4}$.

The bound (2.12) follows from the identity $H_\Delta(\tilde{w}) = \sum d_i H_{\lambda_i}(\tilde{w})$ (see (2.2)), the spectral bound (2.11) and the mean-value bound (1.4) for the coefficients d_i. The low spectrum contribution for $|\lambda_i| \le \mathcal{S}$ is bounded by an argument based on the Sobolev restriction theorem (see Section 7.2.2) . We also use the fact that there are no contribution to $H_\Delta(\tilde{w})$ coming from the discrete series since the vector $\tilde{w}$ is ΔK-invariant.

2.7.1. *Proof of the geometric bound (2.10).* The inequality (2.10) easily follows from the pointwise bound on X. Namely, in the automorphic realization, the vector $w_n = e_n \otimes e'_{-n}$ is represented by a function whose restriction $u_n = r_\Delta(\nu_E(w_n))$ to the diagonal is *non-negative* (see also Section 6.2)

$$u_n(x) = \nu(e_n)(x) \cdot \bar{\nu}(e'_{-n})(x) = |\nu(e_n)(x)|^2 \geq 0.$$

From this we see that

$$H_\Delta(w_T) = \int_X |u_n(x)|^2 d\mu_X \leq \int_X |u_n(x) + u_{n+2}(x)|^2 d\mu_X = H_\Delta(\tilde{w}_T)\ . \qquad \square$$

2.7.2. *Sketch of proof of the spectral bounds* (2.6) *and* (2.11). Proof of these bounds is carried out by the standard application of the stationary phase method and the Van der Corput lemma. It constitutes the main *technical* bulk of the paper. We will use the explicit form of the kernel defining Hermitian forms H_λ in the model realizations of representations V, V' and V_λ. Namely, we use the standard realization of these representations in the space $C^\infty_{even}(S^1)$ of even functions on S^1 (see [BR2] and Section 2.1). Under this identification, the basis $\{e_n\}_{n\in 2\mathbb{Z}}$ becomes the standard basis of exponents $\{e_n = e^{int}\}$, where $0 \leq t < 2\pi$ is the standard parameter on S^1.

In [BR2], Section 5, we described how to write down an invariant functional for principal series representations. Namely, let $V = V_\tau$, $V' = V_{\tau'}$ with $\tau,\ \tau' \in i\mathbb{R}$. In the circle model of representation V_τ, $V_{\tau'}$, V_λ, the following kernel on the space $V_\tau \otimes V_{\tau'} \otimes V_\lambda \simeq C^\infty((S^1)^3)$ defines an invariant functional kernel on $(S^1)^3$:

$$K_\lambda(x,y,z) = |\sin(x-y)|^{\frac{-1-\tau-\tau'+\lambda}{2}} |\sin(x-z)|^{\frac{-1-\tau+\tau'-\lambda}{2}} |\sin(y-z)|^{\frac{-1+\tau-\tau'-\lambda}{2}}\ ,$$

where $x,\ y,\ z \in S^1$. We denote this functional by l^{mod}_λ. Using the kernel $K_\lambda(x,y,z)$, we can define the Hermitian forms H_λ on $E \simeq C^\infty(S^1 \times S^1)$ by the corresponding oscillatory integral (over $(S^1)^4$; see Section 8.2). This allows us to use the stationary phase method in the proof of bounds (2.6) and (2.11).

Here appears the main difference between test vectors w_n and $\tilde{w}_n$. It manifests itself in the form of the oscillating integrals computing $H_\lambda(w_n)$ and $H_\lambda(\tilde{w}_n)$. Namely, both of these integrals have the same phase function which has a *degenerate* critical point. The main difference between them is that for the vector w_T the corresponding integral has a non-zero amplitude at this critical point (this gives the crucial lower bound (2.6)) and for $\tilde{w}_T$ the amplitude vanishes at the critical point (resulting in bounds (2.11)).

In fact, we will use the values of $H_\lambda(w)$ only for ΔK-invariant vectors $w \in E$. This considerably simplifies our computations since we can reduce them to two repeated integrations in one variable and use the stationary phase method in *one variable.*

Remarks. 1. The existence of vectors satisfying spectral conditions (2.6) and (2.11) allows us to shorten the summation over the spectrum, comparatively to the range of the summation in the convexity bound (1.4). This is necessary if one wants to deduce a subconvexity bound from the Bessel inequality of Hermitian forms (2.3) since the convexity bound (1.4) is essentially *sharp* (see [Re1]). This approach to the subconvexity is reminiscent of the classical amplification method introduced by Selberg (see [Mi], [MiV] for the review of the state of the art subconvexity results). Usually one uses a variant of a trace formula to control the so-called off-diagonal terms arising after shortening the sum. In our approach there is no use of the Selberg or the Kuznetsov trace formulas. Instead, we use the hidden symmetries of the diagonal form H_Δ.

2. The origin of our exponent $5/3 = 2(1/2 + 1/3)$ in the main Theorem 1.3 (i.e., the bound (1.5)) is directly related to the exponent $1/3$ in the well-known properties of the Airy function. In fact, we reduce the proof of the crucial lower bound (2.6) to the asymptotic of the Airy integral (see Proposition 9.1).

3. After obtaining results presented in this paper, we realized that there exists another possible approach to bounds for triple and other periods of automorphic functions. It is based on the notion of strong Gelfand pairs (see [Gr] and references therein). This approach is presented in [Re2].

There is one technical complication in the approach based on Gelfand pairs, though. We where not able to produce the desired family of test vectors which is also $\Delta K \times \Delta K$-invariant. Without this property one has to consider terms in the spectral decomposition (2.2) coming from the discrete series representations. It is more cumbersome to study model trilinear functionals on discrete series as these representations do not have nice geometric models. As a result, in this paper we use another property of the form H_Δ , the extra positivity provided by the Cauchy-Schwartz inequality (see Section 2.7.1), instead of the associated Gelfand pairs structure. We hope to return to this subject elsewhere.

3. Representation-theoretic setting

3.1. We recall the standard connection between Maass forms and representation theory of $PGL_2(\mathbb{R})$ (see [G6]). Most of the material in the next three sections is taken from [BR2], where it is discussed in more detail.

3.1.1. *Automorphic space.* Let $\mathbb{H}$ be the upper half plane with the hyperbolic metric of constant curvature -1. The group $SL_2(\mathbb{R})$ acts on $\mathbb{H}$ by fractional linear transformations. This action allows to identify the group $PSL_2(\mathbb{R})$ with the group of all orientation preserving motions of $\mathbb{H}$. For reasons explained below (see Remark 4.2), we would like to work with the group G of all motions of $\mathbb{H}$; this group is isomorphic to $PGL_2(\mathbb{R})$. Hence throughout the paper we consider the group $G = PGL_2(\mathbb{R})$ and denote by K its standard maximal compact subgroup $K = PO(2)$. We have natural identification $G/K = \mathbb{H}$.

We fix a discrete co-compact subgroup $\Gamma \subset G$ and set $X = \Gamma \setminus G$. We fix the unique G-invariant measure μ_X on X of total mass one. The group G acts on X (from the right) and hence on the space of functions on X. Let $L^2(X) = L^2(X, d\mu_X)$ be the space of square integrable functions and $(\Pi_X, G, L^2(X))$ the corresponding unitary representation. We will denote by P_X the Hermitian form on $L^2(X)$ given by the inner product.

3.1.2. *Automorphic representations.* Let (π, G, V) be an irreducible smooth Fréchet representation of G (see [Ca] where they are called smooth representations of moderate growth).

Definition. An *automorphic structure* on (π, V) is a continuous G-morphism $\nu : V \to C^\infty(X)$.

We call an automorphic representation a pair (π, ν) of a representation and the automorphic structure on it. In this paper we always assume that (π, V) is irreducible, admissible and also assume that (π, V) is unitary. This means that V is equipped with a G-invariant positive definite Hermitian form P, and V is the space of smooth vectors in the completion L of V with respect to P. An automorphic structure $\nu : V \to C^\infty(X)$ is assumed to be normalized, i.e., we assume that $P = \nu^*(P_X)$.

3.1.3. *Automorphic representations and Maass forms.* Let $(\pi_\lambda, G, V_\lambda)$ be a representation of the generalized principal series corresponding to $\lambda \in \mathbb{C}$. The space V_λ is the space of smooth even homogeneous functions on $\mathbb{R}^2 \setminus 0$ of the homogeneous degree $\lambda - 1$ (which means that $f(ax, ay) = |a|^{\lambda-1} f(x, y)$ for all $a \in \mathbb{R} \setminus 0$) with the action of $GL(2, \mathbb{R})$ given by $\pi_\lambda(g)f(x, y) = f(g^{-1}(x, y))|\det g|^{(\lambda-1)/2}$ (see [G5]).

In explicit computations it is often convenient to pass from the plane model to a circle model. Namely, the restriction of functions in V_λ to the unit circle $S^1 \subset \mathbb{R}^2$ defines an isomorphism of the space V_λ with the space $C^\infty_{even}(S^1)$ of even smooth functions on S^1, so we can think about vectors in V_λ as functions on S^1. The constant function 1 on S^1 corresponds to the standard unit K-invariant vector $e_\lambda \in V_\lambda$. We normalize the invariant Hermitian for P by the condition $P(e_\lambda) = 1$. For $\lambda \in i\mathbb{R}$, this corresponds to the standard Hermitian form $\langle f, g\rangle_{V_\lambda} = 1/2\pi \int_{S^1} f\bar{g}d\theta$ on (even) functions on S^1.

Suppose $\nu : V \to C^\infty(X)$ is an automorphic structure on V_λ. Then $\phi_\lambda = \nu(e_\lambda) \in C^\infty(X)^K = C^\infty(Y)$ is a Maass form with the eigenvalue $\mu = \frac{1-\lambda^2}{4}$.

This construction, which is due to Gelfand and Fomin, gives a one-to-one correspondence between Maass forms and class one automorphic representations (and more generally between automorphic forms and automorphic representations of G). We refer to [G6] for a more detailed discussion (see also [BR2]).

3.1.4. *Decomposition of the representation* $(\Pi_X, G, L^2(X))$*.* It is well-known that for a compact X, the representation $(\Pi_X, G, L^2(X))$ decomposes into a direct (infinite) sum of irreducible representations of G with finite multiplicities (see [G6]). We will fix one such decomposition and call it the automorphic spectrum of X. We can write

$$L^2(X) = (\oplus_i L_i) \oplus (\oplus_\kappa L_\kappa) ,$$

where L_i are irreducible representations corresponding to Maass forms (including the trivial representation), and L_κ are irreducible representations of discrete series.

For us it will be convenient to write this decomposition as the following decomposition of the Hermitian form P_X on $C^\infty(X)$

$$P_X = \sum_i P_i \ + \ \sum_\kappa P_\kappa \ , \tag{3.1}$$

where $P_i = pr_i^*(P_X)$ and $P_\kappa = pr_\kappa^*(P_X)$.

4. Triple products

We introduce now our main object of study.

4.1. Automorphic triple products. Suppose we are given three automorphic representations (π_j, V_j, ν_j), $j = 1, 2, 3$ of G

$$\nu_j : V_j \to C^\infty(X) \ .$$

We define the G-invariant trilinear form $l^{aut}_{V_1,V_2,V_3} : V_1 \otimes V_2 \otimes V_3 \to \mathbb{C}$ by the formula

$$l^{aut}_{V_1,V_2,V_3}(v_1 \otimes v_2 \otimes v_3) = \int_X \phi_{v_1}(x)\phi_{v_2}(x)\phi_{v_3}(x)d\mu_X \ ,$$

where $\phi_{v_j} = \nu_j(v_j) \in C^\infty(X)$ for any $v_j \in V_j$.

