**Problem assignment 4.**

**Introduction to Differential Geometry.**

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**[P] 1.** Let $\pi : X \to Y$ be a morphism of domains, $a$ a point of $X$ and $b = \pi(x) \in Y$. Consider the differential $T = D\pi : T_a X \to T_b Y$.

(i) Suppose $T$ is invertible. Show that $\pi$ is a local diffeomorphism (i.e. there exist local coordinate systems $(x^1, ..., x^n)$ on $X$ and $(y^1, ..., y^n)$ on $Y$ such that $\pi^*(y^i) = x^i$).

(ii) Suppose $T$ is imbedding. Show that locally $\pi$ is isomorphic to the standard linear imbedding (in coordinates this means $\pi^*(y^i) = x^i$ for $i = 1, ..., n = \dim X$, and $\pi^*(y^i) = 0$ for $n < i \leq m = \dim Y$).

(iii) Suppose $T$ is onto. Show that locally $\pi$ is isomorphic to the standard linear projection (in coordinates $\pi^*(y^i) = x^i$).

(iv) Suppose that the differential $D\pi$ has constant rank $k$ at all points of $X$ close to $a$. Show that then $\pi$ is locally isomorphic to a linear morphism (in coordinates $\pi^*(y^i) = x^i$ for $i = 1, ..., k$ and $\pi^*(y^i) = 0$ for $i > k$).

**[P] 2.** Consider Euclidean space $V = \mathbf{R}^3$ with coordinates $(x, y, z)$. Denote by $S$ the unit sphere in $V$ defined by equation $x^2 + y^2 + z^2 = 1$. Let B denote the Riemannian metric on $S$ induced by the standard metric on $V$.

Let $N = (0, 0, 1) \in S$ be the north pole. Consider functions $(x, y)$ as coordinate system on $S$ near point $N$.

(i) Write down the metric $B$ in these coordinates (**Hint**. Write $B$ as $dx^2 + dy^2 + dz^2$).

(ii) Compute the gradient $\mathrm{grad}(x)$ of the function $x$ with respect to the Riemannian metric $B$. Compute $\mathrm{grad}(y)$.

(iii) Let $f$ be a smooth function on $S$. Using coordinate system $(x, y)$ write down formula for computing the differential $df$ and the gradient $\mathrm{grad}(f)$.

**[P] 3.** Let $c(t)$ be a parameterized curve in $\mathbf{R}^n$. Introduce the natural parameter $s$ (which is called length) on this curve so that $ds/dt = ||dc(t)/dt||$ and consider the curve $c(s)$ parameterized by $s$. By definition the vector $v(s) = dc/ds$ has length 1 at every point $s$.

Show that the vector of derivative $dv(s)/ds$ is orthogonal to $v$ at every point.

**[P] 4.** Consider 1-form $\alpha = ydx + xdy + zdz$ on $\mathbf{R}^3$.

Compute the integral of $\alpha$ along the curve $\Gamma$ given by $c(t) = (\sin t, t^3, t^2)$ where $0 \leq t \leq 1$.

**[P] 5.** Consider the unit circle $\Gamma \subset \mathbf{R}^2$ and choose anti-clockwise orientation on it. Compute the integral $\oint_\Gamma \alpha$, where $\alpha = ydx \in \Omega^1(\mathbf{R}^2)$.

**[P] 6.** Let $A$ be an associative algebra (for example, algebra of endomorphisms of some linear space $V$). Let us define a new operation $[\ ,\ ]$ on $A$ by $[a, b] = ab - ba$. It is clear that $[,]$ is a bilinear skew-symmetric operation.

(i) Show that $[,]$ satisfies Jacobi identity

$[a, [b, c]] + [c, [a, b]] + [b, [c, a]] = 0.$

Show that for any $a \in A$ the operator $\partial = Ad_a : A \to A$ defined by $Ad_a(x) = [a, x]$ is a derivation for both multiplication $(a, b) \mapsto ab$ and commutator operation $(a, b) \mapsto [a, b]$.

**[P] 7.** Let $\xi, \eta$ be two vector fields on a domain $U$. Let us choose a coordinate system $(x^i)$ on $U$ and write these fields in these coordinates $\xi = \sum u^i \partial_i, \eta = \sum v^j \partial_j$.

Write explicit formula for the commutator vector field $[\xi, \eta]$.

**8.** Let $V$ be a finite dimensional vector space, $B$ a symmetric bilinear form on $V$. Denote by $\nu : V \to V^*$ the corresponding operator.

Choose a basis $(e_i)$ of $V$ and denote by $(f^i)$ the dual basis of $V^*$.

Show that the matrix of the operator $\nu$ with respect to these bases coincides with the matrix $M_{ij} = B(e_i, e_j)$ of the form $B$.

Suppose that the form $B$ is nondegenerate and denote by $B^*$ the corresponding form on $V^*$. Show that the matrix $M^*$ of the form $B^*$ is the inverse matrix to $M$.

($\Box$)**9.** Let $U$ be a domain, $A = S(U)$ the algebra of smooth functions on $U$. Every function $f \in A$ defines a linear operator $\bar{f} : A \to A$ given by multiplication by $f$.

Let $End(A)$ be the algebra of all linear operators $T : A \to A$. We define subspaces $D^{-1} \subset D^0 \subset D^1 \subset ... \subset D^k \subset ... \subset End(A)$ using inductive procedure

$D^{-1} = 0$, $D^k = \{T \in End(A) | [\bar{f}, T] \in D^{k-1}$ for all $f \in A\}$.

The space $D^k$ is called the space of **differential operators of degree** $\leq k$. The space $D = D(U) := \bigcup D^k$ is called the space of **differential operators** on $U$.

(i) Show that $D^k \cdot D^l \subset D^{k+l}$. In particular, this shows that the space of differential operators is an algebra.

(ii) Show that $D^0$ consists of operators $\bar{f}$ for $f \in A$; in other words $D^0 = S(U)$.

(iii) Show that $D^1 = S(U) \oplus Vect(U)$.

(iv) Introduce a coordinate system $(x^1, ..., x^n)$ on $U$. Show that every differential operator T can be uniquely written in the form $T = \sum f_\alpha \partial^\alpha$ where the sum is taken over all multi indexes $\alpha = (i_1, ..., i_n)$, $f_\alpha \in S(U)$ and $\partial^\alpha := \partial_1^{i_1} \cdot ... \cdot \partial_n^{i_n}$.