Problem assignment 6

Algebraic Geometry and Commutative Algebra

Joseph Bernstein December 11, 2002

1. Let $X \subset \mathbf{P}^n$ be a closed subset of dimension $d > 0$. Show that for generic hyperplane $H \subset \mathbf{P}^n$ the intersection $Y = X \cap H$ is a variety of dimension $d - 1$.

2. Let $\pi : X \to Y$ be a morphism of algebraic varieties. Suppose that we know that all fibers of π are finite. Show that $\dim X \leq \dim Y$.

Definition. Let X be an affine algebraic variety and $A = \mathcal{O}(X)$. For every point $x \in X$ we define the **fiber** $F|_x$ of F at the point x to be the vector space corresponding to the sheaf $\nu^*(\mathcal{F})$, where $\nu : pt \to X$ is the morphism corresponding to x and $\mathcal{F}$ is the localized sheaf of $\mathcal{O}_X$-modules corresponding to F

3. (i) Show that $F|_x = F/\mathfrak{m}_x F$, where $\mathfrak{m}_x$ is the maximal ideal of the point x.

(ii) Show that if F is finitely generated then the set of points $x \in X$ where F has nonzero fibers coincides with the support of the corresponding sheaf $\mathcal{F}$.

Show that this is not necessarily true if F is not finitely generated.

4. Let $\mathcal{F}$ be a coherent $\mathcal{O}$-module on X.

(i) Show that the function $x \mapsto d(x) = \dim \mathcal{F}|_x$ on X is lower semi continuous. Show that it is constructible.

(ii) Show that the sheaf $\mathcal{F}$ is locally free iff the function $d(x)$ is locally constant.

5. Let A be a commutative algebra, I, J ideals in A. Consider A-modules $M = A/I$ and $N = A/J$.

Compute (describe) tensor product $M \otimes_A N$.

6. Let $\mathbf{C}$ be complex numbers with the usual topology, $\mathbf{C}^* = \mathbf{C} \setminus 0$.

(i) Let $\mathcal{O}$ be the sheaf of holomorphic functions on $\mathbf{C}^*$. Describe its stalk at 1.

(ii) Let $j : \mathbf{C}^* \to \mathbf{C}$ be the natural open imbedding. Consider the sheaf $j_*(\mathcal{O})$ on $\mathbf{C}$ and describe its stalks at 1 and 0.

7. In problem 6 let $d : \mathcal{O} \to \mathcal{O}$ be a morphism of sheaves given by derivative with respect to parameter z. Fix a complex number λ and set $a = (zd - \lambda) : \mathcal{O} \to \mathcal{O}$. Denote by K and C the kernel and cokernel of morphism a.

(i)Describe the stalks of sheaves K and C.

(ii) Describe the stalks at 0 of the sheaves $j_*(K)$, $j_*(C)$.