Let (π, V, ν) and (π', V', ν') be two *fixed* automorphic representations of class one. For any automorphic representation $(\pi_i, V_{\lambda_i}, \nu_i)$ of class one, we have the automorphic trilinear functional

$$l^{aut}_{V,V',V_{\lambda_i}} : V \otimes V' \otimes V_{\lambda_i} \to \mathbb{C} \ .$$

In particular, the triple periods c_i in (1.2) can be expressed in terms of this form as

$$c_i = l^{aut}_{V,V',V_{\lambda_i}}(e \otimes e' \otimes e_{\lambda_i}) \ , \tag{4.1}$$

where $e \in V$, $e' \in V'$, $e_{\lambda_i} \in V_{\lambda_i}$, are standard K-fixed unit vectors.

4.2. Uniqueness of triple products. The central fact about invariant trilinear functionals is the following uniqueness result:

Theorem. *Let (π_j, V_j), where $j = 1, 2, 3$, be three irreducible smooth admissible representations of G. Then* $\dim \mathrm{Hom}_G(V_1 \otimes V_2 \otimes V_3, \mathbb{C}) \le 1$.

Remark. The uniqueness statement was proven by A. Oksak in [O] for the group $SL(2,\mathbb{C})$ and the proof could be adopted for $PGL_2(\mathbb{R})$ as well (see also [Mo] and [Lo]). For the p-adic $GL(2)$, more refined results were obtained by D. Prasad (see [P]). He also proved the uniqueness when at least one representation is a discrete series representation of $GL_2(\mathbb{R})$.

There is no uniqueness of trilinear functionals for representations of $SL_2(\mathbb{R})$ (the space is two-dimensional). This is the reason why we prefer to work with $PGL_2(\mathbb{R})$.

We note, however, that the absence of uniqueness does not pose any serious problem for the method we present. All what is really needed for our method is the fact that the space of invariant functionals is finite dimensional.

4.3. Model triple products. In Section 8.1, we use an explicit model for representations (π, V), (π', V') and (π_i, V_i) to construct a model invariant trilinear functional. The model functional will be given by an explicit formula. We call it the *model triple product* and denote it by $l^{mod}_{V,V',V_{\lambda_i}}$, or simply $l^{mod}_{\lambda_i}$, if π and π' are fixed.

These model functionals are defined for any three irreducible unitary representation of principal series of G, even if these are not automorphic.

By the uniqueness principle for representations π, π', π_i, there exists a constant $a_i = a_{V,V',V_{\lambda_i}}$ such that:

$$l^{aut}_{V,V',V_i} = a_i \cdot l^{mod}_{V,V',V_{\lambda_i}} \ . \tag{4.2}$$

The constant a_i depends on the automorphic realization of abstract representations π, π' and π_{λ_i}, and on the choice of the model functional $l^{mod}_{\lambda_i} = l^{mod}_{V,V',V_{\lambda_i}}$.

From now on we will work with the coefficients $d_i = |a_i|^2$.

4.3.1. *Exponential decay.* Relations (4.1) and (4.2) give rise to a formula for the triple product coefficients c_i

$$c_i = l^{aut}_{\lambda_i}(e \otimes e' \otimes e_{\lambda_i}) = a_i \cdot l^{mod}_{\lambda_i}(e \otimes e' \otimes e_{\lambda_i}) \ .$$

Let us explain how one can deduce the exponential decay for the coefficients c_i using this identity.

The value of the model triple product functional $l^{mod}_{\lambda_i}(e \otimes e' \otimes e_{\lambda_i})$ constructed in Section 8.1 is given by an explicit integral. In [BR2], Appendix A, we evaluated this integral in terms of the standard Euler Γ-function by a direct computation in the model and showed that $|l^{mod}_{\lambda}(e_\tau \otimes e_{\tau'} \otimes e_\lambda)|^2 = 1/\gamma(\lambda)$, where $\gamma(\lambda)$ is as in Section 1.3. After applying the Stirling formula to that expression, one sees that it has an exponential decay in $|\lambda|$. Hence, in order to obtain bounds on the coefficients c_i, one needs to bound coefficients $d_i = |a_i|^2$. In [BR2] we showed that the coefficients d_i are at most polynomial. This explains the exponential decay of coefficients c_i. We note that the coefficients d_i encode deep arithmetic information, e.g., special values of L-functions.

5. Hermitian forms

5.1. Hermitian forms and trilinear coefficients d_i. We explain now how to obtain bounds for the coefficients d_i

Our method is based on the fact that these coefficients appear in the spectral decomposition of some geometrically defined Hermitian form on the space E which is essentially the tensor product of spaces V and V'. This form plays a crucial role in what follows.

More precisely, denote by L and L' the Hilbert completions of spaces V and V', consider the unitary representation $(\Pi, G\times G, L\otimes L')$ of the group $G\times G$ and denote by E its smooth part; so E is a smooth completion of $V\otimes V'$.

Denote by $\mathcal{H}(E)$ the (real) vector space of continuous Hermitian forms on E and by $\mathcal{H}^+(E)$ the cone of nonnegative Hermitian forms.

We will describe several classes of Hermitian forms on E; some of them have spectral description, others are described geometrically.

Let W be a smooth unitary admissible representation of G. Any G-invariant functional $l : V\otimes V'\otimes W\to\mathbb{C}$ defines a G-intertwining morphism $T^l : V\otimes V'\to W^*$ which extends to a G-morphism

$$T^l : E\to\overline{W}\ ,$$

where we have identified the complex conjugate space $\overline{W}$ with the smooth part of the space W^* (see Section 6.1).

The standard Hermitian form (scalar product) $P=P_W$ on the space W induces the Hermitian form $\bar{P}$ on $\overline{W}$. Using the operator T^l we define the Hermitian form H^l on the space E by $H^l=(T^l)^*(\bar{P})$, i.e., $H^l(u)=\bar{P}(T^l(u))$ for any $u\in E$.

Remark. We note that if the representation of G in the space W is irreducible and $l\neq 0$, then starting with the Hermitian form H^l, we can reconstruct the space W, the functional l, and the morphism T^l uniquely up to an isomorphism.

5.1.1. *Forms* H_λ. Let us introduce a special notation for the particular case we are interested in. For any $\lambda\in i\mathbb{R}\cup(0,1)$, consider the class one representation $W=V_\lambda$, choose the model trilinear functional $l_\lambda^{mod} : V\otimes V'\otimes V_\lambda\to\mathbb{C}$ described in Section 8.1 and denote the corresponding Hermitian form on E by H_λ^{mod} or simply by H_λ. Accordingly, let H_i^{aut} be the form corresponding to the automorphic functional. We have $H_i^{aut}=d_i\cdot H_{\lambda_i}^{mod}$, where $d_i=|a_i|^2=|a_{V,V',V_i}|^2$ are as in (4.2). This is the definition of the coefficients d_i we are going to work with.

5.2. Diagonal form H_Δ**.** Consider the space $C^\infty(X\times X)$. The diagonal $\Delta : X\to X\times X$ gives rise to the restriction morphism $r_\Delta : C^\infty(X\times X)\to C^\infty(X)$. We define a nonnegative Hermitian form H_Δ on $C^\infty(X\times X)$ by $H_\Delta=(r_\Delta)^*(P_X)$, i.e.,

$$H_\Delta(u)=P_X(r_\Delta(u))=\int_X|r_\Delta(u)|^2d\mu_X$$

for any $u\in C^\infty(X\times X)$.

We say that H_Δ is the *diagonal form.*

We now consider the spectral decomposition of the Hermitian for H_Δ (for a detailed discussion, see [BR2]). Using the spectral decomposition (3.1) $P_X = \sum_i P_i + \sum_\kappa P_\kappa$ we can write $H_\Delta = \sum_i H_i^{aut} + \sum_\kappa H_\kappa^{aut}$. We have seen before that $H_i^{aut} = d_i H_{\lambda_i}$. Hence we have the following spectral identity (which is a version of the Parseval identity)

$$H_\Delta = \sum_i d_i H_{\lambda_i} + \sum_\kappa H_\kappa^{aut} .$$

Here the summation on the right is over *all* irreducible unitary automorphic representations appearing in the decomposition of $L^2(X)$ (see (3.1)). The first sum is over the class one automorphic representations (including the trivial one) and the second sum is over the discrete series automorphic representations.

Remark. For most of the proof we will need just the inequality (the Bessel inequality)

$$\sum_i d_i H_{\lambda_i} \leq H_\Delta . \tag{5.1}$$

In order to avoid computations with discrete series, we consider only vectors $w \in E$ which are ΔK-invariant under the natural diagonal action of $\Delta G \subset G \times G$ on E. For such vectors, the inequality (5.1) becomes the equality

$$H_\Delta(w) = \sum_i H_i^{aut}(w) = \sum_i d_i H_{\lambda_i}(w) . \tag{5.2}$$

Here the summation is over all automorphic representations of class one.

This follows from the simple fact that for a ΔK-invariant vector $w \in E$, the restriction onto the diagonal ΔX of the automorphic realization $\nu \otimes \bar{\nu}(w)$ is a K-invariant function on X, and hence orthogonal to discrete series representations appearing in $L^2(X)$.

6. L^4-norm of K-types

In this section we prove Theorem 2.6. We assume, for simplicity, that the representation V is a representation of the principal series.

6.1. Complex conjugate representation. Our proof of Theorem 2.6 is spectral, it is based on the basic spectral identity (2.2) applied to the case when the representation V' coincides with the complex conjugate $\bar{V}$ of the representation V.

We recall that for any complex vector space V we can define the complex conjugate space $\bar{V}$. By definition, $\bar{V}$ is the same real vector space as V, i.e., we have a canonical bijection $c : V \to \bar{V}$, and the structure of the complex vector space is given by $\lambda c(v) = c(\bar{\lambda} v)$, $\lambda \in \mathbb{C}$. In particular, c is an antilinear bijection.

The complex conjugate representation $(\bar{\pi}, G, \bar{V})$ naturally corresponds to any representation (π, G, V); unitary structure on V defines a unitary structure on $\bar{V}$.

Let us note that for $\tau \in i\mathbb{R}$, the representation $\bar{V}_\tau$ is *canonically* isomorphic to the representation $V_{\bar{\tau}}$ when we consider them as spaces of functions on $\mathbb{R}^2\backslash 0$ (see Section 3.1.3). The isomorphism is given by the complex conjugation $c(v) = \bar{v}$.

An Hermitian form on a space V gives rise to the morphism $V \to V^+$, where $V^+ := \overline{(V^*)}$ is the complex conjugate of the dual space.

6.2. Complex conjugate representation in automorphic picture. Suppose now that we fixed an automorphic structure $\nu : V \to C^\infty(X)$ on the representation V. Then it defines the canonical automorphic structure $\bar{\nu} : \bar{V} \to C^\infty(X)$ on the complex conjugate representation by the formula $\bar{\nu}(c(v)) = \overline{\nu(v)}$.

We will consider the representation $E = V \otimes \bar{V}$ of the group $G \times G$ and denote by $\nu_E = E \to C^\infty(X \times X)$ the corresponding automorphic structure on E (here $\nu_E = \nu \otimes \bar{\nu}$). We have the following basic claim (compare with 2.7.1).

