Problem assignment 3.

Introduction to Differential Geometry.

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[P] 1. Taylor expansion. Let $W \subset \mathbf{R}^n$ be a domain. Consider a function f of class C^k on W. Show that for every point $a \in W$ one can find a polynomial function P on $\mathbf{R}^n$ of degree k which approximates f near point a up to order k (this means that the difference $e = f - P$ is $o(||x - a||^k)$.

Show that the polynomial P is uniquely defined. Write a formula for its coefficients.

[P] 2. Cut off functions. (i) Show that there exists a smooth function $u(t)$ in one variable such that $u(t) = 0$ for $t \leq 0$ and $u(t) > 0$ for $t > 0$. (**Hint.** Take $u(t) = \exp(-t^{-2})$ for $t > 0$).

(ii) Show that there exists a smooth non-negative function $v(t)$ which vanishes for $|t| > 1$ such that $v(0) = 1$.

(iii) Show that there exists a smooth non-negative monotone function $w(t)$ such that $w(t) = 0$ for $t < 0$ and $w(t) = 1$ for $t > 1$.

(iv) Construct a smooth non-negative function $y(t)$ such that $y(t) = 0$ for $|t| > 1$ and $y(t) = 1$ for $|t| < 0.99$.

[P] 3. Extension of smooth functions. (i) Let f be a smooth function on the ball $B(a, 1)$ of radius 1 in $\mathbf{R}^n$. Show that f can be extended from a smaller ball $B(a, 0.9)$ to the whole space $\mathbf{R}^n$ as a smooth function.

(ii) Consider a domain U and its open subdomain W. Suppose we are given a smooth function f defined on W. Show that for every compact subset $C \subset W$ we can find a smooth function h on U which coincides with f on some neighborhood of C.

4. Let u be the function from problem 2(i). Construct the functions h, f

$h(x, y) = u(3x^2 - y) \cdot u(y - x^2)$ and $f(x, y) = h(x, y)/h(x, 2x^2)$, $f(0, 0) = 0$.

Show that the restriction of the function f to any straight line is smooth but f is not a continuous function on $\mathbf{R}^2$.

[P] 5. Let f be a smooth function on a domain $U \subset \mathbf{R}^n$.

(i) Show that we can write $f = f(0) + \sum g_i \cdot x_i$ where x_i are coordinate functions on $\mathbf{R}^n$ and g_i are smooth functions on U. (First consider the case when U is a ball B centered at 0).

(ii) Show how to compute the values $g_i(0)$.

6. Prove the following statement. Let $f = f(x, y)$ be a function on a domain $U \subset \mathbf{R}^2$. Suppose we know that the mixed partial derivatives f_{xy} and f_{yx} exist everywhere in U and are continuous at a point $a \in U$. Then they are equal at this point. (Look for hints on the website of the course).

[P] 7. Let $p : U \to W$ be a morphism (i.e. a smooth map) of domains, $a \in U$ and $b = p(a) \in W$. Suppose that p has a section (i.e. a morphism $s : W \to U$ such that $p \circ s = Id$) such that $s(b) = a$.

Show that locally near points a and b we can choose coordinates x_i on U and y_j on W in which morphisms p and s have the standard form

$p(x_1, ..., x_n) = (x_1, ..., x_m)$ and $s(y_1, ..., y_m) = (y_1, ..., y_m, 0, ..., 0)$.

[P] 8. Consider a smooth function $f(x, y)$ defined in a neighborhood of a point $a = (0, 0)$. Suppose that $f_x(a) = 0$ and $f_{xx}(a) = 1$.

Show that if we choose a small neighborhood U of a then for small y the function $u(y) = \min\{f(x, y)|(x, y) \in U\}$ is a smooth function.

Can you compute the derivative of u at the point $y = 0$?