Claim. *For any vector $v \in V$ consider the vector $w = v \otimes \bar{v} = v \otimes c(v) \in E$ and the corresponding function $\nu_E(w)$ on $X \times X$. Then the restriction $u = r_\Delta(\nu_E(w))$ of this function to the diagonal ΔX is a non-negative function on X, and $H_\Delta(w) = ||u||^2_{L^2(X)} = ||\nu(v)||^4_{L^4(X)}$.*

This follows from the observation that $u(x) = \nu(v)(x) \cdot \overline{\nu(v)(x)} = |\nu(v)(x)|^2$. □

6.3. K-types. We assume that $V = V_\tau$ is a representation of the principal series. All the necessary computations will be done in the circle model $V_\tau \simeq C^\infty_{even}(S^1)$ (i.e., we realize a vector in V as a smooth function f of the angular parameter $t \in \mathbb{R}$ such that $f(t+\pi) = f(t)$). The invariant unitary Hermitian form on V is given by $||f||^2 = \frac{1}{\pi}\int_0^\pi |f(t)|^2 dt$.

Let $e_n = \exp(int)$, where $n \in 2\mathbb{Z}$, be an orthonormal basis of K-types in the space V_τ (all weights are even since we work with the group $G = PGL_2(\mathbb{R})$).

Consider the space $\bar{V}_\tau$. We have a natural identification $\bar{V}_\tau \simeq V_{-\tau}$ induced by the realization of these spaces as spaces of functions on $\mathbb{R}^2 \setminus 0$.

We denote by $\{e'_n = \overline{e_{-n}}\}_{n\in 2\mathbb{Z}}$ the corresponding complex conjugate basis for $\bar{V}_\tau \simeq V_{-\tau}$. Under the natural identification $V_{-\tau} \simeq C^\infty_{even}(S^1)$, we have $e'_n = \exp(int)$ as before.

6.4. Test vectors. In the Introduction (see formula (2.9)) we defined two families of test vectors central for our proof of the subconvexity. We repeat this construction.

For any $n \in 2\mathbb{Z}$, $n \geq 0$, we consider two vectors in $E = E_\tau = V_\tau \otimes V_{-\tau}$ given by

$$w_n = e_n \otimes e'_{-n} \ ,$$

and

$$\tilde{w}_n = w_n + w_{n+2} \ .$$

We note that in the model $V_\tau \otimes V_{-\tau} \simeq C^\infty_{even,even}(S^1 \times S^1)$ these vectors are represented by the functions $w_n(x,y) = e^{in(x-y)}$ and $\tilde{w}_n(x,y) = (1 + e^{i2(x-y)})e^{in(x-y)}$.

In Section 2.7.1 we have proven the basic geometric bound (2.10) for these vectors

$$H_\Delta(w_n) \leq H_\Delta(\tilde{w}_n) \ . \tag{$\star$}$$

6.5. Main Proposition. Our main claim is the following

Proposition. *There exists a positive constant D such that*

$$H_\Delta(\tilde{w}_n) \leq D \ , \tag{$\natural$}$$

for all n.
We prove this proposition in Section 7.2.

Remark. The bound ($\star$) is of a geometric nature as it concerns the form H_Δ defined on the automorphic space X and appeals to the automorphic realization of V in $C^\infty(X)$. On the other hand, our proof of the bound ($\natural$) is purely *spectral*, despite its geometric appearance.

6.6. Proof of Theorem 2.6. Proposition 6.5 and the geometric bound $H_\Delta(w_n) \leq H_\Delta(\tilde{w}_n)$ (see (2.10)) imply the bound in Theorem 2.6 for L^4-norm of K-types. Namely, from Claim 6.2 we see that

$$||\nu(e_n)||^4_{L^4(X)} = H_\Delta(w_n) \leq H_\Delta(\tilde{w}_n) \leq D \ , \tag{6.1}$$

for some D independent of n. □

Remark. From the proof in the next section it could be seen that the bound for the constant D in Theorem 2.6 (and in Proposition 6.5) that we obtain depends on geometry of the Riemann surface $Y = X/K$ and on the parameter τ of the representation V. Namely, we have the following bound

$$D \leq C \cdot \frac{\mathrm{vol}(Y)}{\mathrm{vol}(B_{i(Y)})} \cdot (1 + |\tau|)^2 \ ,$$

for some absolute constant $C > 0$. Here $B_{i(Y)}$ is a hyperbolic ball of the radius equal to the injectivity radius $i(Y)$ of Y.

7. Proof of Proposition 6.5

7.1. Spectral Lemma. Our proof is based on the following spectral bounds (these are bounds (2.11) from the Introduction).

Recall that we set $\mathcal{S} = 2(|\tau| + |\tau'|) + 1$ (in fact in this section we can assume that $\tau' = -\tau$).

Lemma. *There exists a constant C such that for any $n \in 2\mathbb{Z}$, the following spectral bounds hold*

(II_1): $H_\lambda(\tilde{w}_n) \le C\cdot(1+|n|)^{-1}|\lambda|^{-1} + C|\lambda|^{-3}$ *for all* λ *satisfying* $\mathcal{S} \le |\lambda| \le 4|n|$,
(II_2): $H_\lambda(\tilde{w}_n) \le C|\lambda|^{-3}$ *for all* λ *satisfying* $|\lambda| \ge 4|n|$.

The model Hermitian forms H_λ on E are defined explicitly for every $\lambda \in i\mathbb{R}$ as in Section 8.1. The proof of the lemma amounts to a routine application of the stationary phase method and the van der Corput lemma (see Section 9.3). In fact, the restriction $|\lambda| \ge \mathcal{S}$ is purely technical. One can obtain good bounds for the value of $H_\lambda(\tilde{w}_n)$ for all λ. We will not need this in what follows. The constant C in the lemma above satisfies a bound $C \le C'\mathcal{S}$ for some absolute constant C'.

7.2. Proof of Proposition 6.5. For any given n, the function $\nu_E(\tilde{w}_n)$ is a bounded smooth function on $X \times X$ and hence $H_\Delta(\tilde{w}_n)$ is well-defined. We have to show that it is bounded by some constant D independent of n.

As could be seen from our construction in Section 6.4, vectors $\tilde{w}_n$ are ΔK-invariant. It follows from the discussion in Remark 5.2 that for such vectors, we have the following Parseval identity (5.2)

$$H_\Delta(\tilde{w}_n) = \sum_i H_i^{aut}(\tilde{w}_n) = \sum_i d_i H_{\lambda_i}(\tilde{w}_n)\ .$$

Here the sum is over the spherical spectrum $I = \{\lambda_0, \lambda_1, ...\}$. Let $k_0 \in \mathbb{N}$ be such that $2^{k_0} \le \mathcal{S} \le 2^{k_0+1}$. We decompose the spherical spectrum I as a union of subsets I_{k_0}, I_{k_0+1}, ... (dyadic intervals) according to the absolute value of $|\lambda|$, and estimate the contribution of each of these subsets.

Namely, we consider subsets I_k of the spectrum I defined by $I_{k_0} = \{\lambda \in I \mid |\lambda| < 2^{k_0+1}\}$ and $I_k = \{\lambda \in I \mid |\lambda| \in [2^k, 2^{k+1})\}$ for $k > k_0$.

Notice that all exceptional spectra that correspond to representations of the complementary series and to the trivial representation is contained in the interval I_{k_0} (we call it the low spectrum). All the other intervals contain only imaginary values of λ which correspond to representations of the principal series.

We have $H_\Delta(\tilde{w}_n) = \sum_{k \ge k_0} H_k$, where $H_k = \sum_{\lambda_i \in I_k} d_i H_{\lambda_i}(\tilde{w}_n)$.

7.2.1. *Estimate of* H_k *for* $k > k_0$. The idea of the proof is that on the interval I_k the function $H_\lambda(\tilde{w}_n)$ is more or less constant, so we will not lose much when we replace it by its maximal value.

According to the bound (II_2), Lemma 7.1, we see that for $\lambda \in I_k$ we have a bound $H_\lambda(\tilde{w}_n) \le M_k$ where $M_k = C(n^{-1}2^{-k} + 2^{-3k})$ for k satisfying $2^k < 4n$, and $M_k = C2^{-3k}$ for k satisfying $2^k \ge 4n$. Here C is a universal constant that depends only on τ.

According to the mean-value bound (1.4) we have $\sum_{\lambda_i \in I_k} d_i \leq A\ 2^{2k}$. Hence we arrive at the bound $H_k \leq 2^k A M_k$. This implies that

$$\sum_{k>k_0} H_k \leq AC \left(\sum_{k>0} 2^{-k} + \sum_{2^k<4n} 2^k n^{-1} \right) \leq AC(1+8)\ .$$

7.2.2. *Estimate of the low spectrum contribution* H_{k_0}. We claim that the sum H_{k_0} is bounded by some constant D' which depends only on the geometry of the space Y. In principle we can apply to this case the spectral argument similar to the one described above. However this would lead to some unpleasant computations with the exceptional spectrum. For that reason we prefer to give the following more geometric argument.

The vector $\tilde{w}_n \in E$ is a ΔK-invariant vector. Hence the corresponding function $b = \nu_E(\tilde{w}_n)|_{\Delta X} = |\phi_n|^2 + |\phi_{n+2}|^2$ is a K-invariant function and we can view it as a function on Y. Moreover, we can compute its L^1-norm on Y

$$||b||_{L^1(Y)} = \int_Y \left(|\phi_n|^2 + |\phi_{n+2}|^2\right) dv = 2\ .$$

Consider the subspace $R = \mathrm{span}\{\phi_{\lambda_i} \mid |\lambda_i| < 2^{k_0}\} \subset C^\infty(Y)$. This is a finite-dimensional vector space consisting of smooth functions. Since the space R is finite-dimensional we can bound the supremum norm on this space $||\cdot||_\infty$ by L^2-norm, i.e., there exists a constant C_R such that $||f||_\infty \leq C_R ||f||_{L^2(Y)}$ for all functions $f \in R$.

Claim. $H_{k_0} \leq 4C_R^2$.

Indeed, by definition $H_{k_0} = ||a||^2_{L^2(Y)}$, where the vector $a \in R$ is the orthogonal projection $a = pr_R(b)$ of the vector b onto the subspace $R \subset L^2(Y)$.

Thus we have

$$H_{k_0}^2 = |<a,a>|^2 = |<b,a>|^2 \leq ||b||^2_{L^1(Y)} \cdot ||a||^2_\infty \leq 4C_R^2 \cdot ||a||^2_{L^2(Y)} = 4C_R^2 \cdot H_{k_0}\ .$$

This implies the claim and finishes the proof of the proposition. □

Remarks. 1. The constant C_R from the proof can be effectively bounded in terms of the geometry of the Riemann surface Y and of the parameter $\mathcal{S}$. Namely, consider the second Sobolev norm N on the space $C^\infty(Y)$ given by $N(f)^2 = \int_Y (|\Delta f|^2 + |f|^2) dv$, where Δ is the Laplace operator. The Sobolev embedding theorem tells that $||f||_\infty \leq C_Y N(f)$ for some Sobolev constant C_Y that depends only on the geometry of the surface Y. In particular, one have a simple bound $C_Y^2 \leq \mathrm{vol}(Y)/\mathrm{vol}(B_{i(Y)})$, where $B_{i(Y)}$ is the ball of the injectivity radius $i(Y)$ of Y. We claim that $C_R^2 \leq (2+\mathcal{S}^2)C_Y^2$ and thus $H_{k_0} \leq 4(2+\mathcal{S}^2)C_Y^2$.

Indeed, all the eigenvalues of the Laplace operator Δ on the space R are bounded by $\frac{1}{4} + \mathcal{S}^2$. This implies that for $f \in R$ we have $N(f)^2 \leq (2+\mathcal{S}^2)||f||^2_{L^2}$. Thus we see that $C_R^2 \leq (2+\mathcal{S}^2)C_Y^2$ and hence $H_{k_0} \leq 4(2+\mathcal{S}^2)C_Y^2$.

2. The proof of Proposition 6.5 given above could be easily extended to the case of a general finite co-volume lattice $\Gamma \subset G$. In fact, the only place where we implicitly used compactness of X is in the proof of the mean-value bound (1.4) which we quoted from [BR2]. However, in [BR1] we proved similar bound for a general finite co-volume lattice and cuspidal functions ϕ and ϕ'.

For a general finite co-volume lattice, the spectral decomposition of the Laplace-Beltrami operator on $Y = \Gamma\backslash\mathbb{H}$ is given by a collection of eigenfunctions ϕ_z, where the parameter z runs through some set Z with the Plancherel measure $d\mu$. The spectral set Z has discrete points which correspond to eigenfunctions (Maass forms) $\phi_z \in L^2(Y)$ and continuous part which corresponds to eigenfunctions coming from the unitary Eisenstein series. The collection $\{\phi_z\}_{z\in Z}$ defines a transform $\hat{u}(z) =< u, \phi_z >$ for every $u \in C_c^\infty(Y)$. The main property of this transform is the Plancherel formula $||u||^2_{L^2(Y)} = \int_Z |\hat{u}(z)|^2 d\mu$.

Let us fix two Maass *cusp forms* ϕ and ϕ' on Y. For every $z \in Z$, we define the parameter $\lambda_z \in \mathbb{C}$ and the coefficient d_z in the same way as before. In [KS] the following mean-value bound was obtained (improving on our result in [BR1])

$$\int_{T\leq|\lambda_z|\leq 2T} d_z\ d\mu \leq A\,(\ln(T))^{\frac{3}{2}} \cdot T^2\ .$$

The proof given in present paper, together with the above mean-value bound, gives the following bound for L^4-norm of K-types in a *fixed cuspidal* representation $\nu : V \to C^\infty(X)$

$$||\nu(e_n)||_{L^4(X)} \leq D\,(\ln(2+|n|))^{\frac{3}{2}} \text{ for all n.}$$

This is our analog of Theorem 2.6 for non-uniform lattices. In particular we do not know if L^4-norm of K-types are uniformly bounded for a non-uniform lattice.

The bound on L^4-norm of K-types implies as before that the following subconvexity bound holds for a general finite co-volume lattice

$$\int_{Z_T} d_z\ d\mu \leq B\,(\ln(T))^{\frac{3}{2}} \cdot T^{5/3}\ , \text{ where } Z_T = \{z \in Z \mid |\lambda_z| \in I_T\}\ .$$

The rest of the paper is devoted to the proof of spectral bounds $(II_{1,2})$ from Lemma 7.1 and the lower bound (2.6). This will be done using computations in the explicit model of irreducible representations. As a preparation we start with an explicit construction of model Hermitian forms H_λ.

8. Model trilinear functionals

8.1. Model trilinear functionals. In this section we briefly recall our construction from [BR2] of model trilinear invariant functionals.

For every $\lambda \in \mathbb{C}$, we denote by (π_λ, V_λ) the smooth class one representation of the generalized principle series of the group $G = PGL_2(\mathbb{R})$ described in Section 3.1.3. As a vector space V_λ is isomorphic to the space of smooth even functions $C^\infty_{even}(S^1)$ on S^1.

We describe the *model* invariant trilinear functional using this geometric model. Namely, for three given complex numbers $\tau,\ \tau',\ \lambda$, we explicitly construct a nontrivial trilinear functional $l^{mod} : V_\tau \otimes V_{\tau'} \otimes V_\lambda \to \mathbb{C}$ by means of its kernel. In the circle model, the trilinear functional on the triple $V_\tau,\ V_{\tau'},\ V_\lambda$ is given by the following integral:

$$l^{mod}_{\pi,\pi',\pi_\lambda}(f_1 \otimes f_2 \otimes f_3) = (2\pi)^{-3} \int\limits_{(S^1)^3} f_1(x) f_2(y) f_3(z) K_{\tau,\tau',\lambda}(x,y,z)\, dx dy dz\ ,$$

with the kernel

$$K_{\tau,\tau',\lambda}(x,y,z) \ = \ |\sin(x-y)|^{\frac{-\tau-\tau'+\lambda-1}{2}} |\sin(x-z)|^{\frac{-\tau+\tau'-\lambda-1}{2}} |\sin(y-z)|^{\frac{\tau-\tau'-\lambda-1}{2}}. \quad (8.1)$$

Here x, y, z are the standard angular parameters on the circle S^1. As we verified in [BR2] this defines a non-zero G-invariant functional.

Remark. 1. For an arbitrary representation the integral defining the trilinear functional is often divergent and the functional should be defined using regularization of this integral. There are standard procedures how to make such a regularization (see [G1]). Fortunately, in the case of class one unitary representations, all integrals converge absolutely, so we will not discuss the regularization procedure.

2. We do not have a similar simple formula for the trilinear invariant functional when at least one representation is a representation of discrete series. This is because we do not know a simple "geometric" model for representations of discrete series. As a result it is more cumbersome to carry out explicit computations in that case. Another problem we have to face is that the results of [BR2] have not been extended yet to cover the discrete series.

Nevertheless, we expect our methods to carry out for discrete series as well and to produce corresponding subconvexity bounds, and bound for L^4-norms of K-types.

8.2. Reduction for ΔK-invariant vectors. In what follows, we only need to deal with ΔK-invariant vectors in $E \simeq V_\tau \otimes V_{\tau'}$. For such vectors, we can reduce the integral (8.1) representing the model invariant functional, and hence the Hermitian form H_λ to the integral of one variable.

Namely, let $l^{mod}_\lambda : E \otimes V_\lambda \to \mathbb{C}$ be the model trilinear functional introduced in Section 8.1, $T_\lambda = T^{mod}_\lambda : E \to V_{-\lambda}$ be the corresponding map, and H_λ the model Hermitian form on E obtained from the composition of T_λ with the invariant unitary form on $V_{-\lambda}$. We assume that V_λ is a representation of the principal series since we are only interested in the case when $|\lambda| \geq \mathcal{S}$. In this case, the unitary form on $V_\lambda \simeq C^\infty_{even}(S^1)$ is the standard normalized unitary form on $L^2(S^1)$.

Let $w \in E \simeq C^{\infty}_{even,even}(S^1 \times S^1)$ be a ΔK-invariant vector. Since it is ΔK-invariant it can be represented by a function of one variable $c = x - y$: $w(x, y) = u(c)$, where $u \in C^{\infty}_{even}(S^1)$. We claim that the estimate of $H_\lambda(w)$ could be reduced to an estimate of an integral in one variable. Namely, on the space of ΔK-invariant vectors in E the form H_λ has rank 1, i.e., it is equal to the absolute value squared of some functional b_λ on $C^\infty(S^1)$. More precisely, we have the following

Lemma. *Fix τ, $\tau' \in i\mathbb{R}$ as before and assume that $\lambda \in i\mathbb{R}$. There exists an L^1 function l_λ on S^1 such that for any function $u \in C^{\infty}_{even}(S^1)$ and for the corresponding vector $w(x, y) = u(x - y) \in E$, we have $H_\lambda(w) = |b_\lambda(u)|^2$ where $b_\lambda(u) = \int l_\lambda(c)u(c)dc$.*

Proof. Since the vector w is ΔK-invariant its image $T_\lambda(w) \in V_\lambda$ is proportional to the standard unit K-invariant vector e_λ. The proportionality coefficient $b_\lambda(u)$ equals

$$T_\lambda(w)(0) = (1/2\pi)^2 \int K_{\tau,\tau',\lambda}(x, y, 0)w(x, y)dxdy = 1/2\pi \int_{S^1} l_\lambda(c)u(c)dc \ ,$$

where

$$l_\lambda(c) = \frac{1}{2\pi} \int_{S^1} K_\lambda(y + c, y, 0)dy \tag{8.2}$$

and $K_\lambda(x, y, z)$ is the kernel of the model trilinear functional defined in (8.1).

Thus we see that $H_\lambda(w) = ||T_\lambda(w)||^2 = |b_\lambda(u)|^2$. □

Remark. Uniqueness of trilinear functionals implies that $b_{-\lambda} = a(\lambda) \cdot b_\lambda$ for some scalar $a(\lambda) \in \mathbb{C}^\times$. It is also easy to see that $|a(\lambda)| = 1$.

9. Proof of spectral bounds

9.0.1. *A convention.* In what follows we will study asymptotic behavior for various oscillating integrals. We will consider expansions consisting of a main term and a remainder. We will bound corresponding remainders in terms of C^N-norms.

We will use the following notations. We consider functionals on $C^\infty(\mathbb{R})$ of the form $I_\Lambda(\phi) = \int_{\mathbb{R}} k_\Lambda(x)\phi(x)dx$ where $\phi \in C^\infty(\mathbb{R})$ (usually with compact support). Here $k_\Lambda(x) \in L^1(\mathbb{R})$ is a kernel function depending on a set of parameters $\Lambda \in \mathbb{R}^n$. We consider approximations of such functionals of the form $I_\Lambda(\phi) = I^0_\Lambda(\phi) + RI_\Lambda(\phi)$ where we call $I^0_\Lambda(\phi)$ the main term and $RI_\Lambda(\phi)$ the remainder. Usually, the main term will be given by the stationary phase method (i.e., it will be given by a functional which is a weighted sum of δ-functions at points corresponding to critical points of the phase of k_Λ). We will consider bounds for $RI_\Lambda(\phi)$ in terms for C^N-norms of function ϕ. For $\phi \in C^\infty(a, b)$ and an integer $N \geq 0$, we will denote by $||\phi||_{C^N}$ the C^N-norm of ϕ defined by $||\phi||_{C^N} = \sup\limits_{0\leq m\leq N,\ x\in(a,b)} |\phi^{(m)}(x)|$.

9.1. Estimate of the functional b_λ. In Section 8.2 we have reduced estimates of the form H_λ to the estimates of the functional b_λ. We will be interested in the case when the function u from Lemma 8.2 has a form $u(c) = \phi(c)e^{inc}$, where ϕ is a fixed smooth function and $n \in 2\mathbb{Z}$ is a parameter. We can consider the expression $b_\lambda(u)$ as a functional $F_{\lambda,n}$ on the space $C^\infty(S^1)$ which depends on two parameters λ and n. This functional is given by

$$F_{\lambda,n}(\phi) := \int_{S^1} l_\lambda(c)e^{inc}\phi(c)dc \ . \tag{9.1}$$

The main technical difficulty in evaluating this functional is that we have to give estimates for the values of this functional that are uniform in two parameters λ and n.

Recall that we set $\mathcal{S} = 2(|\tau| + |\tau'|) + 1$ and assume that $|\lambda| \geq \mathcal{S}$. Using the symmetry of functional $F_{\lambda,n}$, we will show that it is enough to consider the case when $n \in 2\mathbb{Z}_+$ and $\lambda = it,\ t \geq \mathcal{S}$. It turns out that under these conditions the functional $F_{\lambda,n}$ is almost proportional to an elementary functional $\phi \mapsto \phi(c_0)$ where $c_0 = \pi/2$.

We have the following

Proposition. *Consider the functional $F_{\lambda,n}$ when $n \in 2\mathbb{Z}_+$ and $\lambda = it,\ t \geq \mathcal{S}$. We have the following estimates of the values of this functional in terms of C^N-norms on $C^\infty(S^1)$. There exists $C > 0$ such that*

(1) *If $t \geq 4n$ then $|F_{\lambda,n}(\phi)| \leq C||\phi||_{C^3} \cdot t^{-\frac{3}{2}}$.*
(2) *If $t < 4n$ we have an approximation $F_{\lambda,n}(\phi) = F^0_{\lambda,n}(\phi) + RF_{\lambda,n}(\phi)$,*

with the main term given by $F^0_{\lambda,n}(\phi) = A(\lambda, n)\phi(c_0)$, for $A(\lambda, n) = t^{-\frac{5}{6}}\mathbb{A}(t^{-\frac{1}{3}}(2n - t))$. The error term satisfies a bound

$$|RF_{\lambda,n}(\phi)| \leq C||\phi||_{C^2} \cdot t^{-\frac{1}{2}}(1 + n)^{-\frac{1}{2}} + C||\phi||_{C^3} \cdot t^{-\frac{3}{2}} \ .$$

Here $\mathbb{A}$ is the classical Airy function (see [Mag], [He, Section 7.6]*).*

We will prove this proposition in Section 10 by carefully estimating the oscillating integral defining the functional $F_{\lambda,n}(\phi)$. For the constant C above we can obtain a bound of the form $C \leq C'\mathcal{S}$ for some absolute constant C'.

9.2. Proof of the spectral bound (2.6)**.** We repeat the construction of the test vector w_T in (2.5). We assume that $V = V_\tau,\ V' = V_{\tau'}$ for some $\tau,\ \tau' \in i\mathbb{R}$. We choose an orthonormal basis $\{e_n\}_{n\in 2\mathbb{Z}}$ in V consisting of K-types and similarly an orthonormal basis $\{e'_n\}$ in V'.

For a given $T \geq \mathcal{S}$, we choose even $n \geq 0$ such that $|T - 2n| \leq 10$, and set $w_T := e_n \otimes e'_{-n}$.

Using the reduction from Section 8.2, we see that the vector $w = w_T$ corresponds to a function $u(c) = e^{inc}$. Hence we have $H_\lambda(w) = |F_{\lambda,n}(\phi)|^2$, where $\phi \equiv 1$.

From (2) in Proposition 9.1 we see that $F_{\lambda,n}(\phi) = A(\lambda, n)\phi(c_0) + RF_{\lambda,n}(\phi)$. In this case we have $|RF_{\lambda,n}(\phi)| \leq C(1 + |n|)^{-1}$, $\phi(c_0) = 1$. The Airy function $\mathbb{A}$ is a smooth *non-vanishing* at 0 function ([Mag], [He, Section 7.6]). Hence there are constants $b,\ c > 0$ such that $|\mathbb{A}(x)| \geq c$ for all $|x| \leq b$. This implies that $|A(it,n)| \geq ct^{-5/6}$ for $|2n-t| \leq bt^{-\frac{1}{3}}$. Hence in the approximation of $F_{\lambda,n}(\phi)$ stated in Proposition 9.1 (2), the main term $A(it,n)\phi(c_0)$ dominates the reminder $RF_{\lambda,n}(\phi)$. The lower bound (2.6) follows. □

9.3. Proof of Lemma 7.1, ($II_{1,2}$). We assume that $V' \simeq \overline{V}$, i.e., $\tau = -\tau'$. Let $n \in 2\mathbb{Z}$ and $\lambda \in i\mathbb{R}$, $|\lambda| \geq \mathcal{S}$, and $\tilde{w} = \tilde{w}_n$ as in Section 6.4. As in Section 9.2, we have $H_\lambda(\tilde{w}) = |F_{\lambda,n}(\tilde{\phi})|^2$, where $\tilde{\phi}(c) = 1 + e^{2ic}$. This time we are looking for a uniform in n upper bound valid for *all* $|\lambda| \geq \mathcal{S}$.

We need to bound the integral $F_{\lambda,n}(\tilde{\phi})$. From the form of integral (9.1) it follows that it is enough to consider the case of $n \geq 0$ and $\mathrm{Im}(\lambda) \geq 0$. Indeed, using the change of variables $c \mapsto -c$ in integral (9.1), we can assume that $n \geq 0$. Considering the complex conjugate to l_λ, we can assume that $\mathrm{Im}(\lambda) \geq 0$.

Hence we can apply Proposition 9.1. We have $\tilde{\phi}(c_0) = 0$, and $F_{\lambda,n}(\tilde{\phi}) = RF_{\lambda,n}(\tilde{\phi})$. Thus estimates in Lemma 7.1 ($II_{1,2}$), directly follow from the Proposition 9.1. □

10. Proof of Proposition 9.1

10.1. Proof of Proposition 9.1. The functional $F_{\lambda,n}$ is defined using the oscillating integral $F_{\lambda,n}(\phi) = \int l_\lambda(c)e^{inc}\phi(c)dc$. One of the difficulties in evaluating this functional is that its kernel function l_λ is not an elementary function.

However, since this function itself is defined by an oscillating integral, we can approximate it by an elementary function k_λ which is the sum of main term contributions from critical points of this oscillating integral.

10.1.1. *Approximation of the kernel l_λ.* We have the following
Lemma. *Fix $\tau,\ \tau' \in i\mathbb{R}$ and $\mathcal{S}$ as before and assume that $\lambda \in i\mathbb{R}$, $|\lambda| \geq \mathcal{S}$. There exists a constant $C > 0$ depending on τ and τ', such that we have the following approximation*

$$l_\lambda(c) = a_\lambda \cdot |\lambda|^{-\frac{1}{2}} k_\lambda(c) + r_\lambda(c) \ , \tag{10.1}$$

where $a_\lambda = e^{i\frac{\pi}{4}} 2^{1+\frac{\lambda}{2}}$ and the kernel $k_\lambda(c)$ is given by an explicit formula $k_\lambda(c) = A(c)m_\lambda(c)$ with

$$A(c) = |\sin(c)|^{\frac{-\tau-\tau'-1}{2}} \quad , \quad m_\lambda(c) = |\sin(c/2)|^{-\frac{\lambda}{2}} |\cos(c/2)|^{\frac{\lambda}{2}} \ , \tag{10.2}$$

and the error term $r_\lambda(u)$ satisfies the bound

$$|r_\lambda(c)| \leq C|\lambda|^{-\frac{3}{2}} |\sin(c)|^{-\frac{1}{2}} |\ln(|\sin(c/2)\cos(c/2)|)| \ . \tag{10.3}$$

We will prove this lemma in Section 10.2. For the constant C above we can obtain a bound of the form $C \leq C'\mathcal{S}$ for some absolute constant C'.

Using this approximation we can approximate the functional $F_{\lambda,n}$ by a simpler functional defined for $n \in 2\mathbb{Z}$ and $\phi \in C^\infty_{even}(S^1)$, by

$$G_{\lambda,n}(\phi) := \int_{S^1} k_\lambda(c)e^{inc}\phi(c)dc = 2\int_0^\pi k_\lambda(c)e^{inc}\phi(c)dc \ . \tag{10.4}$$

The lemma above implies
Corollary. *There exists a constant $C' = C'(\tau,\tau') > 0$ such that*

$$|F_{\lambda,n}(\phi) - a_\lambda|\lambda|^{-\frac{1}{2}} \cdot G_{\lambda,n}(\phi)| \leq C'||\phi||_{L^\infty(S^1)} \cdot |\lambda|^{-3/2} \ , \tag{10.5}$$

for all $|\lambda| \geq \mathcal{S}$.

Hence Proposition 9.1 follows from an appropriate estimate for the functional $G_{\lambda,n}(\phi)$.

10.1.2. *Estimate for $G_{\lambda,n}(\phi)$.* We have the following estimate for the functional $G_{\lambda,n}$ defined in (10.4).

Proposition. *Consider the functional $G_{\lambda,n}$ when $n \in 2\mathbb{Z}_+$ and $\lambda = it,\ t \geq \mathcal{S}$. There exists a constant $C > 0$ depending on τ and τ', such that we have the following estimates*

(1) *If $t \geq 4n$ then $|G_{\lambda,n}(\phi)| \leq C||\phi||_{C^3} \cdot t^{-3}$,*
(2) *If $t < 4n$ then we have an approximation $G_{\lambda,n}(\phi) = G^0_{\lambda,n}(\phi) + RG_{\lambda,n}(\phi)$,*

with the main term given by $G^0_{\lambda,n}(\phi) = A(\lambda,n)\phi(c_0)$, for $A(\lambda,n) = t^{-\frac{1}{3}}\mathbb{A}(t^{-\frac{1}{3}}(2n-t))$. The error term $RG_{\lambda,n}(\phi)$ satisfies a bound

$$|RG_{\lambda,n}(\phi)| \leq C||\phi||_{C^2} \cdot (1+n)^{-1/2} + C||\phi||_{C^3} \cdot t^{-\frac{3}{2}} \ .$$

Here $\mathbb{A}$ is the classical Airy function.

This proposition and bound (10.5) imply Proposition 9.1. This finishes the proof of Proposition 9.1. □

10.2. Proof of Lemma 10.1.1. We prove the claims in the lemma by essentially straightforward application of the stationary phase method in the form explained in Appendix A. In order to estimate the error of this approximation we use the standard integration by parts argument.

To compute the approximation k_λ of l_λ, we consider for fixed $\tau,\ \tau' \in i\mathbb{R}$ and for $|\lambda| \geq \mathcal{S}$, $\lambda \in i\mathbb{R}$, the integral (8.2):

$$\begin{aligned} l_\lambda(c) &= (2\pi)^{-\frac{1}{2}} \int_{S^1} K_{\tau,\tau',\lambda}(y+c,y,0)dy \\ &= (2\pi)^{-\frac{1}{2}}|\sin(c)|^{\frac{-\tau-\tau'+\lambda-1}{2}} \cdot \int_{S^1} |\sin(y+c)|^{\frac{-\tau+\tau'-\lambda-1}{2}}|\sin(y)|^{\frac{\tau-\tau'-\lambda-1}{2}}dy \\ &= |\sin(c)|^{\frac{-\tau-\tau'+\lambda-1}{2}} l'_\lambda(c) \ , \end{aligned}$$

where the kernel $K_{\tau,\tau',\lambda}$ is as in (8.1), and we denote by l'_λ (suppressing the dependence on $\tau,\ \tau'$) the function

$$l'_\lambda(c) = (2/\pi)^{\frac{1}{2}} \int_{t\in\mathbb{R}/\pi\mathbb{Z}} |\sin(t+c/2)|^{\frac{-\tau+\tau'-\lambda-1}{2}} |\sin(t-c/2)|^{\frac{\tau-\tau'-\lambda-1}{2}} dt\ . \tag{10.6}$$

To find the asymptotic of the integrals of the type of $l'_\lambda(c)$ is a problem in classical analysis. We view the integral (10.6) as a one-dimensional integral (in t) with parameters λ and c. We treat such integrals in Appendix A where we show that the main term (i.e., the term $M_\lambda(c)$ below) in the asymptotic of such integrals is given by the stationary phase method with respect to the parameter $\lambda \to \infty$ while the parameter c is *fixed* ($c \neq 0,\ \pi$). In our case, by a straightforward calculation, we find out that there are two non-degenerate critical points of the phase at $t = 0$ and $t = \pi/2$. Hence the main term is a sum of two terms (see equation (10.8)). We estimate the remainder *uniformly* in c for $c \neq 0,\ \pi$. This is done by reducing the problem to the standard Beta type integrals. We explain this reduction in Section A.1.

Proposition A.1 implies that integral (10.6) has the following uniform asymptotic expansion in $\lambda \in i\mathbb{R}$, $|\lambda| \geq \mathcal{S}$ and c ($c \neq 0,\ \pi$) for *fixed* $\tau,\ \tau'$,

$$l'_\lambda(c) = e^{i\frac{\pi}{4}} |\lambda|^{-\frac{1}{2}} \cdot M_\lambda(c) + r'_\lambda(c)\ , \tag{10.7}$$

where the main term $M_\lambda(c)$ comes from stationary points of the phase at $t = 0,\ \pi/2$ and is given by

$$M_\lambda(c) = \left|\sin\left(\frac{c}{2}\right)\right|^{-\lambda} + \left|\cos\left(\frac{c}{2}\right)\right|^{-\lambda}\ ; \tag{10.8}$$

and for $c \neq 0,\ \pi$, the remainder $r'_\lambda(c)$ satisfies the bound

$$|r'_\lambda(c)| \leq C|\lambda|^{-3/2} |\ln(|\sin(c/2)\cos(c/2)|)| \tag{10.9}$$

with a constant $C > 0$ depending on $\tau,\ \tau'$, but not on c and λ.

Let $m_\lambda(c) = |\sin(c/2)|^{-\frac{\lambda}{2}} |\cos(c/2)|^{\frac{\lambda}{2}}$. After elementary manipulations with (10.8), we arrive at

$$\begin{aligned} l_\lambda(c) &= |\sin(c)|^{\frac{-\tau-\tau'+\lambda-1}{2}} l'_\lambda(c) \\ &= e^{i\frac{\pi}{4}} 2^{\frac{\lambda}{2}} |\lambda|^{-\frac{1}{2}} |\sin(c)|^{\frac{-\tau-\tau'-1}{2}} \quad [m_\lambda(c) + m_{-\lambda}(c)] + |\sin(c)|^{\frac{-\tau-\tau'-1}{2}} r'_\lambda(c). \end{aligned}$$

The function l_λ has the period equal to π. We note that $m_\lambda(c+\pi) = m_{-\lambda}(c)$.

In (8.2) we integrate $l_\lambda(c)$ against a function u with a period equal to π. Hence we obtain the asymptotic formula (10.1). □

10.3. Proof of Proposition 10.1.2. The functional $G_{\lambda,n}$ was defined in (10.4) through the kernel k_λ as in (10.1)

$$G_{\lambda,n}(\phi) = \int_{\mathbb{R}/\pi\mathbb{Z}} \phi(c) |\sin(c)|^{\frac{-\tau-\tau'-1}{2}} |\sin(c/2)|^{-\frac{\lambda}{2}} |\cos(c/2)|^{\frac{\lambda}{2}}\ e^{inc}\ dc \tag{10.10}$$

for $\phi \in C^\infty_{even}(S^1)$, $\lambda = it \in i\mathbb{R}$, $t \geq \mathcal{S}$, and all $n \in 2\mathbb{Z}_+$. We consider this integral as a functional on the space of functions $\phi \in C^\infty(S^1)$. This functional depends on "large" parameters λ and n, and on axillary parameters τ and τ'. Our goal is to find a good approximation for values of this functional and give an estimate of the error term.

Let us denote by $S_{\lambda,n}(c) = \frac{\lambda}{2}(-\ln(|\sin(c/2))| + \ln(|\cos(c/2)|)) + inc$ the phase of the oscillating integral (10.10) and by $a(c) = |\sin(c)|^{\frac{-\tau-\tau'-1}{2}}$ its amplitude. Then the functional (10.10) takes the form

$$G_{\lambda,n}(\phi) = \int_{\mathbb{R}/\pi\mathbb{Z}} \phi(c)a(c)e^{S_{\lambda,n}(c)}dc \ . \tag{10.11}$$

A direct computation shows that the critical points of the phase function $S_{\lambda,n}$ are solutions of the equation $\sin(c) = \delta$, where $\delta = 2in/\lambda = 2n/t$. This shows that the functional (10.10) has different asymptotic behavior for different values of parameter δ. Let us list what we can expect; note that we consider only the case $\delta \geq 0$ (i.e., that $n \geq 0$ and $t \geq \mathcal{S}$).

(1) For $\delta > 1$ the phase function $S_{\lambda,n}$ has two critical points of Morse type; in this case we can estimate the integral using the stationary phase method.
(2) When δ approaches 1 these critical points collide at the point $c_0 = \pi/2$. In order to get uniform bounds in this region we use properties of the Airy function.
(3) When $\delta < 1$ the critical points disappear. In this case we will show that the integral (10.10) is rapidly decaying.

Our goal is to show that the functional $G_{\lambda,n}(\phi)$ can be approximated by a functional proportional to the delta function at c_0 (i.e., by $A(\lambda,n)\phi(c_0)$). We will also give explicit uniform bounds for the error term $RG_{\lambda,n}(\phi) = G_{\lambda,n}(\phi) - A(\lambda,n)\phi(c_0)$.

We rewrite the phase function $S_{\lambda,n}$ in the form $S_{\lambda,n}(c) = \frac{\lambda}{2}S_\delta(c)$, where $\delta = 2in/\lambda$. We will think about integrals $G_{\lambda,n}(\phi)$ as a oscillatory integrals with "large" parameter λ and additional parameter δ.

Using the partition of unity we see that to prove the proposition it is enough to consider separately two cases:

(1) The function ϕ is supported in a small neighborhood of the point $c_0 = \pi/2$.
(2) The function ϕ vanishes in a neighborhood of the point $c_0 = \pi/2$.

Case 1. Let ϕ be supported in a small enough neighborhood of the point $c_0 = \pi/2$. We claim that for such ϕ, the following bound holds

$$|G_{it,n}(\phi) - A(it,\delta)\phi(c_0)| \leq C||\phi||_{C^2} \cdot t^{-\frac{2}{3}} \ . \tag{10.12}$$

Here $A(it,\delta) = t^{-\frac{1}{3}}\mathbb{A}(t^{\frac{2}{3}}(\delta-1)) = t^{-\frac{1}{3}}\mathbb{A}(t^{-\frac{1}{3}}(2n-t))$, and $\mathbb{A}$ is the classical Airy function.

The condition $1+\varepsilon \geq \delta \geq 1-\varepsilon$ implies that $n \asymp |\lambda|$. Hence the above bound implies that Proposition 10.1.2 holds for such ϕ.

We now specify the size of the support of ϕ and prove bound (10.12). For any $0.01 > \varepsilon > 0$, there exists a neighborhood $U_\varepsilon \subset [c_0 - 0.1, c_0 + 0.1]$ of the point c_0 which *does not* contain critical points of S_δ for $\delta \notin [1-\varepsilon, 1+\varepsilon]$. We assume that ϕ is supported in this neighborhood for ε to be specified latter. Integration by part implies then that for $\delta \notin [1-\varepsilon, 1+\varepsilon]$, the bound $|G_{\lambda,n}(\phi)| \ll |\lambda|^{-N}$ holds for any $N > 0$. Hence we only need to consider the case $1+\varepsilon \geq \delta \geq 1-\varepsilon$. We claim that in this case there exists a change of variables which transforms the integral $G_{\lambda,n}(\phi)$ to the Airy type integral. Namely, a direct computation shows that $\frac{\partial}{\partial c} S_\delta|_{c_0} = \frac{\partial^2}{\partial c^2} S_\delta|_{c_0} = 0$ and $\frac{\partial^3}{\partial c^3} S_\delta|_{c_0} \neq 0$. (In fact, it is easy to see that the dependence of S_δ on δ is non-degenerate. Namely, the family of functions $\{S_\delta\}$ is a versal deformation of the function $(c-c_0)^3$ in the sense of [Ar].) We now can quote a classical result on oscillating integrals of the Airy type. Namely, Theorem 7.7.18, [He] evidently implies the following claim

Claim. *Let $f \in C^\infty(\mathbb{R}^2)$ be a real valued smooth compactly supported function such that $\frac{\partial f}{\partial x} = \frac{\partial^2 f}{\partial x^2} = 0$ and $\frac{\partial^3 f}{\partial x^3} \neq 0$ at the point $(x, y) = (0, 0)$. Then there exist $\varepsilon > 0$ and smooth real valued functions $a(y)$, $b(y)$ defined on the interval $(-\varepsilon, \varepsilon)$, such that $a(0) = 0,\ b(0) = f(0)$ and*

$$\left| \int u(x) e^{i\omega f(x,y)} dx - e^{i\omega b(y)} \cdot \mathbb{A}\left(a(y)\omega^{\frac{2}{3}}\right) \omega^{-\frac{1}{3}} \cdot u(0) \right| \leq C||u||_{C^2} \cdot \omega^{-\frac{2}{3}} ,$$

for all real $\omega \geq 1$. Here $\mathbb{A}$ is the classical Airy function.

The above claim implies bound (10.12) for $G_{\lambda,n}(\phi)$. Namely, fix $\varepsilon > 0$ such that the above claim is applicable to $f(x, y) = S_{y+1}(x)$ for $y \in [-\varepsilon, \varepsilon]$ (i.e., for $\delta \in [1-\varepsilon, 1+\varepsilon]$). Let U_ε be a neighborhood of the point c_0 which does not contain critical points of S_δ for $\delta \notin [1-\varepsilon, 1+\varepsilon]$. We assume that *supp*$(\phi) \subset U_\varepsilon$. Applying the above claim for $\delta \in [1-\varepsilon, 1+\varepsilon]$, we obtain the bound (10.12).

Case 2. Let ϕ be a function vanishing in a neighborhood of the point $c_0 = \pi/2$. In this case we have upper bounds

$$|G_{\lambda,n}(\phi)| \leq \begin{cases} C'||\phi||_{C^2} \cdot |\lambda|^{-\frac{1}{2}}, & \text{for } \delta > 0.9, \\ C'_N||\phi||_{C^N} \cdot |\lambda|^{-N}, & \text{for } 0 < \delta \leq 0.9, \end{cases} \tag{10.13}$$

for any $N > 0$ and some constants C', C'_N, which could be explicitly bounded in terms of τ and τ'. These bounds immediately follow from the van der Corput lemma and integration by parts as explained in Section B.3. □

Appendix A. Beta integrals

In this appendix we explain how to prove asymptotic expansion for certain oscillating integrals which we call Beta integrals. We use these asymptotic in the proof of Lemma 10.1.1.

A.1. Beta integrals. Fix a function $h \in C^\infty(\mathbb{R})$ such $h(0) = 0$, $h' > 0$. Fix $\sigma,\ \sigma' \in \mathbb{C}$ such that $Re(\sigma),\ Re(\sigma') > -1$ and $Re(\sigma) + Re(\sigma') = -1$. (In fact, in this paper we will need only the case $Re(\sigma) = Re(\sigma') = -\frac{1}{2}$). We consider following integrals

$$\mathrm{H}_{\lambda,c}(\phi) = \int_{\mathbb{R}} |h(t-c)|^{\sigma+\lambda} |h(t+c)|^{\sigma'+\lambda} \phi(t) dt\ , \tag{A.1}$$

where $\phi \in C^\infty(\mathbb{R})$, and $\lambda \in i\mathbb{R}$. We are interested in the *uniform* asymptotic of such integrals in c, $c \neq 0$, and for $|\lambda|$ sufficiently large. Moreover, we will assume that both *supp*(ϕ) (containing 0) and values of c are sufficiently small, depending on the function h.

We write the integral $\mathrm{H}_{\lambda,c}(\phi) = \int_{\mathbb{R}} \phi(t) a_{\sigma,\sigma'}(c;t) e^{\lambda S(c;t)} dt$ in the standard form customary in the stationary phase method. Here $\phi(t)a_{\sigma,\sigma'}(c;t)$ is the amplitude and $S(c;t)$ is the phase in this oscillating integral, both depending on the parameter c and some auxiliary parameters $\sigma,\ \sigma'$ which we consider fixed. For any *fixed* $c \neq 0$ and smooth ϕ of compact support, one can obtain the asymptotic in $|\lambda| \to \infty$ for $\mathrm{H}_{\lambda,c}(\phi)$ from the stationary phase method (see [He, Theorem 7.7.6]). We choose the range of the parameter c and the support of ϕ small enough so that for all $c \neq 0$, the following conditions are satisfied. There exists the unique critical point t_c (in variable t) of the phase $S(c;t)$, this critical point is non-degenerate, and it is disjoint from singularities of the amplitude $a_{\sigma,\sigma'}$ at points $t = \pm c$ (in fact if h is odd, as in our case, then $t_c = 0$ for all $c \neq 0$). We denote by $\mathrm{H}^0_{\lambda,c}(\phi)$ the main term of the contribution from the critical point t_c to the asymptotic of $\mathrm{H}_{\lambda,c}(\phi)$ given by the stationary phase method. (In particular, we will show that for large $|\lambda|$ and fixed c, $|\mathrm{H}^0_{\lambda,c}(\phi)| = A|\lambda|^{-\frac{1}{2}}$ and $|\mathrm{H}_{\lambda,c}(\phi) - \mathrm{H}^0_{\lambda,c}(\phi)| \leq B|\lambda|^{-3/2}$.)

Our aim is to obtain a meaningful bound for the remainder

$$R\mathrm{H}_{\lambda,c}(\phi) = \mathrm{H}_{\lambda,c}(\phi) - \mathrm{H}^0_{\lambda,c}(\phi)\ ,$$

which is *uniform* in λ and c. Recall that we set $\mathcal{S} = 2(|\tau| + |\tau'|) + 1$. We claim the following bound

Proposition. *Fix $h \in C^\infty(\mathbb{R})$ as before. There are constants $C_1,\ C_2 > 0$, and intervals $(-\epsilon,\ \epsilon)$ and $[-d,\ d]$ depending on the function h, such that the remainder satisfies the bound*

$$|R\mathrm{H}_{\lambda,c}(\phi)| \leq C_1 ||\phi||_{C^1} \cdot |\lambda|^{-\frac{3}{2}} + C_2 ||\phi||_{C^2} \cdot |\ln|c|| \cdot |\lambda|^{-2} \tag{A.2}$$

for any $|\lambda| \geq \mathcal{S}$, $c \in (-\epsilon,\ \epsilon)$, $c \neq 0$, and for any smooth function ϕ such that supp$(\phi) \subset [-d,\ d]$.

In fact the method we present allows one to give the asymptotic expansion to any order with the explicit bound on the remainder.

A.2. Proof of Proposition A.1. We show that it is enough to consider the special case of $h(t) = t$. Namely, we claim there exists a smooth change of variables (t, c) to the new set of variables (x, a), where c depends on a only, such that it transforms the kernel function $|h(t-c)|^{\sigma+\lambda}|h(t+c)|^{\sigma'+\lambda}$ to the homogenous kernel $|x-a|^{\sigma+\lambda}|x+a|^{\sigma'+\lambda}$ times some smooth function mildly depending on a.

A.2.1. *Reduction to the special case.* Let $g \in C_c^\infty(\mathbb{R})$ be a function such that $h(t) = tg(t)$ and $g(0) > 0$. We denote by $f(t,c) = h(t-c)h(t+c)$. The necessary change of variables is given by the following lemma.

Lemma. *There exists a change of variables* $(x,a) = (x(t,c), a(t,c))$ *in a neighborhood of the point* $(0,0)$ *such that*

(1) *The variable a is a function of c only,*
(2) $f(t,c) = (x+a)(x-a)$ *in new coordinates, and*
(3) $h(t-c) = (x-a)g_1(x,a)$ *and* $h(t+c) = (x+a)g_2(x,a)$*, where g_1 and g_2 are smooth functions not vanishing near the point* $(0,0)$.

Using this lemma, we can rewrite the integral

$$\mathrm{H}_{\lambda,c}(\phi) = \int_{\mathbb{R}} |h(t-c)|^{\sigma+\lambda}|h(t+c)|^{\sigma'+\lambda}\phi(t)dt \ = \int_{\mathbb{R}} |x-a|^{\sigma+\lambda}|x+a|^{\sigma'+\lambda}\psi(x)dx \ , \quad (A.3)$$

where ψ is a smooth function such that $\psi(0) = \phi(0)$ and C^n-norms of ψ are bounded by those of ϕ. Explicitly $\psi(x) = \phi(t(x,a))|g_1(x,a)|^{\sigma}|g_2(x,a)|^{\sigma'}\left|\frac{\partial x}{\partial t}\right|$.

We introduce integrals

$$H_{\lambda,a}(\psi) = \int_{\mathbb{R}} |x-a|^{\sigma+\lambda}|x+a|^{\sigma'+\lambda}\psi(x)dx \ . \quad (A.4)$$

Lemma A.2.1 implies that $\mathrm{H}_{\lambda,c}(\phi) = H_{\lambda,a}(\psi)$ for an appropriate function ψ (see (A.3)). Here parameters c and a are related via the change of variables in Lemma A.2.1.

The integral $H_{\lambda,a}(\psi)$ also has an asymptotic expansion (in λ for every *fixed* a) with the main term $H^0_{\lambda,a}(\psi)$ given by the stationary phase method at $x = 0$, and a remainder $RH_{\lambda,a}(\psi)$. We want to compare asymptotic expansions of $\mathrm{H}_{\lambda,c}(\phi)$ and of $H_{\lambda,a}(\psi)$. Our considerations are based on the well-known *invariancy* of terms obtained by the stationary phase method (see [Ar], [St]). Namely, we have $H^0_{\lambda,a}(\psi) = \mathrm{H}^0_{\lambda,c}(\phi)$. Since integrals themselves are also equal we have the equality of remainders $RH_{\lambda,a}(\psi) = R\mathrm{H}_{\lambda,c}(\phi)$. Hence, we can use the estimate for the remainder for the integral $H_{\lambda,a}$ which we obtained in (A.8), Corollary A.3. Note that the function h enters into the main term $H^0_{\lambda,a}(\psi)$ and the remainder $RH_{\lambda,a}(\psi)$ via the function ψ.

Parameters a and c belong to a bounded set. Hence C^N-norms of ψ could be bounded independently of a in terms of $||\phi||_{C^N}$ and of $||h||_{C^N}$. This implies that the constant in the bound (A.2) for the remainder $RH_{\lambda,a}(\psi)$ could be chosen independently of c. This finishes the proof of Proposition A.1. □

A.3. Standard Beta integrals. Consider following standard Beta integrals

$$H_{\lambda,\sigma,\sigma'}(\phi)=\int_{\mathbb{R}}|y-1|^{\sigma+\lambda}|y+1|^{\sigma'+\lambda}\phi(y)dy\ , \tag{A.5}$$

where $\phi\in C^\infty(\mathbb{R})$, $\lambda\in i\mathbb{R}$, and $\sigma,\ \sigma'$ are as before. We apply the stationary phase method and the elementary method of integration by parts as described in Section B.1 in order to obtain the following bound.

Let $R=\mathbb{R}\setminus[-0.5,\ 0.5]$ and $\xi=y\frac{\partial}{\partial y}$. The phase function in integral (A.5) has the unique stationary point at $y=0$ which is non-degenerate. Let $H^0_{\lambda,\sigma,\sigma'}(\phi)$ be the main term in the asymptotic of $H_{\lambda,\sigma,\sigma'}(\phi)$ as $|\lambda|\to\infty$ (i.e., $H^0_{\lambda,\sigma,\sigma'}(\phi)=\alpha\phi(0)\cdot|\lambda|^{-\frac{1}{2}}$ with $\alpha=\left(\frac{\pi}{i}\right)^{\frac{1}{2}}$ given by the stationary phase method).

Lemma. *There are constants $C_1,C_2>0$ such that the bound*

$$\left|H_{\lambda,\sigma,\sigma'}(\phi)-H^0_{\lambda,\sigma,\sigma'}(\phi)\right|\leq C_1||\phi||_{C^1([-0.9,0.9])}\cdot|\lambda|^{-\frac{3}{2}}+C_2RH(\phi)\cdot|\lambda|^{-2}$$

holds for any $|\lambda|\geq\mathcal{S}$, and for any smooth compactly supported function ϕ. Here the reminder is given by $RH(\phi)=\int\limits_R\sum\limits_{i=0}^{2}|\xi^i(\phi)|\frac{dy}{|y|}$.

Proof. It is enough to treat separately the case of ϕ supported near zero (e.g., in the interval $[-0.9,0.9]$) and that of ϕ vanishing near zero (e.g., vanishing on $[-0.5,\ 0.5]$).

Case 1. Function ϕ supported near zero. The stationary phase method (see [He, Theorem 7.7.6]) implies that

$$|H_{\lambda,\sigma,\sigma'}(\phi)-H^0_{\lambda,\sigma,\sigma'}(\phi)|\leq C_1||\phi||_{C^1}\cdot|\lambda|^{-\frac{3}{2}}\ , \tag{A.6}$$

with an explicit constant C_1. Such a bound is enough for our purposes.

Case 2. Function ϕ vanishes near zero. We rewrite the integral $H_{\lambda,\sigma,\sigma'}(\phi)$ in the form I_F from (B.1), Appendix B, with

$$F(y;\lambda,\sigma,\sigma')=y|y-1|^{\sigma}|y+1|^{\sigma'}|y-1|^{\lambda}|y+1|^{\lambda}, \tag{A.7}$$

and the form $\omega=dy/y$.

Consider the vector field $\xi=y\frac{\partial}{\partial y}$. A straightforward computation shows that $G:=\xi(F)/F=\lambda(\frac{y}{y+1}+\frac{y}{y-1})+g_{\sigma,\sigma'}(y)$, where the function $g_{\sigma,\sigma'}$ is bounded on the set $R=\mathbb{R}\setminus[-0.5,\ 0.5]$. Hence, for $|\lambda|\geq\mathcal{S}$, the function $H=G^{-1}$ is uniformly bounded in λ and $y\in\mathbb{R}\setminus[-0.5,\ 0.5]$. Moreover, if we make a change of variable $z=y^{-1}$, then the function H and the vector field ξ are smooth on the interval $J=[-1,\ 1]$ (including at zero, after extending H and ξ by continuity). Via compactness, this implies that all functions $\xi^i(H)$ are uniformly bounded (in the coordinate z) on J, and hence are bounded on $\mathbb{R}\setminus[-0.5,\ 0.5]$ (in the original coordinate y). This allows us to estimate the integral $I_F(\phi)$ and finishes the proof of the lemma. □

We will use the bound described in the lemma in order to estimate the integral $H_{\lambda,a}$ as defined in (A.4). Clearly we can reduce the integral $H_{\lambda,a}$ to the standard Beta integral $H_{\lambda,\sigma,\sigma'}$. Namely,

$$H_{\lambda,a}(\psi) = \int_{\mathbb{R}} |x-a|^{\sigma+\lambda}|x+a|^{\sigma'+\lambda}\psi(x)dx = |a|^{\sigma+\sigma'-1+2\lambda}\int_{\mathbb{R}} |y-1|^{\sigma+\lambda}|y+1|^{\sigma'+\lambda}\psi(ay)dy \ .$$

Let $H^0_{\lambda,a}(\psi)$ be the main term in the asymptotic of $H_{\lambda,a}(\psi)$ which is given by the stationary phase method for $a \neq 0$ fixed. Applying the above lemma to the last integral we obtain the following bound.

Corollary. *Let ψ be a compactly supported smooth function. There are constants $C_3, C_4 > 0$, depending on ψ such that the bound*

$$\left|H_{\lambda,a}(\psi) - H^0_{\lambda,a}(\psi)\right| \leq C_3||\psi||_{C^1([-0.9,0.9])}\cdot|\lambda|^{-\frac{3}{2}} + C_4|\ln(a)|\cdot|\lambda|^{-2} \tag{A.8}$$

holds for all $|\lambda| \geq \mathcal{S}$ and $a \in (0,\ 0.1]$.

We have $H^0_{\lambda,a}(\psi) = |a|^{\sigma+\sigma'-1+2\lambda}\alpha|\lambda|^{-\frac{1}{2}}\psi(0)$. Note that we assumed that $Re(\sigma+\sigma'-1+2\lambda) = 0$ and hence $|H^0_{\lambda,a}(\psi)| = |\alpha\psi(0)|\cdot|\lambda|^{-\frac{1}{2}}$.

Proof. Let $supp(\psi) \subset [-A,\ A]$ and denote by $\psi^a(y) = \psi(ay)$. We note that $\sup|\xi^i(\psi^a)| \leq \sup|\xi^i(\psi)|$ for any $a \in (0,\ 0.1]$. Hence we have

$$\begin{aligned}|RH(\psi^a)| &\leq C_1|\lambda|^{-n}\sum_i\int|F|\xi^i(\psi(ay)u(y))||\omega| \\ &\leq C_2|\lambda|^{-n}\sum_i\int_{\frac{1}{2}}^{a^{-1}A}|F||\omega| \leq C_3|\lambda|^{-n}|\ln(a)| \ ,\end{aligned}$$

for any n and for some explicit constants $C_{1,2,3}$ depending on derivatives of ψ. Here we use the fact that $|F|$ is bounded as $y \to \pm\infty$ and that $\omega = dy/y$. ◻

A.3.1. *Proof of Lemma A.2.1.* The proof is based on the theory of normal forms of differentiable functions and on Hadamard's lemma (see [Ar], [Ma]).

Consider a smooth family of functions $f(t,c) = h(t-c)h(t+c) = (t^2-c^2)g(t-c)g(t+c)$, where we view t as a variable and c as a parameter. For $c = 0$ the function $f(t,0)$ is equivalent (under a smooth change of variable t) to the function t^2. The theory of versal deformations then implies that there is a change of variable $x = x(t,c)$ such that $f(x,c) = u(c) + x^2$ for some smooth function u (see [Ar]). On the other hand, the differential of $f(t,c)$ with respect to c vanishes for all t and $f(0,c) < 0$. This implies that we can write $u(c) = -c^2\tilde{u}^2(c)$ with $\tilde{u}(c) > 0$. Hence there exists a new parameter $a = a(c)$ such that $f(x,a) = x^2 - a^2 = (x-a)(x+a)$.

By Hadamard's lemma (see [Ma]) $h(t-c)$ is divisible by $(x-a)$ since these functions have the same zeroes (one of the branches of zero set for the function $f(x,a) = x^2 - a^2$).

Hence we can write $h(t-c) = (x-a)g_1(x,a)$. It is clear that g_1 is invertible near 0. Similarly for the function $h(t+c)$. □

Appendix B. Integration by parts and Van der Corput lemma

B.1. Integration by parts. We want to study integrals of the form

$$I_F(\phi) = \int_{\mathbb{R}} F(y;\lambda,r)\phi(y)\omega \ , \tag{B.1}$$

where ω is a one-form in y, F is a certain kernel depending on a large parameter $\lambda \in i\mathbb{R}$ and on some additional (multi)parameter $r \in \mathbb{R}^m$. We would like to obtain estimates of I_F for $|\lambda| \to \infty$. We are interested in uniform in r estimates given in terms of C^k-norms of the function ϕ (i.e., we want to estimate a C^k-norm of the functional I_F). We have the following elementary method based on the integration by parts.

First we note that there is a trivial estimate by the absolute value: $|I_F(\phi)| \le R_F(\phi)$, where $R_F(\phi) = \int_{\mathbb{R}} |F(y;\lambda,r)\phi(y)||\omega|$. We use the integration by parts to bootstrap this estimate.

Let ξ be a vector field such that

i: $H \cdot \xi(F) = \lambda \cdot F$, where H is a smooth in all parameters function such that
ii: for some $n > 0$, absolute values of functions H, $\xi H, \dots, \xi^n H$ are bounded by a constant $C > 0$, uniformly in all parameters,
iii: $\xi\omega = 0$.

Proposition. *For ξ and H as above, we have the following bound*

$$|I_F(\phi)| \le |\lambda|^{-n} \cdot C^n \sum_{i=0}^{n} R_F(\xi^i\phi) \ . \tag{B.2}$$

Proof. We have the following functional equation

$$I_F(\phi) = -\lambda^{-1} \cdot I_F(\xi(H\phi)) \ . \tag{B.3}$$

Indeed, we have

$$I_F(\xi(H\phi)) = \int F \cdot \xi(H\phi)\omega = -\int \xi(F)H\phi \ \omega = -\lambda \int F\phi\omega = -\lambda I_F(\phi) \ .$$

Iterating this we obtain $I_F(\phi) = (-1)^n|\lambda|^n I_F(D^n(\phi))$, where $D(\phi) = \xi(H\phi)$. Clearly we have $D^n(\phi) = \sum_{0\le i_0,\dots,i_{n+1}\le n} [H^{i_0} \cdot (\xi(H))^{i_1} \cdot (\xi^2(H))^{i_2} \dots (\xi^n(H))^{i_n}] \cdot \xi^{i_{n+1}}(\phi)$, where the summation is over an appropriate set of indexes. Hence we arrive at the desired bound

$$|I_F(\phi)| \le |\lambda|^{-n} \cdot C^n \sum_{i=0}^{n} \int |F||\xi^i(\phi)||\omega| \ . \tag{B.4}$$

□

Let ϕ and f be real valued and smooth in the interval $[a,b]$ functions. Consider the following integral

$$I(\phi,f)=\int_a^b e^{if(x)}\phi(x)dx\ . \tag{B.5}$$

We first consider the special case of $f=t\alpha$, where $t>1$ is a parameter and α such that α' has no zeroes on the support $supp(\phi)\Subset(a,b)$ of ϕ. The bound (B.2) implies the following

Corollary. *The following bound holds*

$$|I(\phi,t\alpha)|\leq C_Nt^{-N} \tag{B.6}$$

for any $N>0$ and a constant C_N depending on α and ϕ.

B.2. Van der Corput lemma. Let $I(\phi,f)$ be as in (B.5). Consider the case when f' has zeroes. For an integer $k\geq 1$ denote by $m_k(f)=\min\limits_{x\in[a,b]}|f^{(k)}(x)|$ and let $M(\phi)=|\phi(b)|+\int_a^b|\phi'(x)|dx$ be the variance of ϕ. We have the following general estimate essentially due to van der Corput (see [St, p. 332]).

Lemma. *Let $k\geq 1$ be such that $m_k(f)>0$. There exists a constant c_k such that the following bound holds*

$$|I(\phi,f)|\leq c_k\cdot m_k(f)^{-\frac{1}{k}}\cdot M(\phi)$$

provided

(1) $k\geq 2$, *or*
(2) $k=1$ *and f' is monotone on $[a,b]$.*

The constant c_k depends only on k and is independent of ϕ, f and of the interval $[a,b]$.

We use this lemma with $k=1$ or 2, so we can assume that c_k is a universal constant.

B.3. Throughout the paper we consider integrals of the form $\int u(x)|x|^{-it}e^{is\cdot g(x)}dx$. In this section we explain how to obtain meaningful upper bounds for these integrals. We claim that the necessary type of bounds follow directly from the integration by parts and from the van der Corput lemma.

Let

$$I(s,t)=\int_{-1}^1 u(x)|x|^{-\frac{1}{2}-it}e^{is\cdot g(x)}dx\ , \tag{B.7}$$

where we assume that $1\leq t\leq s$, g is smooth and monotonic, $0.99<g'(x)<1.01$ (i.e., bounded away from 0 and ∞), $|g^{''}(x)|\leq\frac{1}{2}$ for all x (this insures that there is no degenerate critical points of the phase), and u is smooth of compact support in $(-1,1)$.

There is a simple bound if the phase has no critical points. Let us denote by b the ratio $b = s/t$. Integration by parts shows that if the phase function $bg(x) - \ln|x|$ in the integral (B.7) has no critical points (e.g., $|t| \gg |s|$) then the bound (B.6) reads as

$$|I(s,t)| \leq C_N |t|^{-N} \tag{B.8}$$

for any $N > 0$ and some constant C_N depending on N, u and g.

In the complementary situation we have

Lemma. *Under the above assumptions on g, the following uniform bound holds*

$$|I(s,t)| \leq B s^{-\frac{1}{2}} \ ,$$

where the constant B is independent of s and of t.

Proof. We denote by a the ratio $a = t/s$ and consider the integral over the interval $(0,1)$ (and the similar integral over $(-1,0)$)

$$I(s,a) = \int_0^1 u(x)|x|^{-\frac{1}{2}} e^{is(g(x) - a\ln|x|)} dx \ .$$

We are interested in the uniform (in s) bound for this integral for the values of the parameter a satisfying the bound $s^{-1} \leq a \leq 1$.

In order to apply the van der Corput lemma, we break the interval $(0,1)$ into 4 intervals $J_1 = (2a, 1)$, $J_2 = (\frac{1}{2}a, 2a)$, $J_3 = (\frac{1}{2}s^{-1}, \frac{1}{2}a)$ and $J_4 = (0, \frac{1}{2}s^{-1})$ (for $a \geq \frac{1}{2}$ the first interval is missing). Denote by $f_a(x) = g(x) - a\ln|x|$, $\phi(x) = u(x)|x|^{-\frac{1}{2}}$ and consider the corresponding integrals $I_j(s,a) = \int_{J_j} u(x)|x|^{-\frac{1}{2}} e^{isf_a(x)} dx$.

On the interval J_1 we have $|sf_a'(x)| \geq s$. Hence from the van der Corput lemma (with $k = 1$) we have $|I_1(s,a)| \leq B_1 s^{-1}$.

On the interval J_2 the phase f_a has zero of the first derivative, but satisfies the bound $|sf_a''(x)| > \frac{1}{2}a^{-1}s$ and $M(\phi) \leq 10|a|^{-\frac{1}{2}}$. Hence on the interval J_2 the van der Corput lemma with $k = 2$ implies $|I_2(s,a)| \leq B_2 s^{-\frac{1}{2}}$.

To bound the integral $I_3(s,a)$, we note that $|sf_a'(x)| \geq \frac{1}{2}s$ and that the variation of the amplitude satisfies $M(\phi) \leq |\frac{1}{2}a|^{-\frac{1}{2}} + \int_{\frac{1}{2}s^{-1}}^{\frac{1}{2}a} |x|^{-3/2} dx \leq cs^{\frac{1}{2}}$ on J_3. The van der Corput lemma with $k = 1$ implies that $|I_3(s,a)| \leq B_3 s^{-\frac{1}{2}}$.

Bounding the integral over J_4 by the integral of the absolute value, we see that trivially $|I_4(s,a)| \leq B_4 s^{-\frac{1}{2}}$. □

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